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PHYSICS FOR JEE (MAIN & ADVANCED)

Volume II

THIRD EDITION

MANISH K. SINGHAL

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Physics for JEE (Main & Advanced)

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Adapted by

Manish K. Singhal

WILEY

HALLIDAY / RESNICK / WALKER

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Dedication

This book is dedicated to my loving teacher,
Prof. P.V. Krishnan (retired professor, IIT Delhi), and his wife,
mother ***Krishangi Devi***, whose teachings gave me life.
Whatever I am today, I owe it to them.

Preface

Having experienced teaching at various levels and with a varied group of students, I have experienced the shortcomings faced by most of the students. I can also remember my own experiences of failure and dejection when I used to prepare for competitive exams. Often, I did not find answers to my questions, or I could not find a book that could help me think, nor did I find a collection of problems that stimulate the mind. In this book, finally, I have come to fulfill a vision of a book that has *an international collection of problems, the theory to such depth as we cover at Allen Career Institute, Kota, and international look and feel*. I am happy to share the success of Indian students at International Physics Olympiad (2018) where all the students won gold medal. Three of them (Bhaskar Gupta, Lay Jain, and Nishant Abhangi) happened to be my students. So, you can be confident that this book has been tried, tested, and is successful even for International Physics Olympiad. However, this book is not a substitute for a teacher. I strictly believe in this philosophy and this has been proven over the years: To learn, you need a teacher but learning under a qualified teacher will help you understand this book much better.

Often students find Physics tough or at least challenging. Perhaps it is both because it tries to encourage you to think and it does not follow any stereotyped pattern. A thinking individual will do well in physics. In the modern learning atmosphere, perhaps we are never even encouraged to think. I have made an attempt to cross this barrier and challenge you to think at numerous places. For this purpose, I have scattered numerous checkpoints throughout the chapters.

Those of you who are accustomed to the original Halliday, Resnick, and Walker would be in for a surprise because although this is supposed to be an adaptation, I have freely made the changes as per the needs of the students who are appearing in competitive examinations like Physics Olympiads, JEE Advanced, and so on. So, the net result is that you have an altogether new book, my style. Three new chapters have been added and many have been deleted. All over the world, slowly objective type examinations are replacing subjective papers. So, we have added extensive collection of objective problems on each and every concept covered in chapter.

The book follows a particular format. I introduce a concept and try to illustrate it by the means of a sample problem.

- **Key Idea** explains the problem with a clue to the solution.
- **Reasoning** contains details of the principles of physics used to solve the problem.
- **Calculations** gives the detailed solution.
- The real punch lies in the **Learn** section where many a times the student has to think on the lines of great physicists.

In the exercises that follow at the end of each chapter, I am trying to take the students through the same ideas. The spirit of the book will be marred if the student just focuses on the theory and does not try out the exercises that follow. I have tried to arrange the exercises in increasing order of difficulty. The answers are given at the end of the chapter. If one is not able to solve a particular question, then it will do him good to go back to the illustration and try it out himself.

How to Approach the Question Bank

To gain the most, I would suggest the following sequence:

- **Step 1:** First, go through the text of a section thoroughly with the help of a qualified teacher. Try solving the illustrations yourself and feel the joy of discovering physics.

- **Step 2:** Then go through the subjective exercises pertaining to that section.
- **Step 3:** At the end of that, when you are confident of the basic concepts of the entire chapter, there are practice questions of various types given at the end of problems section.

These problems take you to the realm of JEE, Physics Olympiad, and even beyond. Do not feel discouraged if you have not been able to solve the problems at the first go. Remember, even Einstein and Newton had faced failures in their lives. Failure is a stepping stone to success. I wish you all the best in your endeavors.

My team and I have tried our level best to offer you an error-free book; went through the whole manuscript three to four times. All the questions that have been set in the problems section has been solved twice, once by me personally and then by my colleague. Yet, it may be possible that errors may remain in the book. Comments or suggestions on the content and presentation of the book will be received with great interest and highly appreciated.

Manish Kumar Singhal
Email: singhalmks@gmail.com

Acknowledgments

I have dedicated this book to my loving teacher Prof. P.V. Krishnan and his wife, mother Krishangi Devi. Truly, if they did not give me a spiritual vision of life, I would never even have the boldness of taking up teaching as a profession, much less to venture writing this book. He taught me the real meaning of Science and opened my eyes to a Science which is much more profound and vast than what we are studying here. I encourage all my readers to study his books, “Science and Nescience” and “Wake up! Intellectuals.” These books gave me an entirely new perspective to life, I am sure, they will change yours too.

This book was made possible only by an extensive help rendered by a big group of friends. The leader among them was my friend and colleague Ajit Aggarwal who laboriously went through the entire book and solved each problem personally in detail. I am, indeed, very grateful to Dr. Jearl Walker for allowing me to edit the book the way I wanted.

Thanks!

Thanks are also due to Amit Gupta, author of Vol. I of this book, who introduced me to the idea. Prominent among those who helped me with the books are Kishore Rane of Jaya Vishwambhara Classes, Anantpur; Anurag Mishra, Allen Career Institute, Bhubaneshwar; and Ambarish Srivastava, FIITJEE, Faridabad. The kind of open heartedness which these three individuals showed in sharing their collection of books with me is a rare example in this age of hypocrisy and cynicism.

I am also deeply indebted to Mr. Rajesh Maheshwari of Allen Career Institute, Kota, who gave us such an open learning atmosphere at Kota. His dedication to teaching excellence is something which is unparalleled and would be hard to beat in years to come. The editorial team at Wiley headed by Anjali Chadha, working on the project deserves a gold medal for keeping up with my vision of giving the students gem of a book. My typist Prashant Gupta deserves a special applause for making figures that match the quality of the original book. Special thanks are due to my students of today and past whose thought-provoking questions helped me to explore Physics deeper and deeper. Last, but not the least, I owe a special thanks to Vikas Gupta, MD, Wiley India for putting so much faith on me and risking a big investment for bringing out such a beautiful textbook in India and that too at Indian prices! This project would have never been successful without these people supporting me.

Manish Kumar Singhal

Note to the Student

The new pattern Joint Entrance Examination (JEE) in Engineering was implemented in 2013. It is now held in two parts—JEE (Main) and JEE (Advanced). Hitherto AIEEE is now called JEE (Main) and IIT-JEE is now called JEE (Advanced). After qualifying JEE Main and meeting other specified criteria, a student is eligible to take admission in NITs, IIITs, Centrally funded technical institutions and participating State Government institutions. The 1,50,000 top rankers (including students from all categories) in JEE (Main) are eligible to appear in JEE (Advanced) for admission to the IITs across the country and ISM, Dhanbad.

In this constantly evolving pattern of the entrance examination, your preparation should be pattern-proof. It should not matter to you if the questions asked are Objective Type or Integer Type or Matrix-Match Type. What matters only is the proper understanding of the fundamentals of the subjects. These should be strong enough to handle any type of question.

This is where Resnick, Halliday, and Walker's *Principles of Physics* comes in. This book has been the definitive text for learning Physics concepts and solving problems conceptually for the last four decades. Its unparalleled approach to teaching physics has been appreciated globally by students and teachers alike. And that is why it is probably the best resource for an IIT aspirant like you today.

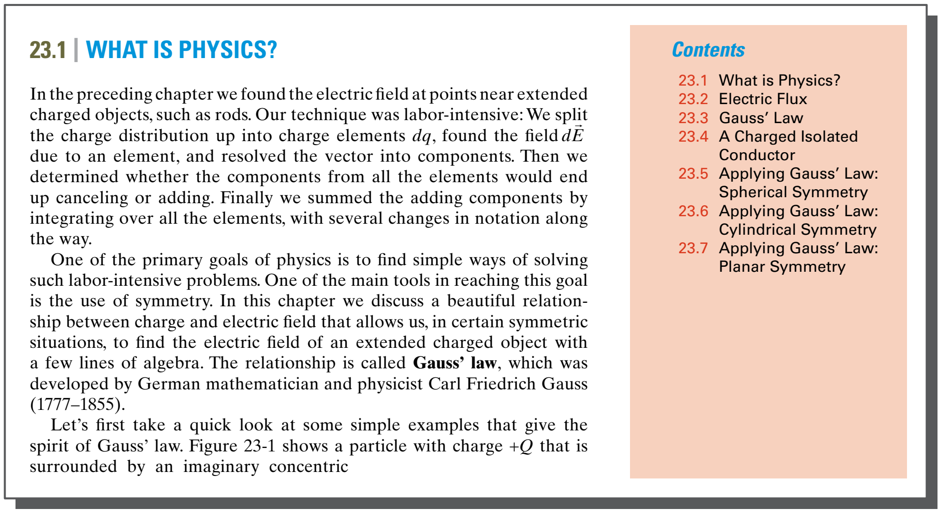
In collaboration with experts from JEE coaching, the original Resnick, Halliday, and Walker's *Principles of Physics* tenth edition, has been customized to give you the tools that will help you crack the toughest entrance exam in India.

We will walk through some of the special book features that will help you derive full advantage of the unique features and elements of this textbook.

A. PEDAGOGY

CHAPTER OPENER

All chapters start with "What is Physics?". This feature sets the context for the physics concepts that will be discussed in the chapter. The brief contents of the chapter are also provided on the opening page for ready reference.



23.1 | WHAT IS PHYSICS?

In the preceding chapter we found the electric field at points near extended charged objects, such as rods. Our technique was labor-intensive: We split the charge distribution up into charge elements dq, found the field $d\vec{E}$ due to an element, and resolved the vector into components. Then we determined whether the components from all the elements would end up canceling or adding. Finally we summed the adding components by integrating over all the elements, with several changes in notation along the way.

One of the primary goals of physics is to find simple ways of solving such labor-intensive problems. One of the main tools in reaching this goal is the use of symmetry. In this chapter we discuss a beautiful relationship between charge and electric field that allows us, in certain symmetric situations, to find the electric field of an extended charged object with a few lines of algebra. The relationship is called **Gauss' law**, which was developed by German mathematician and physicist Carl Friedrich Gauss (1777–1855).

Let's first take a quick look at some simple examples that give the spirit of Gauss' law. Figure 23-1 shows a particle with charge $+Q$ that is surrounded by an imaginary concentric

Contents

23.1 What is Physics?
23.2 Electric Flux
23.3 Gauss' Law
23.4 A Charged Isolated Conductor
23.5 Applying Gauss' Law: Spherical Symmetry
23.6 Applying Gauss' Law: Cylindrical Symmetry
23.7 Applying Gauss' Law: Planar Symmetry

CONCEPT EXPLANATION

Each section starts with clearly defined **Key Concepts** followed by their detailed explanation in the text. The inimitable style of Halliday, Resnick and Walker is obvious in the conceptual strength and real-life examples of the book.

Key Concepts

- The electric flux Φ through a surface is the amount of electric field that pierces the surface.
- The area vector $d\vec{A}$ for an area element (patch element) on a surface is a vector that is perpendicular to the element and has a magnitude equal to the area dA of the element.
- The electric flux $d\Phi$ through a patch element with area vector $d\vec{A}$ is given by a dot product:

$$d\Phi = \vec{E}\cdot d\vec{A}.$$

- The total flux through a surface is given by

$$\Phi = \int \vec{E}\cdot d\vec{A} \quad \text{(total flux)},$$

where the integration is carried out over the surface.

- The net flux through a closed surface (which is used in Gauss' law) is given by

$$\Phi = \int \vec{E}\cdot d\vec{A} \quad \text{(total flux)},$$

where the integration is carried out over the entire surface.

IMPORTANT POINTS

An excess charge placed on an isolated conductor will distribute itself on the surface of that conductor so that all points of the conductor—whether on the surface or inside—come to the same potential. This is true even if the conductor has an internal cavity and even if that cavity contains a net charge.

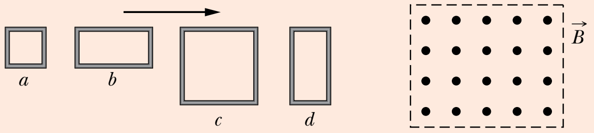
Throughout the book, important tips are given in a box marked with a **star**. This technique highlights the points and also aids retention.

CHECKPOINTS

Reasoning-based conceptual and numerical questions after each section in the chapter that help check your understanding of what has been covered till then.

CHECKPOINT 4

The figure shows four wire loops, with edge lengths of either L or $2L$. All four loops will move through a region of uniform magnetic field $\vec{B}$ (directed out of the page) at the same constant velocity. Rank the four loops according to the maximum magnitude of the emf induced as they move through the field, greatest first.



PROBLEM-SOLVING TACTICS

Tactic 4: Signs of Trouble with Electric Potential When you calculate the potential V at some point P due to a line of charge or any other continuous charge configuration, the signs can cause you trouble. Here is a generic guide to sort out the signs.

If the charge is negative, should the symbols dq and λ represent negative quantities, or should you explicitly show the signs, using $-dq$ and $-\lambda$? You can do either as long as you remember what your notation means, so that when you get to the final step, you can correctly interpret the sign of V.

Another approach, which can be used when the entire charge distribution is of a single sign, is to let the symbols dq and λ represent magnitudes only. The result of the calculation will give you the magnitude of V at P. Then add a sign to V based on the sign of the charge. (If the zero potential is at infinity, positive charge gives a positive potential and negative charge gives a negative potential.)

If you happen to reverse the limits on the integral used to calculate a potential, you will obtain a negative value for V. The magnitude will be correct, but discard the minus sign. Then determine the proper sign for V from the sign of the charge. As an example, we would have obtained a minus sign in Eq. 24-33 if we had reversed the limits in the integral above that equation. We would then have discarded that minus sign and noted that the potential is positive because the charge producing it is positive.

PROBLEM-SOLVING TACTICS

These contain helpful instructions to guide the beginning physics student as to how to approach problem-solving and avoid common errors.

B. SAMPLE PROBLEMS

A thoughtful selection of problems that clearly link the key concepts of the section to their application in problem-solving. The solutions are structured to inculcate reasoning, recall, and retention skills in students.

SAMPLE PROBLEM 24.03

Locating equipotential surfaces

An infinite nonconducting sheet has a surface charge density of $\sigma = 0.10\ \mu\text{C/m}^2$ on one side. How far apart are equipotential surfaces whose potentials differ by 50 V?

KEY IDEA

The electric field produced by an infinite sheet of charge is normal to the sheet and is uniform.

The magnitude of the electric field produced by the infinite sheet of charge is $E = \sigma/2\varepsilon_0$, where σ is the surface charge density. Place the origin of a coordinate system at the sheet and take the x axis to be parallel to the field and positive in the direction of the field. Then the electric potential is

$$V = V_s - \int_0^x E\,dx = V_s - Ex,$$

where V_s is the potential at the sheet. The equipotential surfaces are surfaces of constant x; that is, they are planes that are parallel to the plane of charge. If two surfaces are separated by Δx then their potentials differ in magnitude by

$$\Delta V = E\Delta x = (\sigma/2\varepsilon_0)\Delta x.$$

Calculation: Thus, for $\sigma = 0.10 \times 10^{-6}\ \text{C/m}^2$ and $\Delta V = 50\ V$, we have

$$\Delta x = \frac{2\varepsilon_0 \Delta V}{\sigma} = \frac{2\left(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(50\ \text{V})}{0.10 \times 10^{-6}\ \text{C/m}^2} = 8.8 \times 10^{-3}\ \text{m}.$$

Learn: Equipotential surfaces are always perpendicular to the electric field lines. Figure 24-7(a) depicts the electric field lines and equipotential surfaces for a uniform electric field.

C. ASSESSMENT

The end-of-chapter exercises are designed to be an effective assessment tool for students to evaluate their understanding of the physics concepts and preparedness for the competitive examination. The first section comprises best-in-class application-based physics problems that are part of this legacy textbook. The Practice Questions section is designed, keeping in view the requirements of JEE aspirants.

PROBLEMS

A variety of numerical problems testing different concepts covered within the chapter. All problems are of different levels of difficulty level and have random distribution of concepts. This is done to further sharpen the ability of students to select appropriate concept or formula for application.

PROBLEMS

1. In an oscillating LC circuit with $L = 79$ mH and $C = 4.0\ \mu$F, the current is initially a maximum. How long will it take before the capacitor is fully charged for (a) the first time and (b) the second time?
2. An ac generator with emf $\mathscr{E} = \mathscr{E}_m \sin \omega_d t$, where $\mathscr{E}_m = 25.0$ V and $\omega_d = 377$ rad/s, is connected to a 4.15 μF capacitor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is –12.5 V and increasing in magnitude, what is the current?
3. In Fig. 31-26, a generator with an adjustable frequency of oscillation is connected to resistance $R = 100\ \Omega$, inductances $L_1 = 9.70$ mH and $L_2 = 2.30$ mH, and capacitances $C_1 = 8.40\ \mu$F, $C_2 = 2.50\ \mu$F, and $C_3 = 3.50\ \mu$F. (a) What is the resonant frequency of the circuit? What happens to the resonant frequency if (b) R is increased, (c) L_1 is increased, (d) C_3 is removed from the circuit, and (e) L_2 is removed?

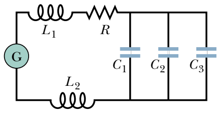
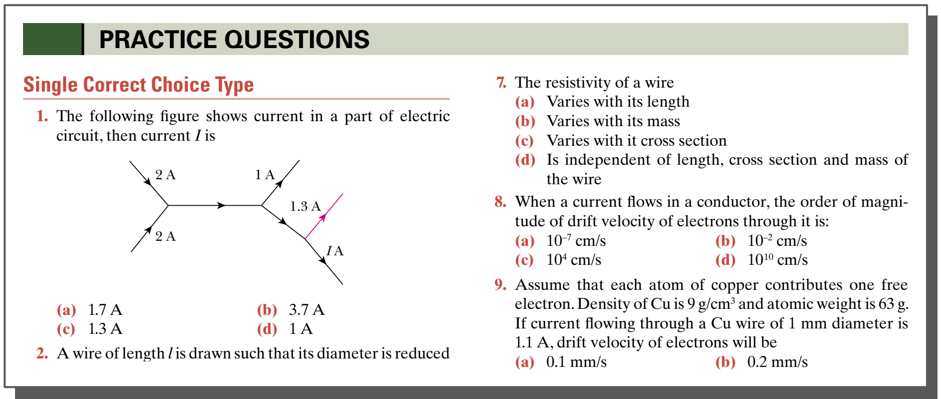


Figure 31-26 Problem 3.

4. In Fig. 31-6, set $R = 400\ \Omega$, $C = 70.0\ \mu$F, $L = 920$ mH, $f_d = 30.0$ Hz, and $\mathscr{E}_m = 72.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.
5. (a) In an RLC circuit, can the amplitude of the voltage across an inductor be greater than the amplitude of the generator emf? (b) Consider an RLC circuit with emf amplitude $\mathscr{E}_m = 10$ V, resistance $R = 5.0\ \Omega$, inductance $L = 1.0$ H, and capacitance $C = 1.0\ \mu$F. Find the amplitude of the voltage across the inductor at resonance.
6. An air conditioner connected to a 125 V rms ac line is equivalent to a 9.20 Ω resistance and a 4.70 Ω inductive reactance in series. Calculate (a) the impedance of the air conditioner and (b) the average rate at which energy is supplied to the appliance.
7. Remove the capacitor from the circuit in Fig. 31-6 and set $R = 400\ \Omega$, $L = 230$ mH, $f_d = 120$ Hz, and $\mathscr{E}_m = 72.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

PRACTICE QUESTIONS

This section is a select yet ample collection of objective type questions, in all formats as asked in the engineering entrance examinations – Single correct choice type, More than one correct choice type, Linked comprehension, Matrix-match (including newly introduced three-column type), and Integer type.



PRACTICE QUESTIONS

Single Correct Choice Type

1. The following figure shows current in a part of electric circuit, then current I is

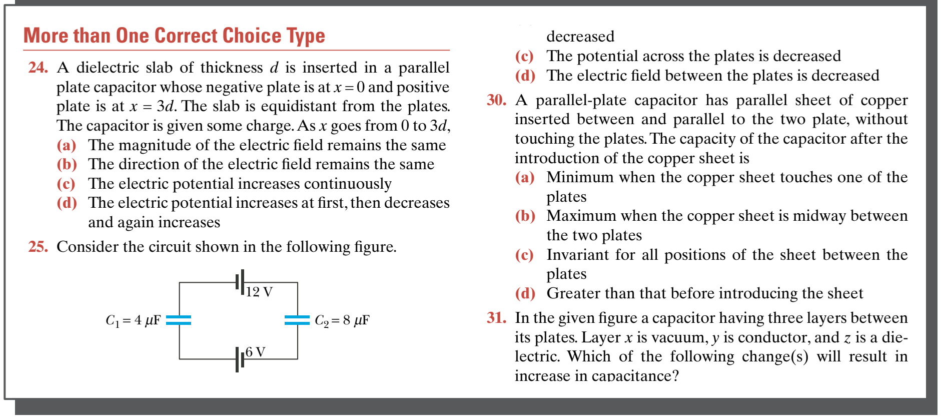
(a) 1.7 A **(b)** 3.7 A
(c) 1.3 A **(d)** 1 A

2. A wire of length l is drawn such that its diameter is reduced

7. The resistivity of a wire
(a) Varies with its length
(b) Varies with its mass
(c) Varies with it cross section
(d) Is independent of length, cross section and mass of the wire

8. When a current flows in a conductor, the order of magnitude of drift velocity of electrons through it is:
(a) 10^{-7} cm/s **(b)** 10^{-2} cm/s
(c) 10^4 cm/s **(d)** 10^{10} cm/s

9. Assume that each atom of copper contributes one free electron. Density of Cu is 9 g/cm^3 and atomic weight is 63 g. If current flowing through a Cu wire of 1 mm diameter is 1.1 A, drift velocity of electrons will be
(a) 0.1 mm/s **(b)** 0.2 mm/s



More than One Correct Choice Type

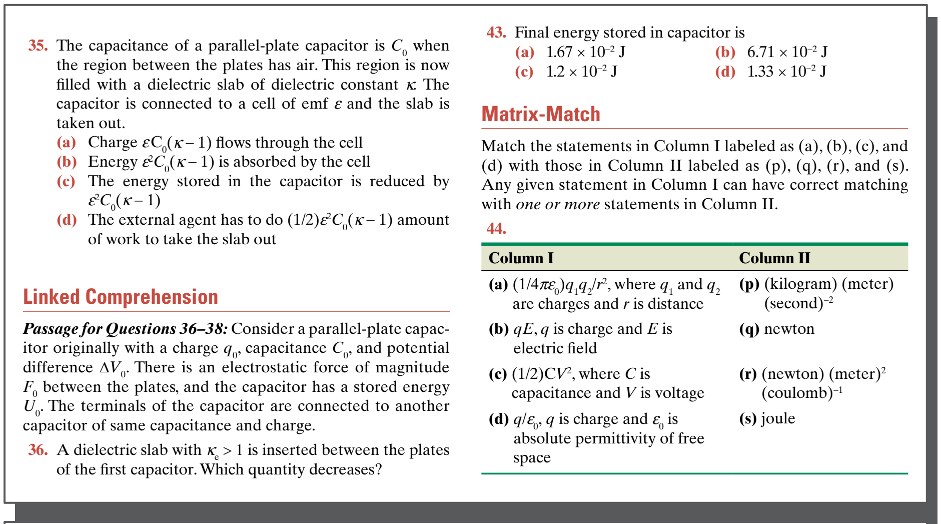
24. A dielectric slab of thickness d is inserted in a parallel plate capacitor whose negative plate is at $x = 0$ and positive plate is at $x = 3d$. The slab is equidistant from the plates. The capacitor is given some charge. As x goes from 0 to $3d$,
(a) The magnitude of the electric field remains the same
(b) The direction of the electric field remains the same
(c) The electric potential increases continuously
(d) The electric potential increases at first, then decreases and again increases

25. Consider the circuit shown in the following figure.

decreased
(c) The potential across the plates is decreased
(d) The electric field between the plates is decreased

30. A parallel-plate capacitor has parallel sheet of copper inserted between and parallel to the two plate, without touching the plates. The capacity of the capacitor after the introduction of the copper sheet is
(a) Minimum when the copper sheet touches one of the plates
(b) Maximum when the copper sheet is midway between the two plates
(c) Invariant for all positions of the sheet between the plates
(d) Greater than that before introducing the sheet

31. In the given figure a capacitor having three layers between its plates. Layer x is vacuum, y is conductor, and z is a dielectric. Which of the following change(s) will result in increase in capacitance?



35. The capacitance of a parallel-plate capacitor is C_0 when the region between the plates has air. This region is now filled with a dielectric slab of dielectric constant κ. The capacitor is connected to a cell of emf ε and the slab is taken out.
(a) Charge $\varepsilon C_0(\kappa - 1)$ flows through the cell
(b) Energy $\varepsilon^2 C_0(\kappa - 1)$ is absorbed by the cell
(c) The energy stored in the capacitor is reduced by $\varepsilon^2 C_0(\kappa - 1)$
(d) The external agent has to do $(1/2)\varepsilon^2 C_0(\kappa - 1)$ amount of work to take the slab out

Linked Comprehension

Passage for Questions 36–38: Consider a parallel-plate capacitor originally with a charge q_0, capacitance C_0, and potential difference ΔV_0. There is an electrostatic force of magnitude F_0 between the plates, and the capacitor has a stored energy U_0. The terminals of the capacitor are connected to another capacitor of same capacitance and charge.

36. A dielectric slab with $\kappa_e > 1$ is inserted between the plates of the first capacitor. Which quantity decreases?

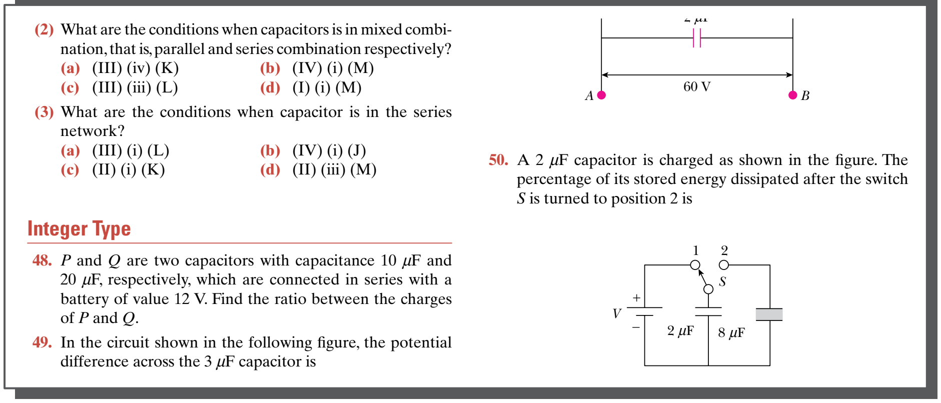
43. Final energy stored in capacitor is
(a) 1.67×10^{-2} J **(b)** 6.71×10^{-2} J
(c) 1.2×10^{-2} J **(d)** 1.33×10^{-2} J

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

44.

Column I	Column II
(a) $(1/4\pi\varepsilon_0)q_1q_2/r^2$, where q_1 and q_2 are charges and r is distance	**(p)** (kilogram) (meter) (second)$^{-2}$
(b) qE, q is charge and E is electric field	**(q)** newton
(c) $(1/2)CV^2$, where C is capacitance and V is voltage	**(r)** (newton) (meter)^2 (coulomb)$^{-1}$
(d) q/ε_0, q is charge and ε_0 is absolute permittivity of free space	**(s)** joule



(2) What are the conditions when capacitors is in mixed combination, that is, parallel and series combination respectively?
(a) (III) (iv) (K) **(b)** (IV) (i) (M)
(c) (III) (iii) (L) **(d)** (I) (i) (M)

(3) What are the conditions when capacitor is in the series network?
(a) (III) (i) (L) **(b)** (IV) (i) (J)
(c) (II) (i) (K) **(d)** (II) (iii) (M)

Integer Type

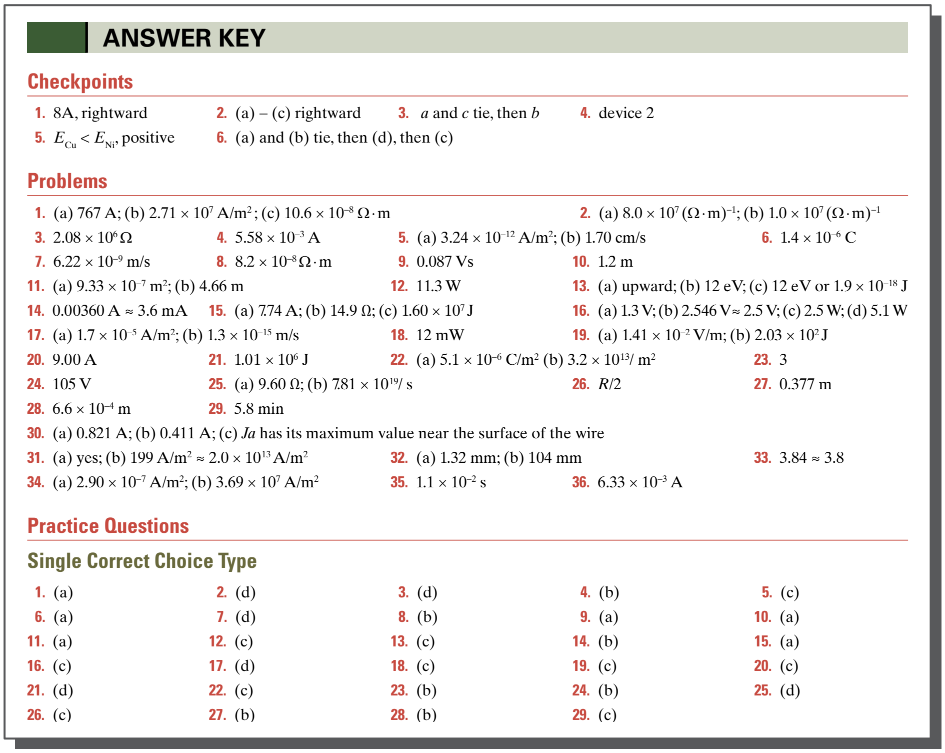
48. P and Q are two capacitors with capacitance 10 μF and 20 μF, respectively, which are connected in series with a battery of value 12 V. Find the ratio between the charges of P and Q.

49. In the circuit shown in the following figure, the potential difference across the 3 μF capacitor is

50. A 2 μF capacitor is charged as shown in the figure. The percentage of its stored energy dissipated after the switch S is turned to position 2 is

ANSWER KEY

Provided at the end of each chapter, it contains answers to all questions including Checkpoints, Problems, and Practice Questions.



ANSWER KEY

Checkpoints

1. 8A, rightward
2. (a) – (c) rightward
3. *a* and *c* tie, then *b*
4. device 2
5. $E_{Cu} < E_{Ni}$, positive
6. (a) and (b) tie, then (d), then (c)

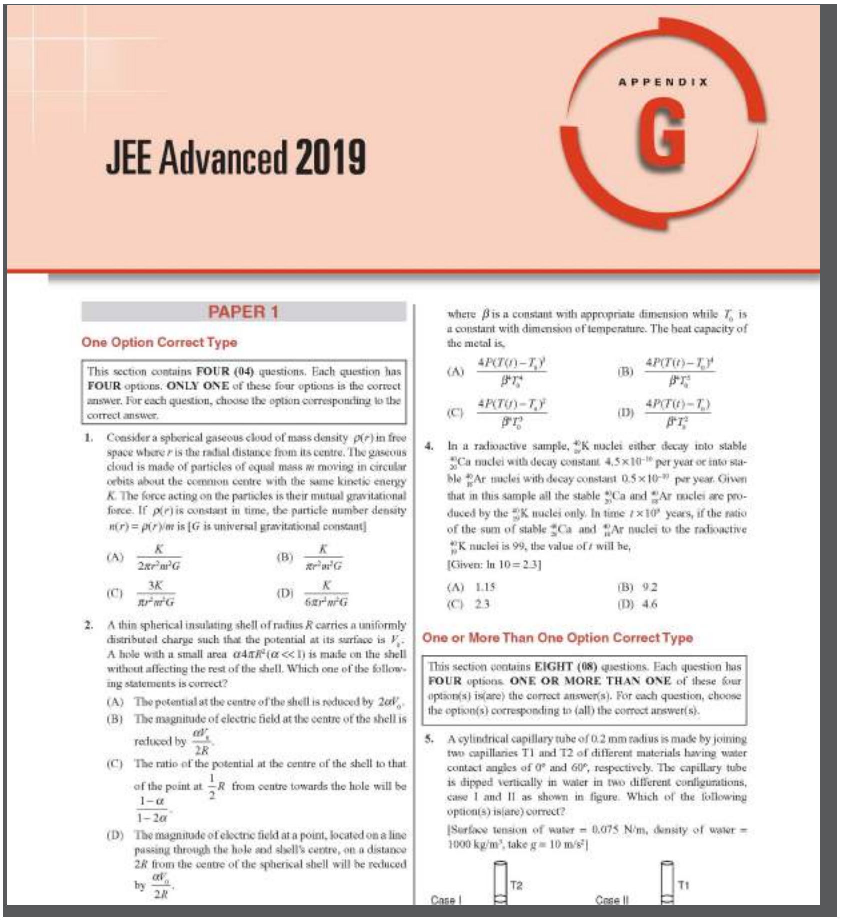
Problems

1. (a) 767 A; (b) 2.71×10^7 A/m^2; (c) $10.6 \times 10^{-8}\ \Omega\cdot$m
2. (a) $8.0 \times 10^7\ (\Omega\cdot\text{m})^{-1}$; (b) $1.0 \times 10^7\ (\Omega\cdot\text{m})^{-1}$
3. $2.08 \times 10^6\ \Omega$
4. 5.58×10^{-3} A
5. (a) 3.24×10^{-12} A/m^2; (b) 1.70 cm/s
6. 1.4×10^{-6} C
7. 6.22×10^{-9} m/s
8. $8.2 \times 10^{-8}\ \Omega\cdot$m
9. 0.087 Vs
10. 1.2 m
11. (a) 9.33×10^{-7} m^2; (b) 4.66 m
12. 11.3 W
13. (a) upward; (b) 12 eV; (c) 12 eV or 1.9×10^{-18} J
14. 0.00360 A ≈ 3.6 mA
15. (a) 7.74 A; (b) 14.9 Ω; (c) 1.60×10^7 J
16. (a) 1.3 V; (b) 2.546 V ≈ 2.5 V; (c) 2.5 W; (d) 5.1 W
17. (a) 1.7×10^{-5} A/m^2; (b) 1.3×10^{-15} m/s
18. 12 mW
19. (a) 1.41×10^{-2} V/m; (b) 2.03×10^2 J
20. 9.00 A
21. 1.01×10^6 J
22. (a) 5.1×10^{-6} C/m^2 (b) 3.2×10^{13}/ m^2
23. 3
24. 105 V
25. (a) 9.60 Ω; (b) 7.81×10^{19}/ s
26. $R/2$
27. 0.377 m
28. 6.6×10^{-4} m
29. 5.8 min
30. (a) 0.821 A; (b) 0.411 A; (c) *Ja* has its maximum value near the surface of the wire
31. (a) yes; (b) 199 A/m^2 ≈ 2.0×10^{13} A/m^2
32. (a) 1.32 mm; (b) 104 mm
33. 3.84 ≈ 3.8
34. (a) 2.90×10^{-7} A/m^2; (b) 3.69×10^7 A/m^2
35. 1.1×10^{-2} s
36. 6.33×10^{-3} A

Practice Questions

Single Correct Choice Type

1. (a)	**2.** (d)	**3.** (d)	**4.** (b)	**5.** (c)
6. (a)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (a)
11. (a)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (a)
16. (c)	**17.** (d)	**18.** (c)	**19.** (c)	**20.** (c)
21. (d)	**22.** (c)	**23.** (b)	**24.** (b)	**25.** (d)
26. (c)	**27.** (b)	**28.** (b)	**29.** (c)	



APPENDIX

G

JEE Advanced 2019

PAPER 1

One Option Correct Type

This section contains **FOUR (04)** questions. Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer. For each question, choose the option corresponding to the correct answer.

1. Consider a spherical gaseous cloud of mass density $\rho(r)$ in free space where r is the radial distance from its centre. The gaseous cloud is made of particles of equal mass m moving in circular orbits about the common centre with the same kinetic energy K. The force acting on the particles is their mutual gravitational force. If $\rho(r)$ is constant in time, the particle number density $n(r) = \rho(r)/m$ is [G is universal gravitational constant]

(A) $\dfrac{K}{2\pi r^2 m^2 G}$ (B) $\dfrac{K}{\pi r^2 m^2 G}$

(C) $\dfrac{3K}{\pi r^2 m^2 G}$ (D) $\dfrac{K}{6\pi r^2 m^2 G}$

2. A thin spherical insulating shell of radius R carries a uniformly distributed charge such that the potential at its surface is V_0. A hole with a small area $\alpha 4\pi R^2 (\alpha \ll 1)$ is made on the shell without affecting the rest of the shell. Which one of the following statements is correct?

(A) The potential at the centre of the shell is reduced by $2\alpha V_0$.
(B) The magnitude of electric field at the centre of the shell is reduced by $\dfrac{\alpha V_0}{2R}$.
(C) The ratio of the potential at the centre of the shell to that of the point at $\dfrac{1}{2}R$ from centre towards the hole will be $\dfrac{1-\alpha}{1-2\alpha}$.
(D) The magnitude of electric field at a point, located on a line passing through the hole and shell's centre, on a distance $2R$ from the centre of the spherical shell will be reduced by $\dfrac{\alpha V_0}{2R}$.

where β is a constant with appropriate dimension while T_0 is a constant with dimension of temperature. The heat capacity of the metal is,

(A) $\dfrac{4P(T(t)-T_0)^3}{\beta^4 T_0^4}$ (B) $\dfrac{4P(T(t)-T_0)^4}{\beta^4 T_0^5}$

(C) $\dfrac{4P(T(t)-T_0)^2}{\beta^4 T_0^3}$ (D) $\dfrac{4P(T(t)-T_0)}{\beta^4 T_0^2}$

4. In a radioactive sample, $^{40}_{19}$K nuclei either decay into stable $^{40}_{20}$Ca nuclei with decay constant 4.5×10^{-10} per year or into stable $^{40}_{18}$Ar nuclei with decay constant 0.5×10^{-10} per year. Given that in this sample all the stable $^{40}_{20}$Ca and $^{40}_{18}$Ar nuclei are produced by the $^{40}_{19}$K nuclei only. In time $t \times 10^9$ years, if the ratio of the sum of stable $^{40}_{20}$Ca and $^{40}_{18}$Ar nuclei to the radioactive $^{40}_{19}$K nuclei is 99, the value of t will be,
[Given: ln 10 = 2.3]

(A) 1.15 (B) 9.2
(C) 2.3 (D) 4.6

One or More Than One Option Correct Type

This section contains **EIGHT (08)** questions. Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) the correct answer(s). For each question, choose the option(s) corresponding to (all) the correct answer(s).

5. A cylindrical capillary tube of 0.2 mm radius is made by joining two capillaries T1 and T2 of different materials having water contact angles of 0° and 60°, respectively. The capillary tube is dipped vertically in water in two different configurations, case I and II as shown in figure. Which of the following option(s) is(are) correct?
[Surface tension of water = 0.075 N/m, density of water = 1000 kg/m^3, take g = 10 m/s^2]

APPENDICES

Comprise useful appendices from the original book and additional appendix of JEE Advanced 2019 paper

ABOUT VOLUME I

The topics covered in Halliday/Resnick/Walker Physics for JEE (Main & Advanced) Volume I, Amit Gupta.

- General Physics
- Mechanics
- Thermal Physics

Contents

Preface vii

Acknowledgments ix

Note to the Student xi

22 Electric Charges and Fields 1

22.1 What is Physics? 1

22.2 Electric Charge 2

22.3 Coulomb's Law 5

22.4 Charge is Quantized 13

22.5 Charge is Conserved 15

22.6 The Electric Field 16

22.7 Electric Field Lines 17

22.8 The Electric Field Due to a Point Charge 18

22.9 The Electric Field Due to an Electric Dipole 21

22.10 The Electric Field Due to a Line of Charge (Continuous Charge Distribution) 23

22.11 The Electric Field Due to a Charged Disk 29

22.12 Electric Field Inside a Conductor 31

22.13 A Point Charge in an Electric Field 32

22.14 A Dipole in an Electric Field 35

22.15 Electrostatic Shielding 39

Review and Summary 40

Problems 41

Practice Questions 47

Answer Key 54

23 Gauss' Law 57

23.1 What is Physics? 57

23.2 Electric Flux 58

23.3 Gauss' Law 63

23.4 A Charged Isolated Conductor 69

23.5 Applying Gauss' Law: Spherical Symmetry 74

23.6 Applying Gauss' Law: Cylindrical Symmetry 78

23.7 Applying Gauss' Law: Planar Symmetry 80

Review and Summary 83

Problems 84

Practice Questions 88

Answer Key 93

24 Electric Potential 97

24.1 What is Physics? 97

24.2 Electric Potential and Electric Potential Energy 98

24.3 Equipotential Surfaces 104

24.4 Calculating the Potential from the Field 105

24.5 Potential Due to a Point Charge 109

24.6 Potential Due to a Group of Point Charges 111

24.7 Potential Due to a Continuous Charge Distribution 113

24.8 Calculating the Field from the Potential 121

24.9 Potential Due to an Electric Dipole 124

24.10 Electric Potential Energy of a System of Charged Particles 129

24.11 Potential of a Charged Isolated Conductor 132

24.12 Earthing 136

24.13 Electrostatic Self-Energy 138

Review and Summary 140

Problems 142

Practice Questions 145

Answer Key 152

25 Capacitance 155

25.1 What is Physics? 155

25.2 Capacitance 155

25.3 Calculating the Capacitance 158

25.4 Capacitors in Parallel and in Series 162

25.5 Energy Stored in an Electric Field 168

25.6 Capacitor with a Dielectric 171

25.7 Dielectrics: An Atomic View 174

25.8 Dielectrics and Gauss' Law 176

Review and Summary 181

Problems 181

Practice Questions 185

Answer Key 191

26 Current and Resistance 193

26.1 What is Physics? 193

26.2 Electric Current 193

26.3 Current Density 196
26.4 Resistance and Resistivity 200
26.5 Ohm's Law 204
26.6 A Microscopic View of Ohm's Law 206
26.7 Power in Electric Circuits 209
26.8 Semiconductors 210
26.9 Superconductors 211
Review and Summary 212
Problems 213
Practice Questions 216
Answer Key 222

27 Circuits 223
27.1 What is Physics? 223
27.2 Work, Energy, and EMF 223
27.3 Calculating the Current in a Single-Loop Circuit 225
27.4 Potential Difference Between Two Points 229
27.5 Multiloop Circuits 233
27.6 Electricity-Related Instruments 243
27.7 *RC* Circuits 248
Review and Summary 253
Problems 254
Practice Questions 258
Answer Key 265

28 Magnetic Force 269
28.1 What is Physics? 269
28.2 What Produces a Magnetic Field? 269
28.3 The Definition of $\vec{B}$ 274
28.4 A Circulating Charged Particle 277
28.5 Cyclotrons 281
28.6 Crossed Fields: Discovery of the Electron 283
28.7 The Hall Effect 287
28.8 Magnetic Force on a Current-Carrying Wire 289
28.9 Torque on a Current Loop 292
28.10 The Magnetic Dipole Moment 294
Review and Summary 299
Problems 300
Practice Questions 304
Answer Key 311

29 Magnetic Fields Due to Currents 315
29.1 What is Physics? 315
29.2 Calculating the Magnetic Field Due to a Current 315
29.3 Magnetic Field of Moving Charged Particles 326
29.4 Force Between Two Parallel Currents 329
29.5 Ampere's Law 331
29.6 Solenoids and Toroids 336
29.7 Limitations of Ampere's Law 340
29.8 A Note About the Frames of Reference 341
Review and Summary 341
Problems 342
Practice Questions 346
Answer Key 355

30 Electromagnetic Induction 357
30.1 What is Physics? 357
30.2 Gauss' Law of Magnetic Fields 357
30.3 Two Experiments 359
30.4 Faraday's Law of Induction 360
30.5 Lenz's Law 363
30.6 Motional EMF 366
30.7 Induction and Energy Transfers 369
30.8 Induced Electric Fields 374
30.9 Inductors and Inductance 379
30.10 Self-Induction 380
30.11 *RL* Circuits 383
30.12 Energy Stored in a Magnetic Field 387
30.13 Mutual Induction 392
Review and Summary 396
Problems 397
Practice Questions 403
Answer Key 414

31 Electromagnetic Oscillations and Alternating Current 417
31.1 What is Physics? 417
31.2 *LC* Oscillations, Qualitatively 417
31.3 The Electrical–Mechanical Analogy 421
31.4 *LC* Oscillations, Quantitatively 422
31.5 Alternating Current 425
31.6 Three Simple Circuits 427
31.7 The Series *RLC* Circuit 435
31.8 Power in Alternating-Current Circuits 440
31.9 Transformers 445
Review and Summary 449
Problems 450
Practice Questions 453
Answer Key 460

32 Electromagnetic Waves 463
32.1 What is Physics? 463
32.2 Maxwell's Rainbow 463
32.3 The Traveling Electromagnetic Wave, Qualitatively 465
32.4 The Traveling Electromagnetic Wave, Quantitatively 468
32.5 Energy Transport and the Poynting Vector 471
32.6 Radiation Pressure 474
32.7 Polarization 474
32.8 Reflection and Refraction 480
32.9 Polarization by Reflection 484
32.10 Polarization by Scattering 485
Review and Summary 486
Problems 487
Practice Questions 489
Answer Key 495

33 Geometrical Optics: Reflection 497
33.1 What is Physics? 497
33.2 Principle of Rectilinear Propagation of Light 497

33.3 Laws of Reflection 499
33.4 Formation of Image by a Mirror 503
33.5 Formation of Images by Spherical Mirrors 509
Review and Summary 520
Problems 521
Practice Questions 523
Answer Key 527

34 Geometrical Optics: Refraction 529
34.1 What is Physics? 529
34.2 Snell's Laws of Refraction 529
34.3 Total Internal Reflection 539
34.4 Spherical Refracting Surfaces 545
34.5 Thin Spherical Lenses 551
34.6 Lens Mirror Combination 563
34.7 Prism 565
34.8 Chromatic Dispersion 570
Review and Summary 576
Problems 576
Practice Questions 580
Answer Key 591

35 Interference and Diffraction 595
35.1 What is Physics? 595
35.2 Light as a Wave 596
35.3 Diffraction 601
35.4 Young's Interference Experiment 602
35.5 Coherence 611
35.6 Intensity in Double-Slit Interference 611
35.7 Modifications of Young's Double-Slit Experiment 618
35.8 Interference from Thin Films 622
35.9 Diffraction and the Wave Theory of Light 630
35.10 Diffraction by a Single Slit: Locating the Minima 631
35.11 Intensity in Single-Slit Diffraction, Quantitatively 635
35.12 Diffraction by a Circular Aperture 636
35.13 Diffraction by a Double Slit 638
Review and Summary 642
Problems 643
Practice Questions 648
Answer Key 659

36 Relativity 663
36.1 What is Physics? 663
36.2 The Postulates 664
36.3 Measuring an Event 665
36.4 The Relativity of Simultaneity 666
36.5 The Relativity of Time 669
36.6 The Relativity of Length 674
36.7 The Lorentz Transformation 677
36.8 Electricity and Magnetism 681
36.9 The Relativity of Velocities 683
36.10 Doppler Effect for Light 684
36.11 A New Look at Momentum 686
36.12 A New Look at Energy 687
Review and Summary 692
Problems 693
Practice Questions 697
Answer Key 701

37 Photons and Matter Waves 705
37.1 What is Physics? 705
37.2 The Photon, the Quantum of Light 706
37.3 The Photoelectric Effect 708
37.4 Photons Have Momentum 714
37.5 Light as a Probability Wave 716
37.6 Electrons and Matter Waves 718
37.7 Schrödinger's Equation 722
37.8 Heisenberg's Uncertainty Principle 724
37.9 Reflection from a Potential Step 726
37.10 Tunneling Through a Potential Barrier 727
Review and Summary 729
Problems 730
Practice Questions 732
Answer Key 738

38 Hydrogen Atom 741
38.1 What is Physics? 741
38.2 String Waves and Matter Waves 741
38.3 Energies of a Trapped Electron 742
38.4 Wave Functions of a Trapped Electron 747
38.5 The Bohr Model of the Hydrogen Atom 751
38.6 Quantum Numbers for the Hydrogen Atom (Optional) 763
Review and Summary 768
Problems 769
Practice Questions 771
Answer Key 777

39 All About Atoms 779
39.1 What is Physics? 779
39.2 Some Properties of Atoms 780
39.3 Angular Momentum, Magnetic Dipole Moments 782
39.4 The Pauli Exclusion Principle 785
39.5 Building the Periodic Table 786
39.6 X-Rays and the Ordering of the Elements 788
39.7 Lasers and Laser Light 794
39.8 How Lasers Work 795
Review and Summary 798
Problems 799
Practice Questions 802
Answer Key 808

40 The Nucleus 811
40.1 What is Physics? 811
40.2 Discovering the Nucleus 811
40.3 Some Nuclear Properties 814
40.4 Radioactive Decay 821
40.5 Alpha Decay 825
40.6 Beta Decay 827

40.7 Gamma Decay 830
40.8 Radioactive Dating 831
40.9 Nuclear Models (Optional) 833
40.10 Nuclear Fission: The Basic Process 836
40.11 Thermonuclear Fusion: The Basic Process 838
40.12 Thermonuclear Fusion in the Sun and Other Stars 840
Review and Summary 841
Problems 842
Practice Questions 846
Answer Key 855

Appendix A The International System of Units (SI) A1
Appendix B Some Fundamental Constants of Physics A3
Appendix C Some Astronomical Data A5
Appendix D Conversion Factors A7
Appendix E Mathematical Formulas A11
Appendix F Properties of the Elements A15
Appendix G JEE Advanced 2019 A19

CHAPTER

22

Electric Charges and Fields

22.1 | WHAT IS PHYSICS?

You are surrounded by devices that depend on the physics of electromagnetism, which is the combination of electric and magnetic phenomena. This physics is at the root of computers, television, radio, telecommunications, household lighting, and even the ability of food wrap to cling to a container. This physics is also the basis of the natural world. Not only does it hold together all the atoms and molecules in the world, it also produces lightning, auroras, and rainbows.

The physics of electromagnetism was first studied by the early Greek philosophers, who discovered that if a piece of amber is rubbed and then brought near bits of straw, the straw will jump to the amber. We now know that the attraction between amber and straw is due to an electric force. The Greek philosophers also discovered that if a certain type of stone (a naturally occurring magnet) is brought near bits of iron, the iron will jump to the stone. We now know that the attraction between magnet and iron is due to a magnetic force.

From these modest origins with the Greek philosophers, the sciences of electricity and magnetism developed separately for centuries—until 1820, in fact, when Hans Christian Oersted found a connection between them: an electric current in a wire can deflect a magnetic compass needle. Interestingly enough, Oersted made this discovery, a big surprise, while preparing a lecture demonstration for his physics students.

The new science of electromagnetism was developed further by workers in many countries. One of the best was Michael Faraday, a truly gifted experimenter with a talent for physical intuition and visualization. That talent is attested to by the fact that his collected laboratory notebooks do not contain a single equation. In the mid-nineteenth century, James Clerk Maxwell put Faraday's ideas into mathematical form, introduced many new ideas of his own, and put electromagnetism on a sound theoretical basis.

Our discussion of electromagnetism is spread through the next 16 chapters. We begin with electrical phenomena, and our first step is to discuss the nature of electric charge and electric force.

In this chapter, we will further (1) define electric field, (2) discuss how to calculate it for various arrangements of charged particles and objects, and (3) discuss how an electric field can affect a charged particle (as in making it move).

Contents

22.1 What is Physics?
22.2 Electric Charge
22.3 Coulomb's Law
22.4 Charge is Quantized
22.5 Charge is Conserved
22.6 The Electric Field
22.7 Electric Field Lines
22.8 The Electric Field Due to a Point Charge
22.9 The Electric Field Due to an Electric Dipole
22.10 The Electric Field Due to a Line of Charge (Continuous Charge Distribution)
22.11 The Electric Field Due to a Charged Disk
22.12 Electric Field Inside a Conductor
22.13 A Point Charge in an Electric Field
22.14 A Dipole in an Electric Field
22.15 Electrostatic Shielding

22.2 | ELECTRIC CHARGE

Key Concepts

- The strength of a particle's electrical interaction with objects around it depends on its electric charge (usually represented as q), which can be either positive or negative. Particles with the same sign of charge repel each other, and particles with opposite signs of charge attract each other.
- An object with equal amounts of the two kinds of charge is electrically neutral, whereas one with an imbalance is electrically charged and has an excess charge.
- Conductors are materials in which a significant number of electrons are free to move. The charged particles in nonconductors (insulators) are not free to move.

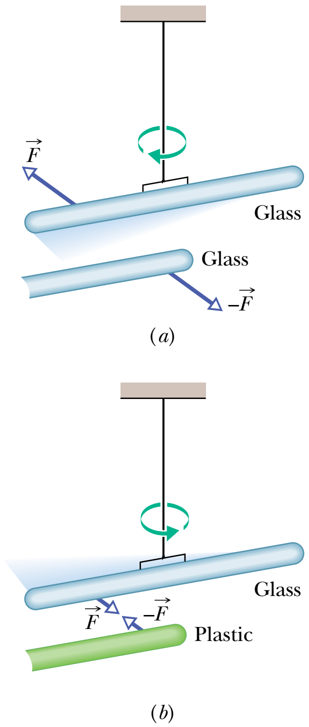


Figure 22-1 (*a*) The two glass rods were each rubbed with a silk cloth and one was suspended by thread. When they are close to each other, they repel each other. (*b*) The plastic rod was rubbed with fur. When brought close to the glass rod, the rods attract each other.

Here are two demonstrations that seem to be magic, but our job here is to make sense of them. After rubbing a glass rod with a silk cloth (on a day when the humidity is low), we hang the rod by means of a thread tied around its center (Fig. 22-l*a*). Then we rub a second glass rod with the silk cloth and bring it near the hanging rod. The hanging rod magically moves away. We can see that a force repels it from the second rod, but how? There is no contact with that rod, no breeze to push on it, and no sound wave to disturb it.

In the second demonstration we replace the second rod with a plastic rod that has been rubbed with fur. This time, the hanging rod moves toward the nearby rod (Fig. 22-1*b*). Like the repulsion, this attraction occurs without any contact or obvious communication between the rods.

In the first demonstration, the force on the hanging rod was *repulsive*, and in the second, *attractive*. After a great many investigations, scientists figured out that the forces in these types of demonstrations are due to the *electric charge* that we set up on the rods when they are in contact with silk or fur. Electric charge is an intrinsic property of the fundamental particles that make up objects such as the rods, silk, and fur. That is, charge is a property that comes automatically with those particles wherever they exist.

Two Types. There are two types of electric charge, named by the American scientist and statesman Benjamin Franklin as positive charge and negative charge. He could have called them anything (such as cherry and walnut), but using algebraic signs as names comes in handy when we add up charges to find the net charge. In most everyday objects, such as a mug, there are about equal numbers of negatively charged particles and positively charged particles, and so the net charge is zero, the charge is said to be *balanced*, and the object is said to be *electrically neutral* (or just *neutral* for short).

Excess Charge. Normally you are approximately neutral. However, if you live in regions where the humidity is low, you know that the charge on your body can become slightly unbalanced when you walk across certain carpets. Either you gain negative charge from the carpet (at the points of contact between your shoes with the carpet) and become negatively charged, or you lose negative charge and become positively charged. Either way, the extra charge is said to be an *excess charge*. You probably don't notice it until you reach for a door handle or another person. Then, if your excess charge is enough, a spark leaps between you and the other object, eliminating your excess charge. Such sparking can be annoying and even somewhat painful. Such *charging* and *discharging* does not happen in humid conditions because the water in the air *neutralizes* your excess charge about as fast as you acquire it.

Two of the grand mysteries in physics are (1) *why* does the universe have particles with electric charge (what is it, really?) and (2) *why* does electric charge come in two types (and not, say, one type or three types). We just do not know. Nevertheless, with lots of experiments similar to our two demonstrations scientists discovered that

Particles with the same sign of electrical charge repel each other, and particles with opposite signs attract each other.

In a moment we shall put this rule into quantitative form as Coulomb's law of *electrostatic force* (or *electric force*) between charged particles. The term *electrostatic* is used to emphasize that, relative to each other, the charges are either stationary or moving only very slowly.

Demos. Now let's get back to the demonstrations to understand the motions of the rod as being something other than just magic. When we rub the glass rod with a silk cloth, a small amount of negative charge moves from the rod to the silk (a transfer like that between you and a carpet), leaving the rod with a small amount of excess positive charge. (Which way the negative charge moves is not obvious and requires a lot of experimentation.) We *rub* the silk over the rod to increase the number of contact points and thus the amount, still tiny, of transferred charge. We hang the rod from the thread so as to *electrically isolate* it from its surroundings (so that the surroundings cannot neutralize the rod by giving it enough negative charge to rebalance its charge). When we rub the second rod with the silk cloth, it too becomes positively charged. So when we bring it near the first rod, the two rods repel each other (Fig. 22-2*a*).

Next, when we rub the plastic rod with fur, it gains excess negative charge from the fur. (Again, the transfer direction is learned through many experiments.) When we bring the plastic rod (with negative charge) near the hanging glass rod (with positive charge), the rods are attracted to each other (Fig. 22-2*b*). All this is subtle. You cannot see the charge or its transfer, only the results. The way of producing charge by rubbing is called *charging by friction*.

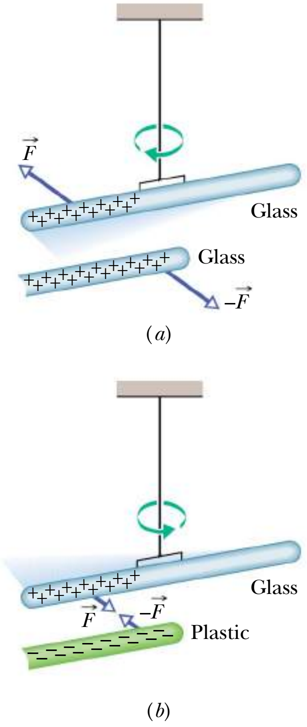


Figure 22-2 (*a*) Two charged rods of the same sign repel each other. (*b*) Two charged rods of opposite signs attract each other. Plus signs indicate a positive net charge, and minus signs indicate a negative net charge.

Let us perform another experiment. (1) Bring two metal spheres, *A* and *B*, supported on insulating stands, in contact as shown in Fig. 22-3*a*. (2) Bring a positively charged rod near one of the spheres, say *A*, taking care that it does not touch the sphere. The free electrons in the spheres are attracted toward the rod. This leaves an excess of positive charge on the rear surface of sphere *B*. Both kinds of charges are bound in the metal spheres and cannot escape. They, therefore, reside on the surfaces, as shown in Fig. 22-3*b*. The left surface of sphere *A* has an excess of negative charge and the right surface of sphere *B* has an excess of positive charge. However, not all the electrons in the spheres have accumulated on the left surface of *A*. The process is called *induction of charge* and it happens almost instantly. The accumulated charges remain on the surface as shown, until the glass rod is held near the sphere. If the rod is removed, the charges are not acted by any outside force and they redistribute to their original neutral state. (3) Separate the spheres by a small distance while the glass rod is still held near sphere *A*, as shown in Fig. 22-3*c*. The two spheres are found to be oppositely charged and attract each other. (4) Remove the rod. The charges on spheres rearrange themselves as shown in Fig. 22-3*d*. Now, separate the spheres quite apart. The charges on them get uniformly distributed over them, as shown in Fig. 22-3*e*.

In this process, the metal spheres will each be equal and oppositely charged. This is *charging by induction*. The positively charged glass rod does not lose any of its charge, contrary to the process of charging by friction.

If you do not have two spheres, but have only one, then also you can do this induction experiment. Connect the sphere to Earth while you bring the positively charged glass rod near the conducting sphere. What will happen is

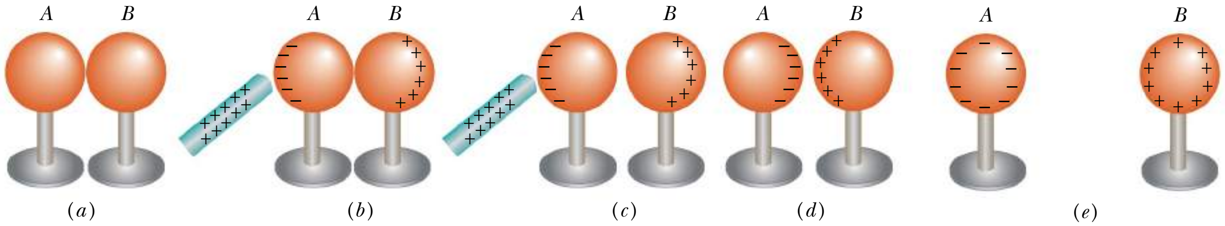


Figure 22-3 Charging by induction.

that the conducting sphere will be negatively charged by gaining some electrons from the Earth. The Earth, as you will soon realize, is a gigantic conductor. Our demonstrations reveal the following:

Charges with the same electrical sign repel each other, and charges with opposite electrical signs attract each other.

A simple device, called **electroscope**, can be used to test the presence of an electric charge (Fig. 22-4*a*). It is made up of a pair of thin strips (or leaves) made of gold, suspended from a small metallic rod in an insulating enclosure; the rod is connected to a metal ball on top of the enclosure. When the metal ball is charged, the charge spreads out over the metallic parts including strips (Fig. 22-4*b*), and the two strips separate due to repulsion between the like charges on them (Fig. 22-4*c*).

When a charged rod is brought near the ball of the electroscope, the strips separate even without any contact between the rod and the electroscope (Fig. 22-4*d*). Also, when the distance between the rod and the electroscope is increased again, the strips drop back, showing that no charge has been transferred from the rod to the electroscope.

The reason for the strips to separate even though the electroscope remains neutral is that the negative charges on the rod repel the free electrons in the metallic parts of the electroscope. The free electrons are pushed as far away as possible from the rod and pile up in the strips. This redistribution of charge is nearly instantaneous. The top of the electroscope develops a positive charge due to electron deficit and the strips develop excess negative charges. The negative charges on the strips are responsible for the repulsion between them. The degree of divergence gives a measure of the amount of charge.

Conductors and Insulators

We can classify materials generally according to the ability of charge to move through them. **Conductors** are materials through which charge can move rather freely; examples include metals (such as copper in common lamp wire), the human body, and tap water. **Nonconductors**—also called **insulators**—are materials through which charge cannot move freely; examples include rubber (such as the insulation on common lamp wire), plastic, glass, and chemically pure water. **Semiconductors** are materials that are intermediate between conductors and insulators; examples include silicon and germanium in computer chips. **Superconductors** are materials that are *perfect* conductors, allowing charge to move without *any* hindrance. In these chapters we discuss only conductors and insulators.

Conducting Path. Here is an example of how conduction can eliminate excess charge on an object. If you rub a copper rod with wool, charge is transferred from the wool to the rod. However, if you are holding the rod while also touching a faucet, you cannot charge the rod in spite of the transfer. The reason is that you, the rod, and the faucet are all conductors connected, via the plumbing, to Earth's surface, which is a huge conductor. Because the excess charges put on the rod by the wool repel one another, they move away from one another by moving first through the rod, then through you, and then through the faucet and plumbing to reach Earth's surface, where they can spread out. The process leaves the rod electrically neutral.

In thus setting up a pathway of conductors between an object and Earth's surface, we are said to *ground* the object, and in neutralizing the object (by eliminating an unbalanced positive or negative charge), we are said to

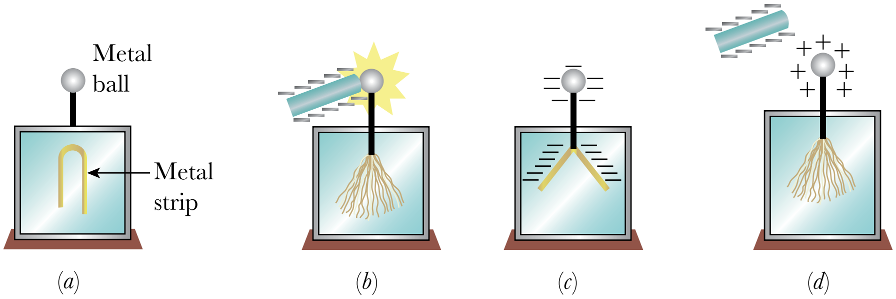


Figure 22-4 (*a*) Electroscope. (*b*) When the ball of an electroscope is charged. (*c*) The two metal strips of the electroscope move apart. (*d*) When charged rod is brought near the metal ball.

discharge the object. If instead of holding the copper rod in your hand, you hold it by an insulating handle, you eliminate the conducting path to Earth, and the rod can then be charged by rubbing (the charge remains on the rod), as long as you do not touch it directly with your hand.

Charged Particles. The properties of conductors and insulators are due to the structure and electrical nature of atoms. Atoms consist of positively charged *protons*, negatively charged *electrons*, and electrically neutral *neutrons.* The protons and neutrons are packed tightly together in a central *nucleus.*

The charge of a single electron and that of a single proton have the same magnitude but are opposite in sign. Hence, an electrically neutral atom contains equal numbers of electrons and protons. Electrons are held near the nucleus because they have the electrical sign opposite that of the protons in the nucleus and thus are attracted to the nucleus. Were this not true, there would be no atoms and thus no you.

When atoms of a conductor like copper come together to form the solid, some of their outermost (and so most loosely held) electrons become free to wander about within the solid, leaving behind positively charged atoms (*positive ions*). We call the mobile electrons *conduction electrons.* There are few (if any) free electrons in a nonconductor.

Induced Charge. The experiment of Fig. 22-5 demonstrates the mobility of charge in a conductor. A negatively charged plastic rod will attract either end of an isolated neutral copper rod. What happens is that many of the conduction electrons in the closer end of the copper rod are repelled by the negative charge on the plastic rod. Some of the conduction electrons move to the far end of the copper rod, leaving the near end depleted in electrons and thus with an unbalanced positive charge. This positive charge is attracted to the negative charge in the plastic rod. Although the copper rod is still neutral, it is said to have an *induced charge*, which means that some of its positive and negative charges have been separated due to the presence of a nearby charge. The process of charging a neutral body by bringing a charged body near it without making a contact between the two bodies is known as *charging by induction*.

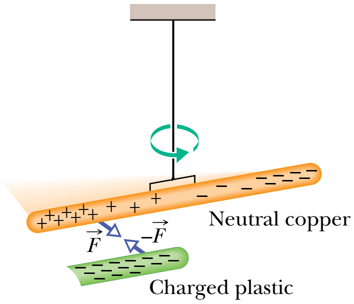


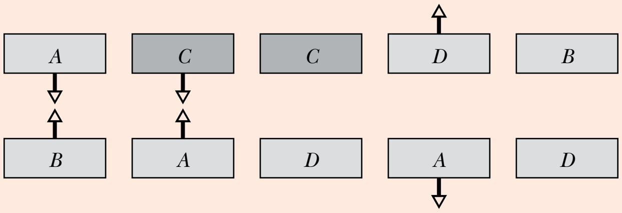
Figure 22-5 A neutral copper rod is electrically isolated from its surroundings by being suspended on a nonconducting thread. Either end of the copper rod will be attracted by a charged rod. Here, conduction electrons in the copper rod are repelled to the far end of that rod by the negative charge on the plastic rod. Then that negative charge attracts the remaining positive charge on the near end of the copper rod, rotating the copper rod to bring that near end closer to the plastic rod.

Similarly, if a positively charged glass rod is brought near one end of a neutral copper rod, induced charge is again set up in the neutral copper rod but now the near end gains conduction electrons, becomes negatively charged, and is attracted to the glass rod, while the far end is positively charged.

Note that only conduction electrons, with their negative charges, can move; positive ions are fixed in place. Thus, an object becomes positively charged only through the *removal of negative charges.* The charges are transmitted from one place to another, depending on the tendency of the substance to acquire electrons.

CHECKPOINT 1

The figure shows five pairs of plates: *A*, *B*, and *D* are charged plastic plates and *C* is an electrically neutral copper plate. The electrostatic forces between the pairs of plates are shown for three of the pairs. For the remaining two pairs, do the plates repel or attract each other?



22.3 | COULOMB'S LAW

Key Concepts

- Electric current i is the rate dq/dt at which charge passes a point:

$$i = \frac{dq}{dt}.$$

- Coulomb's law describes the electrostatic force (or electric force) between two charged particles. If the particles have charges q_1 and q_2, are separated by distance r, and are at rest (or moving only slowly) relative to each other, then the magnitude of the force acting on each due to the other is given by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2} \quad \text{(Coulomb's law)},$$

where $\varepsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2$ is the permittivity constant. The ratio $1/4\pi\varepsilon_0$ is often replaced with the electrostatic constant (or Coulomb constant) $k = 8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2$.

- The electrostatic force vector acting on a charged particle due to a second charged particle is either directly toward the second particle (opposite signs of charge) or directly away from it (same sign of charge).
- If multiple electrostatic forces act on a particle, the net force is the vector sum (not scalar sum) of the individual forces.
- Shell theorem 1: A charged particle outside a shell with charge uniformly distributed on its surface is attracted or repelled as if the shell's charge were concentrated as a particle at its center.
- Shell theorem 2: A charged particle inside a shell with charge uniformly distributed on its surface has no net force acting on it due to the shell.
- Charge on a conducting spherical shell spreads uniformly over the (external) surface.

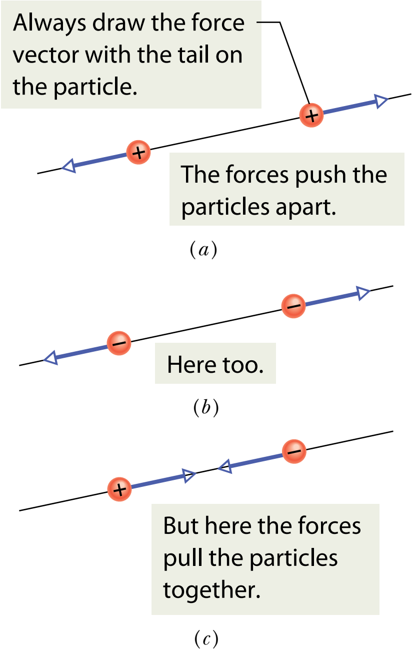

Always draw the force vector with the tail on the particle.
+
+
The forces push the particles apart.
(a)
−
−
Here too.
(b)
+
−
But here the forces pull the particles together.
(c)

Figure 22-6 Two charged particles repel each other if they have the same sign of charge, either (*a*) both positive or (*b*) both negative. (*c*) They attract each other if they have opposite signs of charge.

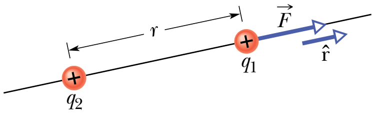


Figure 22-7 The electrostatic force on particle 1 can be described in terms of a unit vector along an axis through the two particles, radially away from particle 2.

Now we come to the equation for Coulomb's law, but first a caution. This equation works for only charged particles (and a few other things that can be treated as particles). For extended objects, with charge located in many different places, we need more powerful techniques. So, here we consider just charged particles and not, say, two charged cats.

If two charged particles are brought near each other, they each exert an **electrostatic force** on the other. The direction of the force vectors depends on the signs of the charges. If the particles have the same sign of charge, they repel each other. That means that the force vector on each is directly away from the other particle (Figs. 22-6*a* and *b*). If we release the particles, they accelerate away from each other. If, instead, the particles have opposite signs of charge, they attract each other. That means that the force vector on each is directly toward the other particle (Fig. 22-6*c*). If we release the particles, they accelerate toward each other.

The equation for the electrostatic forces acting on the particles is called **Coulomb's law** after Charles-Augustin de Coulomb, whose experiments in 1785 led him to it. Let's write the equation in vector form and in terms of the particles shown in Fig. 22-7, where particle 1 has charge q_1 and particle 2 has charge q_2. (These symbols can represent either positive or negative charge.) Let's also focus on particle 1 and write the force acting on it in terms of a unit vector $\hat{r}$ that points along a radial axis extending through the two particles, radially away from particle 2. (As with other unit vectors, has a magnitude of exactly 1 and no unit; its purpose is to point, like a direction arrow on a street sign.) With these decisions, we write the electrostatic force as

$$\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r} \quad \text{(Coulomb's law)}, \tag{22-1}$$

where r is the separation between the particles and k is a positive constant called the *electrostatic constant* or the *Coulomb constant*. (We'll discuss k below.)

Let's first check the direction of the force on particle 1 as given by Eq. 22-1. If q_1 and q_2 have the same sign, then the product q_1q_2 gives us a positive result. So, Eq. 22-1 tells us that the force on particle 1 is in the direction of $\hat{r}$. This is correct, because particle 1 is being repelled from particle 2. Next, if q_1 and q_2 have opposite signs, the product q_1q_2 gives us a negative result. So, now Eq. 22-1 tells us that the force on particle 1 is in the direction opposite $\hat{r}$. This is also correct because particle 1 is being attracted toward particle 2.

For using the formula for Coulomb's law in vector form, we should substitute the values of charges with sign, so that the answer automatically gives us the direction of the forces. We sometimes use the vector form of the law but most of times we take only the magnitude of charges and decide the direction of the force based on the fact that like charges repel and unlike charges attract.

An Aside. Here is something that is very curious. The form of Eq. 22-1 is the same as that of Newton's equation (Eq. 13-3) for the gravitational force between two particles with masses m_1 and m_2 and separation r:

$$\vec{F} = G\frac{m_1 m_2}{r^2}\,\hat{\mathrm{r}} \quad \text{(Newton's law)}, \tag{22-2}$$

where G is the gravitational constant. Although the two types of forces are wildly different, both equations describe inverse square laws (the $1/r^2$ dependences) that involve a product of a property of the interacting particles—the charge in one case and the mass in the other. However, the laws differ in that gravitational forces are always attractive but electrostatic forces may be either attractive or repulsive, depending on the signs of the charges. This difference arises from the fact that there is only one type of mass but two types of charge.

Unit. The SI unit of charge is the **coulomb**. For practical reasons having to do with the accuracy of measurements, the coulomb unit is derived from the SI unit *ampere* for electric current i. We shall discuss current in detail in Chapter 26, but here let's just note that current i is the rate dq/dt at which charge moves past a point or through a region:

$$i = \frac{dq}{dt} \quad \text{(electric current)}. \tag{22-3}$$

Rearranging Eq. 22-3 and replacing the symbols with their units (coulombs C, amperes A, and seconds s) we see that

$$1\ \mathrm{C} = (1\ \mathrm{A})(1\ \mathrm{s}).$$

Force Magnitude. For historical reasons (and because doing so simplifies many other formulas), the electrostatic constant k in Eq. 22-1 is often written as $1/4\pi\varepsilon_0$. Then the magnitude of the electrostatic force in Coulomb's law becomes

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2} \quad \text{(Coulomb's law)}. \tag{22-4}$$

The constants in Eqs. 22-1 and 22-4 have the value

$$k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}. \tag{22-5}$$

The quantity ε_0, called the **permittivity constant**, sometimes appears separately in equations and is

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}. \tag{22-6}$$

Working a Problem. Note that the charge magnitudes appear in Eq. 22-4, which gives us the force magnitude. So, in working problems in this chapter, we use Eq. 22-4 to find the magnitude of a force on a chosen particle due to a second particle and we separately determine the direction of the force by considering the charge signs of the two particles.

Multiple Forces. As with all forces in this book, the electrostatic force obeys the principle of superposition. Suppose we have n charged particles near a chosen particle called particle 1; then the net force on particle 1 is given by the vector sum

$$\vec{F}_{1,\mathrm{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}, \tag{22-7}$$

in which, for example, $\vec{F}_{14}$ is the force on particle 1 due to the presence of particle 4.

To be able to apply this to problem solving, let us state this in words. If you want to know the net force acting on a chosen charged particle that is surrounded by other charged particles, first clearly identify that chosen particle and then find the force on it due to each of the other particles. Draw those force vectors in a free-body diagram of the chosen particle, with the tails anchored on the particle. Then add all those forces *as vectors* according to the rules of Chapter 3, not as scalars. The result is the net force (or resultant force) acting on the particle.

Superposition, however, cannot be taken as an obvious consequence of law of vector addition. It states two things: (i) the net force on any one charge is unaffected by the presence of other charges and (ii) the net force on it can be found by vector addition of the forces due to each of these changes, as if they were acting alone.

PROBLEM-SOLVING TACTICS

Tactic 1: Symbols Representing Charge Here is a general guide to the symbols representing charge. If the symbol q, with or without a subscript, is used in a sentence when no electrical sign has been specified, the charge can be either positive or negative. Sometimes the sign is explicitly shown, as in the notation $+q$ or $-q$.

When more than one charged object is being considered, their charges might be given as multiples of a charge magnitude. For example, the notation $+2q$ means a positive charge with magnitude twice that of some reference charge of magnitude q, and $-3q$ means a negative charge with magnitude three times that of the reference charge of magnitude q.

Tactic 2: Drawing Electrostatic Force Vectors Suppose you are given a diagram of charged particles, such as Fig. 22-8*a*, and you are asked to find the net electrostatic force on one of them. In this case, you should draw a free-body diagram showing only the particle of concern and the forces that particle experiences, as in Fig. 22-8*b*. If, instead, you choose to superimpose those forces on the given diagram showing all the particles, be sure to draw the force vectors with either their tails (preferably) or their heads on the particle of concern. If you draw the vectors elsewhere in the diagram, you invite confusion – and confusion is guaranteed if you draw the vectors of the particles causing the forces on the particle of concern.

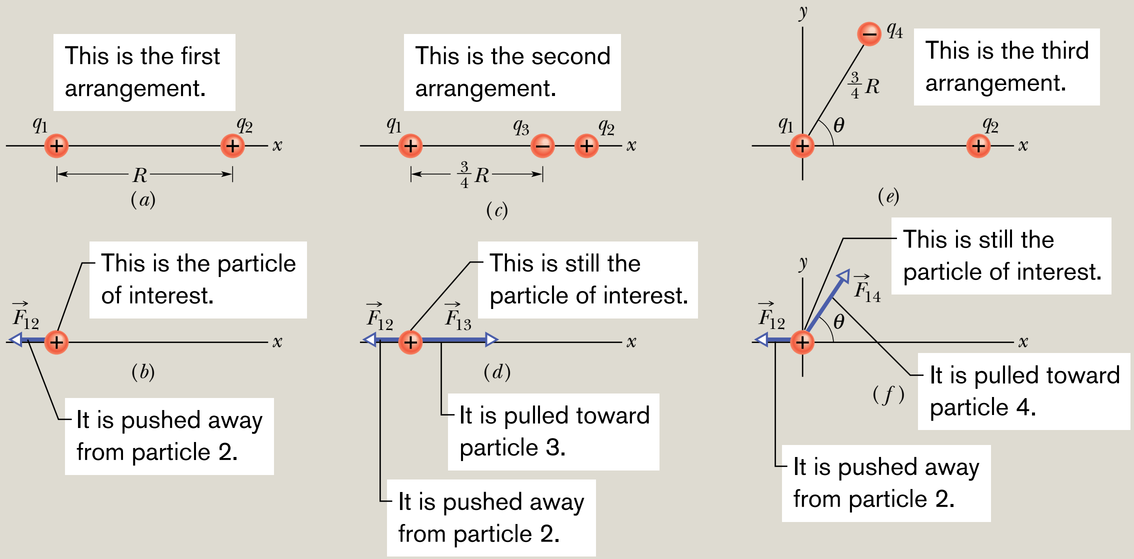

This is the first arrangement.
q_1
q_2
x
R
(a)
This is the second arrangement.
q_1
q_3
q_2
x
$\frac{3}{4}R$
(c)
y
q_4
This is the third arrangement.
$\frac{3}{4}R$
q_1
θ
q_2
x
(e)
This is the particle of interest.
$\vec{F}_{12}$
x
(b)
It is pushed away from particle 2.
This is still the particle of interest.
$\vec{F}_{12}$
$\vec{F}_{13}$
x
(d)
It is pulled toward particle 3.
It is pushed away from particle 2.
This is still the particle of interest.
y
$\vec{F}_{14}$
$\vec{F}_{12}$
θ
x
It is pulled toward particle 4.
(f)
It is pushed away from particle 2.

Figure 22-8 (*a*) Two charged particles of charges q_1 and q_2 are fixed in place on an x axis. (*b*) The free-body diagram for particle 1, showing the electrostatic force on it from particle 2. (*c*) Particle 3 included. (*d*) Free-body diagram for particle 1. (*e*) Particle 4 included. (*f*) Free-body diagram for particle 1.

Shell Theories. Analogous to the shell theories for the gravitational force, we have two shell theories for the electrostatic force:

Shell theory 1. A charged particle outside a shell with charge uniformly distributed on its surface is attracted or repelled as if the shell's charge were concentrated as a particle at its center.

Shell theory 2. A charged particle inside a shell with charge uniformly distributed on its surface has no net force acting on it due to the shell.

(In the first theory, we assume that the charge on the shell is much greater than the particle's charge. Thus the presence of the particle has negligible effect on the distribution of charge on the shell.)

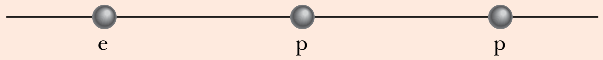
Spherical Conductors

If excess charge is placed on a spherical shell that is made of conducting material, the excess charge spreads uniformly over the (external) surface. For example, if we place excess electrons on a spherical metal shell, those electrons repel one another and tend to move apart, spreading over the available surface until they are uniformly distributed. That arrangement maximizes the distances between all pairs of the excess electrons. According to the first shell theorem, the shell then will attract or repel an external charge as if all the excess charge on the shell were concentrated at its center.

If we remove negative charge from a spherical metal shell, the resulting positive charge of the shell is also spread uniformly over the surface of the shell. For example, if we remove n electrons, there are then n sites of positive charge (sites missing an electron) that are spread uniformly over the shell. According to the first shell theorem, the shell will again attract or repel an external charge as if all the shell's excess charge were concentrated at its center.

CHECKPOINT 2

The figure shows two protons (symbol p) and one electron (symbol e) on an axis. On the central proton, what is the direction of (a) the force due to the electron, (b) the force due to the other proton, and (c) the net force?



SAMPLE PROBLEM 22.01

Finding the net force due to two other particles

This sample problem actually contains three examples and in each we have the same charged particle 1. First there is a single force acting on it. Then there are two forces, but they are just in opposite directions. Then there are again two forces but they are in very different directions.

(a) Figure 22-8*a* shows two positively charged particles fixed in place on an x axis. The charges are $q_1 = 1.60 \times 10^{-19}$ C and $q_2 = 3.20 \times 10^{-19}$ C, and the particle separation is $R = 0.0200$ m. What are the magnitude and direction of the electrostatic force $\vec{F}_{12}$ on particle 1 from particle 2?

KEY IDEAS

Because both particles are positively charged, particle 1 is repelled by particle 2, with a force magnitude given by Eq. 22-4. Thus, the direction of $\vec{F}_{12}$ force on particle 1 is *away from* particle 2, in the negative direction of the x axis, as indicated in the free-body diagram of Fig. 22-8*b*.

Two particles: Using Eq. 22-4 with separation R substituted for r, we can write the magnitude F_{12} of this force as

$$F_{12} = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{R^2}$$

$$= (8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2) \times \frac{(1.60 \times 10^{-19}\ \text{C})(3.20 \times 10^{-19}\ \text{C})}{(0.0200\ \text{m})^2}$$

$$= 1.15 \times 10^{-24}\ \text{N}.$$

Thus, force $\vec{F}_{12}$, has the following magnitude and direction (relative to the positive direction of the x axis):

$$1.15 \times 10^{-24}\,\text{N} \quad \text{and} \quad 180°. \qquad \text{(Answer)}$$

We can also write $\vec{F}_{12}$ in unit-vector notation as

$$\vec{F}_{12} = -(1.15 \times 10^{-24}\ \text{N})\,\text{i}. \qquad \text{(Answer)}$$

(b) Figure 22-8*c* is identical to Fig. 22-8*a* except that particle 3 now lies on the x axis between particles

1 and 2. Particle 3 has charge $q_3 = -3.20 \times 10^{-19}$ C and is at a distance (3/4)R from particle 1. What is the net electrostatic force $\vec{F}_{1,\text{net}}$ on particle 1 due to particles 2 and 3?

KEY IDEA

The presence of particle 3 does not alter the electrostatic force on particle 1 from particle 2. Thus, force $\vec{F}_{12}$ still acts on particle 1. Similarly, the force $\vec{F}_{13}$ that acts on particle 1 due to particle 3 is not affected by the presence of particle 2. Because particles 1 and 3 have charge of opposite signs, particle 1 is attracted to particle 3. Thus, force $\vec{F}_{13}$ is directed *toward* particle 3, as indicated in the free-body diagram of Fig. 22-8*d*.

Three particles: To find the magnitude of $\vec{F}_{13}$, we can rewrite Eq. 22-4 as

$$\begin{aligned} F_{13} &= \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_3|}{(\frac{3}{4}R)^2} \\ &= (8.99 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2) \\ &\quad \times \frac{(1.60 \times 10^{-19}\ \text{C})(3.20 \times 10^{-19}\ \text{C})}{(\frac{3}{4})^2(0.0200\ \text{m})^2} \\ &= 2.05 \times 10^{-24}\ \text{N}. \end{aligned}$$

We can also write $\vec{F}_{13}$ in unit-vector notation:

$$\vec{F}_{13} = (2.05 \times 10^{-24}\ \text{N})\,\hat{\text{i}}.$$

The net force $\vec{F}_{1\text{.net}}$ on particle 1 is the vector sum of $\vec{F}_{12}$, and $\vec{F}_{13}$; that is, from Eq. 22-7, we can write the net force $\vec{F}_{1\text{.net}}$ on particle 1 in unit-vector notation as

$$\begin{aligned} \vec{F}_{1,\text{net}} &= \vec{F}_{12} + \vec{F}_{13} \\ &= -(1.15 \times 10^{-24}\ \text{N})\hat{\text{i}} + (2.05 \times 10^{-24}\ \text{N})\hat{\text{i}} \\ &= (9.00 \times 10^{-25}\ \text{N})\hat{\text{i}}. \qquad \text{(Answer)} \end{aligned}$$

Thus, $\vec{F}_{1\text{.net}}$ has the following magnitude and direction (relative to the positive direction of the x axis):

9.00×10^{-25} N and 0°. (Answer)

(c) Figure 22-8*e* is identical to Fig. 22-8*a* except that particle 4 is now included. It has charge $q_4 = -3.20 \times 10^{-19}$ C, is at a distance $\frac{3}{4}R$ from particle 1, and lies on a line that makes an angle $\theta = 60°$ with the x axis. What is the net electrostatic electrostatic force $\vec{F}_{1\text{.net}}$ on particle 1 due to particles 2 and 4?

KEY IDEA

The net force $\vec{F}_{1\text{.net}}$ is the vector sum of $\vec{F}_{1\text{.net}}$ and a new force $\vec{F}_{14}$ acting on particle 1 due to particle 4. Because particles 1 and 4 have charge of opposite signs, particle 1 is attracted to particle 4. Thus, force $\vec{F}_{14}$ on particle 1 is directed *toward* particle 4, at angle $\theta = 60°$, as indicated in the free-body diagram of Fig. 22-8*f*.

Four particles: We can rewrite Eq. 22-4 as

$$\begin{aligned} F_{14} &= \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_4|}{(\frac{3}{4}R)^2} \\ &= (8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2) \\ &\quad \times \frac{(1.60 \times 10^{-19}\ \text{C})(3.20 \times 10^{-19}\ \text{C})}{(\frac{3}{4})^2(0.0200\ \text{m})^2} \\ &= 2.05 \times 10^{-24}\ \text{N}. \end{aligned}$$

Then from Eq. 22-7, we can write the net force on $\vec{F}_{1\text{.net}}$ particle 1 as

$$\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{14}.$$

Because the forces $\vec{F}_{12}$ and $\vec{F}_{14}$ are not directed along the same axis, we *cannot* sum simply by combining their magnitudes. Instead, we must add them as vectors, using one of the following methods.

Method 1. *Summing directly on a vector-capable calculator.* For $\vec{F}_{12}$, we enter the magnitude 1.15×10^{-24} and the angle 180° For $\vec{F}_{14}$, we enter the magnitude 2.05×10^{-24} and the angle 60°. Then we add the vectors.

Method 2. *Summing in unit-vector notation.* First we rewrite $\vec{F}_{14}$ as

$$\vec{F}_{14} = (F_{14}\cos\theta)\,\hat{\text{i}} + (F_{14}\sin\theta)\,\hat{\text{j}}.$$

Substituting 2.05×10^{-24} N for F_{14} and 60° for θ, this becomes

$$\vec{F}_{14} = (1.025 \times 10^{-24}\ \text{N})\,\hat{\text{i}} + (1.775 \times 10^{-24}\ \text{N})\,\hat{\text{j}}.$$

Then we sum:

$$\begin{aligned} \vec{F}_{1\text{,net}} &= \vec{F}_{12} + \vec{F}_{14} \\ &= -(1.15 \times 10^{-24}\ \text{N})\hat{\text{i}} \\ &\quad + (1.025 \times 10^{-24}\ \text{N})\hat{\text{i}} + (1.775 \times 10^{-24}\ \text{N})\hat{\text{j}} \\ &\approx (-1.25 \times 10^{-25}\ \text{N})\hat{\text{i}} + (1.78 \times 10^{-24}\ \text{N})\hat{\text{j}}. \qquad \text{(Answer)} \end{aligned}$$

Method 3. *Summing components axis by axis.* The sum of the x components gives us

$$\begin{aligned} F_{1,\text{net},x} &= F_{12,x} + F_{14,x} = F_{12} + F_{14}\cos 60° \\ &= -1.15\times10^{-24}\ \text{N} + (2.05\times10^{-24}\ \text{N})(\cos 60°) \\ &= -1.25\times10^{-25}\ \text{N}. \end{aligned}$$

The sum of the y components gives us

$$\begin{aligned} F_{1,\text{net},y} &= F_{12,y} + F_{14,y} = 0 + F_{14}\sin 60° \\ &= (2.05\times10^{-24}\ \text{N})(\sin 60°) \\ &= 1.78\times10^{-24}\ \text{N}. \end{aligned}$$

The net force $\vec{F}_{1\text{.net}}$ has the magnitude

$$F_{1,\text{net}} = \sqrt{F^2_{1,\text{net},x} + F^2_{1,\text{net},y}} = 1.78\times10^{-24}\ \text{N}. \qquad \text{(Answer)}$$

To find the direction of $\vec{F}_{1.\text{ net}}$, we take

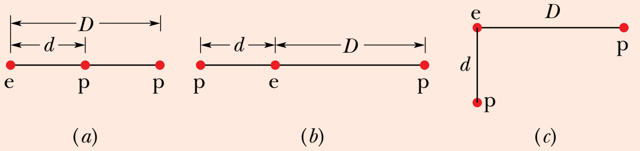
$$\theta = \tan^{-1}\frac{F_{1,\text{net},y}}{F_{1,\text{net},x}} = -86.0°.$$

However, this is an unreasonable result because $\vec{F}_{1\text{.net}}$ must have a direction between the directions of $\vec{F}_{12}$ and $\vec{F}_{14}$. To correct θ, we add 180°, obtaining

$$-86.0° + 180° = 94.0° \qquad \text{(Answer)}$$

CHECKPOINT 3

The figure here shows three arrangements of an electron e and two protons p. (a) Rank the arrangements according to the magnitude of the net electrostatic force on the electron due to the protons, largest first. (b) In situation *c*, is the angle between the net force on the electron and the line labeled *d* less than or more than 45°?



SAMPLE PROBLEM 22.02

Equilibrium of two forces on a particle

Figure 22.9*a* shows two particles fixed in place: a particle of charge $q_1 = +8q$ at the origin and a particle of charge $q_2 = -2q$ at $x = L$. At what point (other than infinitely far away) can a proton be placed so that it is in *equilibrium* (the net force on it is zero)? Is that equilibrium *stable* or *unstable*? (That is, if the proton is displaced, do the forces drive it back to the point of equilibrium or drive it farther away?)

KEY IDEA

If $\vec{F}_1$ is the force on the proton due to charge q_1 and $\vec{F}_1$ is the force on the proton due to charge q_2, then the point we seek is where Thus, $\vec{F}_1 + \vec{F}_2 = 0$. Thus,

$$\vec{F}_1 = -\vec{F}_2. \qquad (22\text{-}8)$$

This tells us that at the point we seek, the forces acting on the proton due to the other two particles must be of equal magnitudes

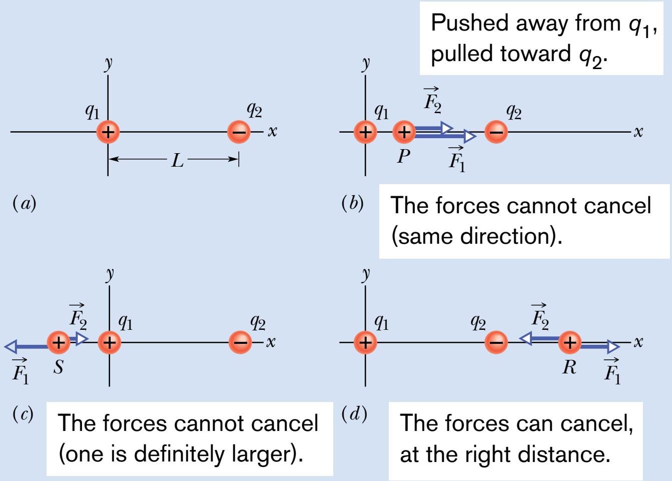
$$\vec{F}_1 = \vec{F}_2, \qquad (22\text{-}9)$$

and that the forces must have opposite directions.

Reasoning: Because a proton has a positive charge, the proton and the particle of charge q_1 are of the same sign, and force $\vec{F}_1$ on the proton must point away from q_1. Also, the proton and the particle of charge q_2 are of opposite signs, so force $\vec{F}_2$ on the proton must point toward q_2. "Away from q_1" and "toward q_2" can be in opposite directions only if the proton is located on the x axis.

If the proton is on the x axis at any point between q_1 and q_2, such as point P in Fig. 22-9*b*, then $\vec{F}_1$ and $\vec{F}_2$ are in the same direction and not in opposite directions as required. If the proton is at any point on the x axis to the left of q_1, such as point S in Fig. 22-9*c*, then $\vec{F}_1$ and $\vec{F}_2$ are in opposite directions. However, Eq. 22-4 tells us that $\vec{F}_1$ and $\vec{F}_2$ cannot have equal magnitudes there: F_1 must be greater than F_2, because F_1 is produced by a closer charge (with lesser r) of greater magnitude ($8q$ versus $2q$).

Finally, if the proton is at any point on the x axis to the right of q_2, such as point R in Fig. 22-9*d*, then $\vec{F}_1$ and $\vec{F}_2$ are again in opposite directions. However, because now



Pushed away from q_1, pulled toward q_2.

The forces cannot cancel (same direction).

The forces cannot cancel (one is definitely larger).

The forces can cancel, at the right distance.

Figure 22-9 (*a*) Two particles of charges q_1 and q_2 are fixed in place on an x axis, with separation L. (*b*)–(*d*) Three possible locations P, S, and R for a proton. At each location, $\vec{F}_1$ is the force on the proton from particle 1 and $\vec{F}_2$ is the force on the proton from particle 2.

the charge of greater magnitude (q_1) is *farther* away from the proton than the charge of lesser magnitude, there is a point at which F_1 is equal to F_2. Let x be the coordinate of this point, and let q_p be the charge of the proton.

Calculations: With Eq. 22-4, we can now rewrite Eq. 22-9:

$$\frac{1}{4\pi\varepsilon_0}\frac{8qq_\text{p}}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{2qq_\text{p}}{(x-L)^2}. \tag{22-10}$$

(Note that only the charge magnitudes appear in Eq. 22-10. We already decided about the directions of the forces in drawing Fig. 22-9*d* and do not want to include any positive or negative signs here.) Rearranging Eq. 22-10 gives us

$$\left(\frac{x-L}{x}\right)^2 = \frac{1}{4}.$$

After taking the square roots of both sides, we find

$$\frac{x-L}{x} = \frac{1}{2}$$

and

$$x = 2L. \qquad \text{(Answer)}$$

The equilibrium at $x = 2L$ is unstable; that is, if the proton is displaced leftward from point R, then F_1 and F_2 both increase but F_2 increases more (because q_2 is closer than q_1), and a net force will drive the proton farther leftward. If the proton is displaced rightward, both F_1 and F_2 decrease but F_2 decreases more, and a net force will then drive the proton farther rightward. In a stable equilibrium, if the proton is displaced slightly, it returns to the equilibrium position.

SAMPLE PROBLEM 22.03

Magnitude of charge on two identical spheres hanging in equilibrium

Two identical, small, charged spheres, each having a mass of 4.0×10^{-2} kg hang in equilibrium as shown in Fig. 22-10*a*. The length of each string is 1.5 m and the angle θ is 37.0°. Find the magnitude of the charge on each sphere.

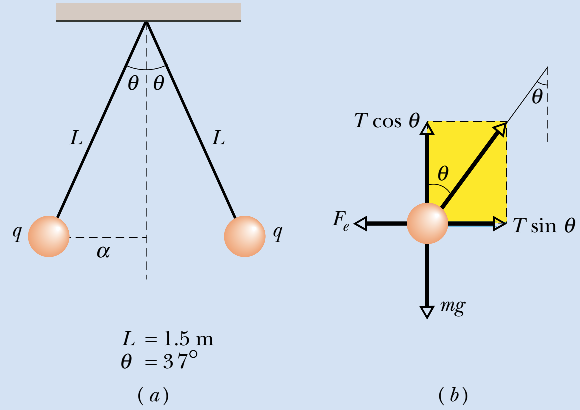


Figure 22-10 (*a*) Two small charged spheres in equilibrium. (*b*) Free body diagram of left sphere.

KEY IDEA

From Fig. 22-10*a*, we can see that the two spheres exert repulsive forces on each other. If they are held close to each other and released, they will move outward from the center and settle into the configuration in Fig. 22-10*a* after the damped oscillations due to air resistance have vanished.

The key phrase "in equilibrium" helps us to classify this as an equilibrium problem, which we approach as we did equilibrium problems in the chapter on Newton's Laws (in Volume 1) with the added feature that one of the forces on a sphere is an electric force. We analyze this problem by drawing the free-body diagram for the left-hand sphere in Fig. 22-10*b*. The sphere is in equilibrium under the application of forces T from the string, the electric force F_e from the other sphere, and the gravitational force mg.

Calculations: Because the sphere is in equilibrium, the forces in the horizontal and vertical directions must separately add up to zero.

$$\sum F_x = T\sin\theta - F_e \tag{22-11}$$

$$\sum F_y = T\cos\theta - mg \tag{22-12}$$

From Eq. 22-12, we see that $T = mg/\cos\theta$; thus, T can be eliminated from Eq. 22-11 if we make this substitution. This gives a value for the magnitude of the electric force F_e:

$$F_e = mg\tan\theta = (4\times10^{-2}\,\text{kg})(10\ \text{m/s}^2)\tan 37° = 0.3\ \text{N}.$$

Considering the geometry of the triangle in Fig. 22-10*a*, we see that $\sin\theta = \alpha/L$. Therefore,

$$a = L\sin\theta = (1.5\ \text{m})\sin 37° = 0.9\,\text{m}$$

The separation of the spheres is $2a$ = 1.8 m. From Coulomb's law (Eq. 22-10), the magnitude of the electric force is

$$F_e = k_e\frac{|q|^2}{r^2}$$

where $r = 2a = 1.8$ m and $|q|$ is the magnitude of the charge on each sphere. (Note that the term $|q|^2$ arises here because the charge is the same on both spheres.) This equation can be solved for $|q|^2$ to give

$$|q|^2 = \frac{F_e r^2}{k_e} = \frac{0.3\times(1.8)^2}{9\times10^9\ \text{Nm}^2/\text{C}^2} = 1.08\times10^{-10}\ \text{C}^2$$

$$|q| = 6\sqrt{3}\,\mu\text{C}$$

Learn: In this problem, we found only the magnitude of the charge $|q|$ on the spheres. There is no way we could find the sign on the charge from the information given. In fact, the sign of the charge is not important. The situation will be the same whether both spheres are positively charged or negatively charged.

Even if the charges on spheres are not equal, the symmetry is not destroyed and the angles remain the same. Newton's third law requires that the electric forces on the two charges be the same, regardless of the equality or nonequality of the charges. The solution to the example remains the same through the calculation of $|q|^2$. In this situation, the value of $1.08 \times 10^{-10}\ \text{C}^2$ corresponds to the product q_1q_2, where q_1 and q_2 are the values of the charges on the two spheres. The symmetry of the problem would be destroyed if the masses of the spheres were not the same. In this case, the strings would make different angles with the vertical and the problem would be more complicated.

CHECKPOINT 4

Assume that charges were initially different. If the charges on them were equally distributed would the angle of divergence increase, decrease, or stay the same?

22.4 | CHARGE IS QUANTIZED

Key Concepts

- Electric charge is quantized (restricted to certain values).
- The charge of a particle can be written as ne, where n is a positive or negative integer and e is the elementary charge, which is the magnitude of the charge of the electron and proton (≈ 1.602×10^{-19} C).

In Benjamin Franklin's day, electric charge was thought to be a continuous fluid—an idea that was useful for many purposes. However, we now know that fluids themselves, such as air and water, are not continuous but are made up of atoms and molecules; matter is discrete. Experiment shows that "electrical fluid" is also not continuous but is made up of multiples of a certain elementary charge. Any positive or negative charge q that can be detected can be written as

$$q = ne, \quad n = \pm 1, \pm 2, \pm 3, \ldots, \tag{22-13}$$

in which e, the **elementary charge**, has the approximate value

$$e = 1.602 \times 10^{-19} \text{ C}. \tag{22-14}$$

The elementary charge e is one of the important constants of nature. The electron and proton both have a charge of magnitude e (Table 22-1). (Quarks, the constituent particles of protons and neutrons, have charges of $\pm e/3$ or $\pm 2e/3$, but they apparently cannot be detected individually. For this and for historical reasons, we do not take their charges to be the elementary charge.)

Table 22-1 The Charges of Three Particles

Particle	Symbol	Charge
Electron	e or e^-	$-e$
Proton	p	$+e$
Neutron	n	0

You often see phrases—such as "the charge on a sphere," "the amount of charge transferred," and "the charge carried by the electron"—that suggest that charge is a substance. (Indeed, such statements have already appeared in this chapter.) You should, however, keep in mind what is intended: *Particles* are the substance and charge happens to be one of their properties, just as mass is.

When a physical quantity such as charge can have only discrete values rather than any value, we say that the quantity is **quantized**. It is possible, for example, to find a particle that has no charge at all or a charge of $+10e$ or $-6e$, but not a particle with a charge of, say, $3.57e$.

The quantum of charge is small. In an ordinary 100 W lightbulb, for example, about 10^{19} elementary charges enter the bulb every second and just as many leave. However, the graininess of electricity does not show up in such large-scale phenomena (the bulb does not flicker with each electron).

In the expression for current (Eq. 22-3), q has discrete values and is hence a non-differentiable function. However, we know that the charges we encounter have a large number of electrons, so q behaves like a continous function. Therefore, for very small or very large values of charge, the current is given, respectively, by

$$i = \frac{\Delta q}{\Delta t} \quad \text{and} \quad i = \frac{dq}{dt}.$$

CHECKPOINT 5

Initially, sphere A has a charge of $-50e$ and sphere B has a charge of $+20e$. The spheres are made of conducting material and are identical in size. If the spheres then touch, what is the resulting charge on sphere A?

SAMPLE PROBLEM 22.04

Mutual electric repulsion in a nucleus

The nucleus in an iron atom has a radius of about 4.0×10^{-15} m and contains 26 protons.

(a) What is the magnitude of the repulsive electrostatic force between two of the protons that are separated by 4.0×10^{-15} m?

KEY IDEAS

The protons can be treated as charged particles, so the magnitude of the electrostatic force on one from the other is given by Coulomb's law.

Calculation: Table 22-1 tells us that the charge of a proton is $+e$. Thus, Eq. 22-4 gives us

$$\begin{aligned} F &= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \\ &= \frac{(8.99 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2)(1.602 \times 10^{-19} \text{ C})^2}{(4.0 \times 10^{-5} \text{ m})^2} \\ &= 14 \text{ N}. \qquad \text{(Answer)} \end{aligned}$$

No explosion: This is a small force to be acting on a macroscopic object like a cantaloupe, but an enormous force to be acting on a proton. Such forces should explode the nucleus of any element but hydrogen (which has only one proton in its nucleus). However, they don't, not even in nuclei with a great many protons. Therefore, there must be some enormous attractive force to counter this enormous repulsive electrostatic force.

(b) What is the magnitude of the gravitational force between those same two protons?

KEY IDEAS

Because the protons are particles, the magnitude of the gravitational force on one from the other is given by Newton's equation for the gravitational force (Eq. 22-2).

Calculation: With $m_p (= 1.67 \times 10^{-27}$ kg) representing the mass of a proton, Eq. 22-2 gives us

$$
\begin{aligned}
F &= G\frac{m_p^2}{r^2} \\
&= \frac{(6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2})(1.67\times10^{-27}\ \mathrm{kg})^2}{(4.0\times10^{-15}\ \mathrm{m})^2} \\
&= 1.2\times10^{-35}\ \mathrm{N}. \qquad \text{(Answer)}
\end{aligned}
$$

Weak versus strong: This result tells us that the (attractive) gravitational force is far too weak to counter the repulsive electrostatic forces between protons in a nucleus. Instead, the protons are bound together by an enormous force called (aptly) the *strong nuclear force*—a force that acts between protons (and neutrons) when they are close together, as in a nucleus.

Although the gravitational force is many times weaker than the electrostatic force, it is more important in largescale situations because it is always attractive. This means that it can collect many small bodies into huge bodies with huge masses, such as planets and stars, that then exert large gravitational forces. The electrostatic force, on the other hand, is repulsive for charges of the same sign, so it is unable to collect either positive charge or negative charge into large concentrations that would then exert large electrostatic forces.

22.5 | CHARGE IS CONSERVED

Key Concepts

- The net electric charge of any isolated system is always conserved.
- If two charged particles undergo an annihilation process, they have opposite signs of charge.
- If two charged particles appear as a result of a pair production process, they have opposite signs of charge.

If you rub a glass rod with silk, a positive charge appears on the rod. Measurement shows that a negative charge of equal magnitude appears on the silk. This suggests that rubbing does not create charge but only transfers it from one body to another, upsetting the electrical neutrality of each body during the process. This hypothesis of **conservation of charge**, first put forward by Benjamin Franklin, has stood up under close examination, both for large-scale charged bodies and for atoms, nuclei, and elementary particles. No exceptions have ever been found. Thus, we add electric charge to our list of quantities—including energy and both linear momentum and angular momentum—that obey a conservation law.

Important examples of the conservation of charge occur in the *radioactive decay* of nuclei, in which a nucleus transforms into (becomes) a different type of nucleus. For example, a uranium-238 nucleus (^{238}U) transforms into a thorium-234 nucleus (^{234}Th) by emitting an *alpha particle.* Because that particle has the same makeup as a helium-4 nucleus, it has the symbol ^{4}He. The number used in the name of a nucleus and as a superscript in the symbol for the nucleus is called the *mass number* and is the total number of the protons and neutrons in the nucleus. For example, the total number in ^{238}U is 238. The number of protons in a nucleus is the *atomic number* Z, which is listed for all the elements in Appendix F. From that list we find that in the decay

$$^{238}\mathrm{U} \rightarrow {}^{234}\mathrm{Th} + {}^{4}\mathrm{He}, \tag{22-15}$$

the *parent* nucleus ^{238}U contains 92 protons (a charge of $+92e$), the *daughter* nucleus ^{234}Th contains 90 protons (a charge of $+90e$), and the emitted alpha particle ^{4}He contains 2 protons (a charge of $+2e$). We see that the total charge is +92e before and after the decay; thus, charge is conserved. (The total number of protons and neutrons is also conserved: 238 before the decay and 234 + 4 = 238 after the decay.)

Another example of charge conservation occurs when an electron e^- (charge $-e$) and its antiparticle, the *positron* e^+ (charge $+e$), undergo an *annihilation process,* transforming into two *gamma rays* (high-energy light):

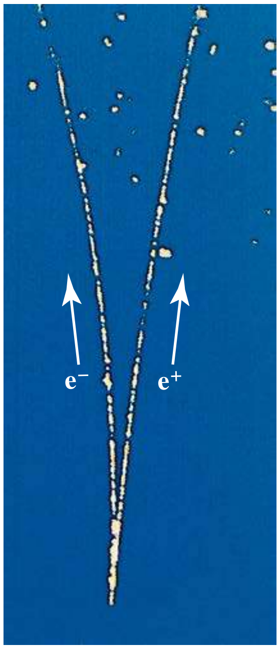
$$e^- + e^+ \rightarrow \gamma + \gamma \quad \text{(annihilation)}. \tag{22-16}$$

In applying the conservation-of-charge principle, we must add the charges algebraically, with due regard for their signs. In the annihilation process of Eq. 22-15 then, the net charge of the system is zero both before and after the event. Charge is conserved.

In *pair production*, the converse of annihilation, charge is also conserved. In this process a gamma ray transforms into an electron and a positron:

$$\gamma \rightarrow e^- + e^+ \quad \text{(pair production)}. \tag{22-17}$$

Figure 22-11 shows such a pair-production event that occurred in a bubble chamber. (This is a device in which a liquid is suddenly made hotter than its boiling point. If a charged particle passes through it, tiny vapor bubbles form along the particle's trail.) A gamma ray entered the chamber from the bottom and at one point transformed into an electron and a positron. Because those new particles were charged and moving, each left a trail of bubbles. (The trails were curved because a magnetic field had been set up in the chamber.) The gamma ray, being electrically neutral, left no trail. Still, you can tell exactly where it underwent pair production—at the tip of the curved V, which is where the trails of the electron and positron begin.



Courtesy Lawrence Berkeley Laboratory

Figure 22-11 A photograph of trails of bubbles left in a bubble chamber by an electron and a positron. The pair of particles was produced by a gamma ray that entered the chamber directly from the bottom. Being electrically neutral, the gamma ray did not generate a telltale trail of bubbles along its path, as the electron and positron did.

22.6 | THE ELECTRIC FIELD

Key Concepts

- A charged particle sets up an electric field (a vector quantity) in the surrounding space. If a second charged particle is located in that space, an electrostatic force acts on it due to the magnitude and direction of the field at its location.
- The electric field $\vec{E}$ at any point is defined in terms of the electrostatic force $\vec{F}$ that would be exerted on a positive test charge q_0 placed there:

$$\vec{E} = \frac{\vec{F}}{q_0}$$

A lot of different fields are used in science and engineering. For example, a *temperature field* for an auditorium is the distribution of temperatures we would find by measuring the temperature at many points within the auditorium. Similarly, we could define a *pressure field* in a swimming pool. Such fields are examples of *scalar fields* because temperature and pressure are scalar quantities, having only magnitudes and not directions.

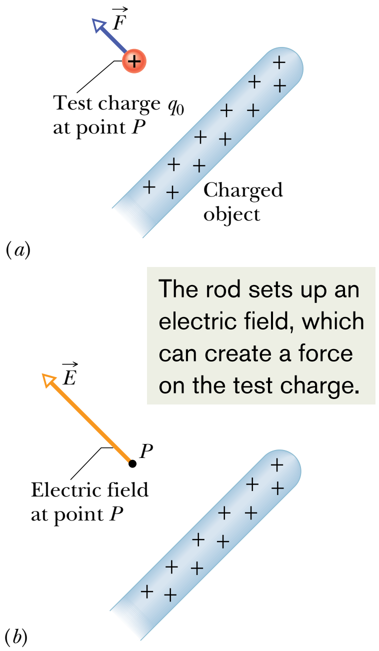
In contrast, an electric field is a *vector field* because it is responsible for conveying the information for a force, which involves both magnitude and direction. This field consists of a distribution of electric field vectors $\vec{E}$, one for each point in the space around a charged object. In principle, we can define $\vec{E}$ at some point near the charged object, such as point P in Fig. 22-12*a*, with this procedure: At P, we place a particle with a small positive charge q_0, called a *test charge* because we use it to test the field. (We want the charge to be small so that it does not disturb the object's charge distribution.) We then measure the electrostatic force $\vec{F}$ that acts on the test charge. The electric field at that point is then

$$\vec{E} = \frac{\vec{F}}{q_0} \quad \text{(electric field)}. \tag{22-18}$$

Because the test charge is positive, the two vectors in Eq. 22-18 are in the same direction, so the direction of $\vec{E}$ is the direction we measure for $\vec{F}$. The magnitude of $\vec{E}$ at point P is F/q_0. As shown in Fig. 22-12b, we always represent an electric field with an arrow with its tail anchored on the point where the measurement is made. (This may sound trivial, but drawing the vectors any other way usually results in errors. Also, another common error is to mix up the terms *force* and *field* because they both start with the letter f. Electric force is a push or pull. Electric field is an abstract property set up by a charged object.) From Eq. 22-16, we see that the SI unit for the electric field is the newton per coulomb (N/C).

We can shift the test charge around to various other points, to measure the electric fields there, so that we can figure out the distribution of the electric field set up by the charged object. That field exists independent of the test charge. It is something that a charged object sets up in the surrounding space (even vacuum), independent of whether we happen to come along to measure it.

For the next several sections, we determine the field around charged particles and various charged objects. First, however, let's examine a way of visualizing electric fields.



The rod sets up an electric field, which can create a force on the test charge.

Figure 22-12 (*a*) A positive test charge q_0 placed at point P near a charged object. An electrostatic force $\vec{F}$ acts on the test charge. (*b*) The electric field $\vec{E}$ at point P produced by the charged object.

22.7 | ELECTRIC FIELD LINES

Key Concepts

- Electric field lines help us visualize the direction and magnitude of electric fields. The electric field vector at any point is tangent to the field line through that point. The density of field lines in that region is proportional to the magnitude of the electric field there. Thus, closer field lines represent a stronger field.
- Electric field lines originate on positive charges and terminate on negative charges. So, a field line extending from a positive charge must end on a negative charge.

Look at the space in the room around you. Can you visualize a field of vectors throughout that space—vectors with different magnitudes and directions? As impossible as that seems, Michael Faraday, who introduced the idea of electric fields in the 19th century, found a way. He envisioned lines, now called **electric field lines**, in the space around any given charged particle or object.

Figure 22-13 gives an example in which a sphere is uniformly covered with negative charge. If we place a positive test charge at any point near the sphere (Fig. 22-13*a*), we find that an electrostatic force pulls on it toward the center of the sphere. Thus at every point around the sphere, an electric field vector points radially inward toward the sphere. We can represent this electric field with electric field lines as in Fig. 22-13*b*. At any point, such as the one shown, the direction of the field line through the point matches the direction of the electric vector at that point.

The rules for drawing electric fields lines are these: (1) At any point, the electric field vector must be tangent to the electric field line through that point and in the same direction. (This is easy to see in Fig. 22-13 where the lines are straight, but we'll see some curved lines soon.) (2) In a plane perpendicular to the field lines, the relative density of the lines represents the relative magnitude of the field there, with greater density for greater magnitude. So for a greater charge, the electric field should be stronger, and hence the number of field lines in the space would be proportionately more.

If the sphere in Fig. 22-13 were uniformly covered with positive charge, the electric field vectors at all points around it would be radially outward and thus so would the electric field lines. So, we have the following rule:

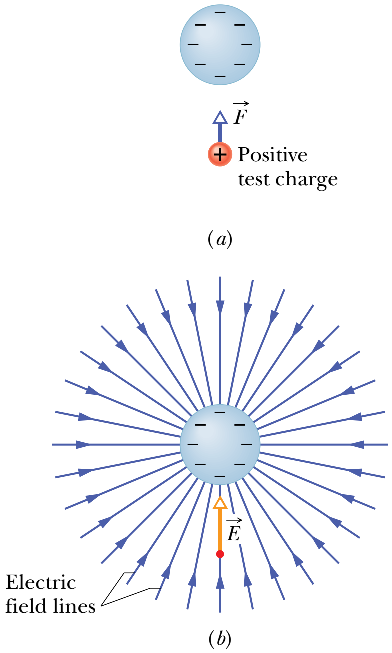


Figure 22-13 (*a*) The electrostatic force $\vec{F}$ acting on a positive test charge near a sphere of uniform negative charge. (*b*) The electric field vector $\vec{E}$ at the location of the test charge, and the electric field lines in the space near the sphere. The field lines extend *toward* the negatively charged sphere. (They originate on distant positive charges.)

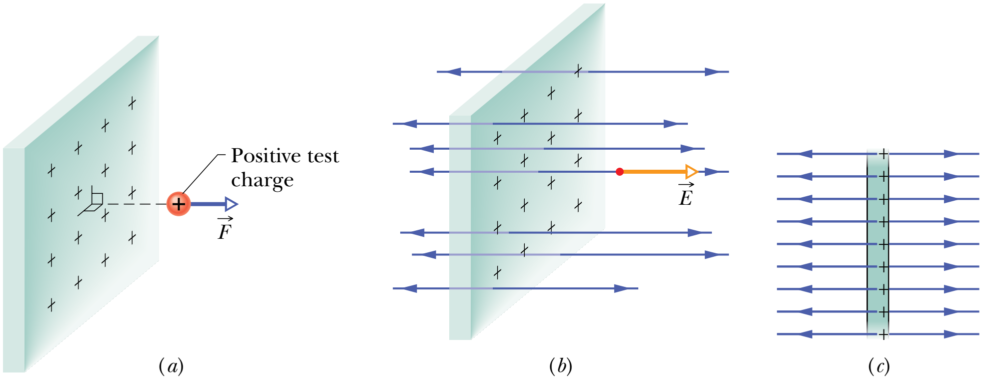


Figure 22-14 (*a*) The force on a positive test charge near a very large, nonconducting sheet with uniform positive charge on one side. (*b*) The electric field vector $\vec{E}$ at the test charge's location, and the nearby electric field lines, extending away from the sheet. (*c*) Side view.

Electric field lines extend away from positive charge (where they originate) and toward negative charge (where they terminate).

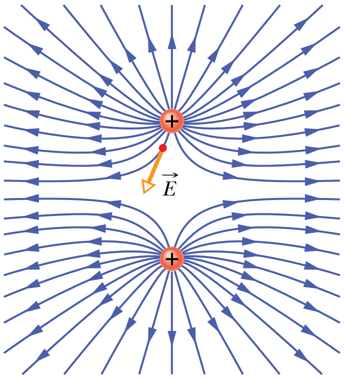


Figure 22-15 Field lines for two particles with equal positive charge. Doesn't the pattern itself suggest that the particles repel each other?

In Fig. 22-13*b*, they originate on distant positive charges that are not shown.

For another example, Fig. 22-14*a* shows part of an infinitely large, nonconducting *sheet* (or plane) with a uniform distribution of positive charge on one side. If we place a positive test charge at any point near the sheet (on either side), we find that the electrostatic force on the particle is outward and perpendicular to the sheet. The perpendicular orientation is reasonable because any force component that is, say, upward is balanced out by an equal component that is downward. That leaves only outward, and thus the electric field vectors and the electric field lines must also be outward and perpendicular to the sheet, as shown in Figs. 22-14*b* and *c*.

Because the charge on the sheet is uniform, the field vectors and the field lines are also. Such a field is a *uniform electric field*, meaning that the electric field has the same magnitude and direction at every point within the field. (This is a lot easier to work with than a *nonuniform field*, where there is variation from point to point.) Of course, there is no such thing as an infinitely large sheet. That is just a way of saying that we are measuring the field at points close to the sheet relative to the size of the sheet and that we are not near an edge.

Figure 22-15 shows the field lines for two particles with equal positive charges. Now the field lines are curved, but the rules still hold: (1) the electric field vector at any given point must be tangent to the field line at that point and in the same direction, as shown for one vector, and (2) a closer spacing means a larger field magnitude. To imagine the full three-dimensional pattern of field lines around the particles, mentally rotate the pattern in Fig. 22-14 around the *axis of symmetry*, which is a vertical line through both particles.

22.8 | THE ELECTRIC FIELD DUE TO A POINT CHARGE

Key Concepts

- The magnitude of the electric field $\vec{E}$ set up by a particle with charge q at distance r from the particle is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2}.$$

- The electric field vectors set up by a positively charged particle all point directly away from the particle. Those set up by a negatively charged particle all point directly toward the particle.
- If more than one charged particle sets up an electric field at a point, the net electric field is the vector sum of the individual electric fields—electric fields obey the superposition principle.

To find the electric field due to a charged particle (often called a *point charge*), we place a positive test charge at any point near the particle, at distance r. From Coulomb's law (Eq. 22-4), the force on the test charge due to the particle with charge q is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{\mathrm{r}}.$$

As previously, the direction of $\vec{F}$ is directly away from the particle if q is positive (because q_0 is positive) and directly toward it if q is negative. From Eq. 22-18, we can now write the electric field set up by the particle (at the location of the test charge) as

$$\vec{E} = \frac{\vec{F}}{q_0} - \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathrm{r}} \quad \text{(charge particle).} \tag{22-19}$$

Let's think through the directions again. The direction of $\vec{E}$ matches that of the force on the positive test charge: directly away from the point charge if q is positive and directly toward it if q is negative.

So, if given another charged particle, we can immediately determine the directions of the electric field vectors near it by just looking at the sign of the charge q. We can find the magnitude at any given distance r by converting Eq. 22-19 to a magnitude form:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2} \quad \text{(charge particle).} \tag{22-20}$$

We write $|q|$ to avoid the danger of getting a negative E when q is negative, and then thinking the negative sign has something to do with direction. Equation 22-3 gives magnitude E only. We must think about the direction separately.

Figure 22-16 gives a number of electric field vectors at points around a positively charged particle, but be careful. Each vector represents the vector quantity at the point where the tail of the arrow is anchored. The vector is not something that stretches from a "here" to a "there" as with a displacement vector.

In general, if several electric fields are set up at a given point by several charged particles, we can find the net field by placing a positive test particle at the point and then writing out the force acting on it due to each particle, such as $\vec{F}_{01}$ due to particle 1. Forces obey the principle of superposition, so we just add the forces as vectors:

$$\vec{F}_0 = \vec{F}_{01} + \vec{F}_{02} + \cdots + \vec{F}_{0n}.$$

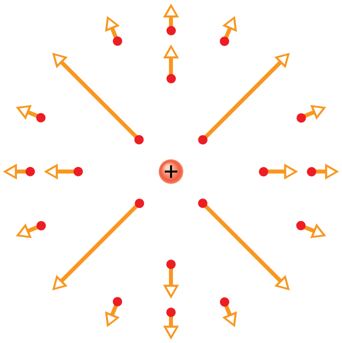


Figure 22-16 The electric field vectors at various points around a positive point charge.

To change over to electric field, we repeatedly use Eq. 22-16 for each of the individual forces:

$$\begin{aligned}\vec{E} &= \frac{\vec{F}_0}{q_0} = \frac{\vec{F}_{01}}{q_0} + \frac{\vec{F}_{02}}{q_0} + \cdots + \frac{\vec{F}_{0n}}{q_0} \\ &= \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n.\end{aligned} \tag{22-21}$$

This tells us that electric fields also obey the principle of superposition. If you want the net electric field at a given point due to several particles, find the electric field due to each particle (such as $\vec{E}_1$ due to particle 1) and then sum the fields as vectors. Figures 22-17*a* and *b* graphically represent the variation of the electric field due to a positive and a negative point charge, respectively, as we move away from the charge. For a positive point charge (Fig. 22-17*a*), the electric field at positive x is positive because it is in the positive direction; and it is negative at negative x because it is in the negative direction. For a negative point charge (Fig. 22.17*b*), the electric field at positive x is negative and it is positive at negative x.

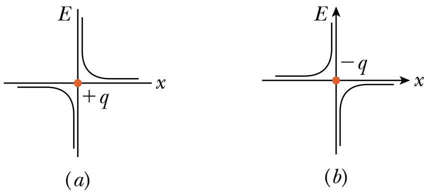
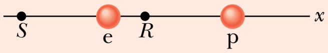


Figure 22-17 Graphical representation of (*a*) Electric field versus x for a positive point charge kept at the origin. (*b*) Electric field versus x for a negative point charge kept at the origin.

CHECKPOINT 6

The figure here shows a proton p and an electron e on an x axis. What is the direction of the electric field due to the electron at (a) point S and (b) point R? What is the direction of the net electric field at (c) point R and (d) point S?



SAMPLE PROBLEM 22.05

Net electric field due to three charged particles

Figure 22-18*a* shows three particles with charges $q_1 = +2Q$, $q_2 = -2Q$, and $q_3 = -2Q$, each a distance d from the origin. What net electric field $\vec{E}$ is produced at the origin?

KEY IDEAS

Charges q_1, q_2, and q_3 produce electric field vectors $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$, respectively, at the origin, and the net electric field is the vector sum $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$. To find this sum, we first must find the magnitudes and orientations of the three field vectors.

Magnitudes and directions: To find the magnitude of $\vec{E}_1$, which is due to q_1, we use Eq. 22-20, substituting d for r and $2Q$ for q and obtaining

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2}.$$

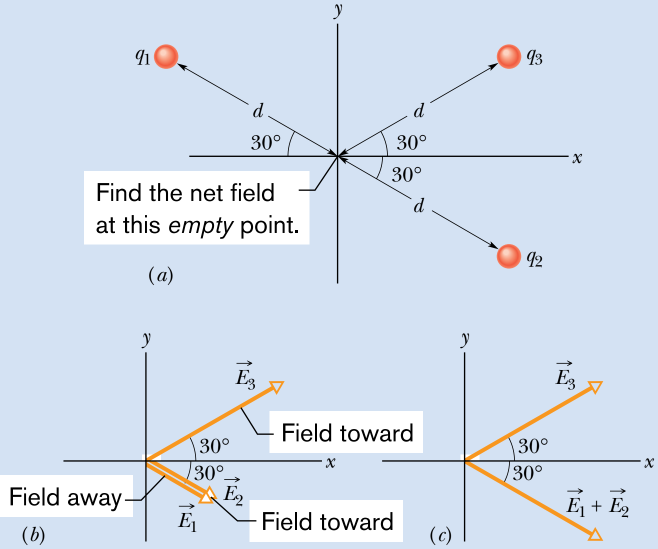


Figure 22-18 (*a*) Three particles with charges q_1, q_2, and q_3 are at the same distance d from the origin. (*b*) The electric field vectors $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$, at the origin due to the three particles. (*c*) The electric field vector $\vec{E}_3$ and the vector sum $\vec{E}_1 + \vec{E}_2$ at the origin.

Similarly, we find the magnitudes of $\vec{E}_2$ and $\vec{E}_3$ to be

$$E_2 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2} \quad \text{and} \quad E_3 = \frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2}.$$

We next must find the orientations of the three electric field vectors at the origin. Because q_1 is a positive charge, the field vector it produces points directly *away* from it, and because q_2 and q_3 are both negative, the field vectors they produce point directly *toward* each of them. Thus, the three electric fields produced at the origin by the three charged particles are oriented as in Fig. 22-18*b*. (*Caution:* Note that we have placed the tails of the vectors at the point where the fields are to be evaluated; doing so decreases the chance of error. Error becomes very probable if the tails of the field vectors are placed on the particles creating the fields.)

Adding the fields: We can now add the fields vectorially just as we add force vectors. However, here we can use symmetry to simplify the procedure. From Fig. 22-18*b*, we see that electric fields $\vec{E}_1$ and $\vec{E}_2$ have the same direction. Hence, their vector sum has that direction and has the magnitude

$$E_1 + E_2 = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2} + \frac{1}{4\pi\varepsilon_0}\frac{2Q}{d^2} = \frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2},$$

which happens to equal the magnitude of field $\vec{E}_3$.

We must now combine two vectors, $\vec{E}_3$ and the vector sum $\vec{E}_1 + \vec{E}_2$, that have the same magnitude and that are oriented symmetrically about the x axis, as shown in Fig. 22-18*c*. From the symmetry of Fig. 22-18*c*, we realize that the equal y components of our two vectors cancel (one is upward and the other is downward) and the equal x components add (both are rightward). Thus, the net electric field $\vec{E}$ at the origin is in the positive direction of the x axis and has the magnitude

$$E = 2E_{3x} = 2E_3 \cos 30°$$

$$= (2)\frac{1}{4\pi\varepsilon_0}\frac{4Q}{d^2}(0.866) = \frac{6.93Q}{4\pi\varepsilon_0 d^2}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 22.06

Net electric field due to two charged particles fixed on an axis

In Fig. 22-19, particle 1 of charge $q_1 = -5.00q$ and a particle 2 of charge $q_2 = +2.00q$ are fixed on the x-axis.

(a) As a multiple of distance L, at what coordinate on the axis is the net electric filed of the particles zero?

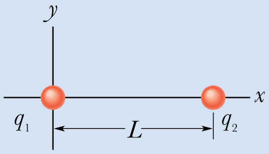


Figure 22-19 Two particles with charges q_1 and q_2 fixed on x-axis.

KEY IDEA

The individual magnitudes $|\vec{E}_1|$ and $|\vec{E}_2|$ are figured from Eq. 22-20, where the absolute value signs for q_2 are unnecessary since this charge is positive. Whether we add the magnitudes or subtract them depends on whether $\vec{E}_1$ is in the same, or opposite, direction as $\vec{E}_2$.

Magnitude and direction: At points left of q_1 (on the $-x$ axis) the fields point in opposite directions, but there is no possibility of cancellation (zero net field) since $|\vec{E}_1|$ is everywhere bigger than $|\vec{E}_2|$ in this region. In the region between the charges ($0 < x < L$) both fields point leftward and there is no possibility of cancellation. At points to the right of q_2 (where $x > L$), $\vec{E}_1$ points leftward and $\vec{E}_2$ points rightward so the net field in this range is

$$\vec{E}_{\text{net}} = (|\vec{E}_2| - |\vec{E}_1|)\hat{\text{i}}$$

Net field zero: Although $|q_1| > q_2$ there is the possibility of $\vec{E}_{\text{net}} = 0$ since these points are closer to q_2 than to q_1. Thus, we look for the zero net field point in the $x > L$ region:

$$|\vec{E}_1| = |\vec{E}_2| \quad \Rightarrow \quad \frac{1}{4\pi\varepsilon_0}\frac{|q_1|}{x^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{(x-L)^2}$$

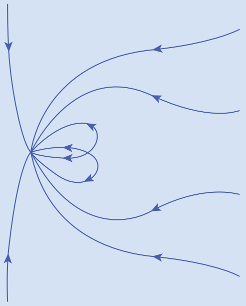
which leads to

$$\frac{x-L}{x} = \sqrt{\frac{q_2}{|q_1|}} = \sqrt{\frac{2}{5}}.$$

Thus, we obtain $x = \dfrac{L}{1-\sqrt{\frac{2}{5}}} \approx 2.72\ L.$

(b) Sketch the net electric field lines between and around the particles.

A sketch of the field lines is shown in the following figure:



22.9 | THE ELECTRIC FIELD DUE TO AN ELECTRIC DIPOLE

Key Concepts

- An electric dipole consists of two particles with charges of equal magnitude q but opposite signs, separated by a small distance d.
- The electric dipole moment $\vec{p}$ has magnitude qd and points from the negative charge to the positive charge.
- The magnitude of the electric field set up by an electric dipole at a distant point on the dipole axis (which runs through both particles) can be written in terms of either the product qd or the magnitude p of the dipole moment:

$$E = \frac{1}{2\pi\varepsilon_0}\frac{qd}{z^3} = \frac{1}{2\pi\varepsilon_0}\frac{p}{z^3},$$

where z is the distance between the point and the center of the dipole.
- Because of the $1/z^3$ dependence, the field magnitude of an electric dipole decreases more rapidly with distance than the field magnitude of either of the individual charges forming the dipole, which depends on $1/r^2$.

Figure 22-20 shows the pattern of electric field lines for two particles that have the same charge magnitude q but opposite signs, a very common and important arrangement known as an **electric dipole**. The particles are separated by distance d and lie along the *dipole axis*, an axis of symmetry around which you can imagine rotating the pattern

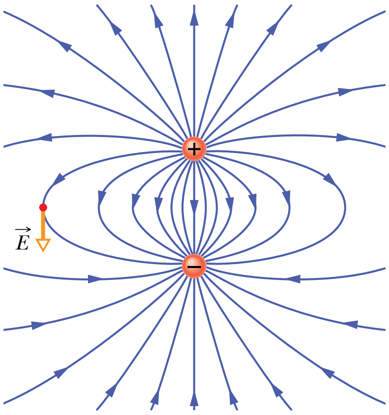


Figure 22-20 The pattern of electric field lines around an electric dipole, with an electric field vector $\vec{E}$ shown at one point (tangent to the field line through that point).

in Fig. 22-20. Let's label that axis as a z axis. Here we restrict our interest to the magnitude and direction of the electric field $\vec{E}$ at an arbitrary point P along the dipole axis, at distance z from the dipole's midpoint.

Figure 22-21*a* shows the electric fields set up at P by each particle. The nearer particle with charge $+q$ sets up field $E_{(+)}$ in the positive direction of the z axis (directly away from the particle). The farther particle with charge $-q$ sets up a smaller field $E_{(-)}$ in the negative direction (directly toward the particle). We want the net field at P, as given by Eq. 22-19. However, because the field vectors are along the same axis, let's simply indicate the vector directions with plus and minus signs, as we commonly do with forces along a single axis. Then we can write the magnitude of the net field at P as

$$\begin{aligned} E &= E_{(+)} - E_{(-)} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r_{(+)}^2} - \frac{1}{4\pi\varepsilon_0}\frac{q}{r_{(-)}^2} \\ &= \frac{q}{4\pi\varepsilon_0\left(z-\frac{1}{2}d\right)^2} - \frac{q}{4\pi\varepsilon_0\left(z+\frac{1}{2}d\right)^2}. \end{aligned} \tag{22-22}$$

After a little algebra, we can rewrite this equation as

$$E = \frac{q}{4\pi\varepsilon_0 z^2}\left(\frac{1}{\left(1-\dfrac{d}{2z}\right)^2} - \frac{q}{\left(1+\dfrac{d}{2z}\right)^2}\right). \tag{22-23}$$

After forming a common denominator and multiplying its terms, we come to

$$E = \frac{q}{4\pi\varepsilon_0 z^2}\frac{2d/z}{\left(1-\left(\dfrac{d}{2z}\right)^2\right)^2} = \frac{q}{2\pi\varepsilon_0 z^3}\frac{d}{\left(1-\left(\dfrac{d}{2z}\right)^2\right)^2}. \tag{22-24}$$

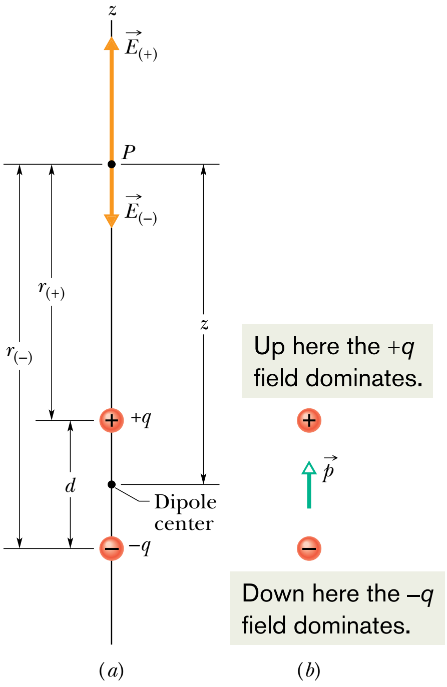


Figure 22-21 (*a*) An electric dipole. The electric field vectors $\vec{E}_{(+)}$ and $\vec{E}_{(-)}$ at point P on the dipole axis result from the dipole's two charges. Point P is at distances $r_{(+)}$ and $r_{(-)}$ from the individual charges that make up the dipole. (*b*) The dipole moment $\vec{p}$ of the dipole points from the negative charge to the positive charge.

We are usually interested in the electrical effect of a dipole only at distances that are large compared with the dimensions of the dipole—that is, at distances such that $z \gg d$. At such large distances, we have $d/2z \ll 1$ in Eq. 22-24. Thus, in our approximation, we can neglect the $d/2z$ term in the denominator, which leaves us with

$$E = \frac{1}{2\pi\varepsilon_0}\frac{qd}{z^3}. \tag{22-25}$$

The product qd, which involves the two intrinsic properties q and d of the dipole, is the magnitude p of a vector quantity known as the **electric dipole moment** $\vec{p}$ of the dipole. (The unit of $\vec{p}$ is the coulomb-meter.) Thus, we can write Eq. 22-25 as

$$E = \frac{1}{2\pi\varepsilon_0}\frac{p}{z^3} \quad \text{(electric dipole)}. \tag{22-26}$$

The direction of $\vec{p}$ is taken to be from the negative to the positive end of the dipole, as indicated in Fig. 22-21*b*. We can use the direction of $\vec{p}$ to specify the orientation of a dipole.

Equation 22-26 shows that, if we measure the electric field of a dipole only at distant points, we can never find q and d separately; instead, we can find only their product. The field at distant points would be unchanged if, for example, q were doubled and d simultaneously halved. Although Eq. 22-26 holds only for distant points along the dipole axis, it turns out that E for a dipole varies as $1/r^3$ for *all* distant points, regardless of whether they lie on the dipole axis; here r is the distance between the point in question and the dipole center.

Inspection of Fig. 22-21 and of the field lines in Fig. 22-20 shows that the direction of $\vec{E}$ for distant points on the dipole axis is always the direction of the dipole moment vector $\vec{p}$. This is true whether point P in Fig. 22-21a is on the upper or the lower part of the dipole axis.

Inspection of Eq. 22-26 shows that if you double the distance of a point from a dipole, the electric field at the point drops by a factor of 8. If you double the distance from a single point charge, however (see Eq. 22-20), the electric field drops only by a factor of 4. Thus the electric field of a dipole decreases more rapidly with distance than does the electric field of a single charge. The physical reason for this rapid decrease in electric field for a dipole is that from distant points a dipole looks like two particles that almost—but not quite—coincide. Thus, because they have charges of equal magnitude but opposite signs, their electric fields at distant points almost—but not quite—cancel each other.

22.10 | THE ELECTRIC FIELD DUE TO A LINE OF CHARGE (CONTINUOUS CHARGE DISTRIBUTION)

Key Concepts

- The equation for the electric field set up by a particle does not apply to an extended object with charge (said to have a continuous charge distribution).
- To find the electric field of an extended object at a point, we first consider the electric field set up by a charge element dq in the object, where the element is small enough for us to apply the equation for a particle. Then we sum, via integration, components of the electric fields $d\vec{E}$ from all the charge elements.
- Because the individual electric fields $d\vec{E}$ have different magnitudes and point in different directions, we first see if symmetry allows us to cancel out any of the components of the fields, to simplify the integration.

So far we have dealt with only charged particles, a single particle or a simple collection of them. We now turn to a much more challenging situation in which a thin (approximately one-dimensional) object such as a rod or ring is charged with a huge number of particles, more than we could ever even count. In this section, we will discuss the electric field caused by a line charge (straight or curved line). In the next section, we consider two-dimensional objects, such as a disk with charge spread over a surface. In the next chapter we tackle three-dimensional objects, such as a sphere with charge spread through a volume.

Figure 22-22 shows a thin ring of radius R with a uniform distribution of positive charge along its circumference. It is made of plastic, which means that the charge is fixed in place. The ring is surrounded by a pattern of electric field lines, but here we restrict our interest to an arbitrary point P on the central axis (the axis through the ring's center and perpendicular to the plane of the ring), at distance z from the center point.

The charge of an extended object is often conveyed in terms of a charge density rather than the total charge. For a line of charge, we use the *linear charge density* λ (the charge per unit length), with the SI unit of coulomb per meter. Table 22-2 shows the other charge densities that we shall be using for charged surfaces and volumes.

So far, we have an equation for the electric field of a particle. (We can combine the field of several particles as we did for the electric dipole to generate a special equation, but we are still basically using Eq. 22-20). Now take a look at the ring in Fig. 22-22. That clearly is not a particle and so Eq. 22-20 does not apply. So we mentally divide the ring into differential elements of charge that are so small

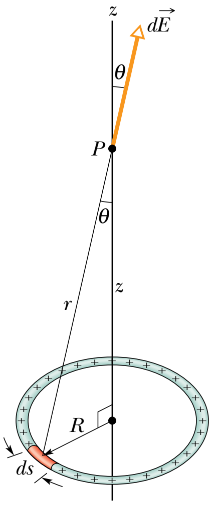


Figure 22-22 A ring of uniform positive charge. A differential element of charge occupies a length ds (greatly exaggerated for clarity). This element sets up an electric field $d\vec{E}$ at point P.

Table 22-2 Some Measures of Electric Charge

Name	Symbol	SI Unit
Charge	q	C
Linear charge density	λ	C/m
Surface charge density	σ	C/m^2
Volume charge density	ρ	C/m^3

that we can treat them as though they *are* particles. Then we *can* apply Eq. 22-20.

We now know to apply Eq. 22-20 to each charge element dq (the front d emphasizes that the charge is very small) and can write an expression for its contribution of electric field $d\vec{E}$ (the front d emphasizes that the contribution is very small). However, each such contributed field vector at P is in its own direction. To add them to get the net field at P we split the vectors into components and then separately sum one set of components and then the other set. However, first we check to see if one set simply all cancels out.

There is a huge number of dq elements in the ring and thus a huge number of $d\vec{E}$ components to add up, even if we can cancel out one set of components. We add up these components by means of integration.

Let's do all this. We arbitrarily pick the charge element shown in Fig. 22-22. Let ds be the arc length of that (or any other) dq element. Then in terms of the linear density λ (the charge per unit length), we have

$$dq = \lambda\, ds. \tag{22-27}$$

An Element's Field. This charge element sets up the differential electric field $d\vec{E}$ at P, at distance r from the element, as shown in Fig. 22-22. Next we rewrite the field equation for a particle (Eq. 22-20) in terms of our new symbols dE and dq, but then we replace dq using Eq. 22-23. The field magnitude due to the charge element is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, ds}{r^2}. \tag{22-28}$$

Notice that the illegal symbol r is the hypotenuse of the right triangle displayed in Fig. 22-22. Thus, we can replace r by rewriting Eq. 22-24 as

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, ds}{(z^2+R^2)}. \tag{22-29}$$

Because every charge element has the same charge and the same distance from point P, Eq. 22-29 gives the field magnitude contributed by each of them. Figure 22-22 also tells us that each contributed $d\vec{E}$ leans at angle θ to the central axis (the z axis) and thus has components perpendicular and parallel to that axis.

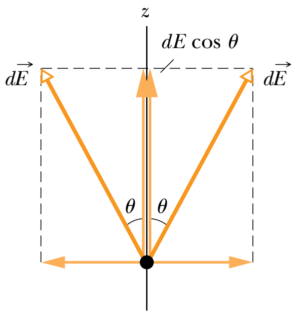


Figure 22-23 The electric fields set up at P by a charge element and its symmetric partner (on the opposite side of the ring). The components perpendicular to the z axis cancel; the parallel components add.

Canceling Components. Now comes the neat part, where we eliminate one set of those components. In Fig. 22-22, consider the charge element on the opposite side of the ring. It too contributes the field magnitude dE but the field vector leans at angle u in the opposite direction from the vector from our first charge element, as indicated in the side view of Fig. 22-23. Thus the two perpendicular components cancel. All around the ring, this cancelation occurs for every charge element and its *symmetric partner* on the opposite side of the ring. So we can neglect all the perpendicular components.

Adding Components. We have another big win here. All the remaining components are in the positive direction of the z axis, so we can just add them up as scalars. Thus we can already tell the direction of the net electric field at P: directly away from the ring. From Fig. 22-23, we see that the parallel components each have magnitude $dE \cos\theta$, but θ is another illegal symbol. We can replace $\cos\theta$ with legal symbols by again using the right triangle in Fig. 22-22 to write

$$\cos\theta = \frac{z}{r} = \frac{z}{(z^2+R^2)^{1/2}}. \tag{22-30}$$

Multiplying Eq. 22-29 by Eq. 22-30 gives us the parallel field component from each charge element:

$$dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{z\lambda}{(z^2+R^2)^{3/2}}\, ds. \tag{22-31}$$

Integrating. Because we must sum a huge number of these components, each small, we set up an integral that moves along the ring, from element to element, from a starting point (call it $s = 0$) through the full circumference ($s = 2\pi R$).

Only the quantity s varies as we go through the elements; the other symbols in Eq. 22-31 remain the same, so we move them outside the integral. We find

$$E = \int dE \cos\theta = \frac{z\lambda}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\int_0^{2\pi R} ds$$

$$= \frac{z\lambda(2\pi R)}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}. \qquad (22\text{-}32)$$

This is a fine answer, but we can also switch to the total charge by using $\lambda = q/(2\pi R)$:

$$E = \frac{qz}{4\pi\varepsilon_0(z^2+R^2)^{3/2}} \quad \text{(charged ring).} \qquad (22\text{-}33)$$

If the charge on the ring is negative, instead of positive as we have assumed, the magnitude of the field at P is still given by Eq. 22-33. However, the electric field vector then points toward the ring instead of away from it.

Let us check Eq. 22-33 for a point on the central axis that is so far away that $z \gg R$. For such a point, the expression $z^2 + R^2$ in Eq. 22-33 can be approximated as z^2, and Eq. 22-33 becomes

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2} \quad \text{(charged ring at large distance).} \qquad (22\text{-}34)$$

This is a reasonable result because from a large distance, the ring "looks like" a point charge. If we replace z with r in Eq. 22-32, we indeed do have the magnitude of the electric field due to a point charge, as given by Eq. 22-20.

Let us next check Eq. 22-33 for a point at the center of the ring—that is, for $z = 0$. At that point, Eq. 22-33 tells us that $E = 0$. This is a reasonable result because if we were to place a test charge at the center of the ring, there would be no net electrostatic force acting on it; the force due to any element of the ring would be canceled by the force due to the element on the opposite side of the ring. By Eq. 22-18, if the force at the center of the ring were zero, the electric field there would also have to be zero.

Note that, even if the charges are nonuniformly distributed, the electric field in x-direction remains the same. But the electric field in y-direction will be non zero. The electric field has a maximum value at $z = R\sqrt{2}$, as can be obtained by differentiating expression for E.

SAMPLE PROBLEM 22.07

Electric field of a uniformly charged straight rod

Figure 22-24 shows a plastic rod having a uniformly distributed charge. The charge per unit length of the rod is λ. Find the electric field due to the rod at a point P which is at a distance r from the rod.

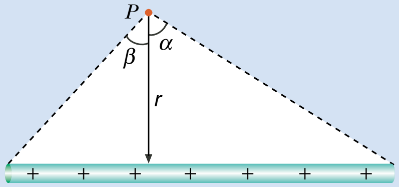


Figure 22-24 A uniformly charged rod is kept along x axis we need to find the electric field at P.

KEY IDEA

Because the rod has a continuous charge distribution, we must find an expression for the electric fields due to differential elements of the rod and then sum those fields via calculus.

Calculations: An element: Consider a differential element having length dx. If the length of a line segment is dx, the charge on it is λdx. Here λ is the charge per unit length of the line (assumed to be constant).

Thus,

$$dq = \lambda dx.$$

The element's field: Our element produces a differential electric field. Treating this element as a point charge, the electric field due to this small charge at a distance r from the line is

$$dE = \frac{k\lambda dx}{(\sqrt{x^2+r^2})^2}.$$

The electric field is away from the charge because the charge is positive (see Fig. 22-25).

Summing: Thus, to find the electric field set up by the rod, we need sum (via integration) the parallel as well as perpendicular components of the differential electric field is setup by all the differential elements of the rod.

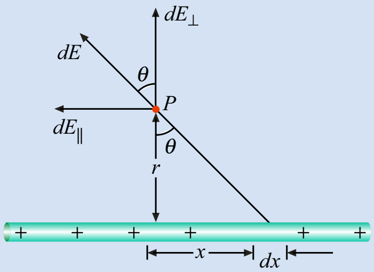


Figure 22-25 A small element of length dx produces an electric field DE at a point P. This electric field is resolved into two components $dE_\perp$ and $dE_\parallel$ as shown.

Let us resolve this electric field in a direction parallel and perpendicular to the line charge.

$$dE_\parallel = dE\sin\theta,$$

$$dE_\perp = dE\cos\theta,$$

$$dE_\parallel = \frac{k\lambda dx}{(x^2+r^2)}\sin\theta.$$

From the geometry of Fig. 22-25, we can say that

$$\frac{x}{r} = \tan\theta,$$

$$x = r\tan\theta.$$

Since we need to replace x everywhere by the variable θ, we need to differentiate this expression with respect to θ.

$$\frac{dx}{d\theta} = r\sec^2\theta,$$

$$dx = r\sec^2\theta d\theta.$$

By replacing this expression for dE, we get

$$dE_\parallel = \frac{k\lambda\times r\sec^2\theta\times\sin\theta}{(r^2+r^2\tan^2\theta)}d\theta,$$

$$dE_\parallel = \frac{k\lambda}{r}\sin\theta d\theta,$$

$$E_\parallel = \frac{k\lambda}{r}\int_{-\beta}^{\alpha}\sin\theta d\theta$$

$$= \frac{k\lambda}{r}[\cos\alpha-\cos(-\beta)] = \frac{k\lambda}{r}[\cos\beta-\cos\alpha].$$

Similarly resolving the electric field in the perpendicular direction, we have

$$E_\perp = \frac{k\lambda}{r}\int_{-\beta}^{\alpha}\cos\theta d\theta$$

$$= \frac{k\lambda}{r}[\sin\alpha+\sin\beta]$$

Note: If the line is infinite which means that it extends from $-\infty$ to $+\infty$, we can assume α = 90° and β = 90°. Then

$$E_\parallel = \frac{k\lambda}{r}[\cos 90°-\cos 90°] = 0$$

$$E_\perp = \frac{k\lambda}{r}[\sin 90°-\sin 90°] = \frac{2k\lambda}{r}.$$

If the line is semi-infinite, extending from $-\infty$ to the foot of the perpendicular, we can assume $\alpha = 90°$ and $\beta = 0°$. Then

$$E_\parallel = \frac{k\lambda}{r}[\cos 0°-\cos 90°] = \frac{k\lambda}{r}$$

$$E_\perp = \frac{k\lambda}{r}[\sin 90°-\sin 0°] = \frac{k\lambda}{r}.$$

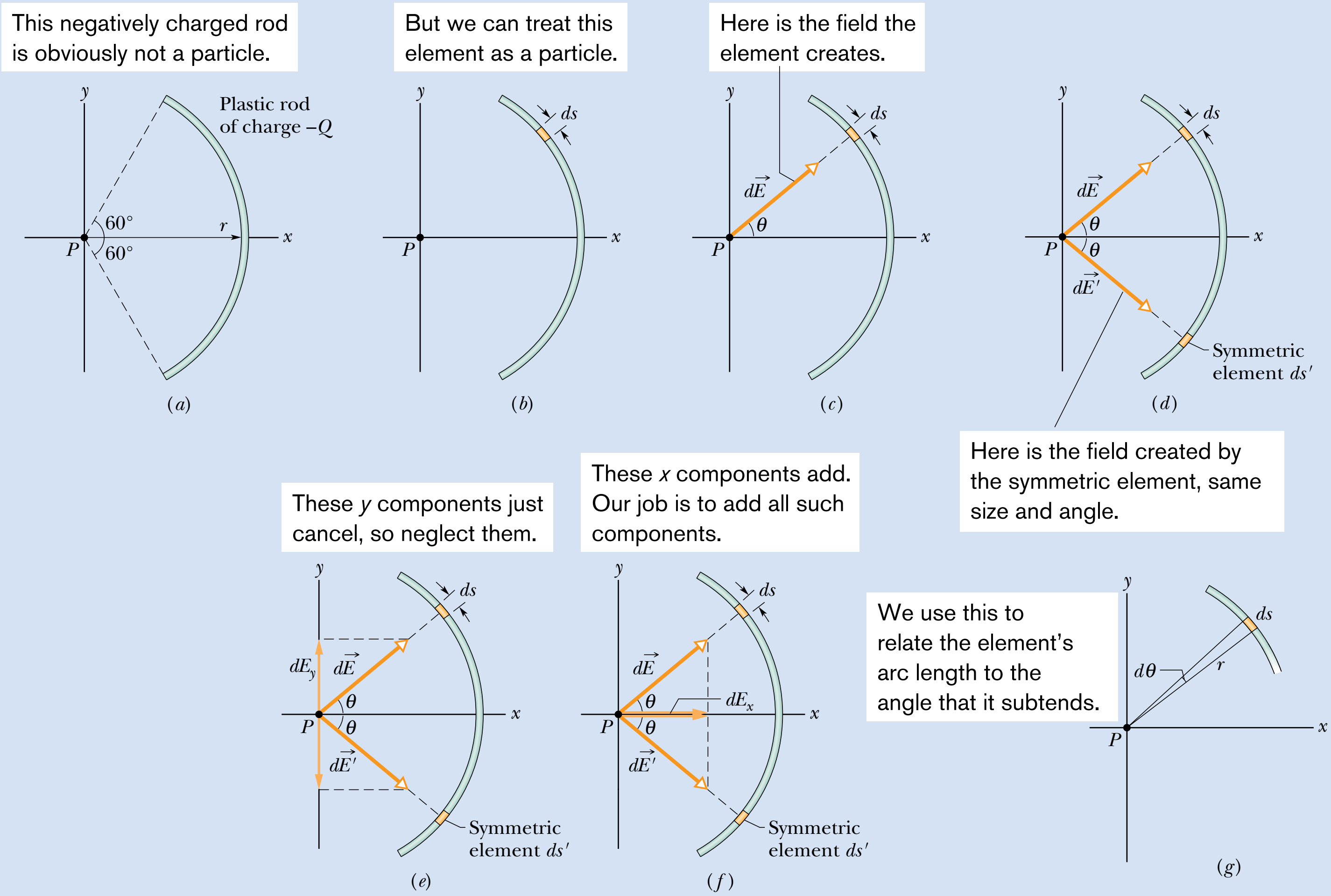
SAMPLE PROBLEM 22.08

Electric field of a charged circular rod

Figure 22-26*a* shows a plastic rod with a uniform charge $-Q$. It is bent in a 120° circular arc of radius r and symmetrically paced across an x axis with the origin at the center of curvature P of the rod. In terms of Q and r, what is the electric field $\vec{E}$ due to the rod at point P?

KEY IDEAS

Because the rod has a continuous charge distribution, we must find an expression for the electric fields due to differential elements of the rod and then sum those fields via calculus.


This negatively charged rod is obviously not a particle.

But we can treat this element as a particle.

Here is the field the element creates.

Here is the field created by the symmetric element, same size and angle.

These y components just cancel, so neglect them.

These x components add. Our job is to add all such components.

We use this to relate the element's arc length to the angle that it subtends.

Figure 22-26 (a) A plastic rod of charge $-Q$ is a circular section of radius r and central angle 120°; point P is the center of curvature of the rod. (b)–(c) A differential element in the top half of the rod, at an angle θ to the x axis and of arc length ds, sets up a differential electric field $d\vec{E}$ at P. (d) An element ds', symmetric to ds about the x axis, sets up a field $d\vec{E}$ at P with the same magnitude. (e)–(f) The field components. (g) Arc length ds' makes an angle $d\theta$ about point P.

An element: Consider a differential element having arc length ds and located at an angle θ above the x axis (Figs. 22-26b and c). If we let λ represent the linear charge density of the rod, our element ds has a differential charge of magnitude

$$dq = \lambda\, ds. \tag{22-35}$$

The element's field: Our element produces a differential electric field $d\vec{E}$ at point P, which is a distance r from the element. Treating the element as a point charge, we can rewrite Eq. 22-3 to express the magnitude of $d\vec{E}$ as

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, ds}{r^2}. \tag{22-36}$$

The direction of $d\vec{E}$ is toward ds because charge dq is negative.

Symmetric partner: Our element has a symmetrically located (mirror image) element ds' in the bottom half of the rod. The electric field $d\vec{E}'$ set up at P by ds' also has the magnitude given by Eq. 22-36, but the field vector points toward ds' as shown in Fig. 22-26d. If we resolve the electric field vectors of ds and ds' into x and y components as shown in Figs. 22-26e and f, we see that their y components cancel (because they have equal magnitudes and are in opposite directions).We also see that their x components have equal magnitudes and are in the same direction.

Summing: Thus, to find the electric field set up by the rod, we need sum (via integration) only the x components of the differential electric fields set up by all the differential elements of the rod. From Fig. 22-26f and

Eq. 22-36, we can write the component dE_x set up by *ds* as

$$dE_x = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^2}\cos\theta\, ds. \qquad (22\text{-}37)$$

Equation 22-37 has two variables, θ and *s*. Before we can integrate it, we must eliminate one variable. We do so by replacing *ds*, using the relation

$$ds = r\, d\theta,$$

in which $d\theta$ is the angle at *P* that includes arc length *ds* (Fig. 22-26*g*). With this replacement, we can integrate Eq. 22-37 over the angle made by the rod at *P*, from $\theta = -60°$ to $\theta = 60°$; that will give us the field magnitude at *P*:

$$\begin{aligned} E &= \int dE_x = \int_{-60°}^{60°} \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^2}\cos\theta\, r\, d\theta \\ &= \frac{\lambda}{4\pi\varepsilon_0 r}\int_{-60°}^{60°}\cos\theta\, d\theta = \frac{\lambda}{4\pi\varepsilon_0 r}\Big[\sin\theta\Big]_{-60°}^{60°} \\ &= \frac{\lambda}{4\pi\varepsilon_0 r}\left[\sin 60° - \sin(-60°)\right] = \frac{1.73\lambda}{4\pi\varepsilon_0 r}. \qquad (22\text{-}38)\end{aligned}$$

(If we had reversed the limits on the integration, we would have gotten the same result but with a minus sign. Since the integration gives only the magnitude of $\vec{E}$, we would then have discarded the minus sign.)

Charge density: To evaluate λ, we note that the full rod subtends an angle of 120° and so is one-third of a full circle. Its arc length is then $2\pi r/3$, and its linear charge density must be

$$\lambda = \frac{\text{charge}}{\text{length}} = \frac{Q}{2\pi r/3} = \frac{0.477Q}{r}.$$

Substituting this into Eq. 22-23 and simplifying give us

$$E = \frac{(1.73)(0.477Q)}{4\pi\varepsilon_0 r^2} = \frac{0.83Q}{4\pi\varepsilon_0 r^2}. \qquad \text{(Answer)}$$

The direction of $\vec{E}$ is toward the rod, along the axis of symmetry of the charge distribution. We can write $\vec{E}$ in unit-vector notation as

$$\vec{E} = \frac{0.83Q}{4\pi\varepsilon_0 r^2}\,\hat{\text{i}}.$$

PROBLEM-SOLVING TACTICS

A Field Guide for Lines of Charge

Here is a generic guide for finding the electric field $\vec{E}$ produced at a point *P* by a line of uniform charge, either circular or straight. The general strategy is to pick out an element *dq* of the charge, find $d\vec{E}$ due to that element, and integrate $d\vec{E}$ over the entire line of charge.

Step 1. If the line of charge is circular, let *ds* be the arc length of an element of the distribution. If the line is straight, run an *x* axis along it and let *dx* be the length of an element. Mark the element on a sketch.

Step 2. Relate the charge *dq* of the element to the length of the element with either $dq = \lambda\, ds$ or $dq = \lambda\, ds$. Consider *dq* and λ to be positive, even if the charge is actually negative. (The sign of the charge is used in the next step.)

Step 3. Express the field $d\vec{E}$ produced at *P* by *dq* with Eq. 22-20, replacing *q* in that equation with either $\lambda\, ds$ or $\lambda\, dx$. If the charge on the line is positive, then at *P* draw a vector $d\vec{E}$ that points directly away from *dq*. If the charge is negative, draw the vector pointing directly toward *dq*.

Step 4. Always look for any symmetry in the situation. If *P* is on an axis of symmetry of the charge distribution, resolve the field $d\vec{E}$ produced by *dq* into components that are perpendicular and parallel to the axis of symmetry. Then consider a second element *dq'* that is located symmetrically to *dq* about the line of symmetry. At *P* draw the vector $d\vec{E}$ that this symmetrical element produces and resolve it into components. One of the components produced by *dq* is a *canceling component;* it is canceled by the corresponding component produced by *dq'* and needs no further attention. The other component produced by *dq* is an *adding component;* it adds to the corresponding component produced by *dq'*. Add the adding components of all the elements via integration.

Step 5. Here are four general types of uniform charge distributions, with strategies for the integral of step 4.

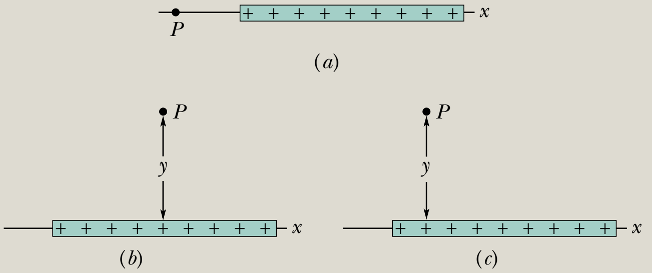


Figure 22-27 (*a*) Point P is on an extension of the line of charge. (*b*) P is on a line of symmetry of the line of charge, at perpendicular distance y from that line. (*c*) Same as (*b*) except that P is not on a line of symmetry.

Ring, with point P on (central) axis of symmetry, as in Fig. 22-22. In the expression for dE, replace r^2 with $z^2 + R^2$, as in Eq. 22-29. Express the adding component of $d\vec{E}$ in terms of θ. That introduces cos θ, but θ is identical for all elements and thus is not a variable. Replace cos θ as in Eq. 22-30. Integrate over s, around the circumference of the ring.

Circular arc, with point P at the center of curvature, as in Fig. 22-26. Express the adding component of $d\vec{E}$ in terms of θ. That introduces either sin θ or cos θ. Reduce the resulting two variables s and θ to one, θ, by replacing ds with $r\ d\ \theta$. Integrate over θ from one end of the arc to the other end.

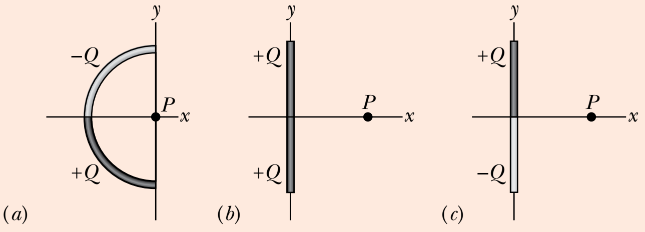
Straight line, with point P on an extension of the line, as in Fig. 22-27*a*. In the expression for dE, replace r with x. Integrate over x, from end to end of the line of charge.

Straight line, with point P at perpendicular distance y from the line of charge, as in Fig. 22-27*b*. In the expression for dE, replace r with an expression involving x and y. If P is on the perpendicular bisector of the line of charge, find an expression for the adding component of $d\vec{E}$. That will introduce either sin θ or cos θ. Reduce the resulting two variables x and θ to one, x, by replacing the trigonometric function with an expression (its definition) involving x and y. Integrate over x from end to end of the line of charge. If P is not on a line of symmetry, as in Fig. 22-27*c*, set up an integral to sum the components dE_x, and integrate over x to find E_x. Also set up an integral to sum the components dE_y, and integrate over x again to find E_y. Use the components E_x and E_y in the usual way to find the magnitude E and the orientation of $\vec{E}$.

Step 6. One arrangement of the integration limits gives a positive result. The reverse gives the same result with a minus sign; discard the minus sign. If the result is to be stated in terms of the total charge Q of the distribution, replace λ with Q/L, in which L is the length of the distribution.

CHECKPOINT 7

The figure here shows three nonconducting rods, one circular and two straight. Each has a uniform charge of magnitude Q along its top half and another along its bottom half. For each rod, what is the direction of the net electric field at point P?



22.11 | THE ELECTRIC FIELD DUE TO A CHARGED DISK

Key Concept

- On the central axis through a uniformly charged disk,

$$E = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)$$

gives the electric field magnitude. Here z is the distance along the axis from the center of the disk, R is the radius of the disk, and σ is the surface charge density.

Now we switch from a line of charge to a surface of charge by examining the electric field of a circular plastic disk, with a radius R and a uniform surface charge density σ (charge per unit area, Table 22-2) on its top surface. The disk sets up a pattern of electric field lines around it, but here we restrict our attention to the electric field at an arbitrary point P on the central axis, at distance z from the center of the disk, as indicated in Fig. 22-28.

We could proceed as in the preceding module but set up a two-dimensional integral to include all of the field contributions from the two-dimensional distribution of charge on the top surface. However, we can save a lot of work with a neat shortcut using our earlier work with the field on the central axis of a thin ring.

We superimpose a ring on the disk as shown in Fig. 22-25, at an arbitrary radius $r \le R$. The ring is so thin that we can treat the charge on it as a charge element dq. To find its small contribution dE to the electric field at point P, we rewrite Eq. 22-31 in terms of the ring's charge dq and radius r:

$$dE = \frac{dq\, z}{4\pi\varepsilon_0 (z^2 + r^2)^{3/2}}. \tag{22-39}$$

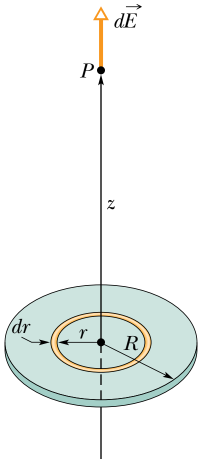


Figure 22-28 A disk of radius R and uniform positive charge. The ring shown has radius r and radial width dr. It sets up a differential electric field $d\vec{E}$ at point P on its central axis.

The ring's field points in the positive direction of the z axis.

To find the total field at P, we are going to integrate Eq. 22-39 from the center of the disk at $r = 0$ out to the rim at $r = R$ so that we sum all the dE contributions (by sweeping our arbitrary ring over the entire disk surface). However, that means we want to integrate with respect to a variable radius r of the ring.

We get dr into the expression by substituting for dq in Eq. 22-39. Because the ring is so thin, call its thickness dr. Then its surface area dA is the product of its circumference $2\pi r$ and thickness dr. So, in terms of the surface charge density σ, we have

$$dq = \sigma dA = \sigma(2\pi r\, dr). \tag{22-40}$$

After substituting this into Eq. 22-39 and simplifying slightly, we can sum all the dE contributions with

$$E = \int dE = \frac{\sigma z}{4\varepsilon_0}\int_0^R (z^2 + r^2)^{-3/2}(2r)dr, \tag{22-41}$$

where we have pulled the constants (including z) out of the integral. To solve this integral, we cast it in the form $\int X^m\, dX$ by setting $X = (z^2 + r^2)$, $m = -\frac{2}{3}$, and $dX = (2r)dr$. For the recast integral we have

$$\int X^m dX = \frac{X^{m+1}}{m+1},$$

and so Eq. 22-41 becomes

$$E = \frac{\sigma z}{4\varepsilon_0}\left[\frac{(z^2 + r^2)^{-1/2}}{-\frac{1}{2}}\right]_0^R. \tag{22-42}$$

Taking the limits in Eq. 22-42 and rearranging, we find

$$E = \frac{z}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) \quad \text{(charged disk)} \tag{22-43}$$

as the magnitude of the electric field produced by a flat, circular, charged disk at points on its central axis. (In carrying out the integration, we assumed that $z \ge 0$.)

If we let $R \to \infty$ while keeping z finite, the second term in the parentheses in Eq. 22-43 approaches zero, and this equation reduces to

$$E = \frac{\sigma}{2\varepsilon_0} \quad \text{(infinite sheet).} \tag{22-44}$$

This is the electric field produced by an infinite sheet of uniform charge located on one side of a nonconductor such as plastic. The electric field lines for such a situation are shown in Fig. 22-13.

We also get Eq. 22-44 if we let $z \to 0$ in Eq. 22-43 while keeping R finite. This shows that at points very close to the disk, the electric field set up by the disk is the same as if the disk were infinite in extent.

22.12 | ELECTRIC FIELD INSIDE A CONDUCTOR

Key Concept

◆ The electric field is zero at all points inside the conductor.

In a conductor (metal), there are free electrons which are contributed by the valence shell of the individual atoms. Because of the metallic bonding between the individual atoms, these electrons are relatively free. Their number is also quite large. These electrons cannot come out of the surface of the metal easily because of a surface barrier. Suppose you put a neutral ideal conducting solid sphere in a region of space in which there is, initially, a uniform electric field. Its influence will be felt by both the free electrons and the bound charges. But only free electrons can move. So, they will move in a direction opposite to the electric field. Soon charges will be redistributed on the conductor surface as shown in Fig. 22-29. This occurs in less than a microsecond.

Accumulation of small number of charges can also build up sizeable field; this field due to the induced charges is in opposite direction to the external electric field and the magnitude is the product of charge and electric field.

Since they are oppositely directed, the two contributions to the electric field inside the conductor tend to cancel each other. Now comes the profound part of the argument: the two contributions to the electric field at any point in the conductor exactly cancel. We know that they have to completely cancel because, if they don't, the free-to-move-charge in the conductor would move as a result of the force exerted on it by the electric field. The force on the charge is always in a direction that causes the charge to be redistributed to positions in which it will create its own electric field that tends to cancel the electric field that caused the charge to move. The point is that the charge will not stop responding to the electric field until the net electric field at every point in the conductor is zero. Thus, we can say that

The electric field is zero at all points inside the conductor.

Remember that any conductor has practically infinite supply of electrons to nullify any practical electric field:

$$\vec{E}_{\text{net}} = \vec{E}_{\text{ext}} + \vec{E}_{\text{in}} = 0.$$

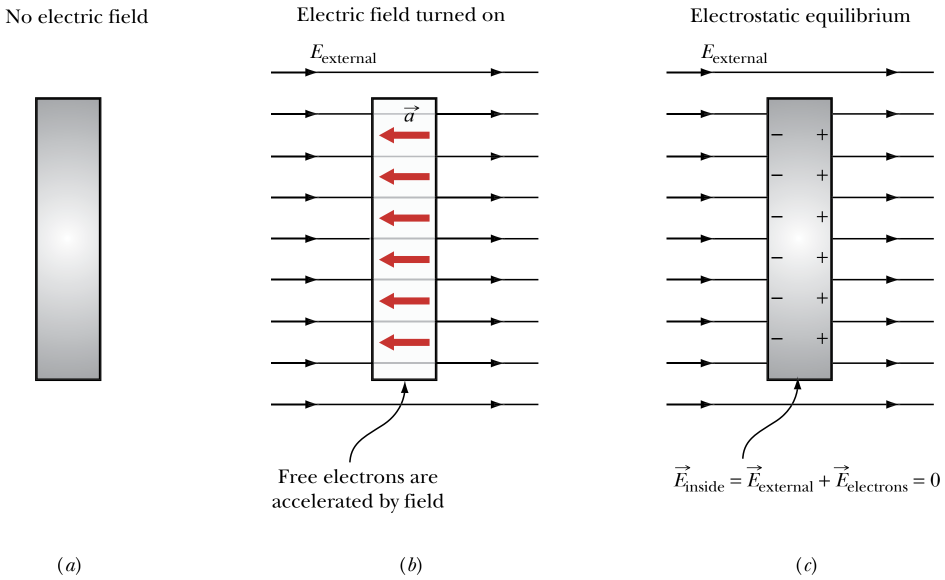

No electric field
Electric field turned on
E_{external}
$\vec{a}$
Free electrons are accelerated by field
Electrostatic equilibrium
E_{external}
$\vec{E}_{\text{inside}} = \vec{E}_{\text{external}} + \vec{E}_{\text{electrons}} = 0$
(a)
(b)
(c)

Figure 22-29 A metallic object contains many so-called "free electrons"–electrons that are not bound to a particular location in the metal. (*a*) When the object is not subject to an electric field, there is no force on the electrons. (*b*) When an external electric field is applied, the field accelerates these free electrons in a direction opposite that of the field. (*c*) The electrons accumulate on the left side leaving positively charged ions on the right. The flow of electrons stops once the electric field set up by the accumulated charges cancels the external field. The number of free electrons in a metal is so large that practically any electric field can be canceled by this redistribution of charges.

SAMPLE PROBLEM 22.09

Electric field at the center of the sphere due to induced charges on the surface

Find the electric field at the center of the sphere due to the induced charges on the surface of the sphere, as shown in Fig. 22-30.

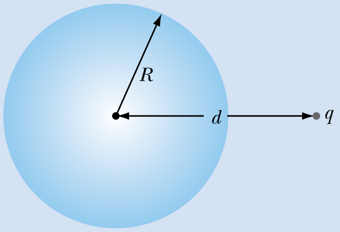


Figure 22-30 A point charge is kept near a conducting sphere.

KEY IDEA

Here, we note that the induced charges will form a certain pattern on the surface of the sphere which cannot be predicted. But we can find the electric field due to q at any point. The net electric field inside the conductor is zero. This is produced by the induced charges and the external charge.

Calculations: At the center this electric field due to q in the positive direction will be (Fig. 22-31):

$$E_Q = \frac{q}{4\pi\varepsilon_0 d^2}.$$

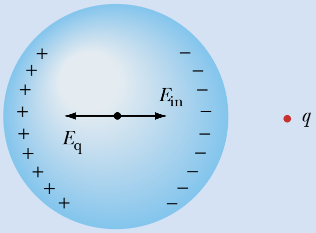


Figure 22-31 Due to the point charge, charges redistribute over the surface of the conductor. These redistributed charges produce electric field E_{in} at the center, as shown.

Since the net electric field at the center has to be zero,

$$E_{\text{induced}} + E_Q = 0 \Rightarrow E_{\text{induced}} = -\frac{q}{4\pi\varepsilon_0 d^2}.$$

Here, the charges will be induced on the surface of the sphere, but they will not be uniformly distributed. So, we cannot apply the formula for the electric field due to a uniformly charged spherical shell.

Learn: When we say the electric field is zero, this implies that electric field should be zero inside the core of a solid metal. These are the points where the free electrons are present within the lattice. These free electrons should not experience any force.

22.13 | A POINT CHARGE IN AN ELECTRIC FIELD

Key Concepts

- If a particle with charge q is placed in an external electric field $\vec{E}$, an electrostatic force $\vec{F}$ acts on the particle:
$$\vec{F} = q\vec{E}.$$
- If charge q is positive, the force vector is in the same direction as the field vector. If charge q is negative, the force vector is in the opposite direction (the minus sign in the equation reverses the force vector from the field vector).

In the preceding sections we worked at the first of our two tasks: given a charge distribution, to find the electric field it produces in the surrounding space. Here we begin the second task: to determine what happens to a charged particle when it is in an electric field set up by other stationary or slowly moving charges.

What happens is that an electrostatic force acts on the particle, as given by

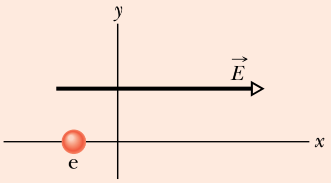
$$\vec{F} = q\vec{E}, \tag{22-45}$$

in which q is the charge of the particle (including its sign) and $\vec{E}$ is the electric field that other charges have produced at the location of the particle. (The field is *not* the field set up by the particle itself; to distinguish the two fields, the field acting on the particle in Eq. 22-45 is often called the *external field.* A charged particle or object is not affected by its own electric field.) Equation 22-45 tells us:

> The electrostatic force $\vec{F}$ acting on a charged particle located in an external electric field $\vec{E}$ has the direction of $\vec{E}$ if the charge q of the particle is positive and has the opposite direction if q is negative.

CHECKPOINT 8

(a) In the figure, what is the direction of the electrostatic force on the electron due to the external electric field shown? (b) In which direction will the electron accelerate if it is moving parallel to the y axis before it encounters the external field? (c) If, instead, the electron is initially moving rightward, will its speed increase, decrease, or remain constant?



SAMPLE PROBLEM 22.10

Motion of a charged particle in a charged field

Beams of high-speed protons can be produced in "guns" using electric fields to accelerate the protons.

(a) What acceleration would a proton experience if the gun's electric field were 2.00×10^4 N/C.

KEY IDEA

The acceleration of the proton is given by Newton's second law: $F = ma$, where F is the electrostatic force.

Calculations: The magnitude of the force acting on the proton is $F = eE$, where E is the magnitude of the electric field. According to Newton's second law, the acceleration of the proton is $a = F/m = eE/m$, where m is the mass of the proton. Thus,

$$a = \frac{F}{m} = \frac{eE}{m}.$$

With $e = 1.6 \times 10^{-19}$ C, $E = 2.00 \times 10^4$ N/C and $m = 1.67 \times 10^{-27}$ kg, we find the acceleration to be

$$\begin{aligned} a &= \frac{eE}{m} \\ &= \frac{(1.60 \times 10^{-19}\ \text{C})(2.00 \times 10^4\ \text{N/C})}{1.67 \times 10^{-27}\ \text{kg}} = 1.92 \times 10^{12}\ \text{m/s}^2. \end{aligned}$$

(b) What speed would the proton attain if the field accelerated the proton through a distance of 1.00 cm?

KEY IDEA

We assume that the proton starts from rest ($v_0 = 0$) and apply the kinematic equation $v^2 = v_0^2 + 2ax$ (or else $x = \frac{1}{2}at^2$ and $v = at$). Thus, the speed of the proton after having traveling a distance x is $v = \sqrt{2ax}$.

Calculations: With $x = 1.00$ cm $= 1.0 \times 10^{-2}$ m, the speed of the proton is

$$\begin{aligned} v &= \sqrt{2ax} \\ &= \sqrt{2(1.92 \times 10^{12}\ \text{m/s}^2)(0.0100\ \text{m})} \\ &= 1.96 \times 10^5\ \text{m/s}. \end{aligned}$$

The time it takes for the proton to attain the final speed is

$$t = \frac{v}{a} = \frac{1.96 \times 10^5\ \text{m/s}}{1.92 \times 10^{12}\ \text{m/s}^2} = 1.02 \times 10^{-7}\ \text{s}.$$

The distance the proton travels can be written as

$$x = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{eE}{m}\right)t^2.$$

Measuring the Elementary Charge

Equation 22-45 played a role in the measurement of the elementary charge e by American physicist Robert A. Millikan in 1910–1913. Figure 22-32 is a representation of his apparatus.When tiny oil drops are sprayed into chamber A, some of them become charged, either positively or negatively, in the process. Consider a drop that drifts downward through the small hole in plate P_1 and into chamber C. Let us assume that this drop has a negative charge q.

If switch S in Fig. 22-32 is open as shown, battery B has no electrical effect on chamber C. If the switch is closed (the connection between chamber C and the positive terminal of the battery is then complete), the battery causes an excess positive charge on conducting plate P_1 and an excess negative charge on conducting plate P_2. The charged plates set up a downward-directed electric field $\vec{E}$ in chamber C. According to Eq. 22-45, this field exerts an electrostatic force on any charged drop that happens to be in the chamber and affects its motion. In particular, our negatively charged drop will tend to drift upward.

By timing the motion of oil drops with the switch opened and with it closed and thus determining the effect of the charge q, Millikan discovered that the values of q were always given by

$$q = ne, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots, \tag{22-46}$$

in which e turned out to be the fundamental constant we call the *elementary charge,* 1.60×10^{-19} C. Millikan's experiment is convincing proof that charge is quantized, and he earned the 1923 Nobel Prize in physics in part for this work. Modern measurements of the elementary charge rely on a variety of interlocking experiments, all more precise than the pioneering experiment of Millikan.

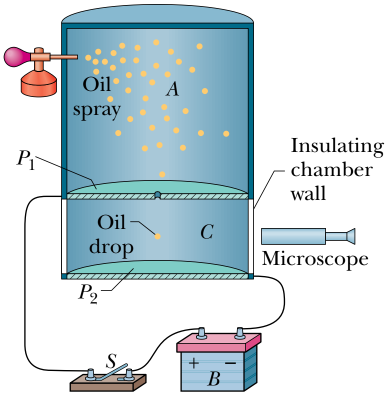


Figure 22-32 The Millikan oil-drop apparatus for measuring the elementary charge e. When a charged oil drop drifted into chamber C through the hole in plate P_1, its motion could be controlled by closing and opening switch S and thereby setting up or eliminating an electric field in chamber C. The microscope was used to view the drop, to permit timing of its motion.

Ink-Jet Printing

The need for high-quality, high-speed printing has caused a search for an alternative to impact printing, such as occurs in a standard typewriter. Building up letters by squirting tiny drops of ink at the paper is one such alternative.

Figure 22-33 shows a negatively charged drop moving between two conducting deflecting plates, between which a uniform, downward-directed electric field $\vec{E}$ has been set up. The drop is deflected upward according to Eq. 22-45 and then strikes the paper at a position that is determined by the magnitudes of $\vec{E}$ and the charge q of the drop.

In practice, E is held constant and the position of the drop is determined by the charge q delivered to the drop in the charging unit, through which the drop must pass before entering the deflecting system. The charging unit, in turn, is activated by electronic signals that encode the material to be printed.

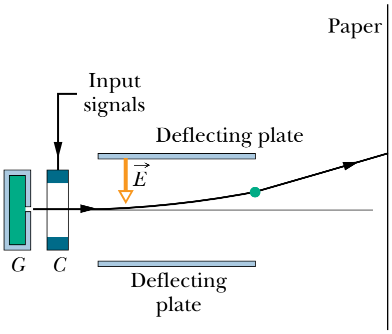
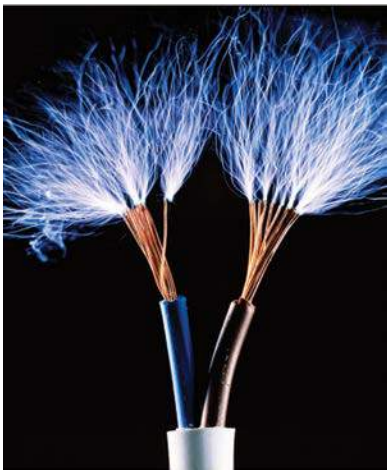


Figure 22-33 Ink-jet printer. Drops shot from generator G receive a charge in charging unit C. An input signal from a computer controls the charge and thus the effect of field $\vec{E}$ on where the drop lands on the paper.

Electrical Breakdown and Sparking

If the magnitude of an electric field in air exceeds a certain critical value E_c, the air undergoes *electrical breakdown,* a process whereby the field removes electrons from the atoms in the air. The air then begins to conduct electric current because the freed electrons are propelled into motion by the field. As they move, they collide with any atoms in their path, causing those atoms to emit light. We can see the paths, commonly called sparks, taken by the freed electrons because of that emitted light. Figure 22-34 shows sparks above charged metal wires where the electric fields due to the wires cause electrical breakdown of the air.



Adam Hart-Davis/Photo Researchers, Inc.

Figure 22-34 The metal wires are so charged that the electric fields they produce in the surrounding space cause the air there to undergo electrical breakdown.

SAMPLE PROBLEM 22.11

Motion of a charged particle in an electric field

Figure 22-35 shows the deflecting plates of an ink-jet printer, with superimposed coordinate axes. An ink drop with a mass m of 1.3×10^{-10} kg and a negative charge of magnitude $Q = 1.5 \times 10^{-13}$ C enters the region between the plates, initially moving along the x axis with speed $v_x = 18$ m/s. The length L of each plate is 1.6 cm. The plates are charged and thus produce an electric field at all points between them. Assume that field $\vec{E}$ is downward directed, is uniform, and has a magnitude of 1.4×10^6 N/C. What is the vertical deflection of the drop at the far edge of the plates? (The gravitational force on the drop is small relative to the electrostatic force acting on the drop and can be neglected.)

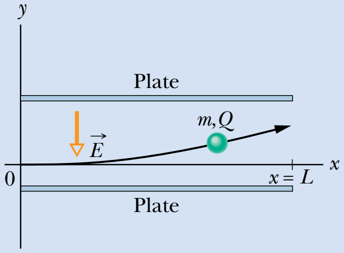


Figure 22-35 An ink drop of mass m and charge magnitude Q is deflected in the electric field of an ink-jet printer.

KEY IDEAS

The drop is negatively charged and the electric field is directed *downward*. From Eq. 22-45, a constant electrostatic force of magnitude QE acts *upward* on the charged drop. Thus, as the drop travels parallel to the x axis at constant speed v_x, it accelerates upward with some constant acceleration a_y.

Calculations: Applying Newton's second law ($F = ma$) for components along the y axis, we find that

$$a_y = \frac{F}{m} = \frac{QE}{m}. \tag{22-47}$$

Let t represent the time required for the drop to pass through the region between the plates. During t the vertical and horizontal displacements of the drop are

$$y = \tfrac{1}{2} a_y t^2 \quad \text{and} \quad L = v_x t, \tag{22-48}$$

respectively. Eliminating t between these two equations and substituting Eq. 22-47 for a_y, we find

$$\begin{aligned} y &= \frac{QEL^2}{2mv_x^2} \\ &= \frac{(1.5 \times 10^{-13}\ \text{C})(1.4 \times 10^6\ \text{N/C})(1.6 \times 10^{-2}\ \text{m})^2}{(2)(1.3 \times 10^{-10}\ \text{kg})(18\ \text{m/s})^2} \\ &= 6.4 \times 10^{-4}\ \text{m} = 0.64\ \text{mm}. \end{aligned} \qquad \text{(Answer)}$$

22.14 | A DIPOLE IN AN ELECTRIC FIELD

Key Concepts

- The torque on an electric dipole of dipole moment $\vec{p}$ when placed in an external electric field $\vec{E}$ is given by a cross product:

$$\vec{\tau} = \vec{p} \times \vec{E}.$$

- A potential energy U is associated with the orientation of the dipole moment in the field, as given by a dot product:

$$U = -\vec{p} \cdot \vec{E}.$$

- If the dipole orientation changes, the work done by the electric field is

$$W = -\Delta U$$

If the change in orientation is due to an external agent, the work done by the agent is $W_a = -W$.

We have defined the electric dipole moment $\vec{p}$ of an electric dipole to be a vector that points from the negative to the positive end of the dipole. As you will see, the behaviour of a dipole in a uniform external electric field $\vec{E}$ can be described completely in terms of the two vectors $\vec{E}$ and $\vec{p}$, with no need of any details about the dipole's structure.

A molecule of water (H_2O) is an electric dipole; Fig. 22-36 shows why. There the black dots represent the oxygen nucleus (having eight protons) and the two hydrogen nuclei (having one proton each). The colored enclosed areas represent the regions in which electrons can be located around the nuclei.

In a water molecule, the two hydrogen atoms and the oxygen atom do not lie on a straight line but form an angle of about 105°, as shown in Fig. 22-36. As a result, the molecule has a definite "oxygen side" and "hydrogen side." Moreover, the 10 electrons of the molecule tend to remain closer to the oxygen nucleus than to the hydrogen nuclei. This makes the oxygen side of the molecule slightly more negative than the hydrogen side and creates an electric dipole moment $\vec{p}$ that points along the symmetry axis of the molecule as shown. If the water molecule is placed in an external electric field, it behaves as would be expected of the more abstract electric dipole of Fig. 22-21.

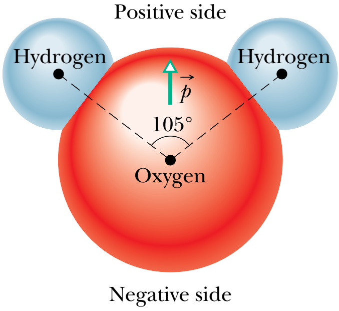
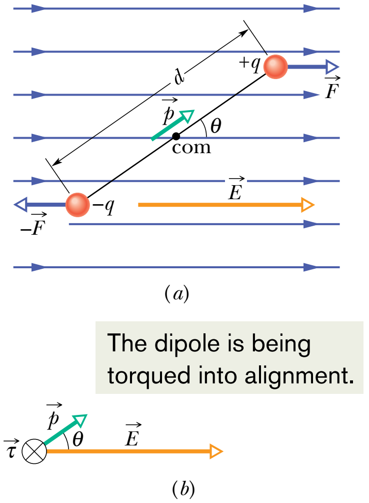


Figure 22-36 A molecule of H_2O, showing the three nuclei (represented by dots) and the regions in which the electrons can be located. The electric dipole moment p: points from the (negative) oxygen side to the (positive) hydrogen side of the molecule.

To examine this behavior, we now consider such an abstract dipole in a uniform external electric field $\vec{E}$, as shown in Fig. 22-37*a*. We assume that the dipole is a rigid structure that consists of two centers of opposite charge, each of magnitude q, separated by a distance d. The dipole moment $\vec{p}$ makes an angle θ with field $\vec{E}$.



The dipole is being torqued into alignment.

Figure 22-37 (*a*) An electric dipole in a uniform external electric field $\vec{E}$. Two centers of equal but opposite charge are separated by distance d. The line between them represents their rigid connection. (*b*) Field $\vec{E}$ causes a torque $\vec{\tau}$ on the dipole. The direction of $\vec{\tau}$ is into the page, as represented by the symbol ⊗.

Electrostatic forces act on the charged ends of the dipole. Because the electric field is uniform, those forces act in opposite directions (as shown in Fig. 22-37*a*) and with the same magnitude $F = qE$. Thus, *because the field is uniform,* the net force on the dipole from the field is zero and the center of mass of the dipole does not move. However, the forces on the charged ends do produce a net torque $\vec{\tau}$ on the dipole about its center of mass. The center of mass lies on the line connecting the charged ends, at some distance x from one end and thus a distance $d - x$ from the other end. From Eq. ($\tau = rF \sin \phi$), we can write the magnitude of the net torque $\vec{\tau}$ as

$$\tau = Fx \sin \theta + F(d - x) \sin \theta = Fd \sin \theta. \qquad (22\text{-}49)$$

We can also write the magnitude of $\vec{\tau}$ in terms of the magnitudes of the electric field E and the dipole moment $p = qd$. To do so, we substitute qE for F and p/q for d in Eq. 22-37, finding that the magnitude of $\vec{\tau}$ is

$$\tau = pE \sin \theta. \qquad (22\text{-}50)$$

We can generalize this equation to vector form as

$$\vec{\tau} = \vec{p} \times \vec{E} \quad \text{(torque on a dipole).} \qquad (22\text{-}51)$$

Vectors $\vec{P}$ and $\vec{E}$ are shown in Fig. 22-37*b*. The torque acting on a dipole tends to rotate $\vec{p}$ (hence the dipole) into the direction of field $\vec{E}$, thereby reducing θ. In Fig. 22-31, such rotation is clockwise. As we discussed in Chapter 10, we can represent a torque that gives rise to a clockwise rotation by including a minus sign with the magnitude of the torque. With that notation, the torque of Fig. 22-37 is

$$\tau = -pE \sin \theta \qquad (22\text{-}52)$$

Potential Energy of an Electric Dipole

Potential energy can be associated with the orientation of an electric dipole in an electric field. The dipole has its least potential energy when it is in its equilibrium orientation, which is when its moment $\vec{p}$ is lined up with the field $\vec{E}$ (then $\vec{\tau} = \vec{p} \times \vec{E} = 0$). It has greater potential energy in all other orientations. Thus the dipole is like a pendulum, which has *its* least gravitational potential energy in *its* equilibrium orientation—at its lowest point. To rotate the dipole or the pendulum to any other orientation requires work by some external agent.

In any situation involving potential energy, we are free to define the zeropotential-energy configuration in an arbitrary way because only differences in potential energy have physical meaning. The expression for the potential energy of an electric dipole in an external electric field is simplest if we choose the potential energy to be zero when the angle θ in Fig. 22-37 is 90°. We then can find the potential energy U of the dipole at any other value of θ

with the expression $\Delta U = -W$ by calculating the work W done by the field on the dipole when the dipole is rotated to that value of θ from 90°. With the aid of expression $W = \int \tau\, d\theta$ and Eq. 22-50, we find that the potential energy U at any angle θ is

$$U = -W = -\int_{90^\circ}^{\theta} \tau\, d\theta = \int_{90^\circ}^{\theta} pE \sin\theta\, d\theta. \qquad (22\text{-}53)$$

Evaluating the integral leads to

$$U = -pE\cos\theta. \qquad (22\text{-}54)$$

We can generalize this equation to vector form as

$$U = -\vec{p}\cdot\vec{E} \quad \text{(potential energy of a dipole).} \qquad (22\text{-}55)$$

Equations 22-54 and 22-55 show us that the potential energy of the dipole is least ($U = -pE$) when $\theta = 0$ ($\vec{p}$ and $\vec{E}$ are in the same direction); the potential energy is greatest ($U = pE$) when $\theta = 180°$ ($\vec{p}$ and $\vec{E}$ are in opposite directions).

When a dipole rotates from an initial orientation θ_i to another orientation θ_f, the work W done on the dipole by the electric field is

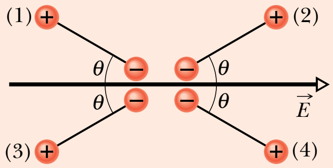
$$W = -\Delta U = -(U_f - U_i), \qquad (22\text{-}56)$$

where U_f and U_i are calculated with Eq. 22-55. If the change in orientation is caused by an applied torque (commonly said to be due to an external agent), then the work W_a done on the dipole by the applied torque is the negative of the work done on the dipole by the field; that is,

$$W_a = -W = (U_f - U_i) \qquad (22\text{-}57)$$

CHECKPOINT 9

The figure shows four orientations of an electric dipole in an external electric field. Rank the orientations according to (a) the magnitude of the torque on the dipole and (b) the potential energy of the dipole, greatest first.



Microwave Cooking

Food can be warmed and cooked in a microwave oven if the food contains water because water molecules are electric dipoles. When you turn on the oven, the microwave source sets up a rapidly oscillating electric field $\vec{E}$ within the oven and thus also within the food. From Eq. 22-51, we see that any electric field $\vec{E}$ produces a torque on an electric dipole moment $\vec{p}$ to align $\vec{p}$ with $\vec{E}$ Because the oven's $\vec{E}$ oscillates, the water molecules continuously flip-flop in a frustrated attempt to align with $\vec{E}$.

Energy is transferred from the electric field to the thermal energy of the water (and thus of the food) where three water molecules happened to have bonded together to form a group. The flip-flop breaks some of the bonds. When the molecules reform the bonds, energy is transferred to the random motion of the group and then to the surrounding molecules. Soon, the thermal energy of the water is enough to cook the food.

SAMPLE PROBLEM 22.12

Torque and energy of an electric dipole in an electric field

A neutral water molecule (H_2O) in its vapor state has an electric dipole moment of magnitude 6.2×10^{-30} C·m.

(a) How far apart are the molecule's centers of positive and negative charge?

KEY IDEA

A molecule's dipole moment depends on the magnitude q of the molecule's positive or negative charge and the charge separation d.

Calculations: There are 10 electrons and 10 protons in a neutral water molecule; so the magnitude of its dipole moment is

$$p = qd = (10e)(d),$$

in which d is the separation we are seeking and e is the elementary charge. Thus,

$$d = \frac{p}{10e} = \frac{6.2\times10^{-30}\ \text{C}\cdot\text{m}}{(10)(1.60\times10^{-19}\ \text{C})}$$
$$= 3.9\times10^{-12}\ \text{m} = 3.9\ \text{pm}. \qquad \text{(Answer)}$$

This distance is not only small, but it is also actually smaller than the radius of a hydrogen atom.

(b) If the molecule is placed in an electric field of 1.5×10^4 N/C, what maximum torque can the field exert on it? (Such a field can easily be set up in the laboratory.)

KEY IDEA

The torque on a dipole is maximum when the angle θ between $\vec{p}$ and $\vec{E}$ is 90°.

Calculation: Substituting $\theta = 90°$ in Eq. 22-35 yields

$$\begin{aligned}\tau &= pE\sin\theta\\ &= (6.2\times10^{-30}\ \text{C}\cdot\text{m})(1.5\times10^4\ \text{N/C})(\sin 90°)\\ &= 9.3\times10^{-26}\ \text{N}\cdot\text{m}. \qquad \text{(Answer)}\end{aligned}$$

(c) How much work must an *external agent* do to rotate this molecule by 180° in this field, starting from its fully aligned position, for which $\theta = 0$?

KEY IDEA

The work done by an external agent (by means of a torque applied to the molecule) is equal to the change in the molecule's potential energy due to the change in orientation.

Calculation: From Eq. 22-42, we find

$$\begin{aligned}W_a &= U_{180°} - U_0\\ &= (-pE\cos 180°) - (-pE\cos 0)\\ &= 2pE = (2)(6.2\times10^{-30}\ \text{C}\cdot\text{m})(1.5\times10^4\ \text{N/C})\\ &= 1.9\times10^{-25}\ \text{J}. \qquad \text{(Answer)}\end{aligned}$$

Why is the Concept of Electric Field Necessary?

It may seem to you from the above discussion that the electric field is a convenient method to find out force on a charged body and it has no utility other than the mathematical convenience. However, it is not true. Electric field is an entity which has its existence along with the charge that produces it. It cannot be seen but its presence can be felt in experiments.

Suppose we consider the force between two distance charges q_1 and q_2 in accelerated motion. The greatest speed with which a signal or information can go from one point to another is c, the speed of light. Thus, the effect of any motion of q_1 on q_2 cannot arise instantaneously. There will be some time delay between the effect (force on q_2) and the cause (motion of q_1). It is precisely seen in Fig. 22-38 that the notion of electric field (strictly, electromagnetic field) is natural and very useful. The field picture is this: The accelerated motion of charge q_1 produces electromagnetic waves, which then propagate with the speed c, reach q_2, and cause a force on q_2. The notion of field elegantly accounts for the time delay. Thus, even though electric and magnetic fields can be detected only by their effects (forces) on charges, they are regarded as physical entities, not merely mathematical constructs. They have independent dynamics of their own, that is, they evolve according to laws of their own. They can also transport energy. Thus, concept of field is now among the central concept in physics.

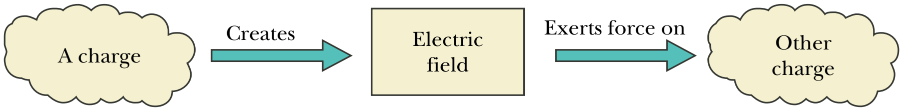


Figure 22-38 The sequence in which a charge feels the presence of another charge.

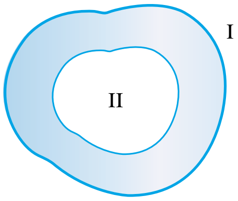


Figure 22-39 A conductor divides the entire space in two regions.

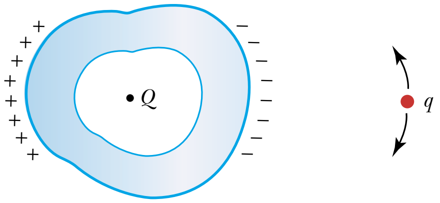


Figure 22-40 The motion of the charge outside the conductor changes the pattern of the induced charge on the surface of the conductor but does not influence the charges inside the cavity.

22.15 | ELECTROSTATIC SHIELDING

A conductor divides the whole space into two parts, (i) outside the conductor and (ii) inside the conductor (Fig. 22-39). We know that the electric field at a point in region II due to charges outside the conductor is cancelled by the charges induced on the surface of the conductor. Suppose that we move the charges outside the conductor. The induced charge pattern will quickly change to nullify the electric field inside the conductor. So, if a charge is kept inside the conductor (obviously in a cavity), it will not feel any net force due to charges outside (Fig. 22-40). In fact, it will not be even able to detect the motion of charges outside. This is called *electrostatic shielding*. This is why we are protected inside a car when lightning strikes.

So, we can say that

The electric field lines cannot penetrate the surface of a conductor.

Similarly, we could say that the motion of charges inside the cavity in a conductor will not be detected by the charges outside the conductor. Some induced charges will appear on the inner surface of the cavity which will cancel the effect of these charges in region I.

SAMPLE PROBLEM 22.13

Electric field at a point due to charge kept inside a thin conducting shell

Find the electric field at A (a) due to q; (b) due to charges induced on the inner surface of the shell and (c) due to charges induced on the outer surface of the shell (Fig. 22-41).

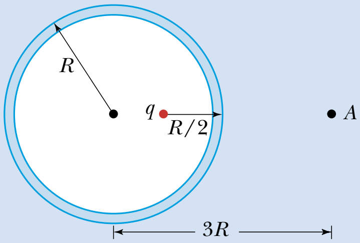


Figure. 22-41 A charge is kept inside a thin conducting shell.

KEY IDEA

First we note that the charge q is not symmetrically located in the shell. Nevertheless, the electric field due to a point charge is easy to calculate.

$$E_q = \frac{q}{4\pi\varepsilon_0 (5R/2)^2} = \frac{q}{25\pi\varepsilon_0 R^2}.$$

For finding the electric field due to the inner shell, we note that since the charge q is not symmetrically placed, the charge distribution on inner surface will not be symmetric. At the points close to q, charge density will be higher and lower at points farther away from the shell. If the charge was distributed uniformly, we could possibly use the formula for the electric field due to a uniformly charged shell at an external point. But this is not the case here. Not only that, we are not even aware of the pattern in which the induced charges will be distributed.

Calculations: This dilemma can be easily resolved by concept of electrostatic shielding. We know that the effect of the charge q will be cancelled by the charges induced on the inner surface of the cavity:

$$\vec{E}_q + \vec{E}_{\text{induced}} = \vec{0}.$$

$$E_{\text{in}} = -\frac{q}{25\pi\varepsilon_0 R^2}.$$

To find the electric field due to the charge on the outer surface, we again do not know the pattern of induced charges on that surface. Without that, we cannot find the electric field at A. Let us use a neat trick. Suppose the charge q was at the center of the shell. What would the charge distribution on the outer surface look like? Yes, it would be uniform. From this position, when we move q, the charge on outer surface will not come to know about it because it has been shielded from cavity. So the charge on the outer surface will still be uniform. Therefore, we can use the formula of the electric field due to a uniform shell:

$$\vec{E}_{\text{outer}} = \frac{kq}{R^2}.$$

So, the net electric field will be

$$\vec{E}_{\text{net}} = \frac{kq}{R^2}.$$

REVIEW AND SUMMARY

Electric Charge The strength of a particle's electrical interaction with objects around it depends on its **electric charge** (usually represented as q), which can be either positive or negative. Particles with the same sign of charge repel each other, and particles with opposite signs of charge attract each other. An object with equal amounts of the two kinds of charge is electrically neutral, whereas one with an imbalance is electrically charged and has an excess charge.

Conductors are materials in which a significant number of electrons are free to move.The charged particles in **nonconductors (insulators)** are not free to move.

Coulomb's Law Electric current i is the rate dq/dt at which charge passes a point:

$$i = \frac{dq}{dt} \quad \text{(electric current).} \tag{22-3}$$

Force Magnitude Coulomb's law describes the electrostatic force (or electric force) between two charged particles. If the particles have charges q_1 and q_2, are separated by distance r, and are at rest (or moving only slowly) relative to each other, then the magnitude of the force acting on each due to the other is given by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1||q_2|}{r^2} \quad \text{(Coulomb's law),} \tag{22-4}$$

where $\varepsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2$ is the **permittivity constant**. The ratio $1/4\pi\varepsilon_0$ is often replaced with the **electrostatic constant (or Coulomb constant)** $k = 8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2$.

The electrostatic force vector acting on a charged particle due to a second charged particle is either directly toward the second particle (opposite signs of charge) or directly away from it (same sign of charge). As with other types of forces, if multiple electrostatic forces act on a particle, the net force is the vector sum (not scalar sum) of the individual forces.

The two shell theories for electrostatics are

Shell theorem 1: A charged particle outside a shell with charge uniformly distributed on its surface is attracted or repelled as if the shell's charge were concentrated as a particle at its center.

Shell theorem 2: A charged particle inside a shell with charge uniformly distributed on its surface has no net force acting on it due to the shell.

Charge on a conducting spherical shell spreads uniformly over the (external) surface.

The Elementary Charge Electric charge is quantized (restricted to certain values). The charge of a particle can be written as ne, where n is a positive or negative integer and e is the elementary charge, which is the magnitude of the charge of the electron and proton ($\approx 1.602 \times 10^{-19}$ C).

Conservation of Charge The net electric charge of any isolated system is always conserved.

Electric Field To explain the electrostatic force between two charges, we assume that each charge sets up an electric field in the space around it.The force acting on each charge is then due to the electric field set up at its location by the other charge.

Definition of Electric Field The *electric field* $\vec{E}$ at any point is defined in terms of the electrostatic force $\vec{F}$ that would be exerted on a positive test charge q_0 placed there is

$$\vec{E} = \frac{\vec{F}}{q_0}. \tag{22-18}$$

Electric Field Lines *Electric field lines* provide a means for visualizing the direction and magnitude of electric fields.The electric field vector at any point is tangent to a field line through that point.The density of field lines in any region is proportional to the magnitude of the electric field in that region. Field lines originate on positive charges and terminate on negative charges.

Field Due to a Point Charge The magnitude of the electric Field $\vec{E}$ set up by a point charge q at a distance r from the charge is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{|q|}{r^2}. \tag{22-20}$$

The direction of $\vec{E}$ is away from the point charge if the charge is positive and toward it if the charge is negative.

Field Due to an Electric Dipole An *electric dipole* consists of two particles with charges of equal magnitude q but opposite sign, separated by a small distance d. Their **electric dipole moment** $\vec{p}$ has magnitude qd and points from the negative charge to the positive charge. The magnitude of the electric field set up by the dipole at a distant point on the dipole axis (which runs through both charges) is

$$E = \frac{1}{2\pi\varepsilon_0}\frac{p}{z^3}, \qquad (22\text{-}26)$$

where z is the distance between the point and the center of the dipole.

Field Due to a Continuous Charge Distribution The electric field due to a *continuous charge distribution* is found by treating charge elements as point charges and then summing, via integration, the electric field vectors produced by all the charge elements to find the net vector.

Field Due to a Charged Disk The electric field magnitude at a point on the central axis through a uniformly charged disk is given by

$$E = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right), \qquad (22\text{-}44)$$

where z is the distance along the axis from the center of the disk, R is the radius of the disk, and s is the surface charge density.

Electric Field inside a Conductor The electric field is zero at all points inside the conductor.

Force on a Point Charge in an Electric Field When a point charge q is placed in an external electric field $\vec{E}$, the electrostatic force $\vec{F}$ that acts on the point charge is

$$\vec{F} = q\vec{E}. \qquad (22\text{-}41)$$

Force $\vec{F}$ has the same direction as $\vec{E}$ if q is positive and the opposite direction if q is negative.

Dipole in an Electric Field When an electric dipole of dipole moment $\vec{p}$ is placed in an electric field $\vec{E}$, the field exerts a torque $\vec{\tau}$ on the dipole:

$$\vec{\tau} = \vec{p} \times \vec{E}. \qquad (22\text{-}51)$$

The dipole has a potential energy U associated with its orientation in the field:

$$U = -\vec{p} \cdot \vec{E}. \qquad (22\text{-}55)$$

This potential energy is defined to be zero when $\vec{p}$ is perpendicular to $\vec{E}$; it is least ($U = -pE$) when $\vec{p}$ is aligned with $\vec{E}$ and greatest ($U = pE$) when $\vec{p}$ is directed opposite $\vec{E}$.

PROBLEMS

1. In Fig. 22-42, four particles form a square. The charges are $q_1 = q_4 = Q$ and $q_2 = q_3 = q$. (a) What is Q/q if the net electrostatic force on particles 2 and 3 is zero? (b) Is there any value of q that makes the net electrostatic force on each of the four particles zero? Explain.

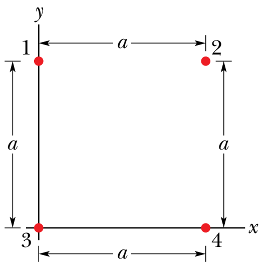


Figure 22-42 Problems 1 and 2.

2. In Fig. 22-42, the particles have charges $q_1 = -q_2 = 300$ nC and $q_3 = -q_4 = 200$ nC, and distance $a = 5.0$ cm. What are the (a) magnitude and (b) angle (relative to the $+x$ direction) of the net force on particle 3?

3. In Fig. 22-43, particle 1 of charge +6.0 μC and particle 2 of charge −2.0 μC are held at separation $L = 10.0$ cm on an x axis. If particle 3 of unknown charge q_3 is to be located such that the net electrostatic force on it from particles 1 and 2 is zero, what must be the (a) x and (b) y coordinates of particle 3?

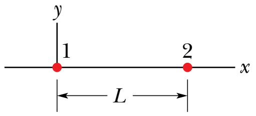


Figure 22-43 Problems 3, 5, and 6.

4. Three particles are fixed on an x axis. Particle 1 of charge q_1 is at $x = -a$, and particle 2 of charge q_2 is at $x = +a$. If their net electrostatic force on particle 3 of charge $+Q$ is to be zero, what must be the ratio q_1/q_2 when particle 3 is at (a) $x = +0.750a$ and (b) $x = +1.50a$?

5. In Fig. 22-43, particle 1 of charge $+q$ and particle 2 of charge $+9.00q$ are held at separation $L = 8.00$ cm on an x axis. If particle 3 of charge q_3 is to be located such that the three particles remain in place when released, what must be the (a) x and (b) y coordinates of particle 3, and (c) the ratio q_3/q?

6. In Fig. 22-43, particles 1 and 2 are fixed in place on an x axis, at a separation of $L = 6.00$ cm. Their charges are $q_1 = +e$ and $q_2 = -27e$. Particle 3 with charge $q_3 = +4e$ is to be placed on the line between particles 1 and 2, so that they produce a net electrostatic force $\vec{F}_{3,\text{net}}$ on it. (a) At what coordinate should particle 3 be placed to minimize the magnitude of that force? (b) What is that minimum magnitude?

7. As a cat rubs its back along a carpet, it acquires a charge of $+8.2 \times 10^{-7}$ C. How many electrons did it lose to the carpet?

8. Figure 22-44 shows electrons 1 and 2 on an x axis and charged ions 3 and 4 of identical charge $-q$ and at identical angles θ. Electron 2 is free to move; the other three particles are fixed in place at horizontal distances R from electron 2 and are intended to hold electron 2 in place. For physically possible values of $q \le 5e$, what are the (a) largest, (b) second largest, and (c) third largest values of θ for which electron 2 is held in place?

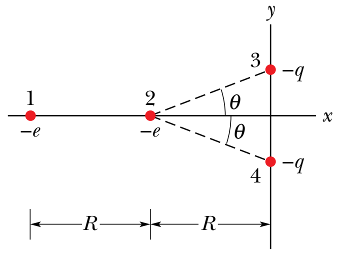


Figure 22-44 Problem 8.

9. The magnitude of the electrostatic force between point charges $q_1 = 26.0\ \mu C$ and $q_2 = 47.0\ \mu C$ is initially 5.70 N. The separation is then changed such that the force magnitude is then 0.570 N. (a) What is the ratio of the new separation to the initial separation? (b) What is the new separation?

10. In the return stroke of a typical lightning bolt, a current of 2.8×10^4 A exists for 20 μs. How much charge is transferred in this event?

11. In Fig. 22-45*a*, particles 1 and 2 have charge 20.0 μC each and are held at separation distance $d = 0.75$ m. (a) What is the magnitude of the electrostatic force on particle 1 due to particle 2? In Fig. 22-45*b*, particle 3 of charge 20.0 μC is positioned so as to complete an equilateral triangle. (b) What is the magnitude of the net electrostatic force on particle 1 due to particles 2 and 3?

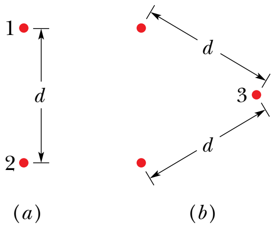


Figure 22-45 Problem 11.

12. Two tiny, spherical water drops, with identical charges of -1.00×10^{-16} C, have a center-to-center separation of 1.20 cm. (a) What is the magnitude of the electrostatic force acting between them? (b) How many excess electrons are on each drop, giving it its charge imbalance?

13. The magnitude of the electrostatic force between two identical ions that are separated by a distance of 10.0×10^{-10} m is 9.25×10^{-10} N. (a) What is the charge of each ion? (b) How many electrons are "missing" from each ion (thus giving the ion its charge imbalance)? (c) What is the force magnitude if the separation is halved?

14. Two identical conducting spheres, fixed in place, attract each other with an electrostatic force of 0.108 N when their center-to-center separation is 50.0 cm. The spheres are then connected by a thin conducting wire. When the wire is removed, the spheres repel each other with an electrostatic force of 0.144 N. Of the initial charges on the spheres, with a positive net charge, what was (a) the negative charge on one of them and (b) the positive charge on the other?

15. Two equally charged particles are held 3.2×10^{-3} m apart and then released from rest. The initial acceleration of the first particle is observed to be 6.0 m/s^2 and that of the second to be 9.0 m/s^2. If the mass of the first particle is 6.3×10^{-7} kg, what are (a) the mass of the second particle and (b) the magnitude of the charge of each particle?

16. The charges and coordinates of two charged particles held fixed in an *xy* plane are $q_1 = +3.0\ \mu C$, $x_1 = 3.5$ cm, $y_1 = 0.50$ cm, and $q_2 = -4.0\ \mu C$, $x_2 = -2.0$ cm, $y_2 = 1.5$ cm. Find the (a) magnitude and (b) direction of the electrostatic force on particle 1 due to particle 2. At what (c) *x* and (d) *y* coordinates should a third particle of charge $q_3 = +6.0\ \mu C$ be placed such that the net electrostatic force on particle 1 due to particles 2 and 3 is zero?

17. Identical isolated conducting spheres 1 and 2 have equal charges and are separated by a distance that is large compared with their diameters (Fig. 22-46*a*). The electrostatic force acting on sphere 2 due to sphere 1 is $\vec{F}$. Suppose now that a third identical sphere 3, having an insulating handle and initially neutral, is touched first to sphere 1 (Fig. 22-46*b*), then to sphere 2 (Fig. 22-46*c*), then to sphere 1 again (not shown), and then finally removed (Fig. 22-46*d*). The electrostatic force that now acts on sphere 2 has magnitude F'. What is the ratio F'/F?

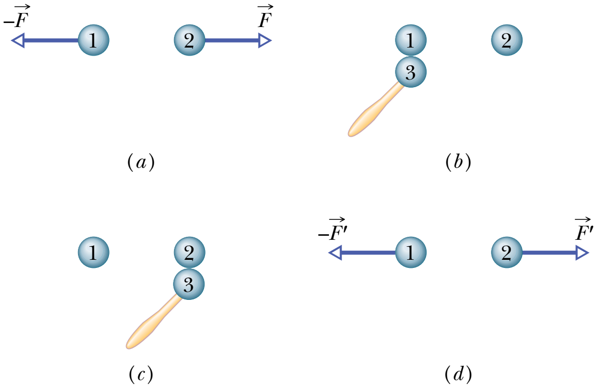


Figure 22-46 Problem 17.

18. In Fig. 22-47, particles 2 and 4, of charge $-e$, are fixed in place on a *y* axis, at $y_2 = -10.0$ cm and $y_4 = 5.00$ cm. Particles 1 and 3, of charge $-e$, can be moved along the *x* axis. Particle 5, of charge $+e$, is fixed at the origin. Initially particle 1 is at $x_1 = -10.0$ cm and particle 3 is at $x_3 = 10.0$ cm. (a) To what *x* value must particle 1 be moved to rotate the direction of the net electric force $\vec{F}_{net}$ on particle 5 by 60° counterclockwise? (b) With particle 1 fixed at its new position, to what *x* value must you move particle 3 to rotate $\vec{F}_{net}$ back to its original direction?

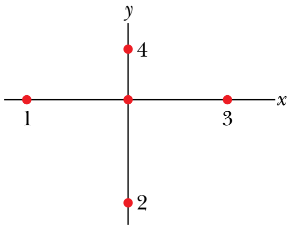


Figure 22-47 Problem 18.

19. In Fig. 22-48*a*, three positively charged particles are fixed on an *x* axis. Particles *B* and *C* are so close to each other that they can be considered to be at the same distance from particle *A*. The net force on particle *A* due to particles *B* and *C* is 2.310×10^{-23} N in the negative direction of the *x* axis. In Fig. 22-48*b*, particle *B* has been moved to the opposite side of *A* but is still at the same distance from it. The net force on *A* is now 2.877×10^{-24} N in the negative direction of the *x* axis. What is the ratio q_C/q_B?

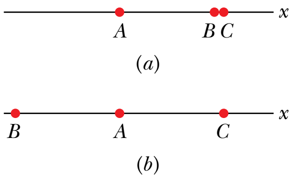


Figure 22-48 Problem 19.

20. In crystals of the salt cesium chloride, cesium ions Cs$^+$ form the eight corners of a cube and a chlorine ion Cl$^-$ is at the cube's center (Fig. 22-49). The edge length of the cube is 0.40 nm. The Cs^+ ions are each deficient by one electron (and thus each has a charge of $-e$), and the Cl^- ion has one excess electron (and thus has a charge of $-e$). (a)What is the magnitude of the net electrostatic force exerted on the Cl^- ion by the eight Cs^+ ions at the corners of the cube? (b) If one of the Cs^+ ions is missing, the crystal is said to have a *defect*; what is the magnitude of the net electrostatic force exerted on the Cl^- ion by the seven remaining Cl^+ ions?

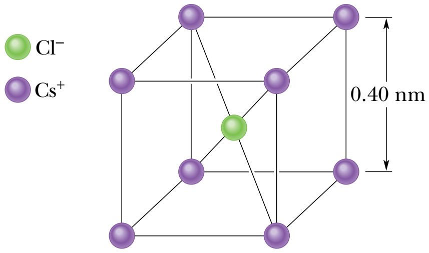


Figure 22-49 Problem 20.

21. Two particles are fixed on an x axis. Particle 1 of charge 50 μC is located at $x = -2.0$ cm; particle 2 of charge Q is located at $x = 3.0$ cm. Particle 3 of charge magnitude 20 μC is released from rest on the y axis at $y = 2.0$ cm. What is the value of Q if the initial acceleration of particle 3 is in the positive direction of (a) the x axis and (b) the y axis?

22. In Fig. 22-50, particles 1 and 2 of charge $q_1 = q_2 = +4e$ are on a y axis at distance $d = 17.0$ cm from the origin. Particle 3 of charge $q_3 = +8e$ is moved gradually along the x axis from $x = 0$ to $x = +5.0$ m. At what values of x will the magnitude of the electrostatic force on the third particle from the other two particles be (a) minimum and (b) maximum? What are the (c) minimum and (d) maximum magnitudes?

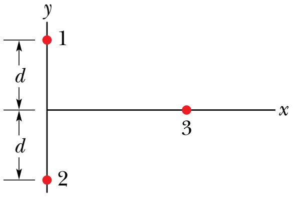


Figure 22-50 Problem 22.

23. In Fig. 22-51*a*, particle 1 (of charge q_1) and particle 2 (of charge q_2) are fixed in place on an x axis, 8.00 cm apart. Particle 3 (of charge $q_3 = +6.00 \times 10^{-19}$ C) is to be placed on the line between particles 1 and 2 so that they produce a net electrostatic force $\vec{F}_{3,\text{net}}$ on it. Figure 22-51 gives the x component of that force versus the coordinate x at which particle 3 is placed. The scale of the x axis is set by $x_s = 8.0$ cm. What are (a) the sign of charge q_1 and (b) the ratio q_2/q_1?

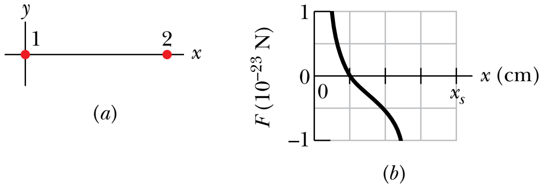


Figure 22-51 Problem 23.

24. A particle of charge 3.00 μC is separated by 0.120 m from a particle of charge -1.50 μC. (a) What is the magnitude of the electrostatic force between them? (b) What must their separation be to reduce that force by an order of magnitude?

25. Figure 22-52 shows an arrangement of four charged particles, with angle $\theta = 35.0°$ and distance $d = 2.00$ cm. Particle 2 has charge $q_2 = +8.00 \times 10^{-19}$ C; particles 3 and 4 have charges $q_3 = q_4 = -1.60 \times 10^{-19}$ C. (a) What is distance D between the origin and particle 2 if the net electrostatic force on particle 1 due to the other particles is zero? (b) If particles 3 and 4 were moved closer to the x axis but maintained their symmetry about that axis, would the required value of D be greater than, less than, or the same as in part (a)?

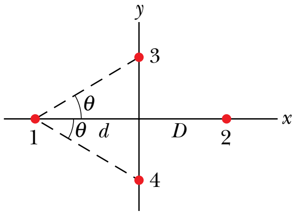


Figure 22-52 Problem 25.

26. In Fig. 22-53, three charged particles lie on an x axis. Particles 1 and 2 are fixed in place. Particle 3 is free to move, but the net electrostatic force on it from particles 1 and 2 happens to be zero. If $2.0L_{23} = L_{12}$, what is the ratio q_1/q_2?

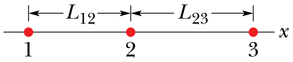


Figure 22-53 Problem 26.

27. Two charged particles are fixed to an x axis: Particle 1 of charge $q_1 = 2.1 \times 10^{-8}$ C is at position $x = 20$ cm and particle 2 of charge $q_2 = -4.00q_1$ is at position $x = 70$ cm. (a) At what coordinate on the axis (other than at infinity) is the net electric field produced by the two particles equal to zero? (b) What is the zero-field coordinate if the particles are interchanged?

28. At what distance along the central perpendicular axis of a uniformly charged plastic disk of radius 0.600 m is the magnitude of the electric field equal to 25% of the magnitude of the field at the center of the surface of the disk?

29. Assume that a honeybee is a sphere of diameter 1.000 cm with a charge of +60.0 pC uniformly spread over its surface. Assume also that a spherical pollen grain of diameter 40.0 μm is electrically held on the surface of the bee because the bee's charge induces a charge of -1.00 pC on the near side of the grain and a charge of +1.00 pC on the far side. (a) What is the magnitude of the net electrostatic force on the grain due to the bee? Next, assume that the bee brings the grain to a distance of 1.000 mm from the tip of a flower's stigma and that the tip is a particle of charge -60.0 pC. (b) What is the magnitude of the net electrostatic force on the grain due to the stigma? (c) Does the grain remain on the bee or move to the stigma?

30. A thin nonconducting rod with a uniform distribution of positive charge Q is bent into a complete circle of radius R (Fig. 22-54). The central perpendicular axis through the ring is a z axis, with the origin at the center of the ring. What is the magnitude of the electric field due to the rod at (a) $z = 0$ and (b) $z = \infty$? (c) In terms of R, at what positive value of z is that magnitude maximum? (d) If $R = 2.00$ cm and $Q = 5.00\ \mu$C, what is the maximum magnitude?

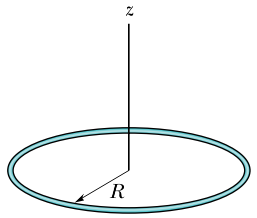


Figure 22-54 Problem 30.

31. Two large parallel copper plates are 8.0 cm apart and have a uniform electric field between them as depicted in Fig. 22-55. An electron is released from the negative plate at the same time that a proton is released from the positive plate. Neglect the force of the particles on each other and find their distance from the positive plate when they pass each other.

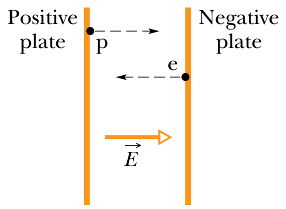


Figure 22-55 Problem 31.

32. Figure 22-56 shows an uneven arrangement of electrons (e) and protons (p) on a circular arc of radius $r = 2.50$ cm, with angles $\theta_1 = 30.0°$, $\theta_2 = 50.0°$, $\theta_3 = 30.0°$, and $\theta_4 = 20.0°$. What are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the net electric field produced at the center of the arc?

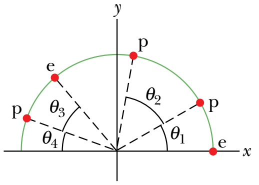


Figure 22-56 Problem 32.

33. Figure 22-57 shows two parallel nonconducting rings with their central axes along a common line. Ring 1 has uniform charge q_1 and radius R; ring 2 has uniform charge q_2 and the same radius R. The rings are separated by distance $d = 4.00R$. The net electric field at point P on the common line, at distance R from ring 1, is zero. What is the ratio q_1/q_2?

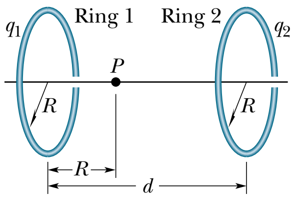


Figure 22-57 Problem 33.

34. In Fig. 22-58, positive charge $q = 9.25$ pC is spread uniformly along a thin nonconducting rod of length $L = 16.0$ cm. What are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the electric field produced at point P, at distance $R = 6.00$ cm from the rod along its perpendicular bisector?

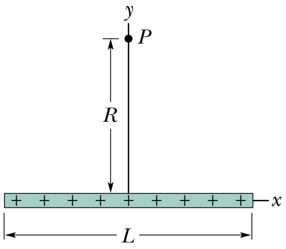


Figure 22-58 Problem 34.

35. In Fig. 22-59, two curved plastic rods, one of charge $+q$ and the other of charge $-q$, form a circle of radius $R = 4.25$ cm in an xy plane. The x axis passes through both of the connecting points, and the charge is distributed uniformly on both rods. If $q = 15.0$ pC, what are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the electric field $\vec{E}$ produced at P, the center of the circle?

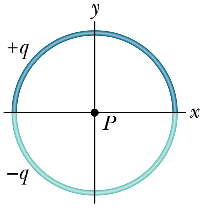


Figure 22-59 Problem 35.

36. A charged particle creates an electric field of magnitude 300 N/C at a point 0.800 m away. What is the difference in the field magnitude between that point and one at 0.400 m?

37. In Fig. 22-60, a nonconducting rod of length L = 8.15 cm has a charge $-q$ = −4.23 fC uniformly distributed along its length. (a) What is the linear charge density of the rod? What are the (b) magnitude and (c) direction (relative to the positive direction of the x axis) of the electric field produced at point P, at distance $a = 6.00$ cm from the rod? What is the electric field magnitude produced at distance $a = 50$ m by (d) the rod and (e) a particle of charge $-q = -4.23$ fC that we use to replace the rod? (At that distance, the rod "looks" like a particle.)

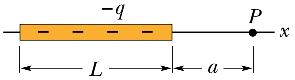


Figure 22-60 Problem 37.

38. In Fig. 22-61, a thin glass rod forms a semicircle of radius $r = 3.00$ cm. Charge is uniformly distributed along the rod, with $+q = 4.50$ pC in the upper half and $-q = -4.50$ pC in the lower half. What are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the electric field $\vec{E}$ at P, the center of the semicircle?

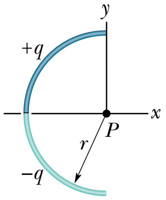


Figure 22-61 Problem 38.

39. Two charged particles are attached to an x axis: Particle 1 of charge -4.00×10^{-7} C is at position $x = -5.00$ cm and particle 2 of charge $+4.00 \times 10^{-7}$ C is at position $x = 10.0$ cm. Midway between the particles, what is their net electric field in unit-vector notation?

40. In Fig. 22-62, the four particles form a square of edge length $a = 5.00$ cm and have charges q_1 = +30 nC, q_2 = −15 nC, q_3 = +15 nC, and q_4 = −30 nC. In unit-vector notation, what net electric field do the particles produce at the square's center?

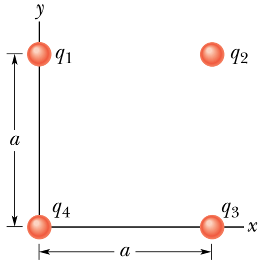


Figure 22-62 Problem 40.

41. An electron with a speed of 2.60×10^8 cm/s enters an electric field of magnitude 1.00×10^3 N/C, traveling along a field line in the direction that retards its motion. (a) How far will the electron travel in the field before stopping momentarily, and (b) how much time will have elapsed?

(c) If the region containing the electric field is 8.00 mm long (too short for the electron to stop within it), what fraction of the electron's initial kinetic energy will be lost in that region?

42. An electron is released from rest on the axis of an electric dipole is 25 nm from the center of the dipole. What is the magnitude of the electron's acceleration if the dipole moment is 3.6×10^{-29} C·m? Assume that 25 nm is much larger than the separation of the charged particles that form the dipole.

43. In Fig. 22-63 the electric field lines on the left have twice the separation of those on the right. (a) If the magnitude of the field at A is 60 N/C, what is the magnitude of the force on a proton at A? (b) What is the magnitude of the field at B?

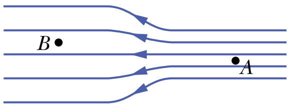


Figure 22-63 Problem 43.

44. Figure 22-64 shows a plastic ring of radius R = 43.0 cm. Two small charged beads are on the ring: Bead 1 of charge +2.00 μC is fixed in place at the left side; bead 2 of charge +6.00 μC can be moved along the ring. The two beads produce a net electric field of magnitude E at the center of the ring. At what (a) positive and (b) negative value of angle θ should bead 2 be positioned such that $E = 2.00 \times 10^5$ N/C?

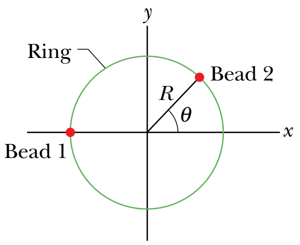


Figure 22-64 Problem 44.

45. Figure 22-65 shows two concentric rings, of radii R and $R' = 4.00R$, that lie on the same plane. Point P lies on the central z axis, at distance D = 2.00R from the center of the rings. The smaller ring has uniformly distributed charge $+Q$. In terms of Q, what is the uniformly distributed charge on the larger ring if the net electric field at P is zero?

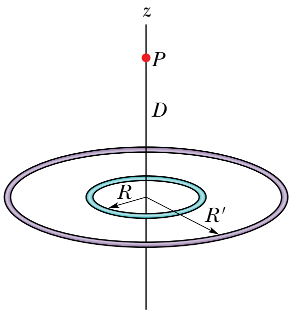


Figure 22-65 Problem 45.

46. A circular plastic disk with radius $R = 2.00$ cm has a uniformly distributed charge $Q = +(2.00 \times 10^6)e$ on one face. A circular ring of width 40 μm is centered on that face, with the center of that width at radius $r = 0.50$ cm. In coulombs, what charge is contained within the width of the ring?

47. Charge is uniformly distributed around a ring of radius R = 4.60 cm, and the resulting electric field magnitude E is measured along the ring's central axis (perpendicular to the plane of the ring). At what distance from the ring's center is E maximum?

48. In Fig. 22-66, the three particles are fixed in place and have charges $q_1 = q_2 = +5e$ and $q_3 = +2e$. Distance a = 3.00 µm. What are the (a) magnitude and (b) direction of the net electric field at point P due to the particles?

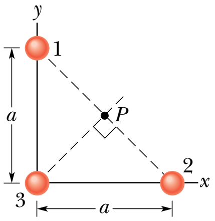


Figure 22-66 Problem 48.

49. In Fig. 22-67, the four particles are fixed in place and have charges $q_1 = q_2 = +5e$, $q_3 = +3e$, and $q_4 = -1e$. Distance d = 8.0 µm. What is the magnitude of the net electric field at point P due to the particles?

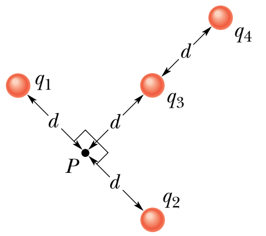


Figure 22-67 Problem 49.

50. An electric dipole consisting of charges of magnitude 1.50 nC separated by 6.20 µm is in an electric field of strength 300 N/C. What are (a) the magnitude of the electric dipole moment and (b) the difference between the potential energies for dipole orientations parallel and perpendicular to $\vec{E}$?

51. An electron is released from rest in a uniform electric field of magnitude 2.00×10^4 N/C. (a) Calculate the acceleration of the electron. (Ignore gravitation.) (b) How much time does the electron take to reach 1.00% of the speed of light?

52. An electron enters a region of uniform electric field with an initial velocity of 30 km/s in the same direction as the electric field, which has magnitude E = 50 N/C. (a) What is the speed of the electron 1.5 ns after entering this region? (b) How far does the electron travel during the 1.5 ns interval?

53. A uniform electric field exists in a region between two oppositely charged plates. An electron is released from rest at the surface of the negatively charged plate and strikes the surface of the opposite plate, 3.0 cm away, in a time 1.5×10^{-8} s. Just as the electron strikes the second plate, what are its (a) momentum magnitude and (b) kinetic energy? (c) What is the magnitude of the electric field $\vec{E}$?

54. An electric dipole consists of charges $+2e$ and $-2e$ separated by 0.85 nm. It is in an electric field of strength 3.4×10^6 N/C. Calculate the magnitude of the torque on the dipole when the dipole moment is (a) parallel to, (b) perpendicular to, and (c) antiparallel to the electric field.

55. At some instant the velocity components of an electron moving between two charged parallel plates are $v_x = 2.5 \times 10^5$ m/s and $v_y = 5.0 \times 10^3$ m/s. Suppose the electric field between the plates is uniform and given by $\vec{E} = (120 \text{ N/C})\hat{\text{j}}$. In unit-vector notation, what are (a) the electron's acceleration in that field and (b) the electron's velocity when its x coordinate has changed by 2.0 cm?

56. A charged particle produces an electric field with a magnitude of 2.0 N/C at a point that is 50 cm away from the particle. What is the magnitude of the field at an additional distance of 25 cm?

57. In Fig. 22-68, a "semi-infinite" nonconducting rod (that is, infinite in one direction only) has uniform linear charge density λ. (a) Show that the electric field $\vec{E}_p$ at point P makes an angle of 45° with the rod and that this result is independent of the distance R. (*Hint:* Separately find the component of $\vec{E}_p$ parallel to the rod and the component perpendicular to the rod.) (b) Find the field magnitude for linear charge density 4.52 nC/m and $R = 3.80$ cm.

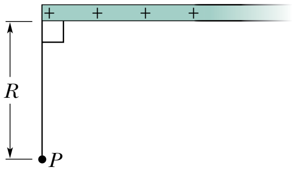


Figure 22-68 Problem 57.

58. Figure 22-69 shows two charged particles on an x axis: $-q = -3.20 \times 10^{-19}$ C at $x = -3.00$ m and $q = 3.20 \times 10^{-19}$ C at $x = +3.00$ m. What are the (a) magnitude and (b) direction (relative to the positive direction of the x axis) of the net electric field produced at point P at $y = 4.00$ m? (c) What is the magnitude of the net field if y is doubled?

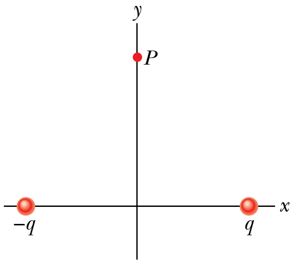


Figure 22-69 Problem 58.

59. Figure 22-70*a* shows a circular disk that is uniformly charged. The central z axis is perpendicular to the disk face, with the origin at the disk. Figure 22-70*b* gives the magnitude of the electric field along that axis in terms of the maximum magnitude E_m at the disk surface. The z axis scale is set by $z_s = 16.0$ cm. What is the radius of the disk?

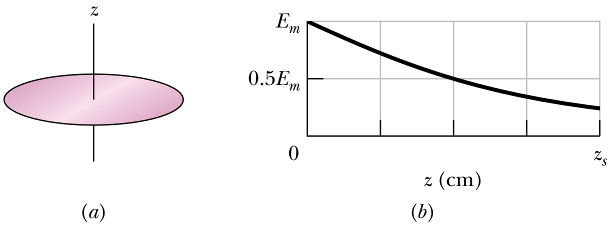


Figure 22-70 Problem 59.

60. In Fig. 22-71, an electron (e) is to be released from rest on the central axis of a uniformly charged disk of radius R. The surface charge density on the disk is +5.00 μC/m². What is the magnitude of the electron's initial acceleration if it is released at a distance (a) R, (b) R/100, and (c) R/1000 from the center of the disk? (d) Why does the acceleration magnitude increase only slightly as the release point is moved closer to the disk?

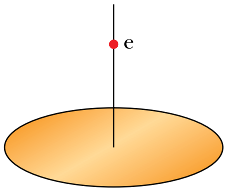


Figure 22-71 Problem 60.

61. Figure 22-72 shows three circular arcs centered on the origin of a coordinate system. On each arc, the uniformly distributed charge is given in terms of $Q = 4.00$ μC. The radii are given in terms of $R = 5.00$ cm. What are the (a) magnitude and (b) direction (relative to the positive x direction) of the net electric field at the origin due to the arcs?

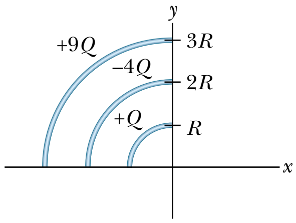


Figure 22-72 Problem 61.

62. Figure 22-73*a* shows a nonconducting rod with a uniformly distributed charge $+Q$. The rod forms a half-circle with radius R and produces an electric field of magnitude E_{arc} at its center of curvature P. If the arc is collapsed to a point at distance R from P (Fig. 22-73*b*), by what factor is the magnitude of the electric field at P multiplied?

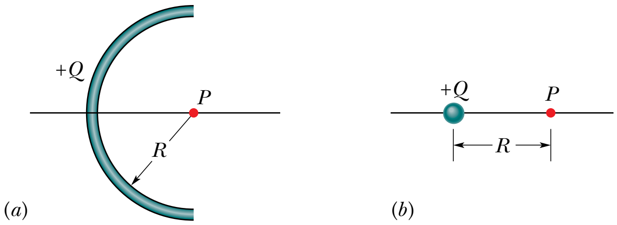


Figure 22-73 Problem 62.

63. Two charged particles lie along the x axis as shown in Fig. 22-74. The particle with charge $q2$ = +8 μC is at x = 6.00 m, and the particle with charge $q1$ = +2 μC is at the origin. Locate the point where the resultant electric field is zero.

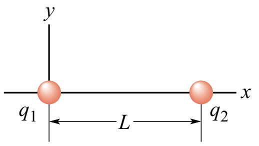


Figure 22-74 Problem 63

64. In the situation described in Problem 63, find the condition that a charge kept on $x = 2$ m may be in a stable equilibrium. If the charge is of magnitude 1 μC and has a mass of 3 g, find the time period of small oscillations of the body from its equilibrium position.

65. A semi-infinite-charged rod is arranged so that its one end coincides with the center of a charged ring as shown in Fig. 22-75. Both of them have a charge per unit length of λ. The radius of ring is R. The electric field of rod exerts force on an element of the ring as shown in Fig. 22-75(*b*). (a) What is the force on the ring due to the rod?

(b) What is the increase in the tension of the ring due to the rod?

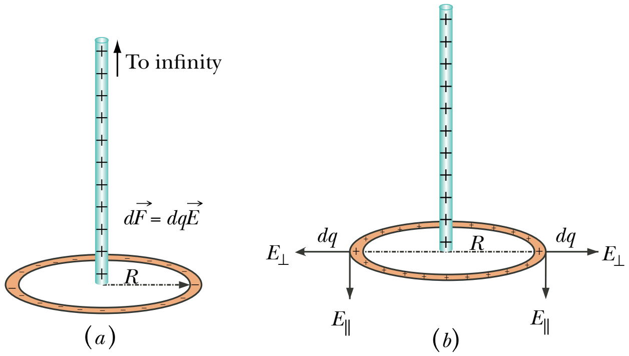


Figure 22-75 Problem 65

66. A positive charge q is placed in front of a conducting solid body at a distance d from its center. (a) Find the electric field at the center of the cube due to the charges appearing on its surface. (b) Does the body have to be spherical for the answer to be true?

67. A conducting spherical shell of outer radius R and inner radius $r = 3R/4$ has no net charge on it. At its center there is a point charge q, and at a distance $2R$ from its center there is a point charge Q (Fig. 22-76). (a) What is the magnitude of the electrostatic force on the charge q at the center due to the shell? (b) What is the magnitude of net force on it?

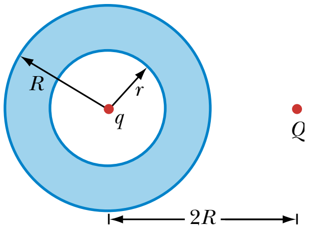


Figure 22-76 Problem 67.

68. A pellet with charge $+Q$ is centered inside a spherical cavity in a conducting ball. The conducting ball has a charge $-2Q$ on it. The cavity in the ball is off-center, as shown in Fig. 22-77. (a) How much charge resides on the inner surface of the cavity? (b) Is the charge on the cavity surface distributed uniformly or nonuniformly? Why? (c) How much charge resides on the outer surface of the conducting ball? (d) Is the charge on the ball's outer surface distributed uniformly or nonuniformly?

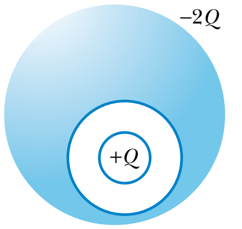


Figure 22-77 Problem 68.

PRACTICE QUESTIONS

Single Correct Choice Type

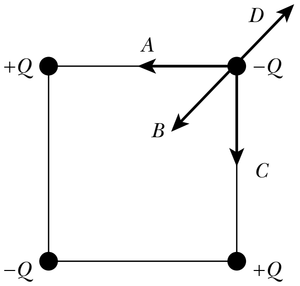
1. Complete the following statement: When a glass rod is rubbed with silk cloth, the rod becomes positively charged as

(a) positive charges are transferred from the silk to the rod.
(b) negative charges are transferred from the rod to the silk.
(c) positive charges are created on the surface of the rod.
(d) negative charges are transferred from the silk to the rod.

2. An aluminum nail has an excess charge of $+3.2\ \mu$C. How many electrons must be added to the nail to make it electrically neutral?

(a) 2.0×10^{13} **(b)** 3.2×10^{16}
(c) 2.0×10^{19} **(d)** 3.2×10^{6}

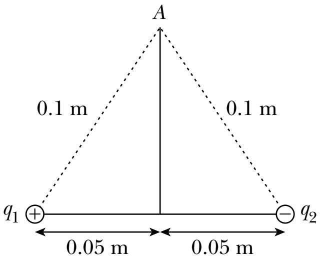
3. Four point charges, each of the same magnitude, with varying signs are arranged at the corners of a square as shown in the figure below. Which of the arrows labeled A, B, C, and D gives the correct direction of the net force that acts on the charge at the upper right corner?



(a) A **(b)** B
(c) C **(d)** D

4. Four point charges $q_A = 2\ \mu$C, $q_B = -5\ \mu$C, $q_C = 2\ \mu$C and $q_D = -5\ \mu$C are located at the corners A, B, C, D of a square $ABCD$ of side 10 cm (see figure). What is the force on a charge of 1 μC placed at the center of the square?

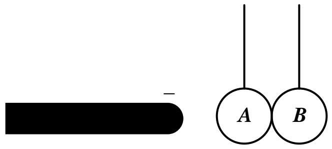
(a) 2×10^3 N **(b)** 4×10^3 N
(c) 8×10^3 N **(d)** 0



5. Assuming only electrostatic forces are present, electric field lines in the space surrounding a charge distribution show

(a) Only the directions in which static charges would accelerate when at points on those lines.
(b) Only the direction in which moving charges would accelerate when at points on those lines.
(c) Tangents to the directions in which either static or moving charges would accelerate when passing through points on those lines.
(d) The paths static or moving charges would take.

6. Two uncharged conducting spheres, ***A*** and ***B***, are suspended from insulating threads so that they touch each other as shown in the figure. While a negatively charged rod is held *near, but not touching* sphere ***A***, someone moves ball ***B*** away from ***A***. How will the spheres be charged, *if at all*?



Sphere *A*	Sphere *B*
(a) 0	+
(b) –	+
(c) 0	0
(d) +	–

7. A plate carries a charge of –3.0 μC, while a rod carries a charge of +2.0 μC. How many electrons must be transferred from the plate to the rod, so that both objects have the same charge?
(a) 6.3×10^{12} electrons
(b) 1.2×10^{13} electrons
(c) 8.0×10^{12} electrons
(d) 1.6×10^{13} electrons

8. Two point charges $q_A = 3\ \mu$C and $q_B = -3\ \mu$C are located 20 cm apart in vacuum. What is the electric field at the mid-point *O* of the line *AB* joining the two charges?
(a) 1.2×10^6 N/C
(b) 5.4×10^6 N/C
(c) 6.9×10^6 N/C
(d) 9.9×10^6 N/C

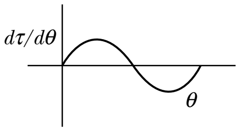
9. Which one of the following statements is true concerning the strength of the electric field between two oppositely charged parallel plates?
(a) It is zero midway between the plates.
(b) It is a maximum midway between the plates.
(c) It is a maximum near the positively charged plate.
(d) It is constant between the plates except near the edges.

10. Two tiny conducting spheres are identical and carry charges of –20.0 μC and +50.0 μC. They are separated by a distance of 2.50 cm. The spheres are brought into contact and then separated to a distance of 2.50 cm. Determine the magnitude of the force that each sphere now experiences, and state whether the force is attractive or repulsive.
(a) 1.30×10^2 N, attractive
(b) 3.24×10^3 N, attractive
(c) 3.24×10^3 N, repulsive
(d) 1.44×10^4 N, repulsive

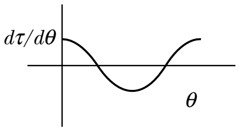
11. A –4.0 μC charge is located 0.30 m to the left of a +6.0 μC charge. What is the magnitude and direction of the electrostatic force on the positive charge?
(a) 2.4 N, to the right
(b) 2.4 N, to the left
(c) 4.8 N, to the right
(d) 4.8 N, to the left

12. Correct plot of $d\tau/d\theta \cdot \theta$ vs θ in case of an electric dipole placed in any uniform electric field is

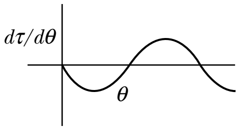
(a)



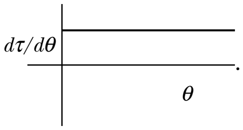
(b)



(c)



(d)



13. Consider three identical metal spheres, ***A***, ***B***, and ***C***. Sphere ***A*** carries a charge of –2.0 μC; sphere ***B*** carries a charge of –6.0 μC; and sphere ***C*** carries a charge of +5.0 μC. Spheres ***A*** and ***B*** are touched together and then separated. Spheres ***B*** and ***C*** are then touched and separated. Does sphere ***C*** end up with an excess or a deficiency of electrons and how many electrons is it?
(a) deficiency, 6×10^{13}
(b) excess, 3×10^{13}
(c) excess, 2×10^{13}
(d) deficiency, 3×10^{12}

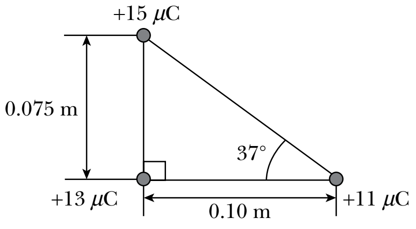
14. An early model for an atom considered to have a positively charged point nucleus of charge *Ze* surrounded by a uniform density of negative charge up to a radius *R*. The atom as a whole is neutral. For this model, what is the electric field at a distance *r* from the nucleus, where $r > R$?
(a) $\dfrac{1}{4\pi\varepsilon_0}\dfrac{Ze}{r^2}$
(b) $\dfrac{1}{4\pi\varepsilon_0}\dfrac{Ze}{R^2}$
(c) $\dfrac{1}{4\pi\varepsilon_0}\dfrac{Ze}{r^2 - R^2}$
(d) 0

15. Two identical charges +*Q* are kept fixed some distance apart. A small particle *P* with charge *q* is placed midway between them. If *P* is given a small displacement Δ, it will undergo simple harmonic motion if
(a) *q* is positive and the given displacement is along the line joining the charges.
(b) *q* is positive and the given displacement is perpendicular to the line joining the charges.
(c) *q* is negative and the given displacement is perpendicular to the line joining the charges.
(d) *q* is negative and the given displacement is along the line.

16. A charge *Q* exerts a 12 N force on another charge *q*. If the distance between the charges is doubled, what is the magnitude of the force exerted on *Q* by *q*?
(a) 3 N **(b)** 6 N
(c) 24 N **(d)** 36 N

17. Four identical metallic objects carry the following charges: +1.6, +6.2, –4.8, and –9.4 μC. The objects are brought simultaneously into contact, so that each touches the others. Then they are separated, what is the final charge on each object?
(a) –6.4 μC **(b)** –1.6 μC
(c) –5.6 μC **(d)** –2.4 μC

18. Three charges are positioned as indicated in the following figure. What are the horizontal and vertical components of the net force exerted on the +15 μC charge by the +11 μC and +13 μC charges?

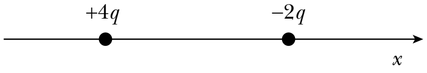


Horizontal	Vertical
(a) 95 N	310 N
(b) 76 N	310 N
(c) 76 N	370 N
(d) 95 N	130 N

19. An electric dipole in a uniform electric field has potential energy U. The magnitude of the torque acting on the dipole due to the field is τ.

(a) U is minimum and τ is zero when the dipole is perpendicular to the field.
(b) U is zero and τ is a maximum when the dipole is perpendicular to the field.
(c) U is a minimum and τ is a maximum when the dipole is perpendicular to the field.
(d) U is a minimum and τ is zero when the dipole is anti-parallel to the field.

20. At which point (or points) in the given figure is the electric field zero N/C for the two point charges shown on the x axis?



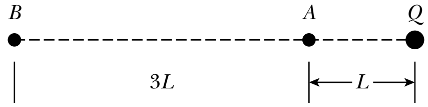
(a) The electric field is never zero in the vicinity of these charges.
(b) The electric field is zero somewhere on the x axis to the left of the +4q charge.
(c) The electric field is zero somewhere on the x axis to the right of the −2q charge.
(d) The electric field is zero somewhere on the x axis between the two charges, but this point is nearer to the −2q charge.

21. A small object, which has a charge $q = 7.5\ \mu$C and mass $m = 9.0 \times 10^{-5}$ kg, is placed in a constant electric field. Starting from rest, the object accelerates to a speed of 2.0×10^{3} m/s in a time of 0.96 s. Determine the magnitude of the electric field.

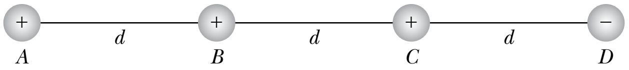
(a) 2.5×10^4 N/C
(b) 4.5×10^4 N/C
(c) 3.4×10^4 N/C
(d) 1.7×10^4 N/C

22. In the following figure, point ***A*** is a distance L away from a point charge Q. Point ***B*** is a distance 4L away from Q. What is the ratio of the electric field at ***B*** to that at ***A***, E_B/E_A?

(a) $\frac{1}{16}$
(b) $\frac{1}{9}$
(c) $\frac{1}{4}$
(d) $\frac{1}{3}$



23. Four point charges have equal magnitudes (see figure). Three are positive, and one is negative, as the drawing shows. They are fixed in place on the same straight line, and adjacent charges are equally separated by a distance d. Consider the net electrostatic force acting on each charge. Calculate the ratio of the largest to the smallest net force.



(a) 2.3
(b) 9.0
(c) 1.5
(d) 5.0

24. What is the magnitude and direction of the electric force on a −1.2 μC charge at a point where the electric field is 2500 N/C and is directed along the +y axis.

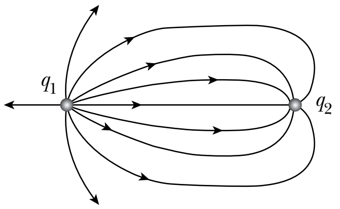
(a) 0.15 N, −y direction
(b) 0.0030 N, −y direction
(c) 0.15 N, +y direction
(d) 0.0030 N, +y direction

25. The membrane surrounding a living cell consists of an inner and an outer wall that are separated by a small space. Assume that the membrane acts like a parallel plate capacitor in which the effective charge density on the inner and outer walls has a magnitude of 7.1×10^{-6} C/m^2. What is the magnitude of the electric field within the cell membrane?

(a) 5.9×10^7 N/C
(b) 8.0×10^5 N/C
(c) 2.0×10^3 N/C
(d) 7.6×10^{-1} N/C

26. The following figure shows the electric field lines in the vicinity of two point charges. Which one of the following statements concerning this situation is true?

(a) q_1 is negative and q_2 is positive.
(b) The magnitude of the ratio (q_2/q_1) is less than one.
(c) Both q_1 and q_2 have the same sign of charge.
(d) The magnitude of the electric field is the same everywhere.

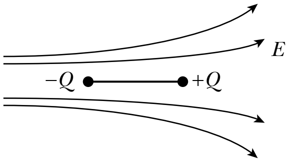


27. There are four charges, each with a magnitude of 2.0 μC. Two are positive and two are negative. The charges are fixed to the corners of a 0.30 m square, one to a corner, in such a way that the net force on any charge is directed toward the center of the square. Find the magnitude of the net electrostatic force experienced by any charge.

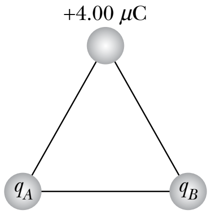
(a) 0.20 N
(b) 0.57 N
(c) 0.37 N
(d) 0.16 N

28. A rigid electric dipole is free to move in the electric field represented in the figure. Which one of the following phrases most accurately describes the initial motion of the dipole if it is released from rest in the position shown?

(a) It moves to the left.
(b) It moves to the right.
(c) It does not move at all.
(d) It moves toward to the top of the page.

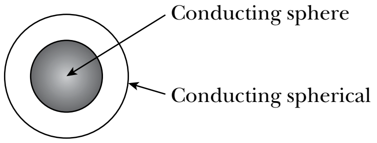


29. The following figure shows an equilateral triangle, each side of which has a length of 2.00 cm. Point charges are fixed to each corner, as shown. The 4.0 μC charge experiences a net force due to the charges q_A and q_B. This net force points vertically downward and has a magnitude of 405 N. Determine the magnitudes and algebraic signs of the charges q_A and q_B.



q_A	q_B
(a) −2.60 μC	−2.60 μC
(b) +5.20 μC	−5.20 μC
(c) −4.50 μC	−4.50 μC
(d) −2.60 μC	−4.50 μC

30. A conducting sphere carries a net charge of −6 μC. The sphere is located at the center of a conducting spherical shell that carries a net charge of +2 μC as shown in the figure. Determine the excess charge on the outer surface of the spherical shell.



(a) −4 μC
(b) +4 μC
(c) −8 μC
(d) +8 μC

31. Two identical conducting spheres carry charges of +5.0 μC and −1.0 μC, respectively. The centers of the spheres are initially separated by a distance L. The two spheres are brought together so that they are in contact. The spheres are then returned to their original separation L. What is the ratio of the magnitude of the electric force on either sphere *after* the spheres are touched to that *before* they were touched?

(a) $\frac{1}{1}$
(b) $\frac{9}{5}$
(c) $\frac{4}{5}$
(d) $\frac{5}{1}$

32. A charge $+q$ is located at the origin, while an identical charge is located on the x axis at $x = +0.50$ m. A third charge of $+2\,q$ is located on the x axis at such a place that the net electrostatic force on the charge at the origin doubles, its direction remaining unchanged. Where should the third charge be located?

(a) +0.50 m
(b) +0.71 m
(c) −0.50 m
(d) −0.71 m

33. A charge of −3.00 μC is fixed at the center of a compass. Two additional charges are fixed on the circle of the compass (radius = 0.100 m). The charges on the circle are −4.00 μC at the position due north and +5.00 μC at the position due east. What is the magnitude and direction of the net electrostatic force acting on the charge at the center? Specify the direction relative to due east.

(a) 3.60 N, 51.3° north of east
(b) 34.6 N, 36.3° north of east
(c) 14.4 N, 44.2° north of east
(d) 17.3 N, 38.7° south of east

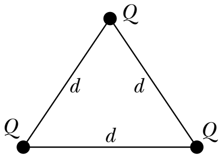
34. Four point charges have the same magnitude of 2.4×10^{-12} C and are fixed to the corners of a square that is 4.0 cm on a side. Three of the charges are positive and one is negative. Determine the magnitude of the net electric field that exists at the center of the square.

(a) 14 N/C
(b) 27 N/C
(c) 42 N/C
(d) 54 N/C

35. At a distance r_1 from a point charge, the magnitude of the electric field created by the charge is 248 N/C. At a distance r_2 from the charge, the field has a magnitude of 132 N/C. Find the ratio r_2/r_1.

(a) 0.730
(b) 0.939
(c) 1.12
(d) 1.37

36. Three identical point charges, Q, are placed at the vertices of an equilateral triangle as shown in the figure. The length of each side of the triangle is d. Determine the magnitude and direction of the total electrostatic force on the charge at the top of the triangle.



(a) $\frac{Q^2\sqrt{3}}{4\pi\varepsilon_0 d^2}$, directed upward

(b) $\frac{Q^2\sqrt{3}}{4\pi\varepsilon_0 d^2}$, directed downward

(c) $\frac{Q^2}{2\pi\varepsilon_0 d^2}$, directed upward

(d) $\frac{Q^2}{2\pi\varepsilon_0 d^2}$, directed downward

37. One mole of a substance contains 6.02×10^{23} protons and an equal number of electrons. If the protons could somehow be separated from the electrons and placed in very small, individual containers separated by 1.00×10^3 m, what would be the magnitude of the electrostatic force exerted by one box on the other?

(a) 8.7×10^8 N

(b) 9.5×10^9 N

(c) 2.2×10^{10} N

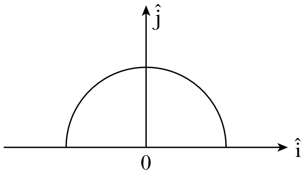
(d) 8.3×10^{13} N

38. Two identically charged spheres suspended from a common point by two massless strings of length l are initially a distance d ($d = 1$) apart because of their mutual repulsion. The charge begins to leak from both the spheres at a constant rate. As a result, the charges approach each other with a velocity v. Then as a function of distance x between them:

(a) $v \propto x^{-1}$ **(b)** $v \propto x^{1/2}$

(c) $v \propto x$ **(d)** $v \propto x^{-1/2}$

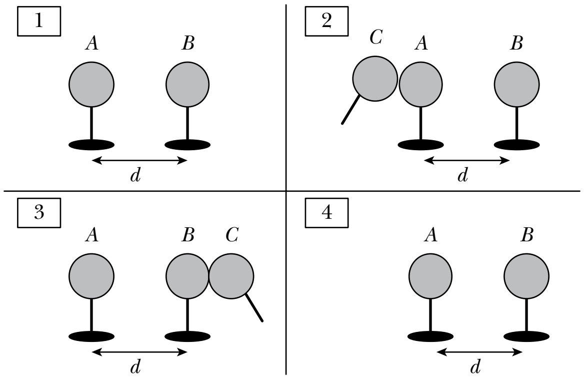
39. A thin semi-circular ring of radius r has a positive charge q distributed uniformly over it as shown in the figure. The net field $\vec{E}$ at the centre O is



(a) $\dfrac{q}{2\pi^2\varepsilon_0 r^2}\hat{j}$ **(b)** $\dfrac{q}{4\pi^2\varepsilon_0 r^2}\hat{j}$

(c) $-\dfrac{q}{4\pi^2\varepsilon_0 r^2}\hat{j}$ **(d)** $-\dfrac{q}{2\pi^2\varepsilon_0 r^2}\hat{j}$

40. In Frame **1**, two identical conducting spheres, ***A*** and ***B***, carry equal amounts of excess charge that have the same sign. The spheres are separated by a distance d; and sphere ***A*** exerts an electrostatic force on sphere B that has a magnitude F. A third sphere, ***C***, which is handled only by an insulating rod, is introduced in Frame **2**. Sphere ***C*** is identical to ***A*** and ***B*** except that it is *initially uncharged*. Sphere ***C*** is touched first to sphere ***A***, in Frame **2**, and then to sphere ***B***, in Frame **3**, and is finally removed in Frame **4** (see figure).



Determine the magnitude of the electrostatic force that sphere ***A*** exerts on sphere ***B*** in Frame **4**.

(a) $\dfrac{F}{2}$ **(b)** $\dfrac{F}{3}$

(c) $\dfrac{3F}{4}$ **(d)** $\dfrac{3F}{8}$

41. A large negatively charged object was placed on an insulated table. A neutral metallic ball rolled straight toward the object but stopped before touching it. A second neutral metallic ball rolled along the same path as the first ball, struck the first ball driving it a bit closer to the negatively charged object and stopped. After all balls stopped rolling, the first ball was closer to the negatively charged object than the second ball. At no time did either ball touch the charged object. Which statement is correct concerning the final charge on each ball?

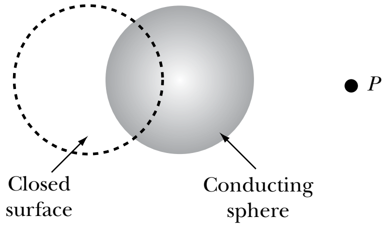
(a) The first ball is positive and the second negative.

(b) The first ball is negative and the second positive.

(c) Both balls remain neutral.

(d) Both balls are positive.

42. The following figure shows a closed surface which intersects a conducting sphere. If a positive charge is placed at the point P, the flux of the electric field through the closed surface



(a) will remain zero. **(b)** will become positive.

(c) will become negative. **(d)** will become undefined.

43. If we seal a pipe with two metal end caps around a point charge Q, the electric field outside the pipe will be

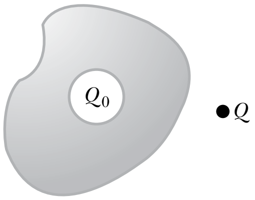
(a) identical to the field of an isolated point charge.

(b) identically zero, because the metal shields the field.

(c) nonzero but dependent on where the charge is within the pipe.

(d) nonzero, but independent of where the charge is within the pipe.

44. A charge Q_0 is placed at the center of spherical cavity of a conductor and another charge Q is placed outside the conductor as shown in the figure. The correct statement is:



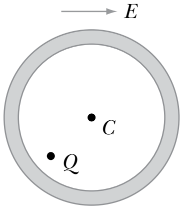
(a) Force on Q_0 and Q are equal and opposite.

(b) Force on Q_0 is zero and force on Q is nonzero.

(c) Force on Q and Q_0 is zero.

(d) Force on Q_0 is nonzero and force on Q is zero.

45. A positive point charge Q is kept, as shown in the figure, inside a neutral conducting shell whose center is at C. An external uniform electric field E is applied. Then



(a) Force on Q due to E is zero.
(b) Net force on Q is zero.
(c) Net force acting on Q and conducting shell considered as a system is zero.
(d) Net force acting on the shell due to E is toward right.

More than One Correct Choice Type

46. A particle of mass 2 kg and charge 1 mC is projected vertically with a velocity 10 m/s. There is a uniform horizontal electric field of 10^4 N/C.
(a) The horizontal range of the particle is 10 m.
(b) The time of flight of the particle is 2 s.
(c) The maximum height reached is 5 m.
(d) The horizontal range of the particle is 0 m.

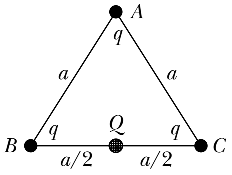
47. Two small spheres with masses m_1 and m_2 hang on weightless, insulating threads with l_1 and l_2. The two spheres carry a charge of q_1 and q_2, respectively. The spheres hang such that they are level with one another and the threads are inclined to the vertical at angles q_1 and q_2. Which of the following conditions is required for q_1 and q_2 to be equal?
(a) $m_1 = m_2$ **(b)** $|q_1| = |q_2|$
(c) $l_1 = l_2$ **(d)** None of these

48. Three non-conducting infinite planes of charge A, B, and C are vertical and parallel to one another. There is a uniform electric field $\vec{E}_\perp$ to the left of plane A and a uniform electric field $\vec{E}_p$ to the right of plane C. The field $\vec{E}_p$ points to the left and the field $\vec{E}_p$ points to the right. The signs of the charges on the plates A, B, and C may be
(a) –, –, – **(b)** +, –, –
(c) +, –, + **(d)** +, +, +

Linked Comprehension

Paragraph for Questions 49 and 50: The given figure shows an equilateral triangle ***ABC***. A positive point charge $+q$ is located at each of the three vertices ***A***, ***B***, and ***C***. Each side of the triangle is of length a.

A point charge Q (that may be positive or negative) is placed at the mid-point between ***B*** and ***C***.



49. Is it possible to choose the value of Q (*that is non-zero*) such that the force on Q is zero? Explain why or why not.
(a) Yes, because the forces on Q are vectors and three vectors can add to zero.
(b) No, because the forces on Q are vectors and three vectors can never add to zero.
(c) Yes, because the electric force at the mid-point between ***B*** and ***C*** is zero whether a charge is placed there or not.
(d) No, because a fourth charge would be needed to cancel the force on Q due to the charge at ***A***.

50. Determine an expression for the magnitude and sign of Q so that the net force on the charge at ***A*** is zero newtons.

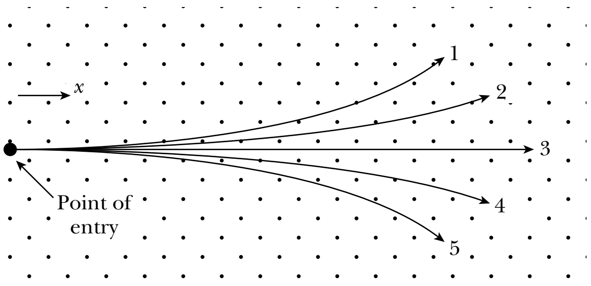
(a) $Q = +q\left(\dfrac{3\sqrt{3}}{4}\right)$

(b) $Q = -q\left(\dfrac{3\sqrt{3}}{4}\right)$

(c) $Q = -q\left(\dfrac{4\sqrt{3}}{3}\right)$

(d) $Q = +q\left(\dfrac{3}{4\sqrt{3}}\right)$

Paragraph for Questions 51–53: The given figure shows five particles are shot from the left into a region that contains a uniform electric field. The numbered lines show the paths taken by the five particles. A negatively charged particle with a charge $-3Q$ follows path **2** while it moves through this field. Do not consider any effects due to gravity.



51. In which direction does the electric field point?
(a) Toward the top of the page.
(b) Toward the left of the page.
(c) Toward the right of the page.
(d) Toward the bottom of the page.

52. Which path would be followed by a helium atom (an electrically neutral particle)?
(a) path **1** **(b)** path **2**
(c) path **3** **(d)** path **4**

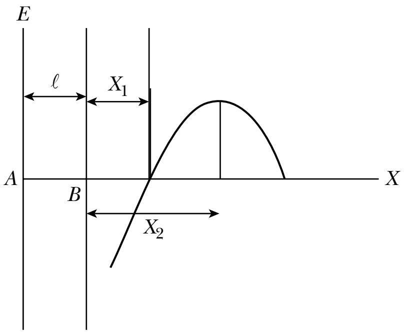
53. Which path would be followed by a charge $+6Q$?
(a) path **1** **(b)** path **5**
(c) path **2** **(d)** path **4**

Matrix-Match

54. An electric dipole is placed in an electric field. Column I is the description of electric field and the angle between the dipole moment $\vec{p}$ and the electric field intensity $\vec{E}$ and Column II gives the effect of the electric field on the dipole. Then,

Column I	Column II
(a) Uniform electric field	**(p)** Force = 0
(b) Electric field due to a point charge	**(q)** Torque = 0
(c) Electric field between the two oppositely charged large plates	**(r)** $\vec{p}\times\vec{E}=0$
(d) Dipole moment parallel to uniformly charged long wire	**(s)** Force ≠ 0

55. Two points, like charges Q_A and Q_B are positioned at points A and B (see figure). The electric field strength to the right of charge Q_B on the line that passes through the two charges varies according to a law that is represented schematically in Figure below accompanying the problem without employing a definite scale. Assume electric field to be positive if its direct ion coincides with (the positive direction on the x-axis. The distance between the charges is l.

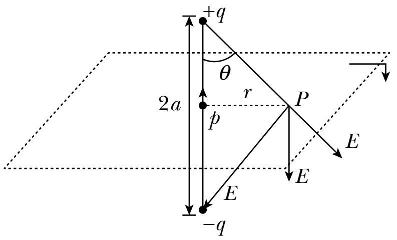
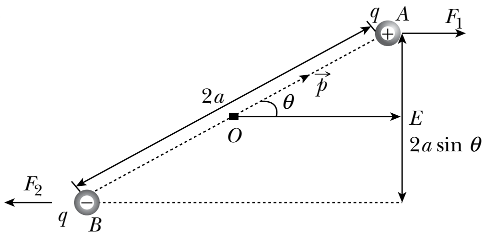


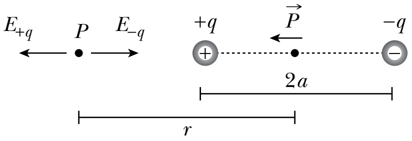
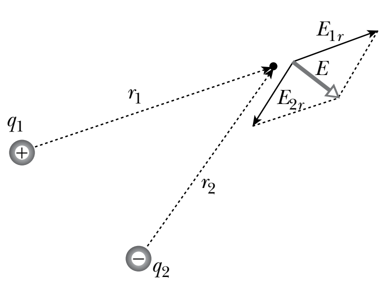
Column I	Column II
(a) Charge Q_A	**(p)** Positive
(b) Charge Q_B	**(q)** Negative
(c) $\lvert Q_A/Q_B\rvert$	**(r)** $\left(\dfrac{l+x_1}{x_1}\right)^2$
(d) x_2	**(s)** $\dfrac{l}{(Q_A/Q_B)^{1/3}-1}$

Directions for Questions 56 and 57: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

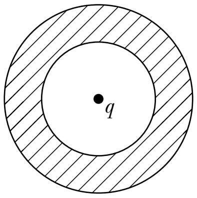
56. An electric dipole is the combination of two equal and opposite charges q and $-q$, respectively, separated by distance $2a$. there are different points where we can find electric field due to the electric dipole. In the given table, column 1 shows different positions of point where we have to find electric field due to dipole, column 2 shows the figure of dipole with different positions of point where we have to find the electric filed and column 3 shows the value or final formula of the electric filed of different positions of the point.

1. When point P lies on the equatorial plane of the electric dipole, electric field due to $+q$ is?
   (a) (III) (ii) (L) **(b)** (IV) (i) (M)
   (c) (III) (iii) (J) **(d)** (I) (i) (M)
2. When point P lies on the axis of the electric dipole, electric field due to $-q$ is?
   (a) (III) (ii) (L) **(b)** (II) (iii) (L)
   (c) (II) (iii) (J) **(d)** (I) (i) (K)
3. When point P lies on the equatorial plan of the electric dipole, electric field due to $-q$ is?
   (a) (I) (i) (J) **(b)** (I) (i) (M)
   (c) (III) (iii) (J) **(d)** (I) (i) (K)

Column I	Column II	Column III
(I) Point P is at a distance r from the center of the dipole	**(i)** 	**(J)** $\dfrac{-q\hat{p}}{4\pi\varepsilon_0(r^2+a^2)}$
(II) P be the point at a distance r from the center of the dipole on side of charge $+q$	**(ii)**  q A F1 2a p θ O E 2a sin θ F2 q B	**(K)** $\dfrac{q\hat{p}}{4\pi\varepsilon_0(r-a)^2}$

Column I	Column II	Column III
(III) *P* be the point at infinity	**(iii)**  E_{+q} P E_{-q} $+q$ $\vec{P}$ $-q$ $2a$ r	**(L)** $\frac{-q\hat{p}}{4\pi\varepsilon_0(r+a)^2}$
(IV) *P* be the point at the center	**(iv)**  E_{1r} E E_{2r} r_1 q_1 r_2 q_2	**(M)** $\frac{q\hat{p}}{4\pi\varepsilon_0(r^2+a^2)}$

57. Consider the figure given below



A charge *Q* is placed inside the Spherical conductor shell.

In the given table, column 1 shows different charges, column 2 shows the positions of the electric charge and column 3 shows the different types of charge distribution.

Column I	Column II	Column III
(I) Charge *q*	**(i)** at the center of the shell	**(J)** Charge distribution is at the inner surface
(II) Charge *Q*	**(ii)** displaced from center	**(K)** Charge distribution is at the outer surface
(III) An excess charge *q*	**(iii)** not given to the shell but just displaced	**(L)** Charge distribution is in the center
(IV) An excess charge *Q*	**(iv)** given to the shell	**(M)** No Charge distribution

1. In which case will the Charge distribution be non-uniform?

(a) (I) (i) (M) **(b)** (IV) (ii) (M)

(c) (II) (ii) (J) **(d)** (I) (iv) (K)

2. In which case will the Charge distribution be uniform?

(a) (III) (iv) (K)

(b) (IV) (i) (M)

(c) (III) (ii) (L)

(d) (I) (i) (M)

3. What happens when the charge is near the conductor shell?

(a) (III) (i) (K)

(b) (IV) (i) (J)

(c) (III) (iii) (M)

(d) (II) (iii) (K)

Integer Type

58. A point charge $q = 1$ C and mass 1 kg is projected with speed 10 m/s in the perpendicular direction of uniform electric field $E = 100$ V/m. What is the value of latus rectum of the path followed by charged particle (in meter)?

59. Two spherical objects are separated by a distance of 1.80×10^{-3} m. The objects are initially electrically neutral and are very small compared to the distance between them. Each object acquires the same negative charge due to the addition of electrons. As a result, each object experiences an electrostatic force that has a magnitude of 4.55×10^{-21} N. How many electrons did it take to produce the charge on one of the objects?

60. An electric dipole made up of a positive and a negative charge, each 1 m C placed 2 cm apart. If the dipole is held in an electric field of 10^5 N/C, then what is the maximum torque the field exerts on the dipole, while turning from a position $\theta = 0°$ to $\theta = 180°$?

ANSWER KEY

Checkpoints

1. *C* and *D* attract; *B* and *D* attract
2. (a) leftward; (b) leftward; (c) leftward
3. (a) *a*, *c*, *b*; (b) less than

4. Increases because $\left(\frac{q_1+q_2}{2}\right)^2 \geqslant q_1q_2$
5. $-15e$ (net charge of $-30e$ is equally shared)
6. (a) rightward; (b) leftward; (c) leftward; (d) rightward (p and e have same charge magnitude, and p is farther)
7. (a) toward positive y; (b) toward positive x; (c) toward negative y
8. (a) leftward; (b) leftward; (c) decrease
9. (a) all tie; (b) 1 and 3 tie, then 2 and 4 tie

Problems

1. (a) -0.354; (b) No
2. (a) 0.26 N; (b) $-32°$
3. (a) 24 cm; (b) 0
4. (a) 49; (b) $-25°$
5. (a) 2.00 cm; (b) 0; (c) -0.563
6. (a) 1.50 cm; (b) 1.64×10^{-23} N
7. 5.1×10^{12}
8. (a) 62.3°; (b) 60°; (c) 56.6°
9. (a) 3.16; (b) 4.39 m
10. 0.56 C
11. (a) 6.39 N; (b) 11.1 N
12. (a) 6.24×10^{-19} N; (b) 625
13. (a) 3.2×10^{-19} C; (b) 2; (c) 3.7×10^{-9} N
14. (a) -0.646 μC; (b) 4.65 μC
15. (a) 4.2×10^{-7} kg; (b) 6.6×10^{-11} C
16. (a) 35 N; (b) 170°; (c) +10 cm; (d) -0.69 cm
17. 0.469
18. (a) -4.02 cm; (b) 4.02 cm
19. 1.285
20. (a) 0; (b) 1.9×10^{-9} N
21. (a) -104 μC; (b) 69.4 μC
22. (a) 0; (b) 12 cm; (c) 0; (d) 2.0×10^{-25} N
23. (a) positive; (b) 9.0
24. (a) 2.81 N; (b) 0.379 m
25. (a) 2.27 cm; (b) less
26. -9.00
27. (a) -30 cm; (b) 1.20 m
28. 0.680 m
29. (a) 3.4×10^{-10} N; (b) 4.1×10^{-8} N; (c) moves to stigma
30. (a) 0; (b) 0; (c) $0.707R$; (d) 4.33×10^{7} N/C
31. 44 μm
32. (a) 2.51×10^{-6} N/C; (b) $-76.4°$ from x axis
33. 0.268
34. (a) 13.9 N/C; (b) +90° counterclockwise from the $+x$ axis
35. (a) 95.1 N/C; (b) $-90°$
36. 900 N/C
37. (a) -5.19×10^{-14} C/m; (b) 4.48×10^{-3} N/C; (c) $-180°$; (d) 1.52×10^{-8} N/C; (e) 1.52×10^{-8} N/C
38. (a) 57.2 N/C; (b) $-90°$ counterclockwise from the ^{+}x axis
39. $-(1.28 \times 10^{6}\ \text{N/C})\hat{\text{i}}$
40. $(-1.53 \times 10^{5}\text{N/C})\hat{\text{j}}$
41. (a) 1.92×10^{-2} m; (b) 14.8 ns; (c) 41.6%
42. 7.3×10^{15} m/s^2
43. (a) 9.6×10^{-18} N; (b) 30 N/C
44. (a) 67.8°; (b) $-67.8°$
45. $-8.00Q$
46. 3.2×10^{-16} C
47. 3.25 cm
48. (a) 639 N/C; (b) 45°
49. $\frac{1}{4\pi\varepsilon_0}\left(\frac{12q}{4d^2}-\frac{3q}{d^2}\right)\hat{\text{j}}$
50. (a) 9.30×10^{-15} C·m; (b) 2.79×10^{-12} J
51. (a) 3.51×10^{15} m/s^2; (b) 0.854 ns
52. (a) 1.7×10^{4} m/s; (b) 3.5×10^{-5} m
53. (a) 3.6×10^{-24} kg·m/s; (b) 7.3×10^{-18} J; (c) 1.5 kN/C
54. (a) 0; (b) 9.2×10^{-22} N·m; (c) 0
55. (a) $-(2.1 \times 10^{13}\ \text{m/s}^2)\hat{\text{j}}$; (b) $(2.5 \times 10^{13}\text{m/s}^2)\hat{\text{i}} - (1.7 \times 10^{6}\text{m/s})\hat{\text{j}}$
56. 0.89 N/C
57. (b) 1.51×10^{3} N/C
58. (a) 1.38×10^{-10} N/C; (b) 180°; (c) 2.77×10^{-11} N/C
59. 13.9 cm
60. (a) 1.45×10^{16} m/s^2; (b) 4.93×10^{16} m/s^2; (c) 4.96×10^{16} m/s^2
61. (a) 1.30×10^{7} N/C; (b) $-45°$
62. 1.57
63. 2 m
64. $\frac{4\pi}{3}$ s
65. (a) $\frac{\lambda^2}{2\varepsilon_0}$; (b) $\frac{\lambda^2}{4\pi\varepsilon_0}$
66. (a) $\frac{q}{4\pi\varepsilon_0 d^2}$; (b) no
67. (a) $\frac{kQq}{4R^2}$ towards right; (b) 0
68. (a) $-Q$; (b) uniformaly; (c) $-Q$; (d) uniformaly

Practice Questions

Single Correct Choice Type

1. (b)
2. (a)
3. (b)
4. (d)
5. (c)
6. (d)
7. (d)
8. (b)
9. (d)
10. (c)
11. (b)
12. (b)
13. (d)
14. (d)
15. (d)
16. (a)
17. (b)
18. (c)
19. (b)
20. (c)

21. (a)
22. (a)
23. (b)
24. (b)
25. (b)
26. (b)
27. (c)
28. (a)
29. (a)
30. (a)
31. (c)
32. (b)
33. (d)
34. (d)
35. (d)
36. (a)
37. (d)
38. (d)
39. (d)
40. (d)
41. (a)
42. (b)
43. (d)
44. (b)
45. (d)

More than One Correct Choice Type

46. (a), (b), (c)
47. (a), (c)
48. (b), (c), (d)

Linked Comprehension

49. (d)
50. (b)
51. (d)
52. (c)
53. (b)

Matrix-Match

54. (a) → (p), (q), (r); (b) → (q), (r), (s); (c) → (p), (r); (d) → (p), (r).
55. (a) → (p), (b) → (q), (c) → (r), (d) → (s).
56. (1) → (d); (2) → (b); (3) → (a)
57. (1) → (c); (2) → (a); (3) → (d)

Integer Type

58. 2
59. 8
60. 2

CHAPTER

23

Gauss' Law

Contents

23.1 What is Physics?
23.2 Electric Flux
23.3 Gauss' Law
23.4 A Charged Isolated Conductor
23.5 Applying Gauss' Law: Spherical Symmetry
23.6 Applying Gauss' Law: Cylindrical Symmetry
23.7 Applying Gauss' Law: Planar Symmetry

23.1 | WHAT IS PHYSICS?

In the preceding chapter we found the electric field at points near extended charged objects, such as rods. Our technique was labor-intensive: We split the charge distribution up into charge elements dq, found the field $d\vec{E}$ due to an element, and resolved the vector into components. Then we determined whether the components from all the elements would end up canceling or adding. Finally we summed the adding components by integrating over all the elements, with several changes in notation along the way.

One of the primary goals of physics is to find simple ways of solving such labor-intensive problems. One of the main tools in reaching this goal is the use of symmetry. In this chapter we discuss a beautiful relationship between charge and electric field that allows us, in certain symmetric situations, to find the electric field of an extended charged object with a few lines of algebra. The relationship is called **Gauss' law**, which was developed by German mathematician and physicist Carl Friedrich Gauss (1777–1855).

Let's first take a quick look at some simple examples that give the spirit of Gauss' law. Figure 23-1 shows a particle with charge $+Q$ that is surrounded by an imaginary concentric sphere. At points on the sphere (said to be a *Gaussian surface*), the electric field vectors have a moderate magnitude (given by $= kQ/r^2$) and point radially away from the particle (because it is positively charged). The electric field lines are also outward and have a moderate density (which, recall, is related to the field magnitude). We say that the field vectors and the field lines *pierce* the surface.

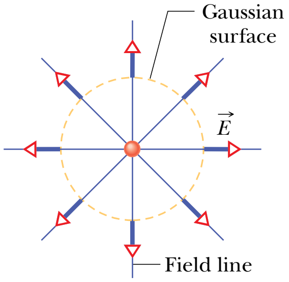


Figure 23-1 Electric field vectors and field lines pierce an imaginary, spherical Gaussian surface that encloses a particle with charge $+Q$.

Figure 23-2 is similar except that the enclosed particle has charge $+2Q$. Because the enclosed charge is now twice as much, the magnitude of the field vectors piercing

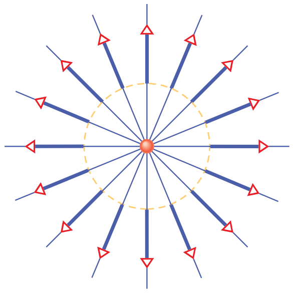


Figure 23-2 Now the enclosed particle has charge $+2Q$.

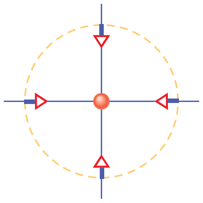


Figure 23-3 Can you tell what the enclosed charge is now?

outward through the (same) Gaussian surface is twice as much as in Fig. 23-1, and the density of the field lines is also twice as much. That sentence, in a nutshell, is Gauss' law.

Guass' law relates the electric field at points on a (closed) Gaussian surface to the net charge enclosed by that surface.

Let's check this with a third example with a particle that is also enclosed by the same spherical Gaussian surface (a *Gaussian sphere*, or *G-sphere*) as shown in Fig. 23-3. What is the amount and sign of the enclosed charge? Well, from the inward piercing we see immediately that the charge must be negative. From the fact that the density of field lines is half that of Fig. 23-1, we also see that the charge must be $0.5Q$. (Using Gauss' law is like being able to tell what is inside a gift box by looking at the wrapping paper on the box.)

The problems in this chapter are of two types. Sometimes we know the charge and we use Gauss' law to find the field at some point. Sometimes we know the field on a Gaussian surface and we use Gauss' law to find the charge enclosed by the surface. However, we cannot do all this by simply comparing the density of field lines in a drawing as we just did. We need a quantitative way of determining how much electric field pierces a surface. That measure is called the electric flux.

23.2 | ELECTRIC FLUX

Key Concepts

- The electric flux Φ through a surface is the amount of electric field that pierces the surface.
- The area vector $d\vec{A}$ for an area element (patch element) on a surface is a vector that is perpendicular to the element and has a magnitude equal to the area dA of the element.
- The electric flux $d\Phi$ through a patch element with area vector $d\vec{A}$ is given by a dot product:

$$d\Phi = \vec{E}\cdot d\vec{A}.$$

- The total flux through a surface is given by

$$\Phi = \int \vec{E}\cdot d\vec{A} \quad \text{(total flux)},$$

where the integration is carried out over the surface.

- The net flux through a closed surface (which is used in Gauss' law) is given by

$$\Phi = \oint \vec{E}\cdot d\vec{A} \quad \text{(total flux)},$$

where the integration is carried out over the entire surface.

The word flux comes from the Latin word meaning "to flow". The idea of flux is first introduced in Fluid Mechanics, where it is defined as the rate of flow (of air or fluid volume) through an area – a *volume flux* in this situation. If we assign a velocity vector to each point in the air stream passing through the loop, then the composite of all vectors is a velocity field and the volume flow rate can be interpreted as the flux of velocity field through the loop. Thus, flux does not necessarily stand for actual flow of something through an area, rather it means the product of an area and the field across the area.

Similar to the flux of velocity, the concept associated with electrostatics is *electric flux*. It is a measure of the number of field lines crossing a given area. (*Note that it is not equal to the number of lines crossing a given area, because the number of lines may vary as these are just required to be proportional to charge producing the electric field.*) The electric field can be estimated with the help of number of field lines per unit area which in turn can be estimated by $E \times$ Area.

Flat Surface, Uniform Field. We begin with a flat surface with area A in a uniform electric field $\vec{E}$. Figure 23-4*a* shows one of the electric field vectors $\vec{E}$ piercing a small square patch with area ΔA (where Δ indicates "small"). Actually, only the x component (with magnitude $E_x = E\cos\theta$ in Fig. 23-4*b*) pierces the patch. The y component merely skims along the surface (no piercing in that) and does not come into play in Gauss' law. The *amount* of electric field piercing the patch is defined to be the **electric flux** $\Delta\Phi$ through it:

$$\Delta\Phi = (E\cos\theta)\Delta A.$$

There is another way to write the right side of this statement so that we have only the piercing component of $\vec{E}$. We define an area vector $\Delta\vec{A}$ that is perpendicular to the patch and that has a magnitude equal to the area ΔA of the patch (Fig. 23-4*c*). Then we can write

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{A},$$

and the dot product automatically gives us the component of $\vec{E}$ that is parallel to $\Delta\vec{A}$ and thus piercing the patch.

To find the total flux Φ through the surface in Fig. 23-4, we sum the flux through every patch on the surface:

$$\Phi = \sum \vec{E}\cdot\Delta\vec{A}. \tag{23-1}$$

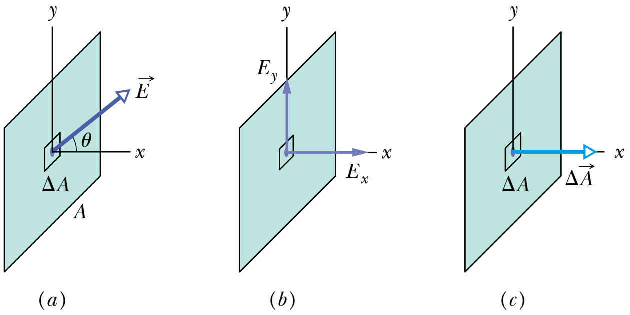


Figure 23-4 (*a*) An electric field vector pierces a small square patch on a flat surface. (*b*) Only the *x* component actually pierces the patch; the *y* component skims across it. (*c*) The area vector of the patch is perpendicular to the patch, with a magnitude equal to the patch's area.

However, because we do not want to sum hundreds (or more) flux values, we transform the summation into an integral by shrinking the patches from small squares with area ΔA to *patch elements* (or *area elements*) with area dA. The total flux is then

$$\Phi = \int \vec{E}\cdot d\vec{A} \quad \text{(total flux)}. \tag{23-2}$$

Now, we can find the total flux by integrating the dot product over the full surface.

Dot Product. We can evaluate the dot product inside the integral by writing the two vectors in unit-vector notation. For example, in Fig. 23-4, $d\vec{A} = dA\hat{\mathrm{i}}$ and $\vec{E}$ might be, say, $(4\hat{\mathrm{i}} + 4\hat{\mathrm{j}})$ N/C. Instead, we can evaluate the dot product in magnitude-angle notation: $E\cos\theta\, dA$. When the electric field is uniform and the surface is flat, the product $E\cos\theta$ is a constant and comes outside the integral. The remaining $\int dA$ is just an instruction to sum the areas of all the patch elements to get the total area, but we already know that the total area is A. So, the total flux in this simple situation is

$$\Phi = (E\cos\theta)A \quad \text{(uniform field, flat surface)}. \tag{23-3}$$

Closed Surface. To use Gauss' law to relate flux and charge, we need a closed surface. Let's use the closed surface in Fig. 23-5 that sits in a nonuniform electric field. (Don't worry. The homework problems involve less complex surfaces.) As before, we first consider the flux through small square patches. However, now we are interested in not only the piercing components of the field but also on whether the piercing is inward or outward (just as we did with Figs. 23-1 through 23-3).

Directions. To keep track of the piercing direction, we again use an area vector $\Delta\vec{A}$ that is perpendicular to a patch, but now we always draw it pointing outward from the surface (*away from the interior*). Then if a field vector pierces outward, it and the area vector are in the same direction, the angle is $\theta = 0$, and $\cos\theta = 1$. Thus, the dot product $\vec{E}\cdot\Delta\vec{A}$ is positive and so is the flux. Conversely, if a field vector pierces inward, the angle is $\theta = 180°$ and $\cos\theta = -1$. Thus, the dot product is negative and so is the flux. If a field vector skims the surface (no piercing), the dot product is zero (because $\cos 90° = 0$) and so is the flux. Figure 23-5 gives some general examples and here is a summary:

> An inward piercing field is negative flux. An outward piercing field is positive flux. A skimming field is zero flux.

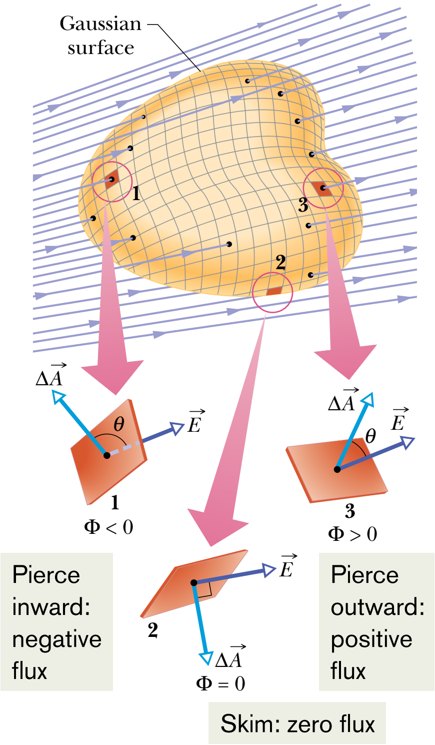


Figure 23-5 A Gaussian surface of arbitrary shape immersed in an electric field. The surface is divided into small squares of area ΔA. The electric field vectors $\vec{E}$ and the area vectors $\Delta\vec{A}$ for three representative squares, marked 1, 2, and 3, are shown.

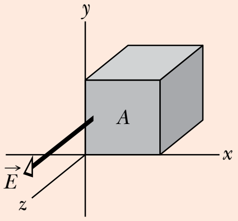
Net Flux. In principle, to find the **net flux** through the surface in Fig. 23-5, we find the flux at every patch and then sum the results (with the algebraic signs included). However, we are not about to do that much work. Instead, we shrink the squares to patch elements with area vectors $d\vec{A}$ and then integrate:

$$\Phi = \oint \vec{E} \cdot d\vec{A} \quad \text{(net flux).} \tag{23-4}$$

The loop on the integral sign indicates that we must integrate over the entire closed surface, to get the *net* flux through the surface (as in Fig. 23-5, flux might enter on one side and leave on another side). Keep in mind that we want to determine the net flux through a surface because that is what Gauss' law relates to the charge enclosed by the surface. (The law is coming up next.) Note that flux is a scalar (yes, we talk about field vectors but flux is the *amount* of piercing field, not a vector itself). The SI unit of flux is the newton–square-meter per coulomb (N·m^2/C).

CHECKPOINT 1

The figure here shows a Gaussian cube of face area A immersed in a uniform electric field $\vec{E}$ that has the positive direction of the z axis. In terms of E and A, what is the flux through (a) the front face (which is in the xy plane), (b) the rear face, (c) the top face, and (d) the whole cube?



SAMPLE PROBLEM 23.01

Flux through a closed cylinder, uniform field

Figure 23-6 shows a Gaussian surface in the form of a closed cylinder (a Gaussian cylinder or G-cylinder) of radius R. It lies in a uniform electric field $\vec{E}$ with the cylinder's central axis (along the length of the cylinder) parallel to the field. What is the net flux Φ of the electric field through the cylinder?

KEY IDEAS

We can find the net flux Φ with Eq. 23-4 by integrating the dot product $\vec{E} \cdot \Delta\vec{A}$ over the cylinder's surface. However, we cannot write out functions so that we can do that with one integral. Instead, we need to be a bit clever: We break up the surface into sections with which we can actually evaluate an integral.

Calculations: We break the integral of Eq. 23-4 into three terms: integrals over the left cylinder cap a, the curved cylindrical surface b, and the right cap c:

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}. \tag{23-5}$$

Pick a patch element on the left cap. Its area vector $d\vec{A}$ must be perpendicular to the patch and pointing away from the interior of the cylinder. In Fig. 23-6, that means the angle between it and the field piercing the patch is 180°. Also, note that the electric field through the end cap is uniform and thus E can be pulled out of the integration. So, we can write the flux through the left cap as

$$\int_a \vec{E} \cdot d\vec{A} = \int E(\cos 180°)\, dA = -E \int dA = -EA,$$

where $\int dA$ gives the cap's area $A(=\pi R^2)$. Similarly, for the right cap, where $\theta = 0$ for all points,

$$\int_c \vec{E} \cdot d\vec{A} = \int E(\cos 90°)\, dA = EA.$$

Finally, for the cylindrical surface, where the angle θ is 90° at all points,

$$\int_b \vec{E} \cdot d\vec{A} = \int E(\cos 90°)\, dA = 0.$$

Substituting these results into Eq. 23-5 leads us to

$$\Phi = -EA + 0 + EA = 0. \qquad \text{(Answer)}$$

The net flux is zero because the field lines that represent the electric field all pass entirely through the Gaussian surface, from the left to the right.

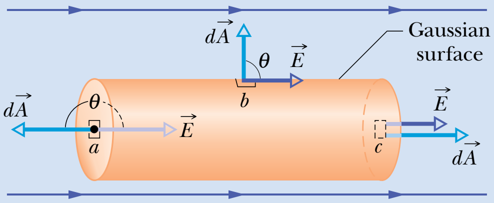


Figure 23-6 A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.

SAMPLE PROBLEM 23.02

Flux through a closed cube, nonuniform field

A *nonuniform* electric field given by $\vec{E} = 3.0x\hat{\text{i}} + 4.0\hat{\text{j}}$ pierces the Gaussian cube shown in Fig. 23-7*a*. (E is in newtons per coulomb and x is in meters.) What is the electric flux through the right face, the left face, and the top face? (We consider the other faces in another sample problem.)

KEY IDEA

We can find the flux Φ through the surface by integrating the scalar product $\vec{E}\cdot d\vec{A}$ over each face.

Right face: An area vector $\vec{A}$ is always perpendicular to its surface and always points away from the interior of a Gaussian surface. Thus, the vector $d\vec{A}$ for any patch element (small section) on the right face of the cube must point in the positive direction of the x axis. An example of such an element is shown in Figs. 23-7*b* and *c*, but we would have an identical vector for any other choice of a patch element on that face. The most convenient way to express the vector is in unit-vector notation,

$$d\vec{A} = dA\hat{\text{i}}.$$

From Eq. 23-4, the flux Φ_r through the right face is then

$$\begin{aligned}\Phi_r &= \int \vec{E}\cdot d\vec{A} = \int (3.0x\hat{\text{i}} + 4.0\hat{\text{j}})\cdot(dA\hat{\text{i}})\\ &= \int [(3.0x)(dA)\hat{\text{i}}\cdot\hat{\text{i}} + (4.0)(dA)\hat{\text{j}}\cdot\hat{\text{i}}]\\ &= \int (3.0x\ dA + 0) = 3.0\int x dA.\end{aligned}$$

We are about to integrate over the right face, but we note that x has the same value everywhere on that face—namely, $x = 3.0$ m. This means we can substitute that constant value for x. This can be a confusing argument. Although x is certainly a variable as we move left to right across the figure, because the right face is perpendicular to the x axis, every point on the face has the same x coordinate. (The y and z coordinates do not matter in our integral.) Thus, we have

$$\Phi_r = 3.0\int (3.0)\, dA = 9.0\int dA.$$

The integral $\int dA$ merely gives us the area $A = 4.0\ \text{m}^2$ of the right face, so

$$\Phi_r = (9.0\ \text{N/C})(4.0\ \text{m}^2) = 36\ \text{N}\cdot\text{m}^2/\text{C}. \qquad \text{(Answer)}$$

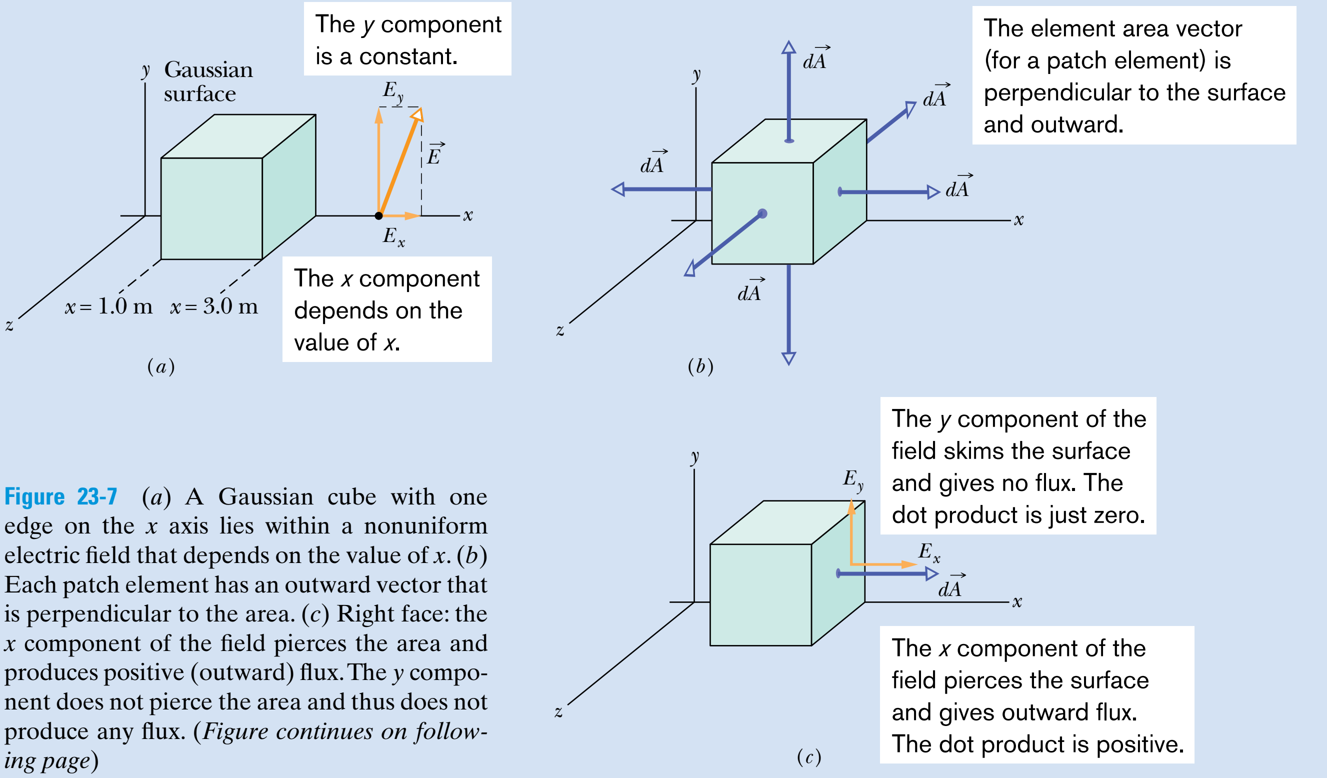
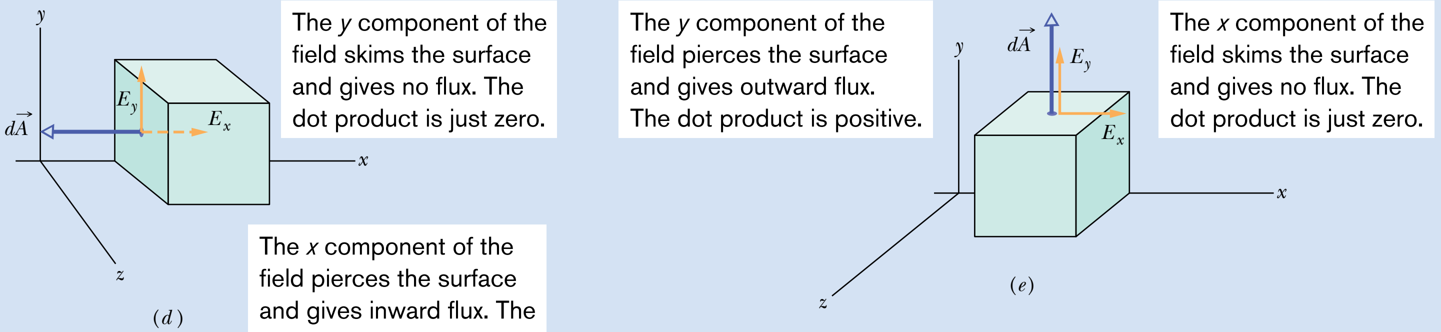

The y component is a constant.
y
Gaussian surface
E_y
$\vec{E}$
x
E_x
z
$x = 1.0$ m
$x = 3.0$ m
The x component depends on the value of x.
(*a*)
$d\vec{A}$
y
$d\vec{A}$
The element area vector (for a patch element) is perpendicular to the surface and outward.
$d\vec{A}$
$d\vec{A}$
x
z
$d\vec{A}$
(*b*)
The y component of the field skims the surface and gives no flux. The dot product is just zero.
y
E_y
E_x
$d\vec{A}$
x
z
The x component of the field pierces the surface and gives outward flux. The dot product is positive.
(*c*)

Figure 23-7 (*a*) A Gaussian cube with one edge on the x axis lies within a nonuniform electric field that depends on the value of x. (*b*) Each patch element has an outward vector that is perpendicular to the area. (*c*) Right face: the x component of the field pierces the area and produces positive (outward) flux. The y component does not pierce the area and thus does not produce any flux. (*Figure continues on following page*)



The y component of the field skims the surface and gives no flux. The dot product is just zero.

The x component of the field pierces the surface and gives inward flux. The dot product is negative.

The y component of the field pierces the surface and gives outward flux. The dot product is positive.

The x component of the field skims the surface and gives no flux. The dot product is just zero.

Figure 23-7 *(Continued from previous page)* (*d*) Left face: the *x* component of the field produces negative (inward) flux. (*e*) Top face: the *y* component of the field produces positive (outward) flux.

Left face: We repeat this procedure for the left face. However, two factors change. (1) The element area vector $d\vec{A}$ points in the negative direction of the x axis, and thus $d\vec{A} = -dA\hat{\mathrm{i}}$ (Fig. 23-7*d*). (2) On the left face, $x = 1.0$ m. With these changes, we find that the flux Φ_l through the left face is

$$\Phi_l = 12\ \mathrm{N{\cdot}m^2/C}. \qquad \text{(Answer)}$$

Top face: Now, $d\vec{A}$ points in the positive direction of the y axis, and thus $d\vec{A} = dA\,\hat{\mathrm{j}}$ (Fig. 23-7*e*). The flux Φ_t is

$$\begin{aligned}\Phi_t &= \int (3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}})\cdot(dA\,\hat{\mathrm{j}})\\ &= \int [(3.0x)(dA)\hat{\mathrm{i}}\cdot\hat{\mathrm{j}} + (4.0)(dA)\hat{\mathrm{j}}\cdot\hat{\mathrm{j}}]\\ &= \int (0 + 4.0\,dA) = 4.0\int dA\\ &= 16\ \mathrm{N\cdot m^2/C}. \qquad \text{(Answer)}\end{aligned}$$

SAMPLE PROBLEM 23.03

Flux through a flat disk

Calculate the flux through the flat disk held perpendicular to the end of a semi-infinite wire of charge per unit length *l*. The radius of the disk is *R* (Fig. 23-8).

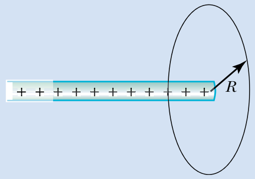


Figure 23-8 Electric flux through a disk held perpendicular to a semi-infinite rod terminating at its center.

KEY IDEA

Here we have a real situation where the electric field is not constant in space but variable. In fact, at different points of the disk, the electric field will not be constant in direction or magnitude. So, obviously we need to resort to integration techniques.

We know that there are two components of the electric field due to a semi-infinite rod. One is in the direction perpendicular to the rod and another is in the direction parallel to the rod. Both these components are drawn at a point on the disk as shown in Fig. 23-9.

Calculation: As can be easily seen, $E_\perp$ is parallel to the surface of the disk at all the points. So, the angle between $d\vec{A}$ and $E_\perp$ is 90° at all the points. So, the flux due to $E_\perp$ at all the points of the disk is

$$d\phi = \vec{E}_\perp \cdot d\vec{A} = E_\perp dA\cos 90^\circ = 0.$$

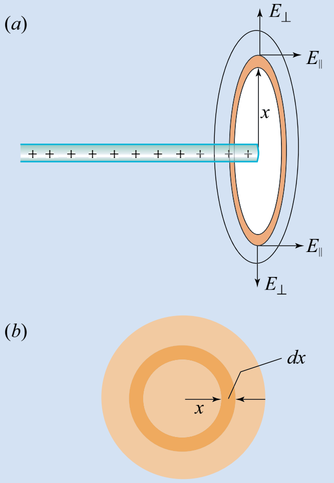


Figure 23-9 (*a*) Pattern of electric field at various points of the disk. (*b*) A small element of the disk.

However, $E_{\parallel}$ is perpendicular to the disk at all the points. So, the angle between $E_{\parallel}$ and $d\vec{A}$ at any point is 0°. At a distance x from the end,

$$E_{\parallel} = \frac{k\lambda}{x}$$

So, considering a small ring of radius x on this disk whose width is dx, the flux through this small ring is

$$d\phi = \vec{E}_{\parallel}\cdot d\vec{A} = \frac{k\lambda}{x}\times 2\pi x dx \times \cos 0^\circ = 2\pi k\lambda dx.$$

Integrating, we obtain,

$$\phi = 2\pi k\lambda R.$$

Learn: Since electric field obeys the law of superposition, electric flux obeys the law of superposition. Thus, the net flux through any area is the sum of the flux due to individual charges. To find the net flux, we should add the flux due to each charge without changing the sign of the flux.

Suppose the disk was having uniform surface charge density σ. The force experienced by every part of the disk would be $dF = E\sigma dA$. But only the component of force perpendicular to the disk would contribute to the net force, other component would cancel. So,

$$F = \int \sigma E dA \cos\theta = \sigma\phi = 2\pi k\sigma\lambda R$$

23.3 | GAUSS' LAW

Key Concepts

- Gauss' law relates the net flux Φ penetrating a closed surface to the net charge q_{enc} enclosed by the surface:

$$\varepsilon_0\Phi = q_{\text{enc}} \quad \text{(Gauss' law).}$$

- Gauss' law can also be written in terms of the electric field piercing the enclosing Gaussian surface:

$$\varepsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{\text{enc}} \quad \text{(Gauss' law).}$$

Gauss' law relates the net flux Φ of an electric field through a closed surface (a Gaussian surface) to the *net* charge q_{enc} that is *enclosed* by that surface. It tells us that

$$\varepsilon_0\Phi = q_{\text{enc}} \quad \text{(Gauss' law).} \qquad (23\text{-}6)$$

By substituting Eq. 23-4, the definition of flux, we can also write Gauss' law as

$$\varepsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{\text{enc}} \quad \text{(Gauss' law).} \qquad (23\text{-}7)$$

Equations 23-6 and 23-7 hold only when the net charge is located in a vacuum or (what is the same for most practical purposes) in air. In Chapter 25, we modify Gauss' law to include situations in which a material such as mica, oil, or glass is present.

In Eqs. 23-6 and 23-7, the net charge q_{enc} is the algebraic sum of all the *enclosed* positive and negative charges, and it can be positive, negative, or zero. We include the sign, rather than just use the magnitude of the enclosed charge, because the sign tells us something about the net flux through the Gaussian surface: If q_{enc} is positive, the net flux is *outward*; if q_{enc} is negative, the net flux is *inward.*

Charge outside the surface, no matter how large or how close it may be, is not included in the term q_{enc} in Gauss' law. The exact form and location of the charges inside the Gaussian surface are also of no concern; the only things that matter on the right side of Eqs. 23-6 and 23-7 are the magnitude and sign of the net enclosed charge. The quantity $\vec{E}$ on the left side of Eq. 23-7, however, is the electric field resulting from *all* charges, both those inside and those outside the Gaussian surface. This statement may seem to be inconsistent, but keep this in mind: The electric field due to a charge outside the Gaussian surface contributes zero net flux *through* the surface, because as many field lines due to that charge enter the surface as leave it.

Let us apply these ideas to Fig. 23-10, which shows two particles, with charges equal in magnitude but opposite in sign, and the field lines describing the electric fields the particles set up in the surrounding space. Four Gaussian surfaces are also shown, in cross section. Let us consider each in turn.

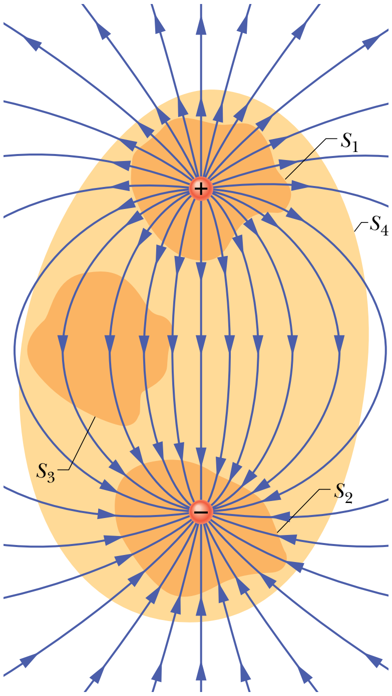


Figure 23-10 Two charges, equal in magnitude but opposite in sign, and the field lines that represent their net electric field. Four Gaussian surfaces are shown in cross section. Surface S_1 encloses the positive charge. Surface S_2 encloses the negative charge. Surface S_3 encloses no charge. Surface S_4 encloses both charges and thus no net charge.

Surface S_1. The electric field is outward for all points on this surface. Thus, the flux of the electric field through this surface is positive, and so is the net charge within the surface, as Gauss' law requires. (That is, in Eq. 23-6, if Φ is positive, q_{enc} must be also.)

Surface S_2. The electric field is inward for all points on this surface. Thus, the flux of the electric field through this surface is negative and so is the enclosed charge, as Gauss' law requires.

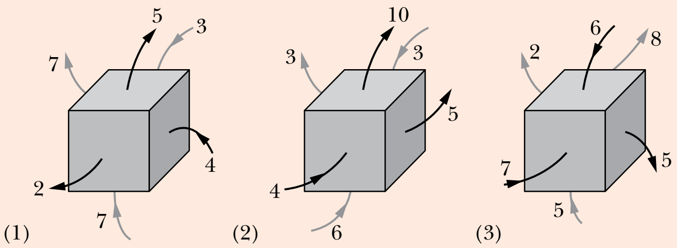
Surface S_3. This surface encloses no charge, and thus $q_{\text{enc}} = 0$. Gauss' law (Eq. 23-6) requires that the net flux of the electric field through this surface be zero. That is reasonable because all the field lines pass entirely through the surface, entering it at the top and leaving at the bottom.

Surface S_4. This surface encloses no *net* charge, because the enclosed positive and negative charges have equal magnitudes. Gauss' law requires that the net flux of the electric field through this surface be zero. That is reasonable because there are as many field lines leaving surface S_4 as entering it.

What would happen if we were to bring an enormous charge Q up close to surface S_4 in Fig. 23-10? The pattern of the field lines would certainly change, but the net flux for each of the four Gaussian surfaces would not change. Thus, the value of Q would not enter Gauss' law in any way, because Q lies outside all four of the Gaussian surfaces that we are considering.

CHECKPOINT 2

The figure shows three situations in which a Gaussian cube sits in an electric field. The arrows and the values indicate the directions of the field lines and the magnitudes (in N·m²/C) of the flux through the six sides of each cube. (The lighter arrows are for the hidden faces.) In which situation does the cube enclose (a) a positive net charge, (b) a negative net charge, and (c) zero net charge?



Formal Derivation of Gauss' Law

Consider a point charge q surrounded by a closed surface of arbitrary shape (Fig. 23-11). The total electric flux through this surface can be obtained by evaluating $\vec{E}\cdot\vec{A}$ for each small area element ΔA and summing over all elements. The flux through each element is

$$\Delta\Phi_E = \vec{E}\cdot\vec{A} = (E\cos\theta)\Delta A = k_e q\frac{\Delta A\cos\theta}{r^2}.$$

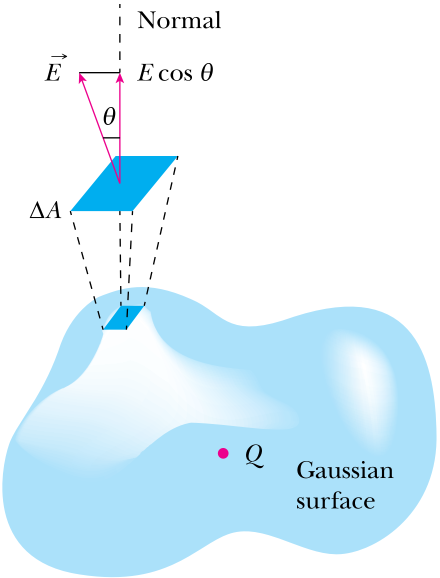


Figure 23-11 An arbitrarily shaped Gaussian surface.

In Fig. 23-11, we can see that the projection of the area element perpendicular to the radius vector is $\Delta A\cos\theta$. Thus, the quantity $(\Delta A\cos\theta)/r^2$ is equal to the solid angle $\Delta\Omega$ that the surface element ΔA subtends at the charge q. We also see that $\Delta\Omega$ is equal to the solid angle subtended by the area element of a spherical surface of radius r (Fig. 23-12). Because the total solid angle at a point is 4π steradians, the total flux through the closed surface is

$$\Delta\Phi_E = kq\oint\frac{dA\cos\theta}{r^2} = kq\oint d\Omega = 4\pi kq = \frac{q}{\varepsilon_0}$$

If we consider a large number of charges within the Gaussian surface, each of them will produce a flux that will give the same expression. The solid angle subtended by a charge outside the Gaussian surface will be zero. Since flux obeys the law of superposition, we can say that the net flux through the Gaussian surface is

$$\Phi_E = \frac{\Sigma Q_{\text{enc}}}{\varepsilon_0},$$

where ΣQ_{enc} refers to the total charge enclosed by the Gaussian surface. Hence, the Gauss's law is derived. Note that this result is independent of the shape of the closed surface and independent of the position of the charge within the surface.

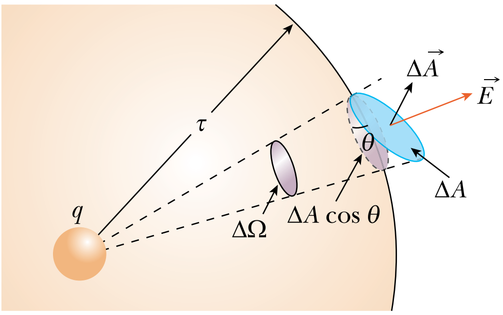


Figure 23-12 The area element ΔA subtends a solid angle $\Delta\Omega = (\Delta A\cos\theta)/r^2$ at charge q, where r is the distance from the charge to the area element, θ is the angle between the electric field E and ΔA for the element, and $E = kq/r^2$ for a point charge.

SAMPLE PROBLEM 23.04

Finding net flux using Gauss' law

Figure 23-13 shows five charged lumps of plastic and an electrically neutral coin. The cross section of a Gaussian surface S is indicated. What is the net electric flux through the surface if $q_1 = q_4 = +3.1$ nC, $q_2 = q_5 = -5.9$ nC and $q_3 = -3.1$ nC?

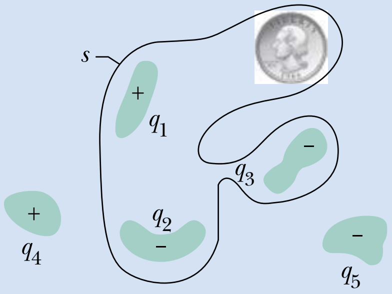


Figure 23-13 Five plastic objects, each with an electric charge, and a coin, which has no net charge. A Gaussian surface, shown in cross section, encloses three of the plastic objects and the coin.

KEY IDEA

The *net* flux Φ through the surface depends on the *net* charge q_{enc} enclosed by surface S.

Calculation: The coin does not contribute to Φ because it is neutral and thus contains equal amounts of positive and negative charge. Charges q_4 and q_5 do not contribute because they are outside surface S. Thus, q_{enc} is $q_1 + q_2 + q_3$ and Eq. 23-6 gives us

$$\Phi = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{q_1 + q_2 + q_3}{\varepsilon_0}$$

$$= \frac{+3.1\times10^{-9}\ \text{C} - 5.9\times10^{-9}\ \text{C} - 3.1\times10^{-9}\ \text{C}}{8.85\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2}$$

$$= -670\ \text{N}\cdot\text{m}^2/\text{C} \qquad \text{(Answer)}$$

The minus sign shows that the net flux through the surface is inward and thus that the net charge within the surface is negative.

Gauss' Law and Coulomb's Law

One of the situations in which we can apply Gauss' law is in finding the electric field of a charged particle. That field has spherical symmetry (the field depends on the distance r from the particle but not the direction). So, to make use of that symmetry, we enclose the particle in a Gaussian sphere that is centered on the particle, as shown in Fig. 23-14 for a particle with positive charge q. Then the electric field has the same magnitude E at any point on the sphere (all points are at the same distance r). That feature will simplify the integration.

The drill here is the same as previously. Pick a patch element on the surface and draw its area vector $d\vec{A}$ perpendicular to the patch and directed outward. From the symmetry of the situation, we know that the electric field $\vec{E}$ at the patch is also radially outward and thus at angle $\theta = 0$ with $d\vec{A}$. So, we rewrite Gauss' law as

$$\varepsilon_0 \oint \vec{E}\cdot d\vec{A} = \varepsilon_0 \oint E\,dA = q_{\text{enc}}. \qquad (23\text{-}8)$$

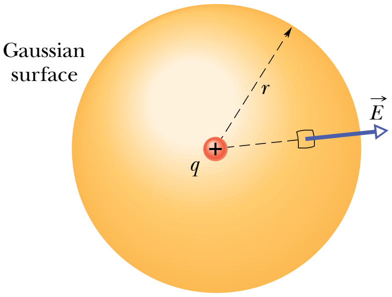


Figure 23-14 A spherical Gaussian surface centered on a particle with charge q.

Here $q_{\text{enc}} = q$. Because the field magnitude E is the same at every patch element, E can be pulled outside the integral:

$$\varepsilon_0 E \oint dA = q. \qquad (23\text{-}9)$$

The remaining integral is just an instruction to sum all the areas of the patch elements on the sphere, but we already know that the total area is $4\pi r^2$. Substituting this, we have

$$\varepsilon_0 E(4\pi r^2) = q \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \qquad (23\text{-}10)$$

This is exactly Eq. 22-4, which we found using Coulomb's law.

☑ CHECKPOINT 3

There is a certain net flux Φ_i through a Gaussian sphere of radius r enclosing an isolated charged particle. Suppose the enclosing Gaussian surface is changed to (a) a larger Gaussian sphere, (b) a Gaussian cube with edge length equal to r, and (c) a Gaussian cube with edge length equal to $2r$. In each case, is the net flux through the new Gaussian surface greater than, less than, or equal to Φ_i?

PROBLEM-SOLVING TACTICS

Tactic 1: Choosing a Gaussian Surface Because the derivation of Eq. 23-10 using Gauss' law is a warm-up for derivations of electric fields produced by other charge configurations, let us go back over the steps involved. We started with a given positive point charge q; we know that electric field lines extend radially outward from q in a spherically symmetric pattern.

To use Gauss' law (Eq. 23-7) to find the magnitude E of the electric field at a distance r, we had to place a hypothetical closed Gaussian surface around q, through a point that is a distance r from q. Then we had to sum via integration the values of $\vec{E}\cdot d\vec{A}$ over the full Gaussian surface. To make this integration as simple as possible, we chose a spherical Gaussian surface (to mimic the spherical symmetry of the electric field). That choice produced three simplifying features. (1) The dot product $\vec{E}\cdot d\vec{A}$ became simple, because at all points on the Gaussian surface the angle between $\vec{E}$ and $d\vec{A}$ is zero, and so at all points we have $\vec{E}\cdot d\vec{A} = E\,dA$. (2) The electric field magnitude E is the same at all points on the spherical Gaussian surface; so, E was a constant in the integration and could be brought out in front of the integral sign. (3) The result was a very simple integration—a summation of the differential areas of the sphere, which we could immediately write as $4\pi r^2$.

Note that Gauss' law holds regardless of the shape of the Gaussian surface we choose to place around charge q_{enc}. However, if we had chosen, say, a cubical Gaussian surface, our three simplifying features would have disappeared and the integration of $\vec{E}\cdot d\vec{A}$ over the cubical surface would have been very difficult. The moral here is to choose the Gaussian surface that most simplifies the integration in Gauss' law.

SAMPLE PROBLEM 23.05

Using Gauss' law to find flux

Find the flux through the disk shown in Fig. 23-15.

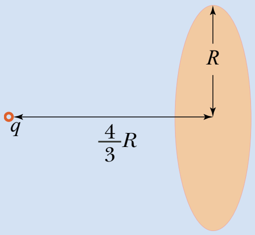


Figure 23-15 Finding flux due to a point charge through a disk. The line joining the charge to the center of the disk is perpendicular to the disk.

KEY IDEA

The electric flux through the disk cannot be found by the equation

$$\phi = \vec{E}\cdot\vec{A} = EA\cos\theta$$

because the electric field due to the point charge at different points of the disk is not the same. Hence, we have to resort to integration methods enumerated earlier. If we wish to use the basic formula, we can divide the disk into small rings as shown in Fig. 23-16 and find the electric field due to charge at all the rings:

$$\phi = \int \vec{E}\cdot d\vec{s}.$$

Calculation: The electric field due to the point charge at the location of the ring shown is given by

$$E = \frac{kq}{(16/9)R^2 + x^2}$$

As we discussed before, the area of the ring is $2\pi x dx$. But the electric field is not normal to the ring. The angle can be found as shown:

$$\cos\theta = \frac{4R/3}{\sqrt{(16R^2/9)+x^2}}$$

$$\tan\alpha = \frac{3}{4}$$

$$\phi = \int \vec{E}\cdot d\vec{s}$$

$$= \int_0^{0.75R} \frac{kq\times(4R/3)\times 2\pi x dx}{[x^2+(16R^2/9)]\times\sqrt{(16R^2/9)+x^2}} = \frac{q}{10\varepsilon_0}$$

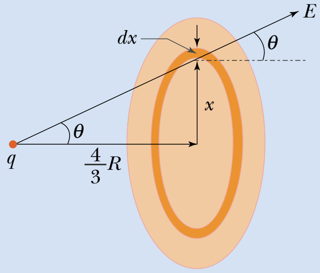


Figure 23-16 Here we divide the entire disk into thin ring and find the flux due to the charge through the thin ring.

Alternative method

KEY IDEA

Let us visualize the electric field lines coming from the point charge. These electric field lines will emanate just as light rays emanate from a point source of light. A part of these electric field lines will pass through the disk (Fig. 23-17). The amount of flux can be calculated by using the concept of solid angle. The total flux due to the point charge through any sphere is given by

$$\phi_T = \frac{q}{\varepsilon_0}.$$

The portion of flux passing through the disk will be a small fraction of it. In fact, since flux through any part of the spherical surface is proportional to the area of the spherical surface, it will also be proportional to the solid angle subtended by this part on the sphere. Since the total solid angle subtended by the sphere on the charge is 4π, the

$$\frac{\phi_{\text{disk}}}{\phi_T} = \frac{\Omega}{4\pi}.$$

Calculation: We can imagine a Gaussian surface centered at the charge and intersecting the disk as shown. This disk makes a cone at the center whose solid angle can be calculated as:

$$\tan\alpha = \frac{3}{4}, \cos\alpha = \frac{4}{5}$$

$$\begin{aligned}\phi_{\text{disk}} &= \frac{q}{\varepsilon_0} \times \frac{\Omega}{4\pi} \\ &= \frac{q}{\varepsilon_0} \times \frac{2\pi}{4\pi}(1-\cos\theta) \\ &= \frac{q}{\varepsilon_0} \times \frac{1}{10}\end{aligned}$$

Learn: Gauss' law is definitely superior in finding the value of flux or the electric field.

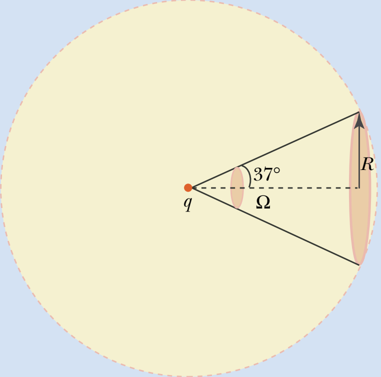


Figure 23-17 The flux through the disk can be calculated by using an imaginary Gaussian surface centered at q. The flux through the disk and the curved portion of sphere will be in proportion to the solid angle Ω subtended by it at q.

SAMPLE PROBLEM 23.06

Using Gauss' law to find the electric field

Figure 23-18 shows, in cross section, a plastic, spherical shell with uniform charge $Q = -16e$ and radius $R = 10$ cm. A particle with charge $q = +5e$ is at the center. What is the electric field (magnitude and direction) at (a) point P_1 at radial distance $r_1 = 6.00$ cm and (b) point P_2 at radial distance $r_2 = 12.0$ cm?

KEY IDEAS

(1) Because the situation in Fig. 23-18*a* has spherical symmetry, we can apply Gauss' law (Eq. 23-7) to find the electric field at a point if we use a Gaussian surface in the form of a sphere concentric with the particle and shell. (2) To find the electric field at a point, we put that point on a Gaussian surface (so that the $\vec{E}$ we want is the $\vec{E}$ in the dot product inside the integral in Gauss' law). (3) Gauss' law relates the net electric flux through a closed surface to the net enclosed charge. Any external charge is not included.

Calculations: To find the field at point P_1, we construct a Gaussian sphere with P_1 on its surface and thus with a radius of r_1. Because the charge enclosed by the Gaussian sphere is positive, the electric flux through the surface must be positive and thus outward. So, the electric field $\vec{E}$ pierces the surface outward and, because of the spherical symmetry, must be *radially* outward, as drawn in Fig. 23-18*b*. That figure does not include the plastic shell because the shell is not enclosed by the Gaussian sphere.

Consider a patch element on the sphere at P_1. Its area vector $d\vec{A}$ is radially outward (it must always be outward from a Gaussian surface). Thus the angle θ between $\vec{E}$ and is zero. We can now rewrite the left side of Eq. 23-7 (Gauss' law) as

$$\varepsilon_0 \oint \vec{E}\cdot d\vec{A} = \varepsilon_0 \oint E\cos 0\, dA = \varepsilon_0 \oint E\, dA = \varepsilon_0 E \oint dA,$$

where in the last step we pull the field magnitude E out of the integral because it is the same at all points on the Gaussian sphere and thus is a constant. The remaining integral is simply an instruction for us to sum the areas of all the patch elements on the sphere, but we already

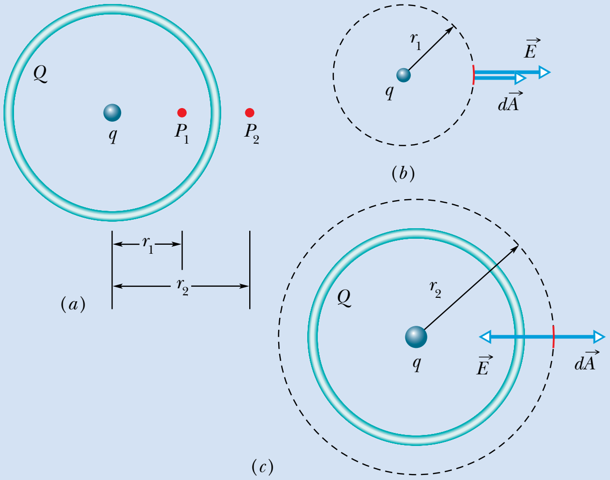


Figure 23-18 (*a*) A charged plastic spherical shell encloses a charged particle. (*b*) To find the electric field at P_1, arrange for the point to be on a Gaussian sphere. The electric field pierces outward. The area vector for the patch element is outward. (*c*) P_2 is on a Gaussian sphere, $\vec{E}$ is inward, and $d\vec{A}$ is still outward.

know that the surface area of a sphere is $4\pi r^2$. Substituting these results, Eq. 23-7 for Gauss' law gives us

$$\varepsilon_0 E 4\pi r^2 = q_{\text{enc}}.$$

The only charge enclosed by the Gaussian surface through P_1 is that of the particle. Solving for E and substituting $q_{\text{enc}} = 5e$ and $r = r_1 = 6.00 \times 10^{-2}$ m, we find that the magnitude of the electric field at P_1 is

$$\begin{aligned} E &= \frac{q_{\text{enc}}}{4\pi\varepsilon_0 r^2} \\ &= \frac{5(1.60 \times 10^{-19}\ \text{C})}{4\pi(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(0.0600\ \text{m})^2} \\ &= 2.00 \times 10^{-6}\ \text{N/C}. \qquad \text{(Answer)} \end{aligned}$$

To find the electric field at P_2, we follow the same procedure by constructing a Gaussian sphere with P_2 on its surface. This time, however, the net charge enclosed by the sphere is $q_{\text{enc}} = q = Q = 5e + (-16e) = -11e$. Because the net charge is negative, the electric field vectors on the sphere's surface pierce inward (Fig. 23-18*c*), the angle θ between $\vec{E}$ and $d\vec{A}$ is 180°, and the dot product is E (cos 180°) $dA = -E\ dA$. Now solving Gauss' law for E and substituting $r = r_2 = 12.00 \times 10^{-2}$ m and the new q_{enc}, we find

$$\begin{aligned} E &= \frac{-q_{\text{enc}}}{4\pi\varepsilon_0 r^2} \\ &= \frac{-[-11(1.60 \times 10^{-19}\ \text{C})]}{4\pi(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(0.120\ \text{m}^2)} \\ &= 1.10 \times 10^{-6}\ \text{N/C}. \qquad \text{(Answer)} \end{aligned}$$

Note how different the calculations would have been if we had put P_1 or P_2 on the surface of a Gaussian cube instead of mimicking the spherical symmetry with a Gaussian sphere. Then angle θ and magnitude E would have varied considerably over the surface of the cube and evaluation of the integral in Gauss' law would have been difficult.

SAMPLE PROBLEM 23.07

Using Gauss' law to find the enclosed charge

What is the net charge enclosed by the Gaussian cube of Sample Problem 23.06?

KEY IDEA

The net charge enclosed by a (real or mathematical) closed surface is related to the total electric flux through the surface by Gauss' law as given by Eq. 23-6 ($\varepsilon_0 \Phi = q_{\text{enc}}$).

Flux: To use Eq. 23-6, we need to know the flux through all six faces of the cube. We already know the flux through the right face ($\Phi_r = 36$ N·m²/C), the left face ($\Phi_l = -12$ N·m²/C), and the top face ($\Phi_t = 16$ N·m²/C).

For the bottom face, our calculation is just like that for the top face *except* that the element area vector $d\vec{A}$ is now directed downward along the y axis (recall, it must be *outward* from the Gaussian enclosure). Thus, we have $d\vec{A} = -dA\hat{j}$, and we find

$$\Phi_b = -16\ \text{N}\cdot\text{m}^2/\text{C}.$$

For the front face we have $d\vec{A} = dA\hat{k}$, and for the back face, $d\vec{A} = -dA\hat{k}$. When we take the dot product of the given electric field $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ with either of these expressions for $d\vec{A}$, we get 0 and thus there is no flux through those faces. We can now find the total flux through the six sides of the cube:

$$\begin{aligned} \Phi &= (36 - 12 + 16 - 16 + 0 + 0)\ \text{N}\cdot\text{m}^2/\text{C} \\ &= 24\ \text{N}\cdot\text{m}^2/\text{C}. \end{aligned}$$

Enclosed charge: Next, we use Gauss' law to find the charge q_{enc} enclosed by the cube:

$$\begin{aligned} q_{\text{enc}} &= \varepsilon_0 \Phi = (8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(24\ \text{N}\cdot\text{m}^2/\text{C}) \\ &= 2.1 \times 10^{-10}\ \text{C}. \qquad \text{(Answer)} \end{aligned}$$

Thus, the cube encloses a *net* positive charge.

23.4 | A CHARGED ISOLATED CONDUCTOR

Key Concepts

- An excess charge on an isolated conductor is located entirely on the outer surface of the conductor.
- The internal electric field of a charged, isolated conductor is zero, and the external field (at nearby points) is perpendicular to the surface and has a magnitude that depends on the surface charge density σ:

$$E = \frac{\sigma}{\varepsilon_0}.$$

Gauss' law permits us to prove an important theorem about conductors:

> If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor.

This might seem reasonable, considering that charges with the same sign repel one another. You might imagine that, by moving to the surface, the added charges are getting as far away from one another as they can. We turn to Gauss' law for verification of this speculation.

Figure 23-19*a* shows, in cross section, an isolated lump of copper hanging from an insulating thread and having an excess charge q. We place a Gaussian surface just inside the actual surface of the conductor.

The electric field inside this conductor must be zero. If this were not so, the field would exert forces on the conduction (free) electrons, which are always present in a conductor, and thus current would always exist within a conductor. (That is, charge would flow from place to place within the conductor.) Of course, there is no such perpetual current in an isolated conductor, and so the internal electric field is zero.

(An internal electric field *does* appear as a conductor is being charged. However, the added charge quickly distributes itself in such a way that the net internal electric field—the vector sum of the electric fields due to all the charges, both inside and outside—is zero. The movement of charge then ceases, because the net force on each charge is zero; the charges are then in *electrostatic equilibrium.*)

If $\vec{E}$ is zero everywhere inside our copper conductor, it must be zero for all points on the Gaussian surface because that surface, though close to the surface of the conductor, is definitely inside the conductor. This means that the flux through the Gaussian surface must be zero. Gauss' law then tells us that the net charge inside the Gaussian surface must also be zero. Then because the excess charge is not inside the Gaussian surface, it must be outside that surface, which means it must lie on the actual surface of the conductor.

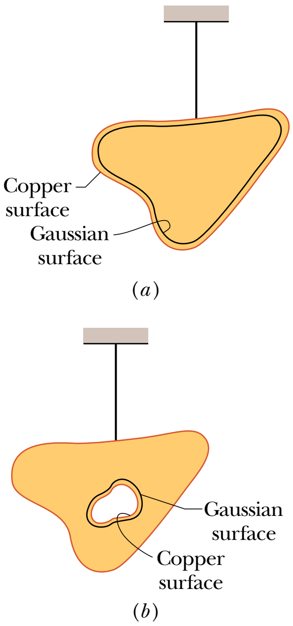


Figure 23-19 (*a*) A lump of copper with a charge q hangs from an insulating thread. A Gaussian surface is placed within the metal, just inside the actual surface. (*b*) The lump of copper now has a cavity within it. A Gaussian surface lies within the metal, close to the cavity surface.

An Isolated Conductor with a Cavity

Figure 23-19*b* shows the same hanging conductor, but now with a cavity that is totally within the conductor. It is perhaps reasonable to suppose that when we scoop out the electrically neutral material to form the cavity, we do not change the distribution of charge or the pattern of the electric field that exists in Fig. 23-19*a*. Again, we must turn to Gauss' law for a quantitative proof.

We draw a Gaussian surface surrounding the cavity, close to its surface but inside the conducting body. Because $\vec{E}=0$ inside the conductor, there can be no flux through this new Gaussian surface. Hence, we have

$$\phi_{\text{Gaussian surface}} = 0 = \frac{q_{\text{induced}}}{\varepsilon_0} \Rightarrow q_{\text{induced}} = 0$$

Therefore, from Gauss' law, that surface can enclose no net charge. We conclude that there is no net charge on the cavity walls; all the excess charge remains on the outer surface of the conductor, as in Fig. 23-19*a*.

Many arguments can be made against this charge configuration. One of the arguments is that the electric field in the lattice cannot be zero with this charge configuration–a necessary condition for electrostatic equilibrium. But a deeper argument lies in the conservative nature of the electric field. Imagine a loop in the region: partly inside the cavity and partly inside the metal. The net value of $\oint \vec{E}\cdot d\vec{l} = 0$ by this nature. But this is not followed by this electric field pattern. To appreciate this, follow the electric field line along the path shown in Fig. 23-20. You can see that in the cavity, $\oint \vec{E}\cdot d\vec{l}$ is positive, but inside the metal $\oint \vec{E}\cdot d\vec{l}$ is zero. However, if there is a charge suspended inside the cavity as in Fig. 23-21*a*, then what should happen?

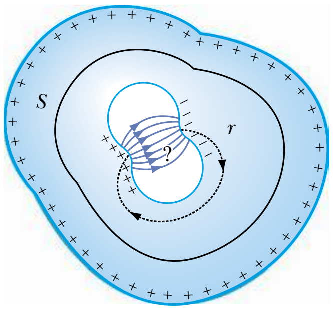


Figure 23-20 A metal conductor having a cavity cannot have a charge distribution on the inner surface of the cavity, as shown.

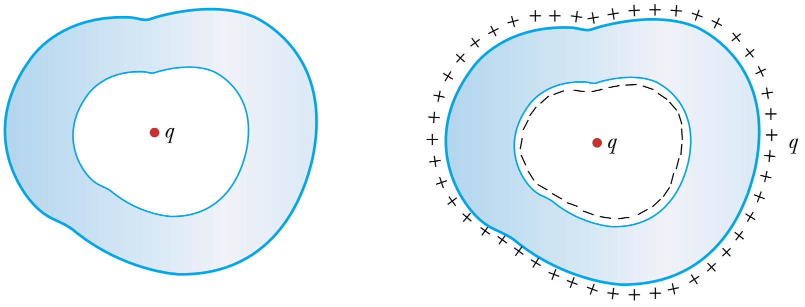


Figure 23-21 (*a*) A charge is suspended inside a cavity in a conductor. (*b*) The pattern of charge distribution on the cavity and outside surface.

We can logically say that some charge will be induced on the inner side of the cavity (Fig. 23-21*b*). It seems that this principle is violated. It is not!

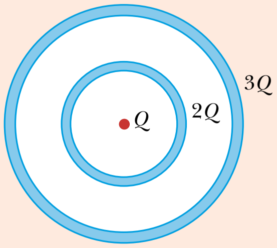
Let us look more closely. When a charge is suspended inside a cavity, charge is induced so that the electric field inside the conductor should become zero. From Gauss' law, the flux within the conductor should be zero, as the field is zero:

$$\phi_T = \frac{q_{\text{induced}} + Q}{\varepsilon_0} = 0.$$

So, an equal and opposite charge is induced on the walls of the cavity. From the previous analysis, we can say that this induced charge will be borrowed from the outer surface of the conductor. So, charge distribution will look like as shown in Fig. 23-21*b*. There will be an electric field inside the cavity in this case. This does not violate our original statement. We only said that the electric field is zero within the core of the *conductor*, not within the cavity inside the conductor.

CHECKPOINT 4

As shown in the figure, a charge of Q is kept in the inner cavity and a charge of $2Q$ is imparted to the inner shell. A charge of $3Q$ is imparted to the outermost shell. (a) Find charge at each of the surfaces. (b) Find $\vec{E}$ in the first annular region and outside.



The Conductor Removed

Suppose that, by some magic, the excess charges could be "frozen" into position on the conductor's surface, perhaps by embedding them in a thin plastic coating, and suppose that then the conductor could be removed completely. This is equivalent to enlarging the cavity of Fig. 23-19*b* until it consumes the entire conductor, leaving only the charges. The electric field would not change at all; it would remain zero inside the thin shell of charge and would remain unchanged for all external points. This shows us that the electric field is set up by the charges and not by the conductor. The conductor simply provides an initial pathway for the charges to take up their positions.

The External Electric Field

You have seen that the excess charge on an isolated conductor moves entirely to the conductor's surface. However, unless the conductor is spherical, the charge does not distribute itself uniformly. Put another way, the surface charge density σ (charge per unit area) varies over the surface of any nonspherical conductor. Generally, this variation makes the determination of the electric field set up by the surface charges very difficult.

However, the electric field just outside the surface of a conductor is easy to determine using Gauss' law. To do this, we consider a section of the surface that is small enough to permit us to neglect any curvature and thus to take the section to be flat. We then imagine a tiny cylindrical Gaussian surface to be partially embedded in the section as shown in Fig. 23-22: One end cap is fully inside the conductor, the other is fully outside, and the cylinder is perpendicular to the conductor's surface.

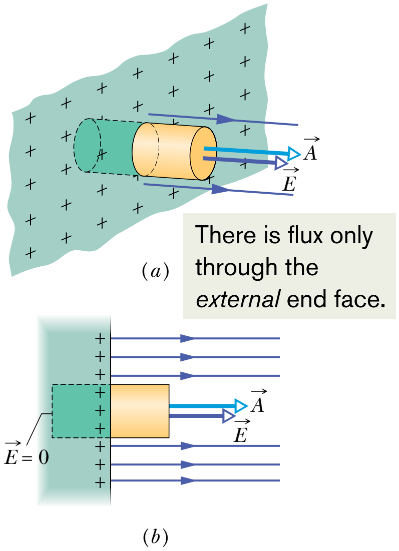


Figure 23-22 (*a*) Perspective view and (*b*) side view of a tiny portion of a large, isolated conductor with excess positive charge on its surface. A (closed) cylindrical Gaussian surface, embedded perpendicularly in the conductor, encloses some of the charge. Electric field lines pierce the external end cap of the cylinder, but not the internal end cap. The external end cap has area A and area vector $\vec{A}$.

The electric field $\vec{E}$ at and just outside the conductor's surface must also be perpendicular to that surface. If it were not, then it would have a component along the conductor's surface that would exert forces on the surface charges, causing them to move. However, such motion would violate our implicit assumption that we are dealing with electrostatic equilibrium. Therefore, $\vec{E}$ is perpendicular to the conductor's surface.

We now sum the flux through the Gaussian surface. There is no flux through the internal end cap, because the electric field within the conductor is zero. There is no flux through the curved surface of the cylinder, because internally (in the conductor) there is no electric field and externally the electric field is parallel to the curved portion of the Gaussian surface. The only flux through the Gaussian surface is that through the external end cap, where $\vec{E}$ is perpendicular to the plane of the cap. We assume that the cap area A is small enough that the field magnitude E is constant over the cap. Then the flux through the cap is EA, and that is the net flux Φ through the Gaussian surface.

The charge q_{enc} enclosed by the Gaussian surface lies on the conductor's surface in an area A. (Think of the cylinder as a cookie cutter.) If σ is the charge per unit area, then q_{enc} is equal to σA. When we substitute σA for q_{enc} and EA for Φ, Gauss' law (Eq. 23-6) becomes

$$\varepsilon_0 EA = \sigma A,$$

from which we find

$$E = \frac{\sigma}{\varepsilon_0} \qquad \text{(conducting surface).} \tag{23-11}$$

Thus, the magnitude of the electric field just outside a conductor is proportional to the surface charge density on the conductor. The sign of the charge gives us the direction of the field. If the charge on the conductor is positive, the electric field is directed away from the conductor as in Fig. 23-22. It is directed toward the conductor if the charge is negative.

The field lines in Fig. 23-22 must terminate on negative charges somewhere in the environment. If we bring those charges near the conductor, the charge density at any given location on the conductor's surface changes, and so does the magnitude of the electric field. However, the relation between σ and E is still given by Eq. 23-11.

Do not be under the mistaken conception that this is the electric field due to the conductor. This is the *net electric field* in the vicinity of the conductor. The interesting fact of nature is that the charge gets redistributed on the surface of the conductor so that the electric field comes out to be the found value. It is quite possible that the charge per unit area is not constant. In that case the electric field will not be having the same magnitude at all the points.

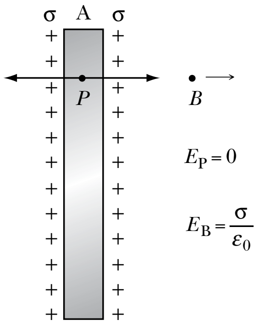


Figure 23-23 The electric field near the surface of an infinite conducting sheet.

To understand this more deeply, let us take the case of a large conducting sheet on which we have kept a charge Q (see Fig. 23-23). By symmetry, this charge will spread on the surface such that the charge per unit area will be the same on both the surfaces say σ. These two surfaces will form two sheets of charge. They will superpose to produce zero electric field inside the conductor and σ/ε_0 outside the conductor. We can see that the charge on each surface will be $Q/2$. So, σ for each surface is $Q/2A$. There will be two infinite sheets of charge. Each of them will produce an electric field of $\sigma/2\varepsilon_0$. Inside the conductor, these two fields will oppose each other, and outside the conductor, these two fields will add up. So, the net electric field outside the conductor comes out to be σ/ε_0.

Let us consider another case where there are two large sheets of conductors rather than one as in Fig. 23-24*a*. One of the sheets is given a charge of Q_1 and another a charge of Q_2. We know that the charges will be redistributed on both the surfaces. Let us assume that the charge on facing surfaces be q_1 and q_2, respectively. Then the charge on the outer surface of the conductors will be $Q_1 - q_1$ and $Q_2 - q_2$, respectively, as can be seen in Fig. 23-24*b*.

To find the values of q_1 and q_2, we have two conditions: (a) The electric field in the first conductor is zero and (b) the electric field in the second conductor is zero. Since the sheets are large, we can assume the charge distribution is uniform. We can also use the fact that the electric field due to a sheet of charge (Note: not a conducting sheet) is $\sigma/2\varepsilon_0$.

There are two conducting sheets having four surfaces as in Fig. 23-24*b*. So, there are four sheets of charges. (Note, that there are four sheets of charges, not the conductors.)

Inside the first conductor, the electric field is given by

$$E_{\text{I}} = \frac{Q_1 - q_1}{2A\varepsilon_0} - \frac{q_1}{2A\varepsilon_0} - \frac{q_2}{2A\varepsilon_0} - \frac{Q_2 - q_2}{2A\varepsilon_0}.$$

Inside the second conductor, the electric field is given by

$$E_{\text{II}} = \frac{Q_1 - q_1}{2A\varepsilon_0} + \frac{q_1}{2A\varepsilon_0} + \frac{q_2}{2A\varepsilon_0} - \frac{Q_2 - q_2}{2A\varepsilon_0}.$$

Both these electric fields are zero. So, we obtain

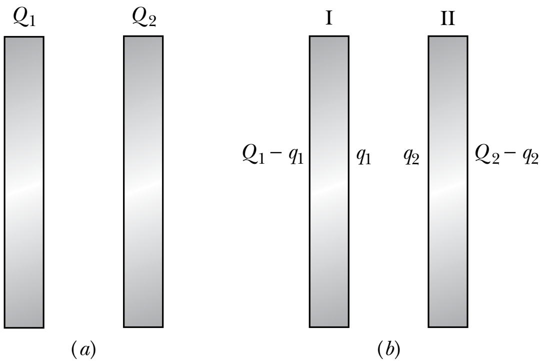


Figure 23-24 (*a*) Two conducting sheets having a charge Q_1 and Q_2. (*b*) The distribution of charges on them.

$$q_1 = -q_2,$$

$$q_1 = \frac{Q_1 - Q_2}{2}.$$

$$Q_{\text{out}} = Q_1 - q_1 = Q_1 - \left(\frac{Q_1 - Q_2}{2}\right) = \left(\frac{Q_1 + Q_2}{2}\right)$$

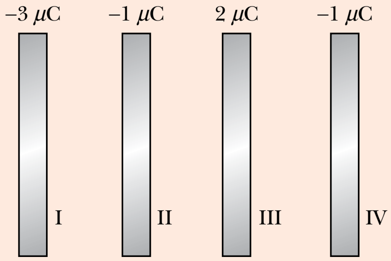
$$Q_{\text{out}} = Q_2 - \left(\frac{Q_1 - Q_2}{2}\right) = \left(\frac{Q_1 + Q_2}{2}\right)$$

This gives an interesting result that the charge on the outside surface is the same and is equal to total charge divided by 2. We can also say that the charge on the facing plates will always be equal and opposite.

If there are many sheets placed parallel to each other, the charge on outermost surfaces will be (total charge)/2 (i.e., $\sum_{i=1}^{n} Q_i/2$, *here* Q_i *is the charge on i*th *sheet. The charge on the inner faces can be found by the law of conservation of charge. The charge on the facing surface will be equal and opposite.*

CHECKPOINT 5

Four large charged conducting sheets kept parallel to each other. Find the charge on each of the eight faces and electric field in region I, II, III, IV as shown in the figures below.



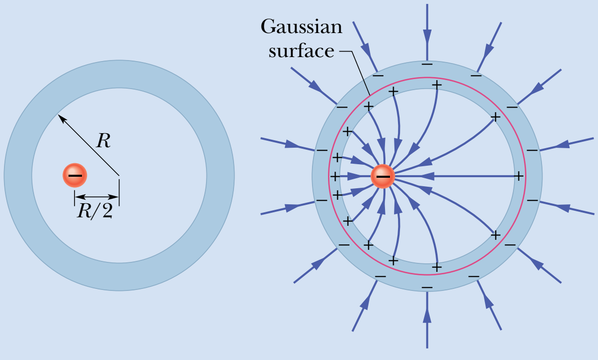
SAMPLE PROBLEM 23.08

Spherical metal shell, electric field and enclosed charge

Figure 23-25*a* shows a cross section of a spherical metal shell of inner radius *R*. A particle with a charge of −5.0 μC is located at a distance *R*/2 from the center of the shell. If the shell is electrically neutral, what are the (induced) charges on its inner and outer surfaces? Are those charges uniformly distributed? What is the field pattern inside and outside the shell?

KEY IDEAS

Figure 23-18*b* shows a cross section of a spherical Gaussian surface within the metal, just outside the inner wall of the shell. The electric field must be zero inside the metal (and thus on the Gaussian surface inside the metal). This means that the electric flux through the Gaussian surface must also be zero. Gauss' law then tells us that the *net* charge enclosed by the Gaussian surface must be zero.



(*a*) (*b*)

Figure 23-25 (*a*) A negatively charged particle is located within a spherical metal shell that is electrically neutral. (*b*) As a result, positive charge is nonuniformly distributed on the inner wall of the shell, and an equal amount of negative charge is uniformly distributed on the outer wall.

Reasoning: With a particle of charge $-5.0\ \mu C$ within the shell, a charge of $+5.0\ \mu C$ must lie on the inner wall of the shell in order that the net enclosed charge be zero. If the particle were centered, this positive charge would be uniformly distributed along the inner wall. However, since the particle is off-center, the distribution of positive charge is skewed, as suggested by Fig. 23-25*b*, because the positive charge tends to collect on the section of the inner wall nearest the (negative) particle.

Because the shell is electrically neutral, its inner wall can have a charge of $+5.0\ \mu C$ only if electrons, with a total charge of $-5.0\ \mu C$, leave the inner wall and move to the outer wall. There they spread out uniformly, as is also suggested by Fig. 23-18*b*. This distribution of negative charge is uniform because the shell is spherical and because the skewed distribution of positive charge on the inner wall cannot produce an electric field in the shell to affect the distribution of charge on the outer wall. Furthermore, these negative charges repel one another.

The field lines inside and outside the shell are shown approximately in Fig. 23-25*b*. All the field lines intersect the shell and the particle perpendicularly. Inside the shell the pattern of field lines is skewed because of the skew of the positive charge distribution. Outside the shell the pattern is the same as if the particle were centered and the shell were missing. In fact, this would be true no matter where inside the shell the particle happened to be located.

23.5 | APPLYING GAUSS' LAW: SPHERICAL SYMMETRY

Key Concepts

- Outside a spherical shell of uniform charge q, the electric field due to the shell is radial (inward or outward, depending on the sign of the charge) and has the magnitude

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad \text{(outside spherical shell)},$$

  where r is the distance to the point of measurement from the center of the shell. The field is the same as though all of the charge is concentrated as a particle at the center of the shell.

- Inside the shell, the field due to the shell is zero.
- Inside a sphere with a uniform volume charge density, the field is radial and has the magnitude

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}r \quad \text{(inside sphere of charge)},$$

  where q is the total charge, R is the sphere's radius, and r is the radial distance from the center of the sphere to the point of measurement.

Here we use Gauss' law to prove the two shell theorems presented without proof in Section 22.1:

> A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at the center of the shell.

Figure 23-26 shows a charged spherical shell of total charge q and radius R and two concentric spherical Gaussian surfaces, S_1 and S_2. If we followed the procedure of Section 23.3 as we applied Gauss' law to surface S_2, for which $r \geq R$, we would find that

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad \text{(spherical shell, field at } r \geq R\text{).} \tag{23-12}$$

This field is the same as one set up by a particle with charge q at the center of the shell of charge. Thus, the force produced by a shell of charge q on a charged particle placed outside the shell is the same as if all the shell's charge is concentrated as a particle at the shell's center. This proves the first shell theorem.

Applying Gauss' law to surface S_1, for which $r < R$, leads directly to

$$E = 0 \quad \text{(spherical shell, field at } r < R\text{),} \tag{23-13}$$

because this Gaussian surface encloses no charge. Thus, if a charged particle were enclosed by the shell, the shell would exert no net electrostatic force on the particle. This proves the second shell theorem.

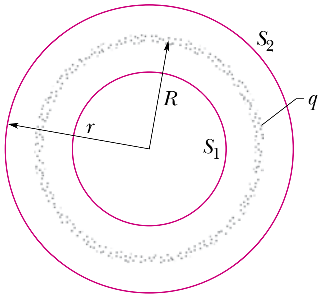


Figure 23-26 A thin, uniformly charged, spherical shell with total charge q, in cross section. Two Gaussian surfaces S_1 and S_2 are also shown in cross section. Surface S_2 encloses the shell, and S_1 encloses only the empty interior of the shell.

> If a charged particle is located inside a shell of uniform charge, there is no electrostatic force on the particle from the shell.

Any spherically symmetric charge distribution, such as that of Fig. 23-27, can be constructed with a nest of concentric spherical shells. For purposes of applying the two shell theorems, the volume charge density ρ should have a single value for each shell but need not be the same from shell to shell. Thus, for the charge distribution as a whole, ρ can vary, but only with r, the radial distance from the center. We can then examine the effect of the charge distribution "shell by shell."

In Fig. 23-27*a*, the entire charge lies within a Gaussian surface with $r > R$. The charge produces an electric field on the Gaussian surface as if the charge were that of a particle located at the center, and Eq. 23-12 holds.

Figure 23-27*b* shows a Gaussian surface with $r < R$. To find the electric field at points on this Gaussian surface, we separately consider the charge inside it and the charge outside it. From Eq. 23-13, the outside charge does not set up a field on the Gaussian surface. From Eq. 23-12, the inside charge sets up a field as though it is concentrated at the center. Letting q' represent that enclosed charge, we can then rewrite Eq. 23-12 as

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2} \quad \text{(spherical distribution, field at } r \leq R\text{).} \tag{23-14}$$

If the full charge q enclosed within radius R is uniform, then q' enclosed within radius r in Fig. 23-27*b* is proportional to q:

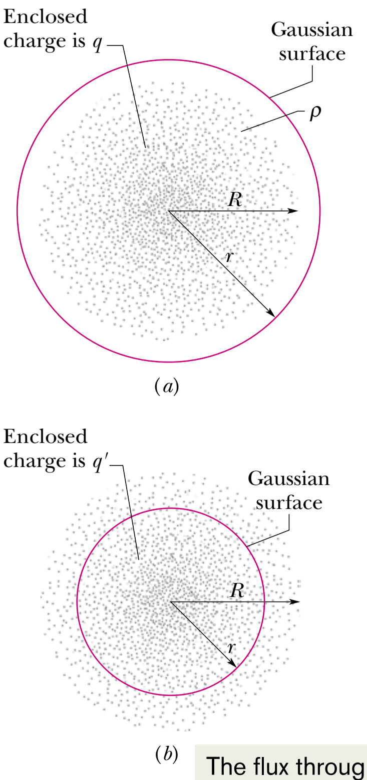
$$\frac{\begin{pmatrix}\text{Charge enclosed by}\\ \text{sphere of radius } r\end{pmatrix}}{\begin{pmatrix}\text{Volume enclosed by}\\ \text{sphere of radius } r\end{pmatrix}} = \frac{\text{Full charge}}{\text{Full volume}} \quad \text{or} \quad \frac{q'}{\frac{4}{3}\pi r^3} = \frac{q}{\frac{4}{3}\pi R^3}. \tag{23-15}$$

This gives us

$$q' = q\frac{r^3}{R^3}. \tag{23-16}$$

Substituting this into Eq. 23-14 yields

$$E = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)r \quad \text{(uniform charge, field at } r \leq R\text{).} \tag{23-17}$$



The flux through the surface depends on only the *enclosed* charge.

Figure 23-27 The dots represent a spherically symmetric distribution of charge of radius R, whose volume charge density ρ is a function only of distance from the center. The charged object is not a conductor, and therefore the charge is assumed to be fixed in position. A concentric spherical Gaussian surface with $r > R$ is shown in (a). A similar Gaussian surface with $r < R$ is shown in (b).

We can also draw the graph of electric field due to a uniformly charged sphere and a uniformly charged shell with the distance from the center (Figs. 23-28 and 23-29).

The idea behind the first shell theorem can be explained without Gauss' law as well. Consider the homogeneous hollow shell with a particle of charge q is placed inside the hollow sphere (not necessarily in the center) as shown in Fig. 23-30*a*. To determine the force exerted by the hollow shell on the particle, consider first the force exerted by a very small patch q_1. The distance from q_1 to q is d_1 and so the magnitude of the electrostatic force exerted by this patch is kq_1q/d_1^2. Extending the cone defined by q and q_1 toward the opposite side of the hollow sphere, we find another very small patch q_2 at a distance d_2 from q. The force exerted by this patch of q is kq_2q/d_2^2. If q is closer to q_1 than to q_2 then patch q_2 has a larger area and so it contains more charge. A larger charge means a larger force, but we also have to account for the larger distance from q_2 to q which reduces the force. If the charge is uniformly distributed over the sphere, then the charge of each patch is proportional to its area. Because the area of the two patches are marked out by similar cones, their areas are proportional to the squares of their distances to q and so

$$\frac{q_1}{q_2} = \frac{d_1^2}{d_2^2}.$$

Rearranging terms we then find $q_2/d_1^2 = q_2/d_2^2$ regardless of the position of q, and so we see that the forces cancel – the two patches exert equal but opposite forces on q (see Fig. 23-30*b*). We can now apply the same arguments to other pairs of small patches on either side of q, each yielding equal and opposite forces on q (Fig. 23-30*c*). The vector sum of all of the forces exerted by all the small patches making up the hollow sphere on q is thus zero.

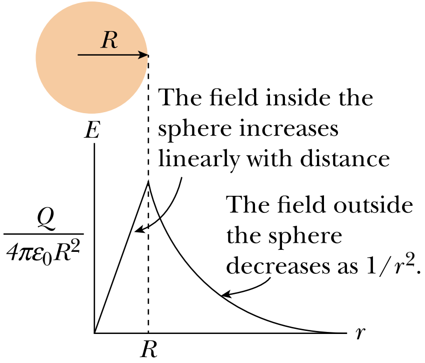


Figure 23-28 The graph of electric field due to a uniformly charged sphere.

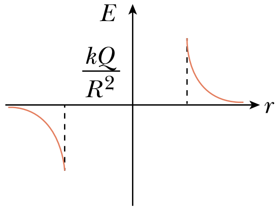
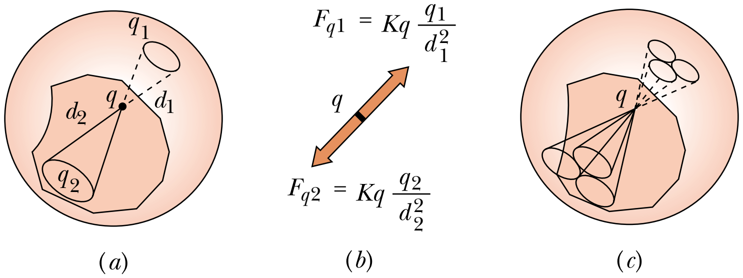


Figure 23-29 Electric field due to a uniformly charged shell.



q_1
q
d_1
d_2
q_2
$F_{q1} = Kq\frac{q_1}{d_1^2}$
q
$F_{q2} = Kq\frac{q_2}{d_2^2}$
q

(*a*) (*b*) (*c*)

Figure 23-30 (*a*) A uniformly charged shell divided into small patches of charges q_1 and q_2 kept inside the shell. (*b*) The small patches exerting force can be approximated by point charges and calculated using Coulomb's law. The forces on q due to these patches are equal and opposite. (*c*) Locating pair of patches that will cancel the force and electric field at point charge q.

An interesting consequence of the shell theorem is that the interaction between two homogeneous spherical bodies having charge distributed uniformly over their surface depends only on the distance between their centers. Assume that we place a point charge q at a distance r from the center of a spherical charge distribution Q as in Fig. 23-31. The field that the point charge experiences is $E = KQ/r^2$, and the force is $F = qE = KQq/r^2$. By Newton's third law, q must exert an equal and opposite force on Q. This force can be understood as being due to the field created by point charge q in the region occupied by spherical charge q, the field around Q. Now, if we replace point charge q by a homogeneous spherical body of same charge q, the field around Q does not change, because of the result we have just proved, and therefore the force on Q remains the same. Again, we use Newton's third law, and conclude that the force on the spherical charge q is still the same.

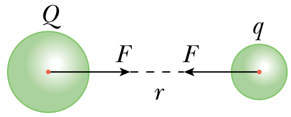


Figure 23-31 The force of interaction between two uniformly charged spherical bodies.

Consequently, two homogeneous spherical charge bodies interact with each other according to Coulomb's law, where r is the distance between their centers.

CHECKPOINT 6

What if the smaller sphere of uniformly distributed charge q was placed inside the larger shell of charge Q. What will be the force on each of them? Consider two cases: (a) The two charge distributions are concentric and (b) the centers of the charges do not coincide.

SAMPLE PROBLEM 23.09

Applying Gauss' law to spherical symmetry

In Fig. 23-32 a nonconducting spherical shell of inner radius $a = 2.00$ cm and outer radius $b = 2.40$ cm has (within its thickness) a positive volume charge density $\rho = A/r$, where A is a constant and r is the distance from the center of the shell. In addition, a small ball of charge $q = 45.0$ fC is located at that center. What value should A have if the electric field in the shell ($a < b < c$) is to be uniform?

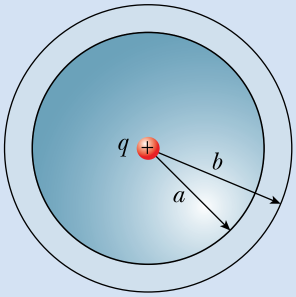


Figure 23-32 A nonconducting spherical shell.

KEY IDEA

Since our system possesses spherical symmetry, to calculate the electric field strength, we may apply Gauss' law and take the Gaussian surface to be in the form of a sphere of radius r.

Finding expression: To find an expression for the electric field inside the shell in terms of A and the distance from the center of the shell, choose A so the field does not depend on the distance. We use a Gaussian surface in the form of a sphere with radius r_g, concentric with the spherical shell and within it ($a < r_g < b$). Gauss' law will be used to find the magnitude of the electric field a distance r_g from the shell center. The charge that is both in the shell and within the Gaussian sphere is given by the integral $q_s = \int \rho\, dV$ over the portion of the shell within the Gaussian surface. Since the charge distribution has spherical symmetry, we may take dV to be the volume of a spherical shell with radius r and infinitesimal thickness dr: $dV = 4\pi r^2 dr$. Thus,

$$q_s = 4\pi \int_a^{r_g} \rho r^2 dr = 4\pi \int_a^{r_g} \frac{A}{r} r^2 dr$$

$$= 4\pi A \int_a^{r_g} r\, dr = 2\pi A (r_g^2 - a^2).$$

The total charge inside the Gaussian surface is

$$q_{\text{enc}} = q + q_s = q + 2\pi A (r_g^2 - a^2).$$

The electric field is radial, so the flux through the Gaussian surface is $\Phi = 4\pi r_g^2 E$, where E is the magnitude of the field. Gauss' law yields

$$\Phi = q_{\text{enc}}/\varepsilon_0 \quad \Rightarrow \quad 4\pi\varepsilon_0 E r_g^2 = q + 2\pi A (r_g^2 - a^2).$$

Calculation: We solve for E:

$$E = \frac{1}{4p\varepsilon_0}\left[\frac{q}{r_g^2} + 2\pi A - \frac{2\pi A a^2}{r_g^2}\right].$$

For the field to be uniform, the first and last terms in the brackets must cancel. They do if $q - 2\pi A a^2 = 0$ or $A = q/2\pi a^2$. With $a = 2.00 \times 10^{-2}$ m and $q = 45.0 \times 10^{-15}$ C, we have $A = 1.79 \times 10^{-11}$ C/m^2.

Learn: The value we have found for A ensures the uniformity of the field strength inside the shell. Using the result found above, we can readily show that the electric field in the region $a \le r \le b$ is

$$E = \frac{2\pi A}{4\pi\varepsilon_0} = \frac{A}{2\varepsilon_0}$$

$$= \frac{1.79 \times 10^{-11} \text{ C/m}^2}{2(8.85 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2)} = 1.01 \text{ N/C}. \quad \text{(Answer)}$$

SAMPLE PROBLEM 23.10

Motion of electron inside uniformly charged sphere

An electron is released from the surface of a uniformly and positively charged sphere (volume charge density r, Fig. 23-33). Show that the electron will execute SHM and find its time period.

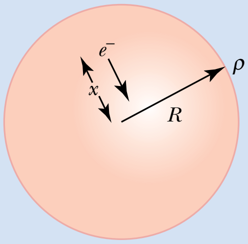


Figure 23-33 An electron inside a uniformly charged sphere.

KEY IDEA

Since the electron will be negatively charged, it will be pulled by the positively charged sphere. The condition for the simple harmonic motion (SHM) is that the force should be proportional to the displacement x from the mean position.

Calculation: At a distance x from the center, the force on the electron is

$$F = -eE = -\frac{\rho x}{3\varepsilon_0}.$$

Since this force is proportional to distance x from the mean position (center) and directed toward it, the motion should be simple harmonic. Solving this, we get

$$a = \frac{-e\rho}{3m\varepsilon_0}x.$$

But in an SHM,

$$a = -\omega^2 x;\ \omega = \sqrt{\frac{e\rho}{3m\varepsilon_0}};\ T = 2\pi\sqrt{\frac{3m\varepsilon_0}{e\rho}}$$

Learn: Here the electron was moving along the diameter. Suppose that this electron was allowed to move only along a chord, starting from rest. In that case also the motion of the electron will be simple harmonic. A component of the force will be balanced by the walls of the tunnel in which electron moves. The net force will come out to be proportional to the distance from the center of the chord which will be the mean position in this case. Even the time period will come out to be the same (Fig. 23-34).

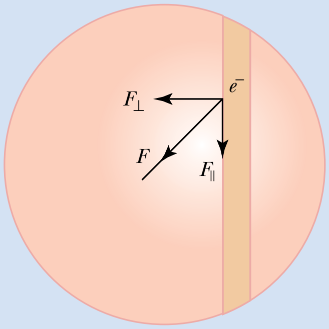


Figure 23-34 An electron inside a smooth tunnel in a uniformly charged sphere.

23.6 | APPLYING GAUSS' LAW: CYLINDRICAL SYMMETRY

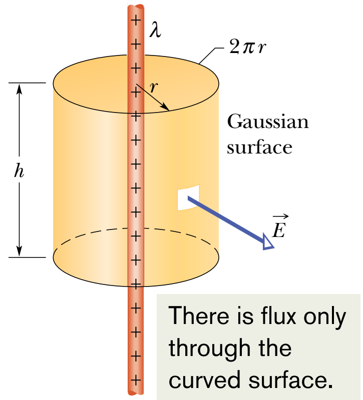
Key Concepts

- The electric field at a point near an infinite line of charge (or charged rod) with uniform linear charge density λ is perpendicular to the line and has magnitude

  $$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{(line of charge)},$$

  where r is the perpendicular distance from the line to the point.

Figure 23-35 shows a section of an infinitely long cylindrical plastic rod with a uniform charge density λ. We want to find an expression for the electric field magnitude E at radius r from the central axis of the rod, outside the rod.



There is flux only through the curved surface.

Figure 23-35 A Gaussian surface in the form of a closed cylinder surrounds a section of a very long, uniformly charged, cylindrical plastic rod.

We could do that using the approach of Chapter 22 (charge element dq, field vector $d\vec{E}$, etc.). However, Gauss' law gives a much faster and easier (and prettier) approach.

The charge distribution and the field have cylindrical symmetry. To find the field at radius r, we enclose a section of the rod with a concentric Gaussian cylinder of radius r and height h. (If you want the field at a certain point, put a Gaussian surface through that point.) We can now apply Gauss' law to relate the charge enclosed by the cylinder and the net flux through the cylinder's surface.

First note that because of the symmetry, the electric field at any point must be radially outward (the charge is positive). That means that at any point on the end caps, the field only skims the surface and does not pierce it. So, the flux through each end cap is zero.

To find the flux through the cylinder's curved surface, first note that for any patch element on the surface, the area vector $d\vec{A}$ is radially outward (away from the interior of the Gaussian surface) and thus in the same direction as the field piercing the patch. The dot product in Gauss' law is then simply $E\,dA\cos 0 = E dA$, and we can pull E out of the integral. The remaining integral is just the instruction to sum the areas of all patch elements on the cylinder's curved surface, but we already know that the total area is the product of the cylinder's height h and circumference $2\pi r$. The net flux through the cylinder is then

$$\Phi = EA\cos\theta = E(2\pi rh)\cos 0 = E(2\pi rh).$$

On the other side of Gauss's law we have the charge q_{enc} enclosed by the cylinder. Because the linear charge density (charge per unit length, remember) is uniform, the enclosed charge is λh. Thus, Gauss' law,

$$\varepsilon_0\Phi = q_{\text{enc}},$$

reduces to

$$\varepsilon_0 E(2\,\pi rh) = \lambda h,$$

yielding

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{(line of charge).} \tag{23-18}$$

This is the electric field due to an infinitely long, straight line of charge, at a point that is a radial distance r from the line. The direction of $\vec{E}$ is radially outward from the line of charge if the charge is positive, and radially inward if it is negative. Equation 23-18 also approximates the field of a *finite* line of charge at points that are not too near the ends (compared with the distance from the line).

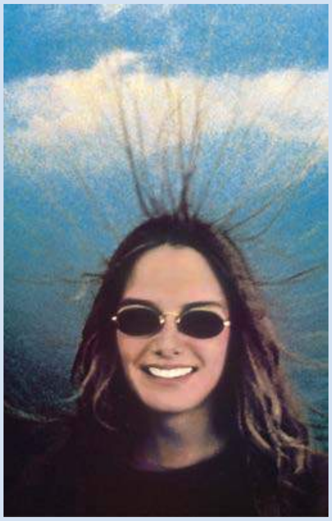
If the rod has a uniform volume charge density ρ, we could use a similar procedure to find the electric field magnitude *inside* the rod. We would just shrink the Gaussian cylinder shown in Fig. 23-35 until it is inside the rod. The charge q_{enc} enclosed by the cylinder would then be proportional to the volume of the rod enclosed by the cylinder because the charge density is uniform.

SAMPLE PROBLEM 23.11

Gauss' law and an upward streamer in a lightning storm

Upward streamer in a lightning storm. The woman in Fig. 23-36 was standing on a lookout platform high in the Sequoia National Park when a large storm cloud moved overhead. Some of the conduction electrons in her body were driven into the ground by the cloud's negatively charged base (Fig. 23-37*a*), leaving her positively charged. You can tell she was highly charged because her hair strands repelled one another and extended away from her along the electric field lines produced by the charge on her.

Lightning did not strike the woman, but she was in extreme danger because that electric field was on the verge of causing electrical breakdown in the surrounding air. Such a breakdown would have occurred



Courtesy NOAA

Figure 23-36 This woman has become positively charged by an overhead storm cloud.

along a path extending away from her in what is called an *upward streamer.* An upward streamer is dangerous because the resulting ionization of molecules in the air suddenly frees a tremendous number of electrons from those molecules. Had the woman in Fig. 23-36 developed an upward streamer, the free electrons in the air would have moved to neutralize her (Fig. 23-37*b*), producing a large, perhaps fatal, charge flow through her body. That charge flow is dangerous because it could have interfered with or even stopped her breathing (which is obviously necessary for oxygen) and the steady beat of her heart (which is obviously necessary for the blood flow that carries the oxygen). The charge flow could also have caused burns.

Let's model her body as a narrow vertical cylinder of height $L = 1.8$ m and radius $R = 0.10$ m (Fig. 23-37*c*). Assume that charge Q was uniformly distributed along the cylinder and that electrical breakdown would have occurred if the electric field magnitude along her body had exceeded the critical value $E_c = 2.4$ MN/C. What value of Q would have put the air along her body on the verge of breakdown?

KEY IDEA

Because $R \ll L$, we can approximate the charge distribution as a long line of charge. Further, because we assume that the charge is uniformly distributed along this line, we can approximate the magnitude of the electric field along the side of her body with Eq. 23-18 ($E = \lambda/2\pi\varepsilon_0 r$).

Calculations: Substituting the critical value E_c for E, the cylinder radius R for radial distance r, and the ratio Q/L for linear charge density λ, we have

$$E_c = \frac{Q/L}{2\pi\varepsilon_0 R}, \quad \text{or} \quad Q = 2\pi\varepsilon_0 RLE_c.$$

Substituting given data then gives us

$$\begin{aligned} Q &= (2\pi)(8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(0.10\ \mathrm{m}) \\ &\quad \times (1.8\ \mathrm{m})(2.4 \times 10^{6}\ \mathrm{N/C}) \\ &= 2.402 \times 10^{-5}\ \mathrm{C} \approx 24\ \mu\mathrm{C}. \qquad \text{(Answer)} \end{aligned}$$

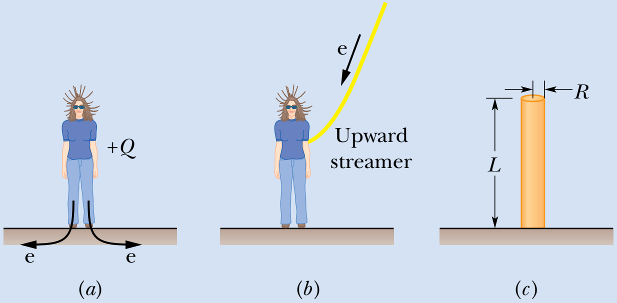


Figure 23-37 (*a*) Some of the conduction electrons in the woman's body are driven into the ground, leaving her positively charged. (*b*) An upward streamer develops if the air undergoes electrical breakdown, which provides a path for electrons freed from molecules in the air to move to the woman. (*c*) A cylinder represents the woman.

23.7 | APPLYING GAUSS' LAW: PLANAR SYMMETRY

Key Concepts

- The electric field due to an infinite nonconducting sheet with uniform surface charge density σ is perpendicular to the plane of the sheet and has magnitude

$$E = \frac{\sigma}{2\varepsilon_0} \quad \text{(nonconducting sheet of charge).}$$

- The external electric field just outside the surface of an isolated charged conductor with surface charge density σ is perpendicular to the surface and has magnitude

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{(external, charged conductor).}$$

Inside the conductor, the electric field is zero.

Nonconducting Sheet

Figure 23-38 shows a portion of a thin, infinite, nonconducting sheet with a uniform (positive) surface charge density σ. A sheet of thin plastic wrap, uniformly charged on one side, can serve as a simple model. Let us find the electric field $\vec{E}$ a distance r in front of the sheet.

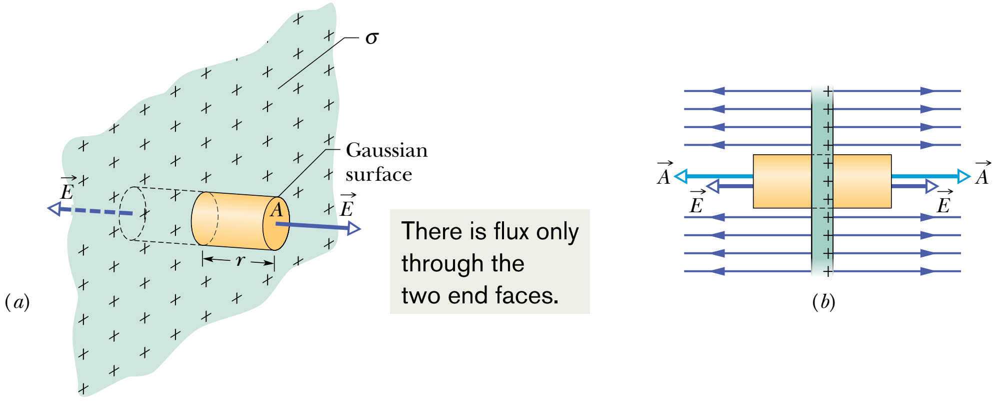


Figure 23-38 (*a*) Perspective view and (*b*) side view of a portion of a very large, thin plastic sheet, uniformly charged on one side to surface charge density σ. A closed cylindrical Gaussian surface passes through the sheet and is perpendicular to it.

A useful Gaussian surface is a closed cylinder with end caps of area A, arranged to pierce the sheet perpendicularly as shown. From symmetry, $\vec{E}$ must be perpendicular to the sheet and hence to the end caps. Furthermore, since the charge is positive, $\vec{E}$ is directed *away* from the sheet, and thus the electric field lines pierce the two Gaussian end caps in an outward direction. Because the field lines do not pierce the curved surface, there is no flux through this portion of the Gaussian surface. Thus $\vec{E} \cdot d\vec{A}$ is simply $E\,dA$; then Gauss' law,

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enc}},$$

Becomes

$$\varepsilon_0(EA + EA) = \sigma A,$$

where σA is the charge enclosed by the Gaussian surface. This gives

$$E = \frac{\sigma}{2\varepsilon_0} \quad \text{(sheet of charge).} \tag{23-19}$$

Since we are considering an infinite sheet with uniform charge density, this result holds for any point at a finite distance from the sheet. Equation 23-19 agrees with Eq. 22-44, which we found by integration of electric field components.

Two Conducting Plates

Figure 23-39*a* shows a cross section of a thin, infinite conducting plate with excess positive charge. From Section 23.3 we know that this excess charge lies on the surface of the plate. Since the plate is thin and very large, we can assume that essentially all the excess charge is on the two large faces of the plate.

If there is no external electric field to force the positive charge into some particular distribution, it will spread out on the two faces with a uniform surface charge density of magnitude σ_1. From Eq. 23-11 we know that just outside the plate this charge sets up an electric field of magnitude $E = \sigma_1/\varepsilon_0$. Because the excess charge is positive, the field is directed away from the plate.

Figure 23-39*b* shows an identical plate with excess negative charge having the same magnitude of surface charge density σ_1. The only difference is that now the electric field is directed toward the plate.

Suppose we arrange for the plates of Figs. 23-39*a* and *b* to be close to each other and parallel (Fig. 23-39*c*). Since the plates are conductors, when we bring them into this arrangement, the excess charge on one plate attracts the excess charge on the other plate, and all the excess charge moves onto the inner faces of the plates as in Fig. 23-39*c*.

With twice as much charge now on each inner face, the new surface charge density (call it σ) on each inner face is twice σ_1. Thus, the electric field at any point between the plates has the magnitude

$$E = \frac{2\sigma_1}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}. \tag{23-20}$$

This field is directed away from the positively charged plate and toward the negatively charged plate. Since no excess charge is left on the outer faces, the electric field to the left and right of the plates is zero.

Because the charges moved when we brought the plates close to each other, the charge distribution of the two-plate system is not merely the sum of the charge distributions of the individual plates.

One reason why we discuss seemingly unrealistic situations, such as the field set up by an infinite sheet of charge, is that analyses for "infinite" situations yield good approximations to many real-world problems. Thus, Eq. 23-19 holds well for a finite nonconducting sheet as long as we are dealing with points close to the sheet and not too near its edges. Equation 23-20 holds well for a pair of finite conducting plates as long as we consider points that are not too close to their edges. The trouble with the edges is that near an edge we can no longer use planar symmetry to find expressions for the fields. In fact, the field lines there are curved (said to be an *edge effect* or *fringing*), and the fields can be very difficult to express algebraically.

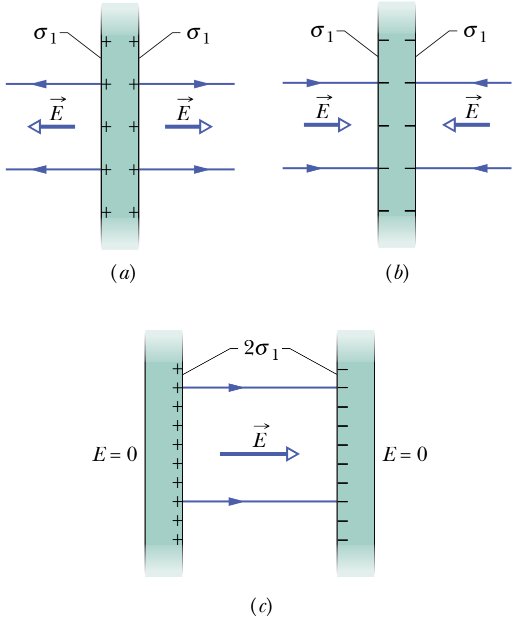
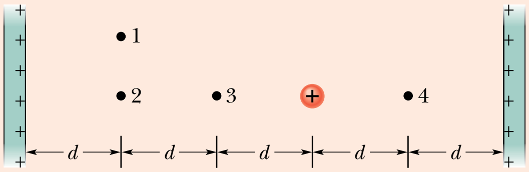


Figure 23-39 (*a*) A thin, very large conducting plate with excess positive charge. (*b*) An identical plate with excess negative charge. (*c*) The two plates arranged so they are parallel and close.

CHECKPOINT 7

The figure shows two large, parallel, nonconducting sheets with identical (positive) uniform surface charge densities, and a sphere with a uniform (positive) volume charge density. Rank the four numbered points according to the magnitude of the net electric field there, greatest first.



SAMPLE PROBLEM 23.12

Electric field near two parallel nonconducting sheets with charge

Figure 23-40*a* shows portions of two large, parallel, nonconducting sheets, each with a fixed uniform charge on one side. The magnitudes of the surface charge densities are $\sigma_{(+)} = 6.8\ \mu\text{C/m}^2$ for the positively charged sheet and $\sigma_{(-)} = 4.3\ \mu\text{C/m}^2$ for the negatively charged sheet.

Find the electric field $\vec{E}$ (a) to the left of the sheets, (b) between the sheets, and (c) to the right of the sheets.

KEY IDEA

With the charges fixed in place (they are on nonconductors), we can find the electric field of the sheets in Fig. 23-40*a* by (1) finding the field of each sheet as if that sheet were isolated and (2) algebraically adding the fields of the isolated sheets via the superposition principle. (We can add the fields algebraically because they are parallel to each other.)

Calculations: At any point, the electric field $\vec{E}_{(+)}$ due to the positive sheet is directed *away* from the sheet and, from Eq. 23-19, has the magnitude

$$E_{(+)} = \frac{\sigma_{(+)}}{2\varepsilon_0} = \frac{6.8\times10^{-6}\ \text{C/m}^2}{(2)(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)}$$
$$= 3.43\times10^5\ \text{N/C}.$$

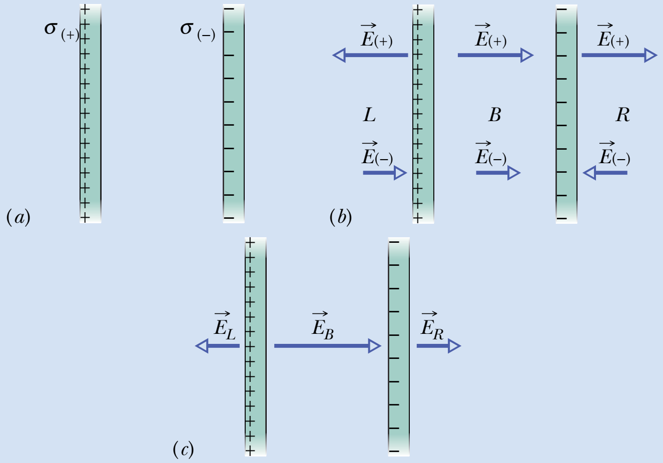


Figure 23-40 (*a*) Two large, parallel sheets, uniformly charged on one side. (*b*) The individual electric fields resulting from the two charged sheets. (*c*) The net field due to both charged sheets, found by superposition.

Similarly, at any point, the electric field $\vec{E}_{(-)}$ due to the negative sheet is directed *toward* that sheet and has the magnitude

$$E_{(-)} = \frac{\sigma_{(-)}}{2\varepsilon_0} = \frac{4.3\times10^{-6}\ \text{C/m}^2}{(2)(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)}$$
$$= 23.43\times10^5\ \text{N/C}.$$

Figure 23-40*b* shows the fields set up by the sheets to the left of the sheets (L), between them (B), and to their right (R).

The resultant fields in these three regions follow from the superposition principle. To the left, the field magnitude is

$$\begin{aligned} E_L &= E_{(+)} - E_{(-)} \\ &= 3.84\times10^5\ \text{N/C} - 2.43\times10^5\ \text{N/C} \\ &= 1.4\times10^5\ \text{N/C}. \qquad \text{(Answer)} \end{aligned}$$

Because $\vec{E}_{(+)}$ is larger than $\vec{E}_{(-)}$, the net electric field $\vec{E}_L$ in this region is directed to the left, as Fig. 23-40*c* shows. To the right of the sheets, the net electric field has the same magnitude but is directed to the right, as Fig. 23-40*c* shows.

Between the sheets, the two fields add and we have

$$\begin{aligned} E_B &= E_{(+)} - E_{(-)} \\ &= 3.84\times10^5\ \text{N/C} + 2.43\times10^5\ \text{N/C} \\ &= 6.3\times10^5\ \text{N/C}. \qquad \text{(Answer)} \end{aligned}$$

The electric field $\vec{E}_B$ is directed to the right.

REVIEW AND SUMMARY

Gauss' Law *Gauss' law* and Coulomb's law are different ways of describing the relation between charge and electric field in static situations. Gauss' law is

$$\varepsilon_0\Phi = q_{\text{enc}} \quad \text{(Gauss' law)}, \tag{23-6}$$

in which q_{enc} is the net charge inside an imaginary closed surface (a *Gaussian surface*) and Φ is the net *flux* of the electric field through the surface:

$$\Phi = \oint \vec{E}\cdot d\vec{A} \quad \text{(electric flux through a Gaussian surface)}. \tag{23-4}$$

Coulomb's law can be derived from Gauss' law.

Applications of Gauss' Law Using Gauss' law and, in some cases, symmetry arguments, we can derive several important results in electrostatic situations. Among these are:

1. An excess charge on an isolated *conductor* is located entirely on the outer surface of the conductor.
2. The external electric field near the *surface of a charged conductor* is perpendicular to the surface and has a magnitude that depends on the surface charge density σ:

   $$E = \frac{\sigma}{\varepsilon_0} \quad \text{(conducting surface)}. \tag{23-11}$$

   Within the conductor, $E = 0$.
3. The electric field due to an *infinite nonconducting sheet* with uniform surface charge density σ is perpendicular to the plane of the sheet and has magnitude

   $$E = \frac{\sigma}{2\varepsilon_0} \quad \text{(sheet of charge)}. \tag{23-19}$$

4. The electric field *outside a spherical shell of charge* with radius R and total charge q is directed radially and has magnitude

   $$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad \text{(spherical shell, for } r \ge R). \tag{23-12}$$

Here r is the distance from the center of the shell to the point at which E is measured. (The charge behaves, for external points, as if it were all located at the center of the sphere.) The field *inside* a uniform spherical shell of charge is exactly zero:

$$E = 0 \quad \text{(spherical shell, for } r < R). \qquad (23\text{-}13)$$

5. The electric field *inside a uniform sphere of charge* is directed radially and has magnitude

$$E = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right) r. \qquad (23\text{-}17)$$

6. The electric field at any point due to an infinite *line of charge* with uniform linear charge density λ is perpendicular to the line of charge and has magnitude

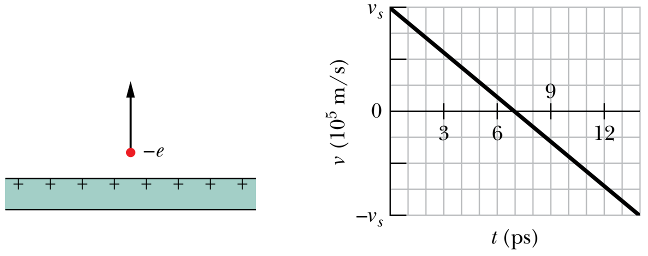
$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{(line of charge)}, \qquad (23\text{-}18)$$

where r is the perpendicular distance from the line of charge to the point.

PROBLEMS

1. An infinite line of charge produces a field of magnitude 1.7×10^4 N/C at distance 9.0 m. Find the field magnitude at distance 2.0 m.

2. In Fig. 23-41*a*, an electron is shot directly away from a uniformly charged plastic sheet, at speed $v_s = 1.6 \times 10^5$ m/s. The sheet is nonconducting, flat, and very large. Figure 23-41*b* gives the electron's vertical velocity component v versus time t until the return to the launch point. What is the sheet's surface charge density?



(*a*) (*b*)

Figure 23-41 Problem 2.

3. An unknown charge sits on a conducting solid sphere of radius 10 cm. If the electric field 30 cm from the center of the sphere has the magnitude 3.0×10^3 N/C and is directed radially inward, (a) what is the net charge on the sphere? and (b) what is the charge density?

4. A charge of uniform linear density 1.5 nC/m is distributed along a long, thin, nonconducting rod. The rod is coaxial with a long conducting cylindrical shell (inner radius = 5.0 cm, outer radius = 10 cm). The net charge on the shell is zero. (a) What is the magnitude of the electric field 15 cm from the axis of the shell? What is the surface charge density on the (b) inner and (c) outer surface of the shell?

5. An isolated conductor has net charge $+10 \times 10^{-6}$ C and a cavity with a particle of charge $q = -4.0 \times 10^{-6}$ C. What is the charge on (a) the cavity wall and (b) the outer surface?

6. Figure 23-42 gives the magnitude of the electric field inside and outside a sphere with a positive charge distributed uniformly throughout its volume. The scale of the vertical axis is set by $E_s = 10 \times 10^7$ N/C. (a) What is the charge on the sphere? (b) What is the field magnitude at $r = 8.0$ m?

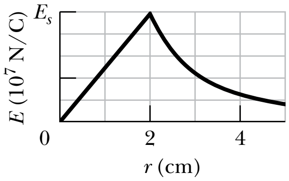


Figure 23-42 Problem 6.

7. In Fig. 23-43, two large, thin metal plates are parallel and close to each other. On their inner faces, the plates have excess surface charge densities of opposite signs and magnitude 2.31×10^{-22} C/m². In unit-vector notation, what is the electric field at points (a) to the left of the plates, (b) to the right of them, and (c) between them?

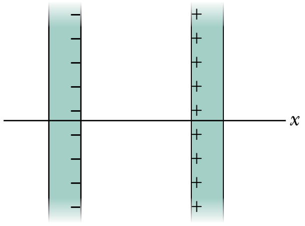


Figure 23-43 Problem 7.

8. In Fig. 23-44, a small circular hole of radius $R = 1.30$ cm has been cut in the middle of an infinite, flat, nonconducting surface that has uniform charge density $\sigma = 4.50$ pC/m². A z axis, with its origin at the hole's center, is perpendicular to the surface. In unitvector notation, what is the electric field at point P at $z = 2.56$ cm?

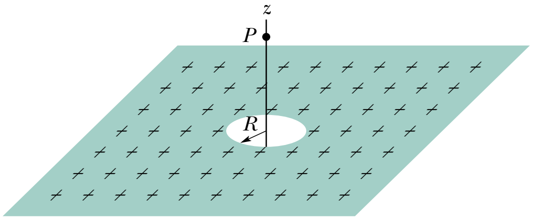


Figure 23-44 Problem 8.

9. Figure 23-45 shows a very large nonconducting sheet that has a uniform surface charge density of $\sigma = -2.00\ \mu\text{C/m}^2$; it also shows a particle of charge $Q = 8.00\ \mu\text{C}$, at distance d from the sheet. Both are fixed in place. If $d = 0.200$ m, at what (a) positive and (b) negative coordinate on the x axis (other than infinity) is the net electric field $\vec{E}_{\text{net}}$ of the sheet and particle zero? (c) If $d = 0.950$ m, at what coordinate on the x axis is $\vec{E}_{\text{net}} = 0$?

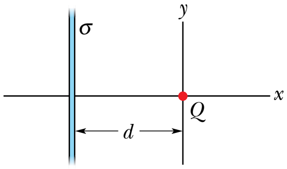


Figure 23-45 Problem 9.

10. Figure 23-46 shows a section of a long, thin-walled metal tube of radius $R = 2.50$ cm, with a charge per unit length of $\lambda = 2.00 \times 10^{-8}$ C/m. What is the magnitude E of the electric field at radial distance (a) $r = R/2.00$ and (b) $r = 2.00R$? (c) Graph E versus r for the range $r = 0$ to $2.00R$.

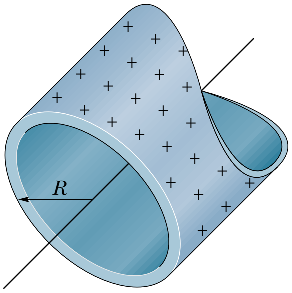


Figure 23-46 Problem 10.

11. The cube in Fig. 23-47 has edge length 1.40 m and is oriented as shown in a region of uniform electric field. Find the electric flux through the right face if the electric field, in newtons per coulomb, is given by (a) $19.0\hat{\text{i}}$ (b) $-2.00\hat{\text{j}}$, and (c) $-20.0\hat{\text{i}} + 4.00\hat{\text{k}}$. (d) What is the total flux through the cube for each field? (e) What is the total flux if the edge length is doubled?

12. At each point on the surface of the cube shown in Fig. 23-47, the electric field is parallel to the z axis. The length of each edge of the cube is 4.0 m. On the top face of the cube the field is $\vec{E} = -34\hat{\text{k}}$ N/C, and on the bottom face it is $\vec{E} = +20\hat{\text{k}}$ N/C Determine the net charge contained within the cube.

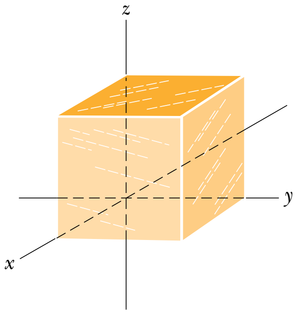


Figure 23-47 Problems 11, 12, and 13.

13. Figure 23-47 shows a Gaussian surface in the shape of a cube with edge length 5.60 m. What are (a) the net flux Φ through the surface and (b) the net charge q_{enc} enclosed by the surface if $\vec{E} = (3.00y\hat{j})$ N/C with y in meters? What are (c) Φ and (d) q_{enc} if $\vec{E} = [-17.0\hat{\text{i}} + (6.00 + 3.00y)\hat{\text{j}}]$ N/C?

14. In Fig. 23-48, a proton is a distance $d/2$ directly above the center of a square of side d. What is the magnitude of the electric flux through the square? (*Hint:* Think of the square as one face of a cube with edge d.)

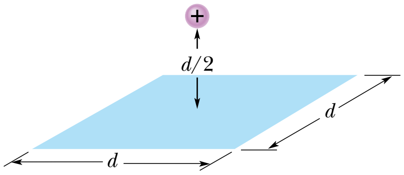


Figure 23-48 Problem 14.

15. A long, nonconducting, solid cylinder of radius 4.0 cm has a nonuniform volume charge density ρ that is a function of radial distance r from the cylinder axis: $\rho = Ar^2$. For $A = 6.3\ \mu\text{C/m}^5$, what is the magnitude of the electric field at (a) $r = 3.0$ cm and (b) $r = 5.0$ cm?

16. A long, straight wire has fixed negative charge with a linear charge density of magnitude 5.2 nC/m. The wire is to be enclosed by a coaxial, thin-walled nonconducting cylindrical shell of radius 1.2 cm. The shell is to have positive charge on its outside surface with a surface charge density σ that makes the net external electric field zero. Calculate σ.

17. Figure 23-49 shows cross sections through two large, parallel, nonconducting sheets with identical distributions of positive charge with surface charge density $\sigma = 2.31 \times 10^{-22}\ \text{C/m}^2$. In unit-vector notation, what is $\vec{E}$ at points (a) above the sheets, (b) between them, and (c) below them?

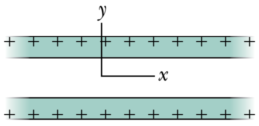


Figure 23-49 Problem 17.

18. In Fig. 23-50, a small, nonconducting ball of mass $m = 7.3$ mg and charge $q = 2.0 \times 10^{-8}$ C (distributed uniformly through its volume) hangs from an insulating thread that makes an angle $\theta = 30°$ with a vertical, uniformly charged nonconducting sheet (shown in cross section). Considering the gravitational force on the ball and assuming the sheet extends far vertically and into and out of the page, calculate the surface charge density σ of the sheet.

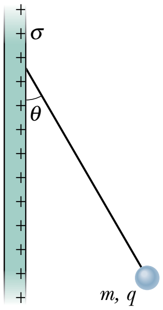


Figure 23-50 Problem 18.

19. A charged particle is held at the center of a spherical shell. Figure 23-51 gives the magnitude E of the electric field versus radial distance r. The scale of the vertical axis is set by $E_s = 5.0 \times 10^7$ N/C. Approximately, what is the net charge on the shell?

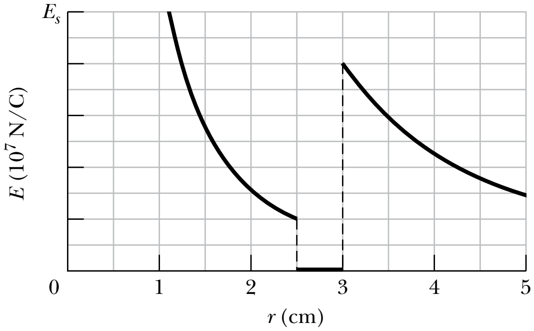


Figure 23-51 Problem 19.

20. Figure 23-52 shows a closed Gaussian surface in the shape of a cube of edge length 2.00 m, with one corner at $x_1 = 5.00$ m, $y_1 = 4.00$ m. The cube lies in a region where the electric field vector is given by $\vec{E} = +23.0\hat{\text{i}} - 2.00y^2\hat{\text{j}} - 16.0\hat{\text{k}}$ N/C, with y in meters. What is the net charge contained by the cube?

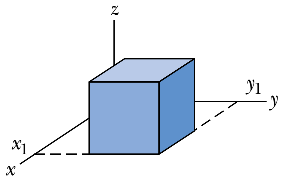


Figure 23-52 Problem 20.

21. Two charged concentric spherical shells have radii 10.0 cm and 15.0 cm. The charge on the inner shell is 7.50×10^{-8} C, and that on the outer shell is 6.33×10^{-8} C. Find the electric field (a) at $r = 12.0$ cm and (b) at $r = 20.0$ cm.

22. Figure 23-53 shows a spherical shell with uniform volume charge density $\rho = 1.56$ nC/m^3, inner radius $a = 10.0$ cm, and outer radius $b = 2.00a$. What is the magnitude of the electric field at radial distances (a) $r = 0$; (b) $r = a/2.00$, (c) $r = a$, (d) $r = 1.50a$, (e) $r = b$, and (f) $r = 3.00b$?

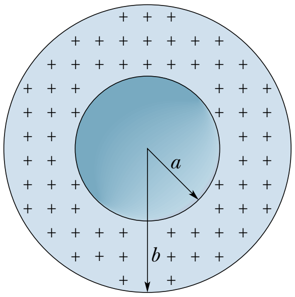


Figure 23-53 Problem 22.

23. In Fig. 23-54, a solid sphere of radius $a = 2.00$ cm is concentric with a spherical conducting shell of inner radius $b = 2.00a$ and outer radius $c = 2.40a$. The sphere has a net uniform charge $q_1 = +\ 2.00$ fC; the shell has a net charge $q_2 = -q_1$. What is the magnitude of the electric field at radial distances (a) $r = 0$, (b) $r = a/2.00$, (c) $r = a$, (d) $r = 1.50a$, (e) $r = 2.30a$, and (f) $r = 3.50a$? What is the net charge on the (g) inner and (h) outer surface of the shell?

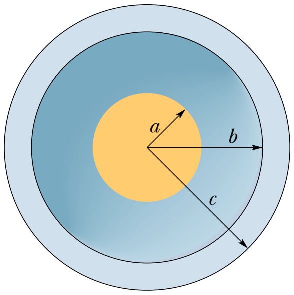


Figure 23-54 Problem 23.

24. In Fig. 23-55, a butterfly net is in a uniform electric field of magnitude $E = 4.5$ mN/C. The rim, a circle of radius $a = 11$ cm, is aligned perpendicular to the field. The net contains no net charge. Find the electric flux through the netting.

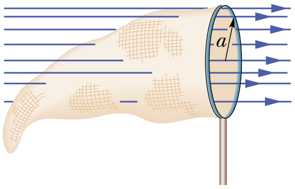


Figure 23-55 Problem 24.

25. The box-like Gaussian surface shown in Fig. 23-56 encloses a net charge of $+32\varepsilon_0$ C and lies in an electric field given by $\vec{E} = [(10.0 + 2.00x)\hat{\text{i}} - 3.00\hat{\text{j}} + bz\hat{\text{k}}]$ N/C, with x and z in meters and b a constant. The bottom face is in the xz plane; the top face is in the horizontal plane passing through $y_2 = 1.00$ m. For $x_1 = 1.00$ m, $x_2 = 4.00$ m, $z_1 = 1.00$ m, and $z_2 = 3.00$ m, what is b?

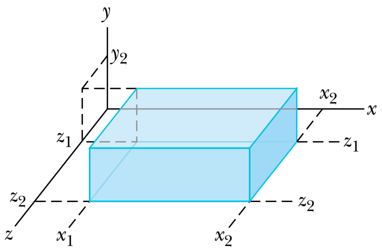


Figure 23-56 Problem 25.

26. In Fig. 23-57, short sections of two very long parallel lines of charge are shown, fixed in place, separated by $L = 10.0$ cm. The uniform linear charge densities are $+6.0\ \mu$C/m for line 1 and $-2.0\ \mu$C/m for line 2. Where along the x axis shown is the net electric field from the two lines zero?

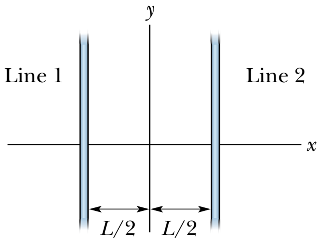


Figure 23-57 Problem 26.

27. Figure 23-58 shows two nonconducting spherical shells fixed in place on an x axis. Shell 1 has uniform surface charge density $+5.0\ \mu$C/m^2 on its outer surface and radius 0.50 cm, and shell 2 has uniform surface charge density $-2.0\ \mu$C/m^2 on its outer surface and radius 2.0 cm; the centers are separated by $L = 6.0$ cm. Other than at $x = \infty$, where on the x axis is the net electric field equal to zero?

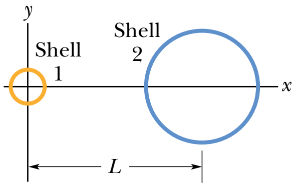


Figure 23-58 Problem 27.

28. A uniformly charged conducting sphere of 0.60 m diameter has surface charge density 5.7 μC/m^2. Find (a) the net charge on the sphere and (b) the total electric flux leaving the surface. (c) What is the net flux through a concentric Gaussian sphere of radius 2.0 m?

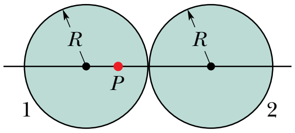


Figure 23-59 Problem 29.

29. Figure 23-59 shows, in cross section, two solid spheres with uniformly distributed charge throughout their volumes. Each has radius R. Point P lies on a line connecting the centers of the spheres, at radial distance $R/4.00$ from the center of sphere 1. If the net electric field at point P is zero, what is the ratio q_1/q_2 of the total charges?

30. A particle of charge $+q$ is placed at one corner of a Gaussian cube. What multiple of q/s_0 gives the flux through (a) each cube face forming that comer and (b) each of the other cube faces?

31. Figure 23-60 shows a closed Gaussian surface in the shape of a cube of edge length 1.50 m. It lies in a region where the nonuniform electric field is given by $\vec{E} = (3.00x + 4.00)\hat{i} + 6.00\hat{j} + 7.00\hat{k}$ N/C, with x in meters. What is the net charge contained by the cube?

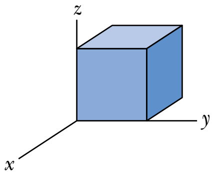


Figure 23-60 Problem 31.

32. A particle of charge 6.3 μC is at the center of a Gaussian cube 92 cm on edge. (a) What is the net electric flux through the surface? (b) What is the net flux if the edge length is doubled?

33. The electric field just above the surface of the charged conducting drum of a photocopying machine has a magnitude E of 1.9×10^5 N/C. What is the surface charge density on the drum?

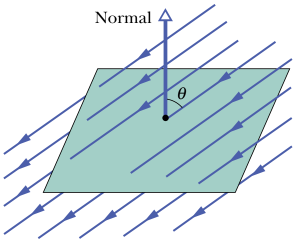


Figure 23-61 Problem 34.

34. The square surface shown in Figure 23-61 measures 6.8 mm on each side. It is immersed in a uniform electric field with magnitude $E = 1800$ N/C and with field lines at an angle of $\theta = 35°$ with a normal to the surface, as shown. Take that normal to be directed "outward," as though the surface were one face of a box. (a) Calculate the electric flux through the surface. (b) If the angle is reduced by a few degrees, does the flux increase, decrease, or remain the same?

35. Figure 23-62 shows a cross section through a very large nonconducting slab of thickness $d = 9.40$ mm and uniform volume charge density $\rho = 1.89$ fC/m³. The origin of an x axis is at the slab's center. What is the magnitude of the slab's electric field at an x coordinate of (a) 0, (b) 2.00 mm, (c) 4.70 mm, and (d) 26.0 mm?

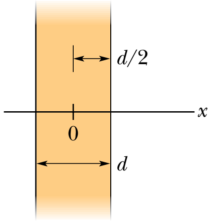


Figure 23-62 Problem 35.

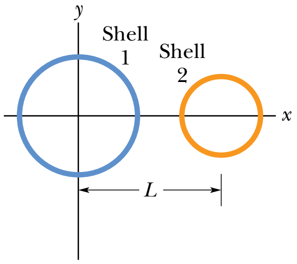


Figure 23-63 Problem 36.

36. Figure 23-63 shows two nonconducting spherical shells fixed in place. Shell 1 has uniform surface charge density +6.0 μC/m² on its outer surface and radius 3.0 cm; shell 2 has uniform surface charge density +4.0 μC/m² on its outer surface and radius 2.0 cm; the shell centers are separated by $L = 12$ cm. In unit-vector notation, what is the net electric field at $x = 2.0$ cm?

37. Figure 23-64*a* shows three plastic sheets that are large, parallel, and uniformly charged. Figure 23-64*b* gives the component of the net electric field along an x axis through the sheets. The scale of the vertical axis is set by $E_s = 3.0 \times 10^5$ N/C. What is the ratio of the charge density on sheet 3 to that on sheet 2?

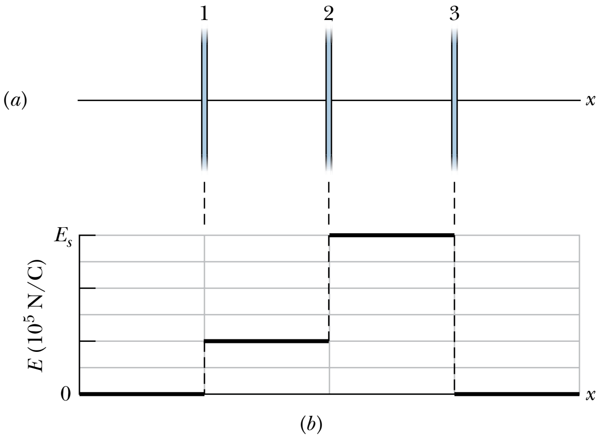


Figure 23-64 Problem 37.

38. *Flux and nonconducting shells.* A charged particle is suspended at the center of two concentric spherical shells that are very thin and made of nonconducting material. Figure 23-65*a* shows a cross section. Figure 23-65*b* gives the net flux Φ through a Gaussian sphere centered on the particle, as a function of the radius r of the sphere. The scale of the vertical axis is set by $\Phi_s = 10 \times 10^5$ N·m²/C (a) What is the charge of the central particle? What are the net charges of (b) shell A and (c) shell B?

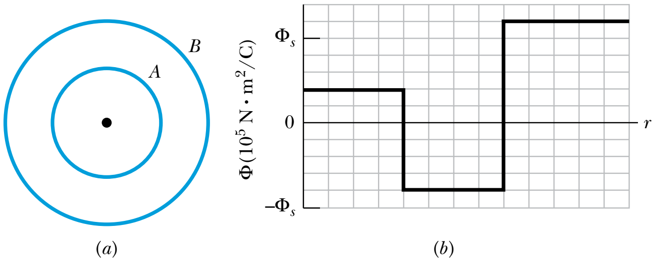


Figure 23-65 Problem 38.

39. Figure 23-66 is a section of a conducting rod of radius $R_1 = 1.30$ mm and length $L = 11.00$ m inside a thin-walled coaxial conducting cylindrical shell of radius $R_2 = 10.0\,R_1$ and the (same) length L. The net charge on the rod is $Q_1 = -5.22 \times 10^{-13}$ C; that on the shell is $Q_1 = -2.00\,Q_1$. What are the (a) magnitude E and (b) direction (radially inward or outward) of the electric field at radial distance $r = 2.00\,R_2$? What are (c) E and (d) the direction at $r = 5.00\,R_1$? What is the charge on the (e) interior and (f) exterior surface of the shell?

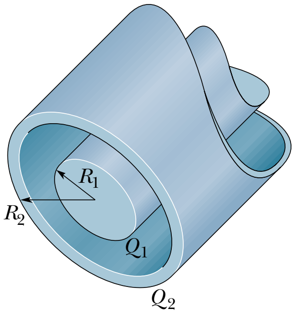


Figure 23-66 Problem 39.

40. An isolated conductor of arbitrary shape has a net charge of $+10 \times 10^{-6}$ C. Inside the conductor is a cavity within which is a point charge $q = +3.0 \times 10^{-6}$ C. What is the charge (a) on the cavity wall and (b) on the outer surface of the conductor?

41. A metal sphere centered at the origin has a radius R and a net charge Q. The electric field at the point $x = 5R$ is E_0. The sphere is replaced by a different metal sphere centered at the origin with radius $2R$ and a net charge Q_0. The field at $x = 5R$ is still E_0. What is the ratio of Q and Q_0?

42. A square metal plate of edge length 8.0 cm and negligible thickness has a total charge of 6.0×10^{-6} C. (a) Estimate the magnitude E of the electric field near the center of the plate (say at a distance of 0.50 mm from the center) assuming that the charge is spread uniformly over the two faces of the plate. (b) Estimate E at a distance of 30 m (large relative to the plate size) assuming that the plate is a point charge.

43. An electron is shot directly toward the center of a large metal plate that has surface charge density -2.0×10^{-6} C/m^2. If the initial kinetic energy of the electron is 1.60×10^{-17} J and if the electron is to stop (due to electrostatic repulsion from the plate) just as it reaches the plate, how far from the plate must the launch point be?

44. Two large parallel conducting plates are 10 cm apart and carry equal but opposite charges on their facing surfaces. An electron placed midway between the plates experiences a force of 3.2×10^{-17} N. What is the potential difference between the plates?

45. Three identical metal plates with large surface areas are kept parallel to each other. The leftmost plate is given a charge Q, the rightmost a charge $-2Q$, and the middle one remains neutral. Find the charge appearing on each of surfaces.

46. If a cavity is made as shown in the uniformly charged sphere (Fig. 23-67), find the electric field at an arbitrary point inside the cavity.

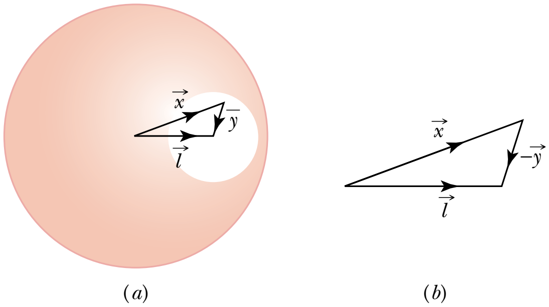
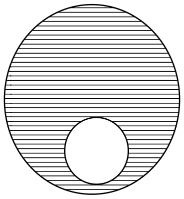


Figure 23-67 Problem 46.

PRACTICE QUESTIONS

Single Correct Choice Type

1. A spherical portion has been removed from a solid sphere having a charge distributed uniformly in its volume as shown in the figure. The electric field inside the emptied space is



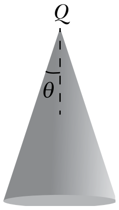
(a) Zero everywhere
(b) Non-zero and uniform
(c) Non-uniform
(d) Zero only at its centre

2. In which of following cases, electric field is not constant?
(a) Inside a uniformly charged spherical shell.
(b) In a spherical cavity inside a uniformly charged sphere.
(c) In front of an infinite sheet of uniform surface charge density.
(d) At a distance x from a point charge q.

3. For a given surface, the Gauss' law is stated as $\oint E \cdot ds = 0$. From this we can conclude that

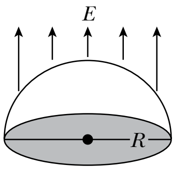
(a) E is necessarily zero on the surface.
(b) E is perpendicular to the surface at every point.
(c) The total flux through the surface is zero.
(d) The flux is only going out of the surface.

4. A charge Q is kept at the apex of the cone as shown in the figure. Find the flux due to this charge through the curved face.



(a) $\frac{Q}{2\varepsilon_0}(1-\cos\theta)$
(b) $\frac{Q}{2\varepsilon_0}(1-\sin\theta)$
(c) Zero
(d) $\frac{Q}{2\varepsilon_0}$

5. Consider a uniform electric field $E = 3\times10^3\hat{\text{i}}$ N/C. What is the flux of this field through a square of 10 cm on a side whose plane is parallel to the yz plane?



(a) 30 $\text{N}\cdot\text{m}^2/\text{C}$
(b) 40 $\text{N}\cdot\text{m}^2/\text{C}$
(c) 50 $\text{N}\cdot\text{m}^2/\text{C}$
(d) 60 $\text{N}\cdot\text{m}^2/\text{C}$

6. A spherical surface completely surrounds a collection of charges. Find the electric flux through the surface if the collection consists of a single $+3.5 \times 10^{-6}$ C charge.

(a) zero $\text{N}\cdot\text{m}^2/\text{C}$
(b) 4.0×10^5 $\text{N}\cdot\text{m}^2/\text{C}$
(c) 1.6×10^5 $\text{N}\cdot\text{m}^2/\text{C}$
(d) 3.1×10^5 $\text{N}\cdot\text{m}^2/\text{C}$

7. If the electric flux entering and leaving an enclosed surface, respectively, are ϕ_1 and ϕ_2, the electric charge inside the surface will be

(a) $(\phi_1 + \phi_2)\varepsilon_0$
(b) $(\phi_1 - \phi_2)\varepsilon_0$
(c) $\frac{\phi_1 + \phi_2}{\varepsilon_0}$
(d) $\frac{\phi_1 - \phi_2}{\varepsilon_0}$

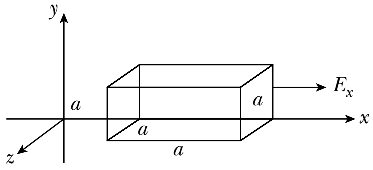
8. A uniformly charged conducting sphere of 2.4 m diameter, has a surface charge density of 80.0 μC/m^2. What is the total electric flux leaving the surface of the sphere?

(a) 1.6×10^8 $\text{N}\cdot\text{m}^2/\text{C}$
(b) 3.2×10^8 $\text{N}\cdot\text{m}^2/\text{C}$
(c) 4.8×10^8 $\text{N}\cdot\text{m}^2/\text{C}$
(d) 6.4×10^8 $\text{N}\cdot\text{m}^2/\text{C}$

9. What is the electric flux passing through a Gaussian surface that surrounds a +0.075 C point charge?

(a) 8.5×10^9 $\text{N}\cdot\text{m}^2/\text{C}$
(b) 1.3×10^7 $\text{N}\cdot\text{m}^2/\text{C}$
(c) 7.2×10^5 $\text{N}\cdot\text{m}^2/\text{C}$
(d) 6.8×10^8 $\text{N}\cdot\text{m}^2/\text{C}$

10. The electric field components in figure are $E_x = b\sqrt{x}$, $E_g = E_z = 0$. Calculate the flux through the cube.

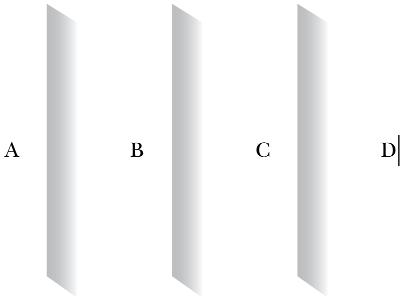


(a) $(\sqrt{2}-1)ba^{5/2}$
(b) $ba^{5/2}$
(c) $\sqrt{2}\,ba^{5/2}$
(d) $(\sqrt{5}-3)ba^{5/2}$

11. A cone of base radius R and height h is located in a uniform electric field $\vec{E}$ parallel to its base. The electric flux entering the cone is

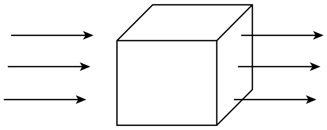
(a) $\frac{1}{2}EhR$
(b) EhR
(c) $2EhR$
(d) $4EhR$

12. Figure given below shows an edge-on view of three extremely large, parallel insulating sheets positively charged with uniform and identical surface charge densities. In which of the regions shown, is the electric field a maximum?



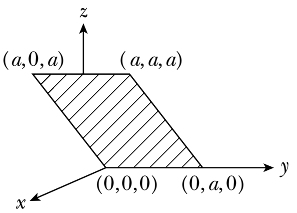
(a) B
(b) C
(c) A and D
(d) The electric field is the same in all four regions of space.

13. A cubical Gaussian surface is placed in a uniform electric field as shown in the figure to the right. The length of each edge of the cube is 1.0 m. The uniform electric field has a magnitude of 5.0×10^8 N/C and passes through the left and right sides of the cube perpendicular to the surface. What is the total electric flux that passes through the cubical Gaussian surface?



(a) 5.0×10^8 $\text{N}\cdot\text{m}^2/\text{C}$
(b) 2.5×10^6 $\text{N}\cdot\text{m}^2/\text{C}$
(c) zero $\text{N}\cdot\text{m}^2/\text{C}$
(d) 3.0×10^9 $\text{N}\cdot\text{m}^2/\text{C}$

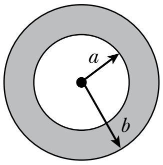
14. Consider an electric field $\vec{E} = E_0\hat{x}$ where E_0 is a constant. The flux through the shaded area (as shown in the figure) due to this field is



(a) $2E_0a^2$ **(b)** $\sqrt{2}E_0a^2$

(c) E_0a^2 **(d)** $\frac{E_0a^2}{\sqrt{2}}$

15. The figure below shows spherical distribution. The charge density varies as $\rho(r) = C/r$ for $a < r < b$ where C is a constant. A point charge q lies at the center of the sphere at $r = 0$. Which of the following is the condition for the electric field to have a constant magnitude in the region $a < r < b$?



(a) $C = 0$ **(b)** $C = q$

(c) $C = \frac{q}{4qa^2}$ **(d)** $C = \frac{qr}{2qa^3}$

16. A uniform electric field with a magnitude of 125000 N/C passes through a rectangle with sides of 2.50 m and 5.00 m. The angle between the electric field vector and the vector normal to the rectangular plane is 65.0°. What is the electric flux through the rectangle?

(a) $1.56 \times 10^6\ \text{N}\cdot\text{m}^2/\text{C}$ **(b)** $1.42 \times 10^5\ \text{N}\cdot\text{m}^2/\text{C}$

(c) $6.60 \times 10^5\ \text{N}\cdot\text{m}^2/\text{C}$ **(d)** $5.49 \times 10^4\ \text{N}\cdot\text{m}^2/\text{C}$

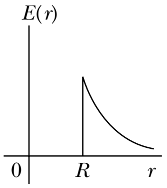
17. A rectangular surface (0.16 m × 0.38 m) is oriented in a uniform electric field of 580 N/C. What is the maximum possible electric flux through the surface?

(a) $46\ \text{N}\cdot\text{m}^2/\text{C}$ **(b)** $220\ \text{N}\cdot\text{m}^2/\text{C}$

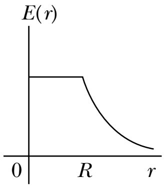
(c) $92\ \text{N}\cdot\text{m}^2/\text{C}$ **(d)** $35\ \text{N}\cdot\text{m}^2/\text{C}$

18. A thin spherical shell of radius R has charge Q spread uniformly over its surface. Which of the following graphs most closely represents the electric field $E(r)$ produced by the shell in the range $0 \le r \le \infty$, where r is the distance from the centre of the shell?

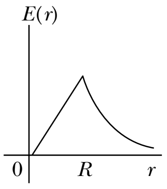
(a)



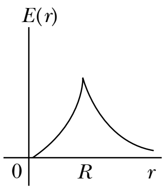
(b)



(c)



(d)



19. A spherically symmetric charge distribution is characterized by a charge density having the following variation:

$$\rho(r) = \rho_0\left(1 - \frac{r}{R}\right) \quad \text{for} \quad r < R$$

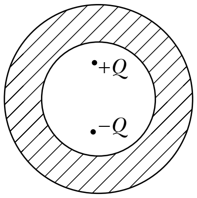
$$\rho(r) = 0 \quad \text{for} \quad r \ge R$$

where r is the distance from the center of the charge distribution and ρ_0 is a constant. The electric field at an internal point ($r < R$) is

(a) $\frac{\rho_0}{4\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4R}\right)$ **(b)** $\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4R}\right)$

(c) $\frac{\rho_0}{3\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4R}\right)$ **(d)** $\frac{\rho_0}{12\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4R}\right)$

20. Shown in the figure are two point charges $+Q$ and $-Q$ inside the cavity of a spherical shell. The charges are kept near the surface of the cavity on opposite sides of the center of the shell. If σ_1 is the surface charge on the inner surface and Q_1 net charge on it and σ_2 the surface charge on the outer surface and Q_2 net charge on it then



(a) $\sigma_1 \ne 0,\ Q_1 \ne 0$; $\sigma_2 \ne 0,\ Q_2 \ne 0$ **(b)** $\sigma_1 \ne 0,\ Q_1 = 0$; $\sigma_2 \ne 0,\ Q_2 = 0$

(c) $\sigma_1 \ne 0,\ Q_1 = 0$; $\sigma_2 = 0,\ Q_2 = 0$ **(d)** $\sigma_1 = 0,\ Q_1 = 0$; $\sigma_2 = 0,\ Q_2 = 0$

21. A total charge of $-6.50\ \mu\text{C}$ is uniformly distributed within a sphere that has a radius of 0.150 m. What is the magnitude and direction of the electric field at 0.300 m from the surface of the sphere?

(a) 2.89×10^5 N/C, radially inward

(b) 9.38×10^5 N/C, radially outward

(c) 6.49×10^5 N/C, radially outward

(d) 1.30×10^6 N/C, radially inward

22. Consider three different closed surfaces. Surface A is a sphere of radius R, surface B is a sphere of radius $2R$, and surface C is a cube with side length R. At the geometric center of each surface is a small ball carrying a positive charge $+Q$; each ball is completely enclosed by the surface. Assuming that there are no other charged objects inside any of the surfaces, rank the surfaces based on the field line flux through them, from largest to smallest.

(a) A > B > C **(b)** A > C > B

(c) B > C > A **(d)** All are equal

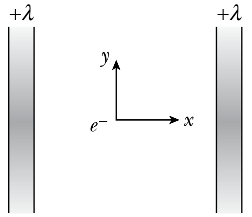
23. A straight, copper wire has a length of 0.50 m and an excess charge of -1.0×10^{-5} C distributed uniformly along its length. Find the magnitude of the electric field at a point located 7.5×10^{-3} m from the midpoint of the wire.

(a) 1.9×10^{10} N/C **(b)** 6.1×10^{13} N/C
(c) 4.8×10^{7} N/C **(d)** 1.5×10^{6} N/C

24. Two spherical shells have a common center. A charge of -1.6×10^{-6} C is spread uniformly over the inner shell, which has a radius of 0.050 m. A charge of $+5.1 \times 10^{-6}$ C is spread uniformly over the outer shell, which has a radius of 0.15 m. Find the magnitude and direction of the electric field at a distance (measured from the common center) of 0.20 m.
(a) 7.9×10^{5} N/C, radially outward
(b) 5.9×10^{5} N/C, radially inward
(c) 3.2×10^{5} N/C, radially outward
(d) zero N/C

More than One Correct Choice Type

25. An electron is placed just in the middle between two long fixed line charges of charge density $+\lambda$ each. The wires are in the xy plane. (Do not consider gravity.)



(a) The equilibrium of the electron will be unstable along x direction.
(b) The equilibrium of the electron will be neutral along y direction.
(c) The equilibrium of the electron will be stable along z direction.
(d) The equilibrium of the electron will be stable along y direction.

26. A non-conducting solid sphere of radius R is uniformly charged. The magnitude of the electric field due to the sphere at a distance r from its center
(a) Increases as r increases for $r < R$
(b) Decreases as r increases for $0 < r < R$
(c) Decreases as r increases for $R < r < \infty$
(d) Is discontinuous at $r = R$

Linked Comprehension

Passage for Questions 27–29: A ball of radius R carries a positive charge whose volume charge density depends only on the distance r from the ball's center as $\rho = \rho_0(1 - r/R)$, where ρ_0 is constant. Assume ε as the permittivity of the ball.

27. The magnitude of the electric field as a function of the distance r outside the ball is given by

(a) $E = \dfrac{\rho_0 R^3}{8\varepsilon r^2}$ **(b)** $E = \dfrac{\rho_0 R^3}{12\varepsilon r^2}$

(c) $E = \dfrac{\rho_0 R^3}{16\varepsilon r^2}$ **(d)** $E = \dfrac{\rho_0 R^3}{24\varepsilon r^2}$

28. The value of distance r_m, inside the ball at which electric field intensity maximum is given by

(a) $r_m = \dfrac{R}{3}$ **(b)** $r_m = \dfrac{3R}{2}$

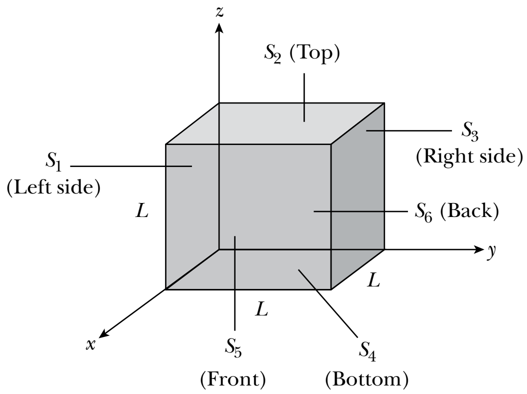
(c) $r_m = \dfrac{2R}{3}$ **(d)** $r_m = \dfrac{4R}{3}$

29. The maximum electric field intensity inside the ball is

(a) $\dfrac{\rho_0 R}{9\varepsilon}$ **(b)** $\dfrac{\rho_0 \varepsilon}{9R}$

(c) $\dfrac{\rho_0 R}{3\varepsilon}$ **(d)** $\dfrac{\rho_0 R}{6\varepsilon}$

Paragraph for Questions 30–32: The cube, as shown in the following figure, has sides of length $L = 10.0$ cm. The electric field is uniform, has a magnitude $E = 4.00 \times 10^3$ N/C, and is parallel to the xy plane at an angle of 37° measured from the $+x$ axis toward the $+y$ axis.



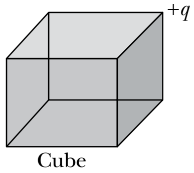
30. Which of the following surfaces have zero flux?
(a) S_1 and S_3 **(b)** S_5 and S_6
(c) S_2 and S_4 **(d)** S_1 and S_2

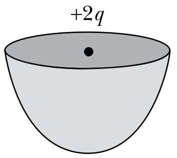
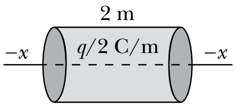
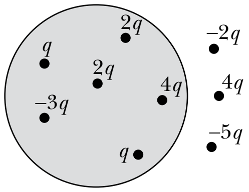
31. Electric flux passing through the surface S_1 is
(a) -24 N·m²/C **(b)** 24 N·m²/C
(c) 32 N·m²/C **(d)** -32 N·m²/C

32. Electric flux passing through the surface S_6 is
(a) -24 N·m²/C **(b)** 24 N·m²/C
(c) 32 N·m²/C **(d)** -32 N·m²/C

Matrix-Match

33. Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one* or *more* statements in Column II.

Column I (Arrangement)	**Column II (Flux through the entire surface of the body)**
(a) 	**(p)** q/ε_0

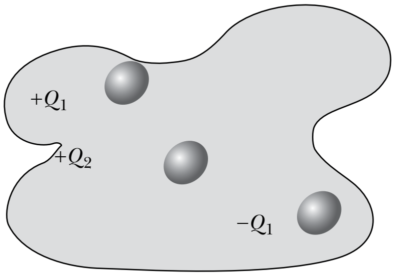
Column I (Arrangement)	Column II (Flux through the entire surface of the body)
(b)  Hemisphere	**(q)** $q/8\varepsilon_0$
(c)  Closed cylinder	**(r)** $q/4\varepsilon_0$
(d)  Sphere	**(s)** $3q/\varepsilon_0$

Directions for Questions 34 and 35: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

34. We can find electric flux through different body shapes by using Gauss's law. In the given table, column 1 shows different body shapes, column 2 shows different arrangements of charges around body and column 3 shows the value or final formula of the electric flux of different body shapes.

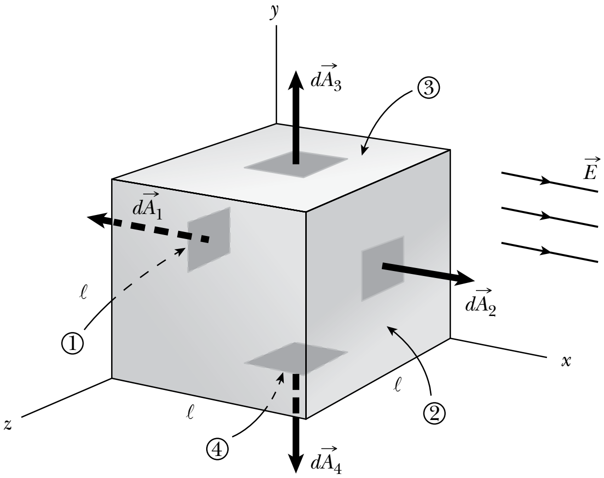
Column I	Column II	Column III
(I) Enclosed body	**(i)** by positive, and negative charges	**(J)** $\phi = \dfrac{Q_1 + Q_2 - Q_3}{\varepsilon_0}$
(II) Hemispherical body	**(ii)** charge is kept at the center of a face	**(K)** $\phi = 0$
(III) Cubical body	**(iii)** charge is kept at the center	**(L)** $\phi_{\text{face}} = \dfrac{Q}{6\varepsilon_0}$
(IV) Cylindrical body	**(iv)** charge is kept at the corner	**(M)** $\phi_{\text{cube}} = \dfrac{Q}{2\varepsilon_0}$

(1) What do you understand from the diagram?



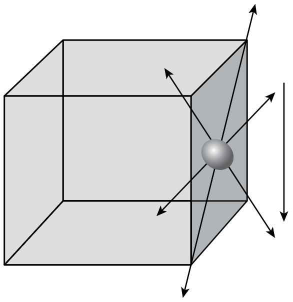
(a) (III) (ii) (L) **(b)** (I) (ii) (J)
(c) (III) (iii) (J) **(d)** (I) (i) (M)

(2) What does the diagram depict?



(a) (III) (iii) (L) **(b)** (II) (iii) (L)
(c) (IV) (iv) (L) **(d)** (I) (i) (K)

(3) What do you understand from the diagram?



(a) (I) (i) (J)
(b) (I) (i) (M)
(c) (III) (iii) (L)
(d) (III) (ii) (M)

35. We can find forces between charges using Coulomb's law. In the given table, Column 1 shows different types of charge combination, Column 2 shows values of charges and Column 3 shows the distance between charges.

Column I	Column II	Column III
(I) Charges with the different values of magnitude and opposite sign	**(i)** $+6 \times 10^{-7}$ C and -6×10^{-7} C	**(J)** at distance of 0.70 m
(II) Charged with the same value of magnitude and opposite sign	**(ii)** $+3.5 \times 10^{-8}$ C and -2.9×10^{-8} C	**(K)** at distance of 50.0 cm
(III) Charges with the different values of magnitude and same sign	**(iii)** -1.0×10^{-6} C and $+4.0 \times 10^{-6}$ C	**(L)** separated by a distance of 0.65 m
(IV) Charged with the same value of magnitude and same sign	**(iv)** 4.0 μC and -4.0 μC	**(M)** separated by a distance of 0.50 m

(1) Determine the electrical force of attraction between two balloons with a value F of 2.16×10^{-5} N?

(a) (I) (ii) (J)
(b) (I) (ii) (L)
(c) (II) (ii) (J)
(d) (I) (iv) (K)

(2) Determine the electrical force of attraction between two balloons with a value F of 1.3×10^{-2} N?

(a) (III) (iv) (K)
(b) (II) (i) (M)
(c) (III) (iii) (L)
(d) (I) (i) (M)

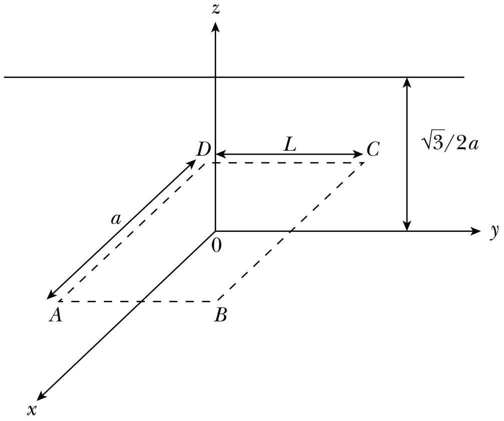
(3) Determine the electrical force of attraction between two balloons with a value F of 0.144 N?

(a) (III) (i) (L)
(b) (IV) (i) (J)
(c) (III) (iii) (K)
(d) (I) (iii) (M)

Integer Type

36. A metal plate 8.0 cm on a side carries a total charge of 6.0 μC. Using the infinite plate approximation, calculate the electric field 0.50 mm above the surface of the plate near the plate's center.

37. An infinitely long uniform line charge distribution of charge per unit length λ lies parallel to the y axis in the yz plane at $z = \sqrt{3}/2a$ (see figure). If the magnitude of the flux of the electric field through the rectangular surface $ABCD$ lying in the xy plane with its centre at the origin is $\lambda L/n\varepsilon_0$ (ε_0 = permittivity of free space), then the value of n is



38. A system consists of a ball of radius $2/\sqrt{\pi a}$ (where 'a' is constant) carrying a spherically symmetric charge and the surrounding space filled with a charge of volume density $\rho = a/r$, where r is the distance from the centre of the ball. Find the ball's charge at which the magnitude of the electric field is independent of r outside the ball. All the values are in MKS system.

ANSWER KEY

Checkpoints

1. (a) $+EA$; (b) $-EA$; (c) 0; (d) 0

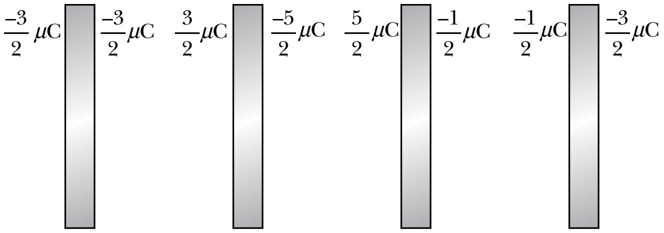
2. (a) 2; (b) 3; (c) 1

3. (a) equal; (b) equal; (c) equal

4. (a) $Q, -Q, 3Q, -3Q, 6Q$; (b) $E_{\mathrm{I}} = \dfrac{kQ}{x^2}$, $E_{\mathrm{II}} = \dfrac{3kQ}{x^2}$; $E_{\mathrm{III}} = \dfrac{6kQ}{x^2}$

5. $E_{\mathrm{I}} = \dfrac{\sigma}{\varepsilon_0} = \dfrac{-3\times10^{-6}}{2A\varepsilon_0}$; $E_{\mathrm{II}} = \dfrac{\sigma}{\varepsilon_0} = \dfrac{-5\times10^{-6}}{2A\varepsilon_0}$

$E_{\mathrm{III}} = \dfrac{-1\times10^{-6}}{2A\varepsilon_0}$; $E_{\mathrm{IV}} = \dfrac{-3\times10^{-6}}{2A\varepsilon_0}$



6. (a) zero; (b) zero

7. 3 and 4 tie, then 2, 1

Problems

1. 7.7×10^4 N/C

2. 2.3 C/m^2

3. (a) -30 nC; (b) 0.24μC/m^2

4. (a) 1.82×10^2 N/C; (b) -4.8×10^{-9} C/m^2; (c) $+2.4 \times 10^{-9}$ C/m^2

5. (a) $+4.0 \times 10^{-6}$ C; (b) $+6.0 \times 10^{-6}$ C

6. (a) 4.4×10^{-6} C; (b) 6.2×10^2 N/C

7. (a) 0; (b) 0; (c) $(-2.61 \times 10^{-11}$ N/C$)\hat{i}$

8. $(0.227$ NC$)\hat{k}$

9. (a) 0.798 m on the positive axis; (b) -0.798 m on the negative axis; (c) $+0.798$ m

10. (a) 0; (b) 7.19×10^3 N/C

11. (a) 0; (b) -3.92 N$\cdot$m^2/C; (c) 0; (d) zero; (e) zero

12. -7.6×10^{-9} C

13. (a) 527 N$\cdot$m^2/C; (b) 4.66×10^{-9} C (c) 527 N$\cdot$m^2/C; (d) 4.66×10^{-9} C

14. $3.014.66 \times 10^{-9}$ N$\cdot$m^2/C

15. (a) 4.8 N/C; (b) 9.1 N/C

16. 6.9×10^{-8} C/m^2

17. (a) $(2.61 \times 10^{-11}$ N/C$)\hat{j}$; (b) 0; (c) $-(2.61 \times 10^{-11}$ N/C$)\hat{j}$

18. 37 nC/m^2

19. 3.3×10^{-6} C

20. -8.5×10^{-10} C

21. (a) 4.68×10^4 N/C; (b) 3.11×10^4 N/C

22. (a) 0; (b) 0; (c) 0; (d) 6.20 N/C; (e) 10.3 N/C; (f) 1.14 N/C

23. (a) 0; (b) 2.25×10^{-2} N/C; (c) 0.0450 N/C; (d) 20.0 m N/C; (e) 0; (f) 0; (g) -2.00 fC; (h) 0

24. 1.7×10^{-4} N$\cdot$m^2/C

25. 3.33 N/C m

26. 10.0 cm

27. -3.9 cm

28. (a) 6.4 μC; (b) 7.3×10^5 N$\cdot$m^2/C; (c) 7.3×10^5 N$\cdot$m^2/C

29. 0.766

30. (a) zero; (b) 0.0417

31. 8.96×10^{-11} C

32. (a) 7.1×10^5 N$\cdot$m^2/C; (b) 7.1×10^5 N$\cdot$m^2/C

33. 1.7×10^6 C/m^2

34. (a) -0.068 N$\cdot$m^2/C; (b) increase

35. (a) 0; (b) 4.27×10^{-7} N/C; (c) 1.00×10^{-6} N/C; (d) 1.00×10^{-6} N/C

36. $-(1.8 \times 10^4$ N/C$)\hat{j}$

37. -1.5

38. (a) 3.5 μC; (b) -11μC; (c) $+18$ μC

39. (a) 3.28×10^{-2} N/C; (b) outward; (c) 0.131N/C; (d) inward; (e) $+5.22 \times 10^{-13}$ C; (f) $+5.22 \times 10^{-13}$ C

40. (a) -3×10^{-6} C; (b) $+1.3 \times 10^{-5}$ C

41. 1

42. (a) 5.3×10^7 N/C; (b) 60 N/C

43. 0.44 mm

44. 20 V

45. $-Q/2; 3Q/2; -3Q/2; 3Q/2; -3Q/2; -Q/2$

46. $\vec{E} = \dfrac{\rho\vec{l}}{3\varepsilon_0}$

Single Correct Choice Type

1. (a)

2. (d)

3. (c)

4. (c)

5. (a)

6. (b)

7. (b)

8. (a)

9. (a)

10. (a)

11. (b)

12. (c)

13. (c)

14. (c)

15. (d)

16. (c)

17. (d)

18. (a)

19. (b)

20. (c)

21. (a)

22. (d)

23. (c)

24. (a)

More than One Correct Choice Type

25. (a), (b), (c)

26. (a), (c)

Linked Comprehension

27. (b)
28. (c)
29. (a)
30. (c)
31. (a)
32. (d)

Matrix-Match

33. (a) → (q); (b) → (p); (c) → (p); (d) → (s)
34. (1) → (b); (2) → (a); (3) → (d)
35. (1) → (b); (2) → (b); (3) → (d)

Integer Type

36. 53
37. 6
38. 8

CHAPTER

24

Electric Potential

Contents

24.1 What is Physics?
24.2 Electric Potential and Electric Potential Energy
24.3 Equipotential Surfaces
24.4 Calculating the Potential from the Field
24.5 Potential Due to a Point Charge
24.6 Potential Due to a Group of Point Charges
24.7 Potential Due to a Continuous Charge Distribution
24.8 Calculating the Field from the Potential
24.9 Potential Due to an Electric Dipole
24.10 Electric Potential Energy of a System of Charged Particles
24.11 Potential of a Charged Isolated Conductor
24.12 Earthing
24.13 Electrostatic Self-Energy

24.1 | WHAT IS PHYSICS?

One goal of physics is to identify basic forces in our world, such as the electric force we discussed in Chapter 22. A related goal is to determine whether a force is conservative—that is, whether a potential energy can be associated with it. The motivation for associating a potential energy with a force is that we can then apply the principle of the conservation of mechanical energy to closed systems involving the force. This extremely powerful principle allows us to calculate the results of experiments for which force calculations alone would be very difficult. Experimentally, physicists and engineers discovered that the electric force is conservative and thus has an associated electric potential energy. In this chapter we first define this type of potential energy and then put it to use.

For a quick taste, let's return to the situation we considered in Chapter 23: In Figure 24-1, particle 1 with positive charge q_1 is located at point P near particle 2 with positive charge q_2. In Chapter 23 we explained how particle 2 is able to push on particle 1 without any contact. To account for the force $\vec{F}$ (which is a vector quantity), we defined an electric field $\vec{E}$ (also a vector quantity) that is set up at P by particle 2. That field exists regardless of whether particle 1 is at P. If we choose to place particle 1 there, the push on it is due to charge q_1 and that pre-existing field $\vec{E}$.

Here is a related problem. If we release particle 1 at P, it begins to move and thus has kinetic energy. Energy cannot appear by magic, so from where does it come? It comes from the electric potential energy U associated with the force between the two particles in the arrangement of Fig. 24-1. To account for the potential energy U (which is a scalar quantity), we define an **electric potential** V (also a scalar quantity) that is set up at P by particle 2. The electric potential exists regardless of whether particle 1 is at P. If we choose to place particle 1 there, the potential energy of the two-particle system is then due to charge q_1 and that pre-existing electric potential V.

Our goals in this chapter are to (1) define electric potential, (2) discuss how to calculate it for various arrangements of charged particles and objects, and (3) discuss how electric potential V is related to electric potential energy U.

Figure 24-1 Particle 1 is located at point P in the electric field of particle 2.

24.2 | ELECTRIC POTENTIAL AND ELECTRIC POTENTIAL ENERGY

Key Concepts

- The electric potential V at a point P in the electric field of a charged object is

$$V = \frac{-W_\infty}{q_0} = \frac{U}{q_0},$$

where W_∞ is the work that would be done by the electric force on a positive test charge q_0 were it brought from an infinite distance to P, and U is the electric potential energy that would then be stored in the test charge–object system.

- If a particle with charge q is placed at a point where the electric potential of a charged object is V, the electric potential energy U of the particle–object system is

$$U = qV.$$

- If the particle moves through a potential difference ΔV, the change in the electric potential energy is

$$\Delta U = q\,\Delta V = q(V_f - V_i).$$

- If a particle moves through a change ΔV in electric potential without an applied force acting on it, applying the conservation of mechanical energy gives the change in kinetic energy as

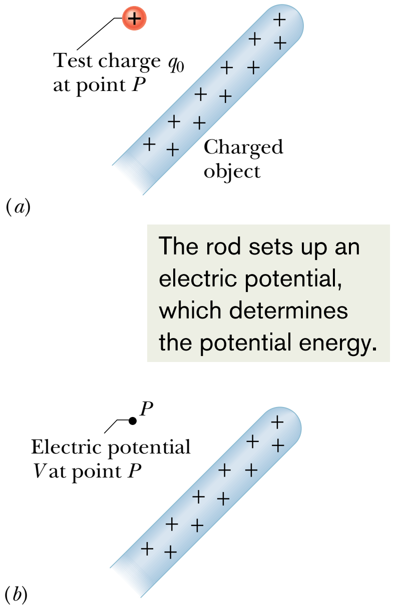
$$\Delta K = -q\,\Delta V.$$

- If, instead, an applied force acts on the particle, doing work W_{app}, the change in kinetic energy is

$$\Delta K = -q\,\Delta V + W_{app}.$$

- In the special case when $\Delta K = 0$, the work of an applied force involves only the motion of the particle through a potential difference:

$$W_{app} = q\,\Delta V.$$


Test charge q_0 at point P
Charged object
(a)
Electric potential V at point P
P
(b)

The rod sets up an electric potential, which determines the potential energy.

Figure 24-2 (*a*) A test charge has been brought in from infinity to point P in the electric field of the rod. (*b*) We define an electric potential V at P based on the potential energy of the configuration in (*a*).

We are going to define the electric potential (or *potential* for short) in terms of electric potential energy, so our first job is to figure out how to measure that potential energy. Back in Chapter 8, we measured gravitational potential energy U of an object by (1) assigning $U = 0$ for a reference configuration (such as the object at table level) and (2) then calculating the work W the gravitational force does if the object is moved up or down from that level. We then defined the potential energy as being

$$U = -W \quad \text{(potential energy).} \tag{24-1}$$

Let's follow the same procedure with our new conservative force, the electric force. In Fig. 24-2*a*, we want to find the potential energy U associated with a positive test charge q_0 located at point P in the electric field of a charged rod. First, we need a reference configuration for which $U = 0$. A reasonable choice is for the test charge to be infinitely far from the rod, because then there is no interaction with the rod. Next, we bring the test charge in from infinity to point P to form the configuration of Fig. 24-2*a*. Along the way, we calculate the work done by the electric force on the test charge. The potential energy of the final configuration is then given by Eq. 24-1, where W is now the work done by the electric force. Let's use the notation W_∞ to emphasize that the test charge is brought in from infinity. The work and thus the potential energy can be positive or negative depending on the sign of the rod's charge.

Next, we define the electric potential V at P in terms of the work done by the electric force and the resulting potential energy:

$$V = \frac{-W_\infty}{q_0} = \frac{U}{q_0}, \quad \text{(electric potential).} \tag{24-2}$$

That is, the *electric potential is the amount of electric potential energy per unit charge when a positive test charge is brought in from infinity*. The rod sets up this potential V at P regardless of whether the test charge (or anything else) happens to be there (Fig. 24-2*b*). From Eq. 24-2 we see that V is a scalar quantity (because there is no direction associated with potential energy or charge) and can be positive or negative (because potential energy and charge have signs).

Repeating this procedure we find that an electric potential is set up at every point in the rod's electric field. In fact, every charged object sets up electric potential V at points throughout its electric field. If we happen to place a particle with, say, charge q at a point where we know the pre-existing V, we can immediately find the potential energy of the configuration:

$$(\text{electric potential energy}) = (\text{particle's charge})\left(\frac{\text{electric potential energy}}{\text{unit charge}}\right),$$

or

$$U = qV, \tag{24-3}$$

where q can be positive or negative.

Two Cautions. (1) The (now very old) decision to call V a *potential* was unfortunate because the term is easily confused with *potential energy*. Yes, the two quantities are related (that is the point here) but they are very different and not interchangeable. (2) Electric potential is a scalar, not a vector.

Language. A potential energy is a property of a system (or configuration) of objects, but sometimes we can get away with assigning it to a single object. For example, the gravitational potential energy of a baseball hit to outfield is actually a potential energy of the baseball–Earth system (because it is associated with the force between the baseball and Earth). However, because only the baseball noticeably moves (its motion does not noticeably affect Earth), we might assign the gravitational potential energy to it alone. In a similar way, if a charged particle is placed in an electric field and has no noticeable effect on the field (or the charged object that sets up the field), we usually assign the electric potential energy to the particle alone.

Units. The SI unit for potential that follows from Eq. 24-2 is the joule per coulomb. This combination occurs so often that a special unit, the *volt* (abbreviated V), is used to represent it. Thus,

$$1 \text{ volt} = 1 \text{ joule per coulomb}.$$

With two unit conversions, we can now switch the unit for electric field from newtons per coulomb to a more conventional unit:

$$\begin{aligned} 1 \text{ N/C} &= \left(1\frac{\text{N}}{\text{C}}\right)\left(\frac{1 \text{ V}}{1 \text{ J/C}}\right)\left(\frac{1 \text{ J}}{1 \text{ N}\cdot\text{m}}\right) \\ &= 1 \text{ V/m}. \end{aligned}$$

The conversion factor in the second set of parentheses comes from our definition of volt given above; that in the third set of parentheses is derived from the definition of the joule. From now on, we shall express values of the electric field in volts per meter rather than in newtons per coulomb.

Motion Through an Electric Field

Change in Electric Potential. If we move from an initial point i to a second point f in the electric field of a charged object, the electric potential changes by

$$\Delta V = V_f - V_i.$$

If we move a particle with charge q from i to f, then, from Eq. 24-3, the potential energy of the system changes by

$$\Delta U = q\,\Delta V = q(V_f - V_i). \tag{24-4}$$

The change can be positive or negative, depending on the signs of q and ΔV. It can also be zero, if there is no change in potential from i to f (the points have the same value of potential). Because the electric force is conservative, the change in potential energy ΔU between i and f is the same for all paths between those points (it is *path independent*).

Work by the Field. We can relate the potential energy change ΔU to the work W done by the electric force as the particle moves from i to f by applying the general relation for a conservative force (Eq. 8-1):

$$W = -\Delta U \quad \text{(work, conservative force)}. \tag{24-5}$$

Next, we can relate that work to the change in the potential by substituting from Eq. 24-4:

$$W = -\Delta U = -q\,\Delta V = -q(V_f - V_i). \tag{24-6}$$

Up until now, we have always attributed work to a force but here can also say that W is the work done on the particle by the electric field (because it, of course, produces the force). The work can be positive, negative, or zero. Because ΔU between any two points is path independent, so is the work W done by the field. (If you need to calculate work for a difficult path, switch to an easier path—you get the same result.)

Suppose you push two bodies together that have charges of the same electrical sign. The work that you must do is stored as electric potential energy in the two-body system (provided the kinetic energy of the bodies does not change). If you later release the charges, you can recover this stored energy, in whole or in part, as kinetic energy of the charged bodies as they rush away from each other.

We define the electric potential energy *of a system of point charges,* held in fixed positions by forces not specified, as follows:

The electric potential energy of a system of fixed-point charges is equal to the work that must be done by an external agent to assemble the system, bringing each charge in from reference configuration.

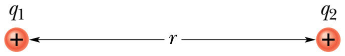


Figure 24-3 Two charges held a fixed distance r apart.

We assume that the charges are stationary both in their initial reference configuration and in their final assembled configuration. Figure 24-3 shows two-point charges q_1 and q_2, separated by a distance r. To find the electric potential energy of this two-charge system, we must mentally build the system, starting with both charges infinitely far away and at rest. When we bring q_1 in from infinity and put it in place, we do no work because no electrostatic force acts on q_1. However, when we next bring q_2 in from infinity and put it in place, we must do work because q_1 exerts an electrostatic force on q_2 during the move.

As before, the work done by the electric field on the charge depends only on their configuration, it does not depend on the path taken in bringing the second charge from infinity to that point. It is emphasized here that the charge number 1 has been fixed and charge 2 has been brought from infinity. It would not make any difference if it is done the other way around. This potential energy is a property of the two charges. It does not belong to a single charge. Electrostatic potential energy is the energy of a system of two or more charges due to their electrostatic interaction. As the force exerted by the charge q_1 is variable, we need to integrate the expression for force.

$$W_{\text{ext}} = \int \vec{F} \cdot d\vec{x}$$

If we bring the charge slowly from infinity, the force required to be exerted is the force required to balance the force due to electric field.

$$F_{\text{ext}} = -qE.$$

By definition, change in potential energy

$$\Delta U = W_{\text{ext}}.$$

So,

$$\Delta U = \int \vec{F} \cdot d\vec{x} = -q\int \vec{E} \cdot d\vec{x}$$

For a two-charge system,

$$\Delta U = -\int \frac{kq_1q_2}{x^2}\, dx.$$

We assume as stated earlier the reference configuration to be one where charges are at infinite separation from each other. Since in any situation, we are only concerned with the change in the potential energy, we could have taken any other point as of zero potential energy and it would not have made a difference to calculations.

$$U_f - U_i = \left.\frac{kq_1q_2}{x}\right|_{\infty}^{r}.$$

Taking U_i to be zero, we get

$$U_f = \frac{kq_1q_2}{r} \tag{24-7}$$

To use this equation, we should remember that charges should be substituted with their sign. But substitute only the magnitude of the distance r between them. Thus, the potential energy is a scalar; however, it may be positive as well as negative depending on the sign of the charges.

What is the use of this new concept? After all we already have an expression for the electric field and from that force can be calculated. From that, we could have analyzed any situation. But actually, to find the new velocity when a charged body moves from one point to another becomes a formidable task in an electric field. In general, the electric field is not constant and so acceleration is not constant. Hence, we cannot apply the equations of motion for finding the velocity, etc.

But if we use the concept of potential energy and that the electric field is a conservative force, we can apply the law of conservation of mechanical energy:

$$U_i + K_i = U_f + K_f. \tag{24-8}$$

Conservation of Energy. If a charged particle moves through an electric field with no force acting on it other than the electric force due to the field, then the mechanical energy is conserved. Let's assume that we can assign the electric potential energy to the particle alone. Then we can write the conservation of mechanical energy of the particle that moves from point i to point f as

$$U_i + K_i = U_f + K_f, \tag{24-9}$$

or

$$\Delta K = -\Delta U. \tag{24-10}$$

Substituting Eq. 24-4, we find a very useful equation for the change in the particle's kinetic energy as a result of the particle moving through a potential difference:

$$\Delta K = -q\,\Delta V = -q(V_f - V_i). \tag{24-11}$$

Work by an Applied Force. If some force in addition to the electric force acts on the particle, we say that the additional force is an *applied force* or *external force*, which is often attributed to an *external agent*. Such an applied force can do work on the particle, but the force may not be conservative and thus, in general, we cannot associate a potential energy with it. We account for that work W_{app} by modifying Eq. 24-9:

(initial energy) + (work by applied force) = (final energy)

or

$$U_i + K_i + W_{\text{app}} = U_f + K_f. \tag{24-12}$$

Rearranging and substituting from Eq. 24-4, we can also write this as

$$\Delta K = -\Delta U + W_{\text{app}} = -q\,\Delta V + W_{\text{app}}. \tag{24-13}$$

The work by the applied force can be positive, negative, or zero, and thus the energy of the system can increase, decrease, or remain the same.

In the special case where the particle is stationary before and after the move, the kinetic energy terms in Eqs. 24-12 and 24-13 are zero and we have

$$W_{\text{app}} = q\,\Delta V \quad (\text{for } K_i = K_f). \tag{24-14}$$

In this special case, the work W_{app} involves the motion of the particle through the potential difference ΔV and not a change in the particle's kinetic energy. By comparing Eqs. 24-6 and 24-14, we see that in this special case, the work by the applied force is the negative of the work by the field:

$$W_{\text{app}} = -W \quad (\text{for } K_i = K_f). \tag{24-15}$$

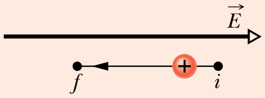
Electron-volts. In atomic and subatomic physics, energy measures in the SI unit of joules often require awkward powers of ten. A more convenient (but non-SI unit) is the e*lectron-volt* (eV), which is defined to be equal to the work required to move a single elementary charge e (such as that of an electron or proton) through a potential difference ΔV of exactly one volt. From Eq. 24-6, we see that the magnitude of this work is $q\,\Delta V$. Thus,

$$\begin{aligned} 1\text{ eV} &= e(1\text{V}) \\ &= (1.602 \times 10^{-19}\text{ C})(1\text{ J/C}) = 1.602 \times 10^{-19}\text{ J}. \end{aligned} \tag{24-16}$$

In essence, you should understand that the term potential is an abstract concept; although it is often used, yet it is only a means to calculate the potential energy which is the central concept. For example, if you touch a wire at 220 V with respect to Earth, you may suffer serious injuries because the amount of charge passing through your body and hence the potential energy associated with it is enormous. But if you touch a Van de Graff generator at even 22 000 V, it may not cause so much damage because the charge involved and hence the potential energy involved may not be so much.

CHECKPOINT 1

In the figure, we move a proton from point *i* to point *f* in a uniform electric field. Is positive or negative work done by (a) the electric field and (b) our force? (c) Does the electric potential energy increase or decrease? (d) Does the proton move to a point of higher or lower electric potential?



CHECKPOINT 2

In the figure of Checkpoint 1, we move the proton from point *i* to point *f* in a uniform electric field directed as shown. (a) Does our force do positive or negative work? (b) Does the proton move to a point of higher or lower potential?

PROBLEM-SOLVING TACTICS

Tactic 1: Electric Potential Energy; Work Done by a Field An electric potential energy is associated with a system of particles as a whole. However, you will see statements (starting with Sample Problem 24.01) that associate it with only one particle within a system. For example, you might read, "An electron in an electric field has a potential energy of 10^{-7} J." Such statements are often acceptable, but you should always keep in mind that the potential energy is actually associated *with a system*—here, the electron plus the charged particles that set up the electric field. Also keep in mind that it makes sense to assign a particular potential energy value, such as 10^{-7} J here, to a particle or even a system *only* if the reference potential energy value is known.

If the potential energy is associated with only one particle within a system, you may read that work is done on the particle *by the electric field*. This means that work is done on the particle by the force due to the charges that set up the field.

Tactic 2: Electric Potential and Electric Potential Energy Electric Potential V and electric potential energy U are quite different quantities and should not be confused.

☞ *Electric potential* is a scalar property associated with an electric field, regardless of whether a charged object has been placed in that field; it is measured in joules per coulomb, or volts.

☞ *Electric potential energy* is an energy of a charged object in an external electric field (or more precisely, an energy of the system consisting of the object and the external electric field); it is measured in joules.

SAMPLE PROBLEM 24.01

Work and potential energy in an electric field

Electrons are continually being knocked out of air molecules in the atmosphere by cosmic-ray particles coming in from space. Once released, each electron experiences an electric force $\vec{F}$ due to the electric field $\vec{E}$ that is produced in the atmosphere by charged particles already on Earth. Near Earth's surface the electric field has the magnitude $E = 150$ N/C and is directed downward. What is the change ΔU in the electric potential energy of a released electron when the electric force causes it to move vertically upward through a distance $d = 520$ m (Fig. 24-4)? Through what potential change does the electron move?

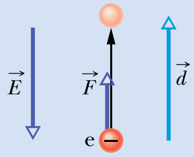


Figure 24-4 An electron in the atmosphere is moved upward through displacement $\vec{d}$ by an electric force $\vec{F}$ due to an electric field $\vec{E}$.

KEY IDEAS

(1) The change ΔU in the electric potential energy of the electron is related to the work W done on the electron by the electric field. Equation 24-5 ($W = -\Delta U$) gives the relation. (2) The work done by a constant force $\vec{F}$ on a particle undergoing a displacement $\vec{d}$ is

$$w = \vec{F} \cdot \vec{d}.$$

(3) The electric force and the electric field are related by the force equation $\vec{F} = q\vec{E}$, where here q is the charge of an electron ($= -1.6 \times 10^{-19}$ C).

Calculations: Substituting the force equation into the work equation and taking the dot product yield

$$W = q\vec{E} \cdot \vec{d} = qEd \cos\theta,$$

where θ is the angle between the directions of $\vec{E}$ and $\vec{d}$. The field $\vec{E}$ is directed downward and the displacement $\vec{d}$ is directed upward; so $\theta = 180°$. We can now evaluate the work as

$$W = (-1.6 \times 10^{-19}\text{ C})(150\text{ N/C})(520\text{ m}) \cos 180°$$
$$= 1.2 \times 10^{-14}\text{ J}.$$

Equation 24-5 then yields

$$\Delta U = -W = -1.2 \times 10^{-14}\text{ J}. \qquad \text{(Answer)}$$

This result tells us that during the 520 m ascent, the electric potential energy of the electron *decreases* by 1.2×10^{-14} J. To find the change in electric potential, we apply Eq. 24-4:

$$\Delta V = \frac{\Delta U}{-q} = \frac{-1.2 \times 10^{-14}\text{ J}}{-1.6 \times 10^{-19}\text{ C}}$$
$$= 7.5 \times 10^{4}\text{ V} = 75\text{ kV}. \qquad \text{(Answer)}$$

This tells us that the electric force does work to move the electron to a *higher* potential.

SAMPLE PROBLEM 24.02

Distance, velocity, closest separation for charged particle projected from ∞ towards fixed particle

A ball of mass of 1 kg and charge 50 μC is directly projected from infinity toward another fixed ball of charge 1/6 μC (as shown in Fig. 24-5). The initial velocity of projection is 10 m/s. Find the distance and its velocity at the closest separation. Assume that the initial line of motion is at a distance of 1 m from the second fixed charge.

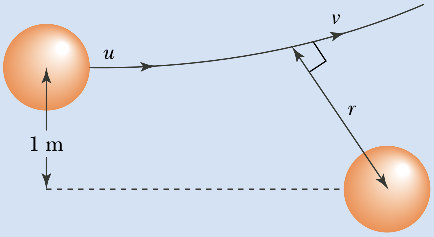


Figure 24-5 A ball of mass of 1 kg and charge 50 μC is projected toward a fixed charge. The initial of motion is at a perpendicular distance of 1 m from fixed charge.

KEY IDEAS

Here, the momentum of the system is not conserved because the second charge is fixed and the nail to which it is tied will exert some force to keep it at rest. But if we take the torque of this force by the nail about the nail, it will be zero $(r_\perp = 0)$.

So, the angular momentum of the system is conserved about the second charge. At the position of closest approach, its component of the velocity toward the second charge will be zero. But the velocity of the first charge will not be zero. Rather, it will take a path as shown in Fig. 24-5 due to repulsion. Its entire velocity will be in a direction perpendicular to the line joining the two charges at this time.

Calculation: $L_i = mux = 1 \times 10 \times 1 = 10\text{ kg m}^2/\text{s}$

$$L_f = mvr = 1 \times v \times r.$$

So, we get

$$v \times r = 10.$$

Also, from the law of conservation of mechanical energy, Solving, we get

Therefore,

$$v = -20 \text{ m/s};\ v = +5 \text{ m/s (which is not possible)}$$

Learn: The charged particle will move with a velocity of 10 m/s after a long time, but it will be deflected from its path. This analysis was used by Rutherford in his famous gold foil experiment to postulate the existence of a massive positively charged nucleus.

24.3 | EQUIPOTENTIAL SURFACES

Key Concept

- The points on an equipotential surface all have the same electric potential. The work done on a test charge in moving it from one such surface to another is independent of the locations of the initial and final points on these surfaces and of the path that joins the points. The electric field $\vec{E}$ is always directed perpendicularly to corresponding equipotential surfaces.

Adjacent points that have the same electric potential form an **equipotential surface,** which can be either an imaginary surface or a real, physical surface. No net work W is done on a charged particle by an electric field when the particle moves between two points i and f on the same equipotential surface. This follows from Eq. 24-6, which tells us that W must be zero if $V_f = V_i$. Because of the path independence of work (and thus of potential energy and potential), $W = 0$ for *any* path connecting points i and f on a given equipotential surface regardless of whether that path lies entirely on that surface.

Figure 24-6 shows a *family* of equipotential surfaces associated with the electric field due to some distribution of charges. The work done by the electric field on a charged particle as the particle moves from one end to the other of paths I and II is zero because each of these paths begins and ends on the same equipotential surface and thus there is no net change in potential. The work done as the charged particle moves from one end to the other of paths III and IV is not zero but has the same value for both these paths because the initial and final potentials are identical for the two paths; that is, paths III and IV connect the same pair of equipotential surfaces.

From symmetry, the equipotential surfaces produced by a charged particle or a spherically symmetrical charge distribution are a family of concentric spheres. For a uniform electric field, the surfaces are a family of planes perpendicular to the field lines. In fact, equipotential surfaces are always perpendicular to electric field lines and thus to $\vec{E}$, which is always tangent to these lines. If $\vec{E}$ were *not* perpendicular to an equipotential surface, it would have a component lying along that surface. This component would then do work on a charged particle as it moved along the surface. However, by Eq. 24-6 work cannot be done if the surface is truly an equipotential surface; the only

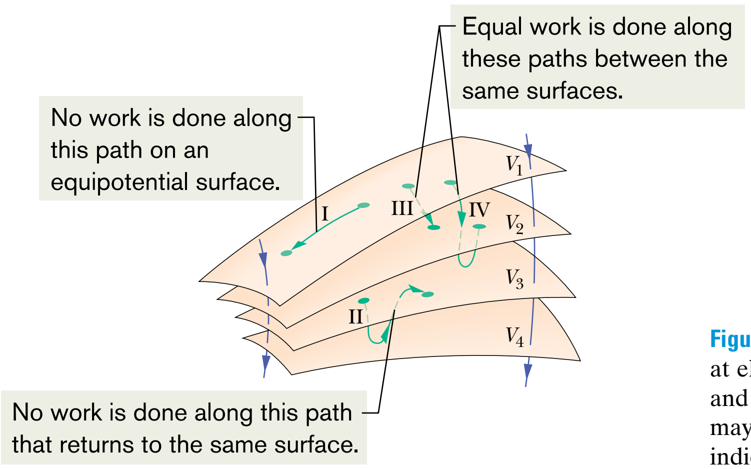

Equal work is done along these paths between the same surfaces.
No work is done along this path on an equipotential surface.
No work is done along this path that returns to the same surface.
I
II
III
IV
V_1
V_2
V_3
V_4

Figure 24-6 Portions of four equipotential surfaces at electric potentials $V_1 = 100$ V, $V_2 = 80$ V, $V_2 = 60$ V, and $V_4 = 40$ V. Four paths along which a test charge may move are shown. Two electric field lines are also indicated.

possible conclusion is that $\vec{E}$ must be everywhere perpendicular to the surface. Figure 24-7 shows electric field lines and cross sections of the equipotential surfaces for a uniform electric field and for the field associated with a charged particle and with an electric dipole.

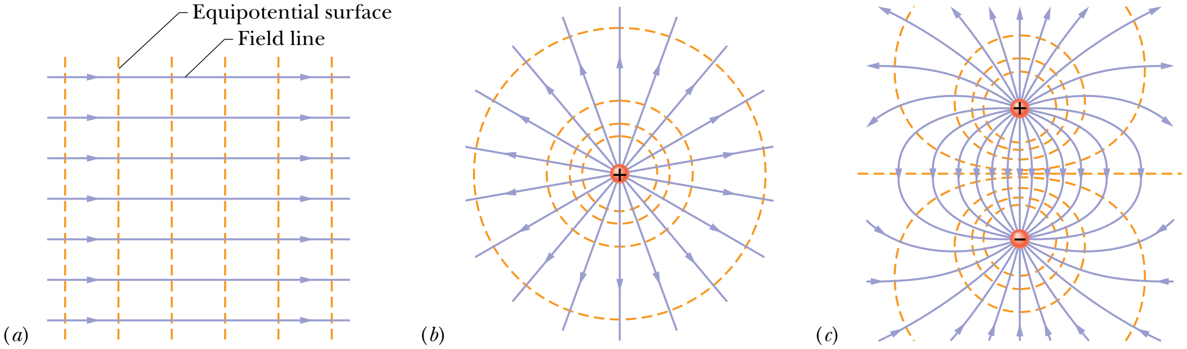


Figure 24-7 Electric field lines (purple) and cross sections of equipotential surfaces (gold) for (*a*) a uniform electric field, (*b*) the field due to a charged particle, and (*c*) the field due to an electric dipole.

SAMPLE PROBLEM 24.03

Locating equipotential surfaces

An infinite nonconducting sheet has a surface charge density of $\sigma = 0.10\ \mu\text{C/m}^2$ on one side. How far apart are equipotential surfaces whose potentials differ by 50 V?

KEY IDEA

The electric field produced by an infinite sheet of charge is normal to the sheet and is uniform.

The magnitude of the electric field produced by the infinite sheet of charge is $E = \sigma/2\varepsilon_0$, where σ is the surface charge density. Place the origin of a coordinate system at the sheet and take the x axis to be parallel to the field and positive in the direction of the field. Then the electric potential is

$$V = V_s - \int_0^x E\,dx = V_s - Ex,$$

where V_s is the potential at the sheet. The equipotential surfaces are surfaces of constant x; that is, they are planes that are parallel to the plane of charge. If two surfaces are separated by Δx then their potentials differ in magnitude by

$$\Delta V = E\Delta x = (\sigma/2\varepsilon_0)\Delta x.$$

Calculation: Thus, for $\sigma = 0.10\times10^{-6}\ \text{C/m}^2$ and $\Delta V = 50\ V$, we have

$$\Delta x = \frac{2\varepsilon_0\Delta V}{\sigma} = \frac{2\left(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2\right)(50\ \text{V})}{0.10\times10^{-6}\ \text{C/m}^2} = 8.8\times10^{-3}\,\text{m}.$$

Learn: Equipotential surfaces are always perpendicular to the electric field lines. Figure 24-7(*a*) depicts the electric field lines and equipotential surfaces for a uniform electric field.

24.4 | CALCULATING THE POTENTIAL FROM THE FIELD

Key Concepts

- The electric potential difference between two points i and f is

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s},$$

where the integral is taken over any path connecting the points. If the integration is difficult along any particular path, we can choose a different path along which the integration might be easier.

- If we choose $V_i = 0$, we have, for the potential at a particular point,

$$V = -\int_i^f \vec{E}\cdot d\vec{s}.$$

- In a uniform field of magnitude E, the change in potential from a higher equipotential surface to a lower one, separated by distance Δx, is

$$\Delta V = -E\Delta x.$$

We can calculate the potential difference between any two points i and f in an electric field if we know the electric field vector $\vec{E}$ all along any path connecting those points. To make the calculation, we find the work done on a positive test charge by the field as the charge moves from i to f, and then use Eq. 24-6.

Consider an arbitrary electric field, represented by the field lines in Fig. 24-8, and a positive test charge q_0 that moves along the path shown from point i to point f. At any point on the path, an electric force $q_0\vec{E}$ acts on the charge as it moves through a differential displacement $d\vec{s}$. We know that the differential work dW done on a particle by a force $\vec{F}$ during a displacement $d\vec{s}$ is given by the dot product of the force and the displacement:

$$dW = \vec{F}\cdot d\vec{s}. \tag{24-17}$$

For the situation of Fig. 24-8, $\vec{F} = q_0\vec{E}$ and Eq. 24-15 becomes

$$dW = q_0\vec{E}\cdot d\vec{s}. \tag{24-18}$$

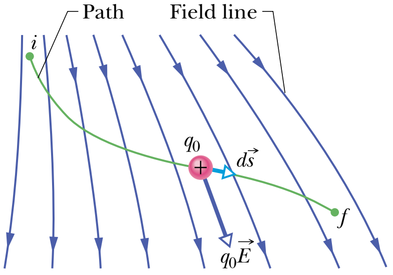


Figure 24-8 A test charge q_0 moves from point i to point f along the path shown in a nonuniform electric field. During a displacement $d\vec{s}$, an electric force $q_0\vec{E}$ acts on the test charge. This force points in the direction of the field line at the location of the test charge.

To find the total work W done on the particle by the field as the particle moves from point i to point f, we sum—via integration—the differential works done on the charge as it moves through all the displacements $d\vec{s}$ along the path:

$$W = q_0\int_i^f \vec{E}\cdot d\vec{s}. \tag{24-19}$$

If we substitute the total work W from Eq. 24-19 into Eq. 24-6, we find

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}. \tag{24-20}$$

Thus, the potential difference $V_f - V_i$ between any two points i and f in an electric field is equal to the negative of the *line integral* (meaning the integral along a particular path) of $\vec{E}\cdot d\vec{s}$ from i to f. However, because the electric force is conservative, all paths (whether easy or difficult to use) yield the same result.

Equation 24-20 allows us to calculate the difference in potential between any two points in the field. If we set potential $V_i = 0$, then Eq. 24-20 becomes

$$V = -\int_i^f \vec{E}\cdot d\vec{s}, \tag{24-21}$$

in which we have dropped the subscript f on V_f. Equation 24-21 gives us the potential V at any point f in the electric field *relative to the zero potential* at point i. If we let point i be at infinity, then Eq. 24-21 gives us the potential V at any point f relative to the zero potential at infinity.

Uniform Field. Let's apply Eq. 24-20 for a uniform field as shown in Fig. 24-9. We start at point i on an equipotential line with potential V_i and move to point f on an equipotential line with a lower potential V_f. The separation between the two equipotential lines is Δx. Let's also move along a path that is parallel to the electric field $\vec{E}$ (and thus perpendicular to the equipotential lines). The angle between $\vec{E}$ and $d\vec{s}$ in Eq. 24-20 is zero, and the dot product gives us

$$\vec{E}\cdot d\vec{s} = E\ ds\cos 0 = E\ ds.$$

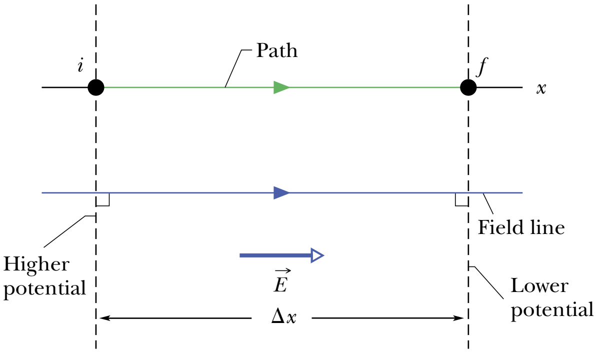


Figure 24-9 We move between points i and f, between adjacent equipotential lines in a uniform electric field $\vec{E}$, parallel to a field line.

Because E is constant for a uniform field, Eq. 24-20 becomes

$$V_f - V_i = -E\int_i^f ds. \tag{24-22}$$

The integral is simply an instruction for us to add all the displacement elements ds from i to f, but we already know that the sum is length Δx. Thus we can write the change in potential $V_f - V_i$ in this uniform field as

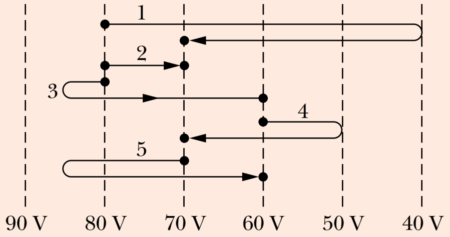
$$\Delta V = -E\,\Delta x \quad \text{(uniform field)} \tag{24-23}$$

This is the change in voltage ΔV between two equipotential lines in a uniform field of magnitude E, separated by distance Δx. If we move in the direction of the field by distance Δx, the potential decreases. In the opposite direction, it increases.

The electric field vector points from higher potential toward lower potential.

CHECKPOINT 3

The figure here shows a family of parallel equipotential surfaces (in cross section) and five paths along which we shall move an electron from one surface to another. (a) What is the direction of the electric field associated with the surfaces? (b) For each path, is the work we do positive, negative, or zero? (c) Rank the paths according to the work we do, greatest first.



SAMPLE PROBLEM 24.04

Finding the potential change from the electric field

(a) Figure 24-10*a* shows two points i and f in a uniform electric field $\vec{E}$. The points lie on the same electric field line (not shown) and are separated by a distance d. Find the potential difference $V_f - V_i$ by moving a positive test charge q_0 from i to f along the path shown, which is parallel to the field direction.

KEY IDEA

We can find the potential difference between any two points in an electric field by integrating $\vec{E}\cdot d\vec{s}$ along a path connecting those two points according to Eq. 24-20.

Calculations: We have actually already done the calculation for such a path in the direction of an electric field line in a uniform field when we derived Eq. 24-23. With slight changes in notation, Eq. 24-23 gives us

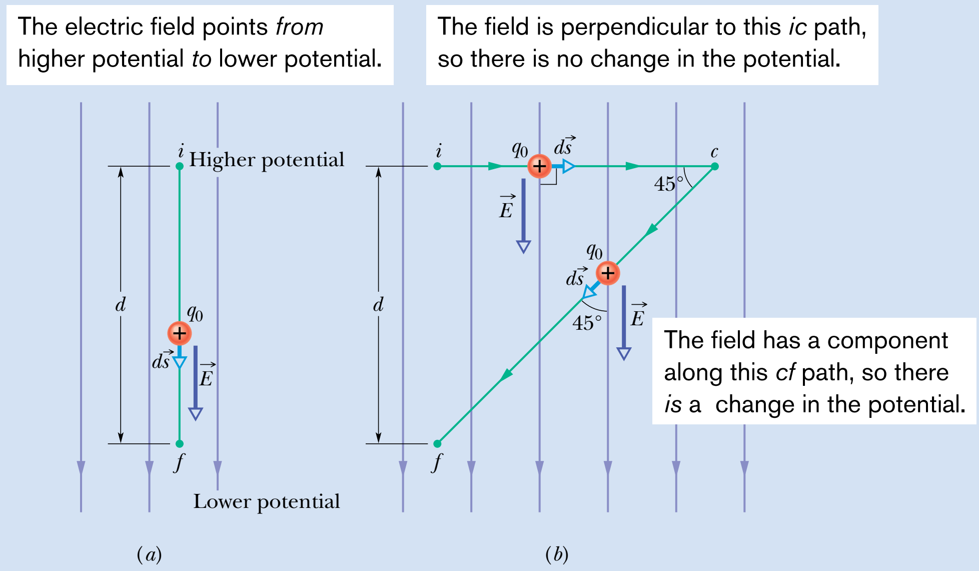
$$V_f - V_i = -Ed. \quad \text{(Answer)}$$

(b) Now find the potential difference $V_f - V_i$ by moving the positive test charge q_0 from i to f along the path icf shown in Fig. 24-10*b*.

Calculations: The Key Idea of (a) applies here too, except now we move the test charge along a path that consists of two lines: ic and cf. At all points along line ic, the displacement $d\vec{s}$ of the test charge is perpendicular to $\vec{E}$. Thus, the angle θ between $\vec{E}$ and $d\vec{s}$ is 90°, and the dot product $\vec{E}\cdot d\vec{s}$ is 0. Equation 24-20 then tells us that points i and c are at the same potential: $V_c - V_i = 0$. Ah, we should have seen this coming. The points are on the same equipotential surface, which is perpendicular to the electric field lines.

For line cf we have $\theta = 45°$ and, from Eq 24-20,

$$V_f - V_i = -\int_c^f \vec{E}\cdot d\vec{s} = -\int_c^f E(\cos 45°)\,ds$$
$$= -E(\cos 45°)\int_c^f ds.$$



The electric field points *from* higher potential *to* lower potential.

The field is perpendicular to this *ic* path, so there is no change in the potential.

The field has a component along this *cf* path, so there *is* a change in the potential.

Figure 24-10 (*a*) A test charge q_0 moves in a straight line from point *i* to point *f*, along the direction of a uniform external electric field. (*b*) Charge q_0 moves along path *icf* in the same electric field.

The integral in this equation is just the length of line *cf*; from Fig. 24-10*b*, that length is *d*/cos 45°. Thus,

$$V_f - V_i = -E(\cos 45°)\frac{d}{\cos 45°} = -Ed. \qquad \text{(Answer)}$$

This is the same result we obtained in (a), as it must be; the potential difference between two points does not depend on the path connecting them. Moral: When you want to find the potential difference between two points by moving a test charge between them, you can save time and work by choosing a path that simplifies the use of Eq. 24-20.

SAMPLE PROBLEM 24.05

Finding electric potential

A graph of the x component of the electric field as a function of x in a region of space is shown in Fig. 24-11. The scale of the vertical axis is set by $E_{xs} = 20.0$ N/C. The y and z components of the electric field are zero in this region. If the electric potential at the origin is 10 V, then:

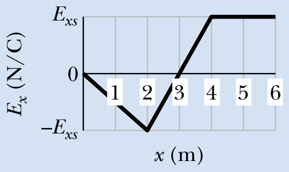


Figure 24-11 A graph of the x component of electric field as a function of x.

(a) What is the electric potential at x = 2.0 m?

KEY IDEA

By Eq. 24-20, the change in potential is the negative of the “area” under the curve. Thus, using the area-of-a-triangle formula, we have

$$V - 10 = -\int_0^{x=2} \vec{E}\cdot d\vec{s} = \frac{1}{2}(2)(20)$$

which yields $V = 30$ V.

(b) What is the greatest positive value of the electric potential for points on the x axis for which $0 \le x \le 6.0$ m?

KEY IDEA

For any region within $0 < x < 3\,\text{m}$, $-\int \vec{E}\cdot d\vec{s}$ is positive, but for any region for which

$x > 3$ m it is negative. Therefore, $V = V_{\text{max}}$ occurs at $x = 3$ m.

$$V - 10 = -\int_0^{x=3} \vec{E}\cdot d\vec{s} = \frac{1}{2}(3)(20)$$

which yields $V_{\text{max}} = 40$ V.

(c) For what value of x is the electric potential zero?

KEY IDEA

In view of our result in part (b), we see that now (to find $V = 0$) we are looking for some $X > 3$ m such that the "area" from $x = 3$ m to $x = X$ is 40 V. Using the formula for a triangle $(3 < x < 4)$ and a rectangle $(4 < x < X)$, we require

$$\frac{1}{2}(1)(20) + (X - 4)(20) = 40.$$

Therefore, $X = 5.5$ m.

SAMPLE PROBLEM 24.06

Finding the potential

Figure 24-12 is a graph of E_x, the x component of the electric field, versus position along the x axis. Find and graph $V(x)$. Assume $V = 0$; V at $x = 0$ m.

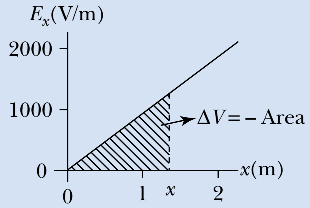


Figure 24-12 Graph of E_x vs. x.

KEY IDEA

The potential difference is the negative of the area under the curve. Ex is positive throughout this region of space, meaning that $\vec{E}$ points in the positive x direction.

Calculation: If we integrate from $x - 0$, then $V_i = V(x - 0) - 0$. The potential for $x > 0$ is the negative of the triangular area under the E_x curve. We can see that $E_x = 1000x$ V/m, where x is in meters (m). Thus,

$$V_i = V(x) = 0 - (\text{area under the } E_x \text{ curve})$$
$$= \frac{1}{2} \times \text{base} \times \text{height} = -\frac{1}{2}(x)(1000x) = -500x^2 \text{ V}.$$

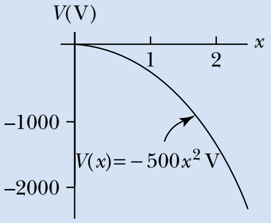


Figure 24-13 Graph V vs. x.

Figure 24-13 shows that the electric potential in this region of space is parabolic, decreasing from 0 V at $x = 0$ m to -2000 V at $x = 2$ m.

Thus, the electric field points in the direction in which V is decreasing.

24.5 | POTENTIAL DUE TO A POINT CHARGE

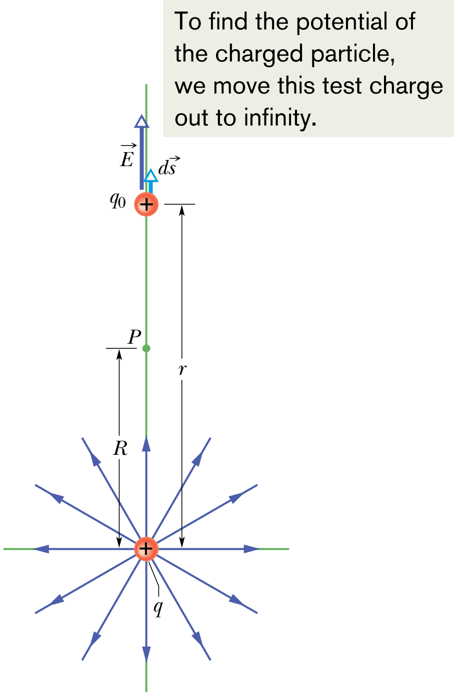
Key Concept

- The electric potential due to a single charged particle at a distance r from that charged particle is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r},$$

where V has the same sign as q.

We now use Eq. 24-20 to derive, for the space around a charged particle, an expression for the electric potential V relative to the zero potential at infinity. Consider a point P at distance R from a fixed particle of positive charge q



To find the potential of the charged particle, we move this test charge out to infinity.

Figure 24-14 The particle with positive charge q produces an electric field $\vec{E}$ and an electric potential V at point P. We find the potential by moving a test charge q_0 from P to infinity. The test charge is shown at distance r from the particle, during differential displacement $d\vec{s}$.

(Fig. 24-14). To use Eq. 24-20, we imagine that we move a positive test charge q_0 from point P to infinity. Because the path we take does not matter, let us choose the simplest one—a line that extends radially from the fixed particle through P to infinity.

To use Eq. 24-20, we must evaluate the dot product

$$\vec{E}\cdot d\vec{s} = E\ \cos\ \theta\ ds. \tag{24-24}$$

The electric field $\vec{E}$ in Fig. 24-14 is directed radially outward from the fixed particle. Thus, the differential displacement $d\vec{s}$ of the test particle along its path has the same direction as $\vec{E}$. That means that in Eq. 24-24, angle $\theta = 0$ and cos $\theta = 1$. Because the path is radial, let us write ds as dr. Then, substituting the limits R and ∞, we can write Eq. 24-20 as

$$V_f - V_i = -\int_R^\infty E\ dr. \tag{24-25}$$

Next, we set $V_f = 0$ (at ∞) and $V_i = V$ (at R). Then, for the magnitude of the electric field at the site of the test charge, we substitute from Eq. 22-3:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \tag{24-26}$$

With these changes, Eq. 24-25 then gives us

$$\begin{aligned} 0 - V &= -\frac{q}{4\pi\varepsilon_0}\int_R^\infty \frac{1}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_R^\infty \\ &= -\frac{1}{4\pi\varepsilon_0}\frac{q}{R}. \end{aligned} \tag{24-27}$$

Solving for V and switching R to r, we then have

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \tag{24-28}$$

as the electric potential V due to a particle of charge q at any radial distance r from the particle.

Although we have derived Eq. 24-28 for a positively charged particle, the derivation holds also for a negatively charged particle, in which case, q is a negative quantity. Note that the sign of V is the same as the sign of q:

> A positively charged particle produces a positive electric potential. A negatively charged particle produces a negative electric potential.

Figure 24-15 shows a computer-generated plot of Eq. 24-28 for a positively charged particle; the magnitude of V is plotted vertically. Note that the magnitude increases as $r \to 0$. In fact, according to Eq. 24-28, V is infinite at $r = 0$, although Fig. 24-14 shows a finite, smoothed-off value there.

Equation 24-28 also gives the electric potential either *outside or on the external surface of* a spherically symmetric charge distribution.

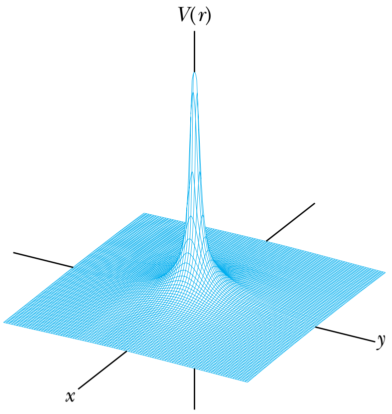


Figure 24-15 A computer-generated plot of the electric potential $V(r)$ due to a positively charged particle located at the origin of an xy plane. The potentials at points in the xy plane are plotted vertically. (Curved lines have been added to help you visualize the plot.) The infinite value of V predicted by Eq. 24-28 for $r = 0$ is not plotted.

PROBLEM-SOLVING TACTICS

Tactic 3: Finding a Potential Difference To find the potential difference ΔV between any two points in the field of an isolated point charge, evaluate Eq. 24-28 at each point and then subtract one result from the other. The value of ΔV will be the same for any choice of reference potential energy because that choice is eliminated by the subtraction.

24.6 | POTENTIAL DUE TO A GROUP OF POINT CHARGES

Key Concept

◆ The potential due to a collection of charged particles is

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}.$$

Thus, the potential is the algebraic sum of the individual potentials, with no consideration of directions.

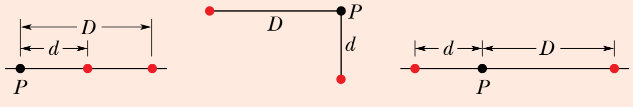
We can find the net electric potential at a point due to a group of charged particles with the help of the superposition principle. Using Eq. 24-28 with the plus or minus sign of the charge included, we calculate separately the potential resulting from each charge at the given point. Then we sum the potentials. Thus, for n charges, the net potential is

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} \quad (n \text{ charged particles}). \tag{24-29}$$

Here q_i is the value of the ith charge and r_i is the radial distance of the given point from the ith charge. The sum in Eq. 24-29 is an *algebraic sum,* not a vector sum like the sum that would be used to calculate the electric field resulting from a group of charged particles. Herein lies an important computational advantage of potential over electric field: It is a lot easier to sum several scalar quantities than to sum several vector quantities whose directions and components must be considered.

CHECKPOINT 4

The figure here shows three arrangements of two protons. Rank the arrangements according to the net electric potential produced at point P by the protons, greatest first.



(a) (b) (c)

SAMPLE PROBLEM 24.07

Net potential of several charged particles

What is the electric potential at point P, located at the center of the square of charged particles shown in Fig. 24-16*a*? The distance d is 1.3 m, and the charges are

$$q_1 = +12\text{ nC},\qquad q_3 = +31\text{ nC},$$
$$q_2 = -24\text{ nC},\qquad q_4 = +17\text{ nC}.$$

KEY IDEA

The electric potential V at point P is the algebraic sum of the electric potentials contributed by the four particles.

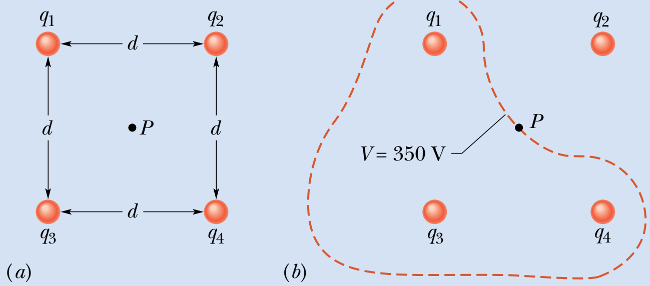


Figure 24-16 (*a*) Four charged particles. (*b*) The closed curve is a (roughly drawn) cross section of the equipotential surface that contains point P.

(Because electric potential is a scalar, the orientations of the particles do not matter.)

Calculations: From Eq. 24-29, we have

$$V = \sum_{i=1}^{4} V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r} + \frac{q_2}{r} + \frac{q_3}{r} + \frac{q_4}{r}\right).$$

The distance r is $d/\sqrt{2}$, which is 0.919 m, and the sum of the charges is

$$\begin{aligned} q_1 + q_2 + q_3 + q_4 &= (12 - 24 + 31 + 17) \times 10^{-9}\text{ C} \\ &= 36 \times 10^{-9}\text{ C}. \end{aligned}$$

Thus, $$V = \frac{(8.99 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(36 \times 10^{-9}\text{ C})}{0.919\text{ m}}$$

$$\approx 350\text{V}. \qquad \text{(Answer)}$$

Close to any of the three positively charged particles in Fig. 24-16*a*, the potential has very large positive values. Close to the single negative charge, the potential has very large negative values. Therefore, there must be points within the square that have the same intermediate potential as that at point P. The curve in Fig. 24-16*b* shows the intersection of the plane of the figure with the equipotential surface that contains point P.

SAMPLE PROBLEM 24.08

Potential is not a vector, orientation is irrelevant

(a) In Fig. 24-17*a*, 12 electrons (of charge $-e$) are equally spaced and fixed around a circle of radius R. Relative to $V = 0$ at infinity, what are the electric potential and electric field at the center C of the circle due to these electrons?

KEY IDEAS

(1) The electric potential V at C is the algebraic sum of the electric potentials contributed by all the electrons. Because electric potential is a scalar, the orientations of the electrons do not matter. (2) The electric field at C is a vector quantity and thus the orientation of the electrons *is* important.

Calculations: Because the electrons all have the same negative charge $-e$ and are all the same distance R from C, Eq. 24-29 gives us

$$V = -12\frac{1}{4\pi\varepsilon_0}\frac{e}{R}. \qquad \text{(Answer)} \qquad (24\text{-}30)$$

Because of the symmetry of the arrangement in Fig. 24-17*a*, the electric field vector at C due to any given electron is canceled by the field vector due to the electron that is diametrically opposite it. Thus, at C,

$$\vec{E} = 0. \qquad \text{(Answer)}$$

Potential is a scalar and orientation is irrelevant.

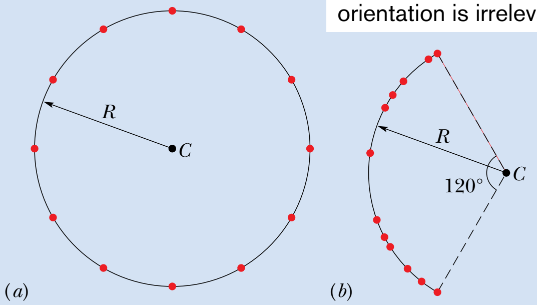


Figure 24-17 (*a*) Twelve electrons uniformly spaced around a circle. (*b*) The electrons nonuniformly spaced along an arc of the original circle.

(b) The electrons are moved along the circle until they are nonuniformly spaced over a 120° arc (Fig. 24-17*b*). At *C*, find the electric potential and describe the electric field.

Reasoning: The potential is still given by Eq. 24-30, because the distance between *C* and each electron is unchanged and orientation is irrelevant. The electric field is no longer zero, however, because the arrangement is no longer symmetric. A net field is now directed toward the charge distribution.

SAMPLE PROBLEM 24.09

Net potential due to two charged particles

Two particles, of charges q_1 and q_2, are separated by distance d in Fig. 24-18 with $V = 0$ at infinity.

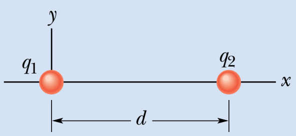


Figure 24-18 Two charged particles separated by distance *d*.

(a) If the particles with charges $q_1 = +5e$ and $q_2 = -15e$ are fixed in place with a separation of d = 24.0 cm, what are the finite (i) positive and (ii) negative values of x at which the net electric potential on the x axis is zero?

KEY IDEA

First, we observe that $V(x)$ cannot be equal to zero for $x > d$. In fact $V(x)$ is always negative for $x > d$. Now we consider the two remaining regions on the x axis: $x < 0$ and $0 < x < d$.

(i) For $0 < x < d$ we have $d_1 = x$ and $d_2 = d - x$. Let

$$V(x) = k\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x} + \frac{-3}{d-x}\right) = 0$$

and solve: $x = d/4$. With $d = 24.0$ cm, we have $x = 6.00$ cm.

(ii) Similarly, for $x < 0$ the separation between q_1 and a point on the x axis whose coordinate is x is given by $d_1 = -x$; while the corresponding separation for q_2 is $d_2 = d - x$. We set

$$V(x) = k\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{-x} + \frac{-3}{d-x}\right) = 0$$

to obtain $x = -d/2$. With $d = 24.0$ cm, we have $x = -12.0$ cm.

(b) If the net electric field due to the particles is zero at $x = d/4$, then locate (in terms of d) any point on the x axis (other than at infinity) at which the electric potential due to the two particles is zero?

KEY IDEA

Since according to the problem statement there is a point in between the two charges on the x axis where the net electric field is zero, the fields at that point due to q_1 and q_2 must be directed opposite to each other. This means that q_1 and q_2 must have the same sign (i.e., either both are positive or both negative). Thus, the potentials due to either of them must be of the same sign. Therefore, the net electric potential cannot possibly be zero anywhere except at infinity.

24.7 | POTENTIAL DUE TO A CONTINUOUS CHARGE DISTRIBUTION

Key Concepts

- For a continuous distribution of charge (over an extended object), the potential is found by (1) dividing the distribution into charge elements dq that can be treated as particles and then (2) summing the potential due to each element by integrating over the full distribution:

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}.$$

- In order to carry out the integration, dq is replaced with the product of either a linear charge density λ and a length element (such as dx), or a surface charge density σ and area element (such as $dx\,dy$).
- In some cases where the charge is symmetrically distributed, a two-dimensional integration can be reduced to a onedimensional integration.

When a charge distribution q is continuous (as on a uniformly charged thin rod or disk), we cannot use the summation of Eq. 24-29 to find the potential V at a point P. Instead, we must choose a differential element of charge dq, determine the potential dV at P due to dq, and then integrate over the entire charge distribution.

Let us again take the zero of potential to be at infinity. If we treat the element of charge dq as a particle, then we can use Eq. 24-28 to express the potential dV at point P due to dq:

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} \quad \text{(positive or negative } dq\text{).} \tag{24-31}$$

Here r is the distance between P and dq. To find the total potential V at P, we integrate to sum the potentials due to all the charge elements:

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}. \tag{24-32}$$

The integral must be taken over the entire charge distribution. Note that because the electric potential is a scalar, there are *no vector components* to consider in Eq. 24-32.

We now examine two continuous charge distributions, a line and a disk.

Line of Charge

In Fig. 24-19*a*, a thin nonconducting rod of length L has a positive charge of uniform linear density λ. Let us determine the electric potential V due to the rod at point P, a perpendicular distance d from the left end of the rod.

We consider a differential element dx of the rod as shown in Fig. 24-19*b*. This (or any other) element of the rod has a differential charge of

$$dq = \lambda\, dx. \tag{24-33}$$

This element produces an electric potential dV at point P, which is a distance $r = (x^2 + d^2)^{1/2}$ from the element (Fig. 24-19*c*). Treating the element as a point charge, we can use Eq. 24-31 to write the potential dV as

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{(x^2+d^2)^{1/2}}. \tag{24-34}$$

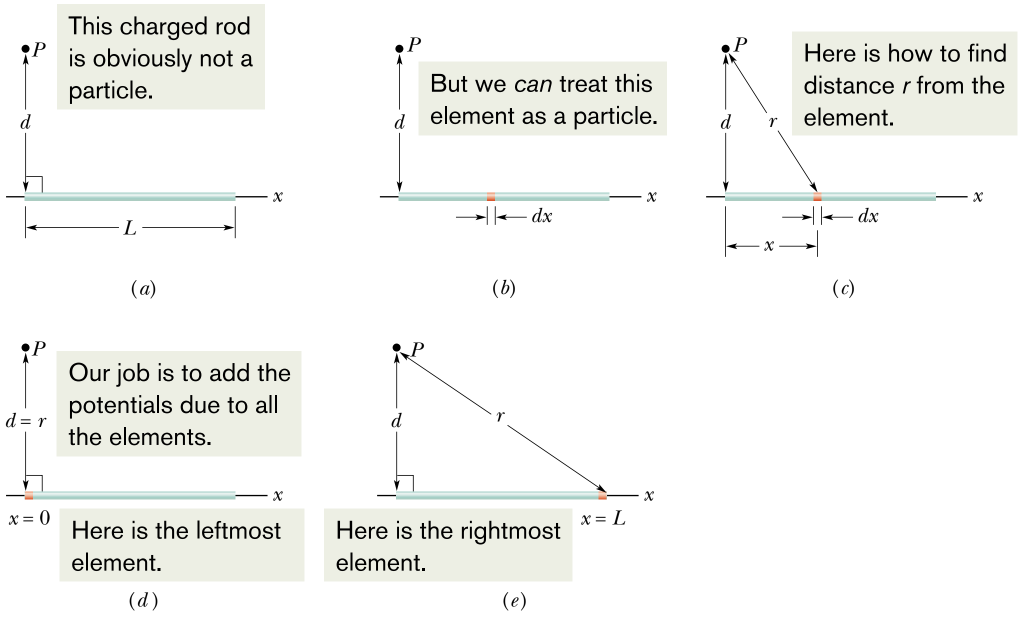

P
This charged rod is obviously not a particle.
d
L
x
(a)
P
But we *can* treat this element as a particle.
d
dx
x
(b)
P
Here is how to find distance *r* from the element.
d
r
dx
x
(c)
P
Our job is to add the potentials due to all the elements.
d = r
x = 0
x
Here is the leftmost element.
(d)
P
d
r
x = L
x
Here is the rightmost element.
(e)

Figure 24-19 (*a*) A thin, uniformly charged rod produces an electric potential V at point P. (*b*) An element can be treated as a particle. (*c*) The potential at P due to the element depends on the distance r. We need to sum the potentials due to all the elements, from the left side (*d*) to the right side (*e*).

Since the charge on the rod is positive and we have taken $V = 0$ at infinity, we know from Section 24.5 that dV in Eq. 24-34 must be positive.

We now find the total potential V produced by the rod at point P by integrating Eq. 24-34 along the length of the rod, from $x = 0$ to $x = L$ (Figs. 24-19*d* and *e*), using integral 17 in Appendix E. We find

$$\begin{aligned} V = \int dV &= \int_0^L \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{(x^2+d^2)^{1/2}}\,dx \\ &= \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{dx}{(x^2+d^2)^{1/2}} \\ &= \frac{\lambda}{4\pi\varepsilon_0}\Big[\ln\Big(x+(x^2+d^2)^{1/2}\Big)\Big]_0^L \\ &= \frac{\lambda}{4\pi\varepsilon_0}\Big[\ln\Big(L+(L^2+d^2)^{1/2}\Big)-\ln d\Big]. \end{aligned}$$

We can simplify this result by using the general relation $\ln A - \ln B = \ln(A/B)$. We then find

$$V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{L+\left(L^2+d^2\right)^{1/2}}{d}\right]. \tag{24-35}$$

Because V is the sum of positive values of dV, it too is positive, consistent with the logarithm being positive for an argument greater than 1.

SAMPLE PROBLEM 24.10

Potential due to line of charge

Figure 24-20 shows a thin plastic rod of length $L = 12.0$ cm and uniform positive charge $Q = 56.1$ fC lying on the x axis. With $V = 0$ at infinity, find the electric potential at point P_1 on the axis, at distance $d = 2.50$ cm of the rod.

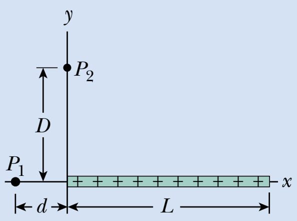


Figure 24-20 A thin plastic rod with uniform positive charge Q lying on the x axis.

KEY IDEA

Consider an infinitesimal segment of the rod, located between x and $x + dx$. It has length dx and contains charge $dq = \lambda\, dx$, where $\lambda = Q/L$ is the linear charge density of the rod. Its distance from P_1 is $d + x$ and the potential it creates at P_1 is

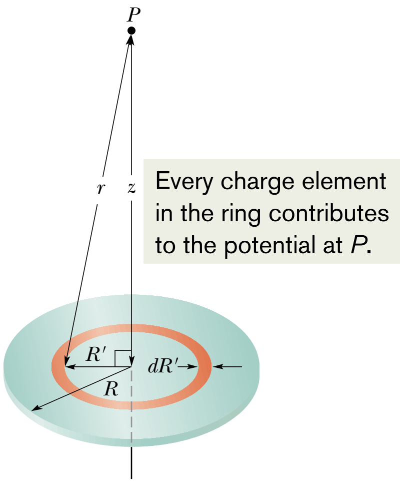
$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{d+x} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{d+x}.$$

Finding potential To find the total potential at P_1, we integrate over the length of the rod and obtain:

$$\begin{aligned} V &= \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{dx}{d+x} = \frac{\lambda}{4\pi\varepsilon_0}\ln(d+x)\Big|_0^L = \frac{Q}{4\pi\varepsilon_0 L}\ln\left(1+\frac{L}{d}\right) \\ &= \frac{(8.99\times10^9\ \mathrm{N\cdot m^2/C^2})(56.1\times10^{-15}\,\mathrm{C})}{0.12\ \mathrm{m}}\ln\left(1+\frac{0.12\ \mathrm{m}}{0.025\ \mathrm{m}}\right) \\ &= 7.39\times10^{-3}\ \mathrm{V}. \end{aligned}$$

Charged Disk

In Section 22.11, we calculated the magnitude of the electric field at points on the central axis of a plastic disk of radius R that has a uniform charge density σ on one surface. Here we derive an expression for $V(z)$, the electric potential at any point on the central axis. Because we have a circular distribution of charge on the disk, we could start with a differential element that occupies angle $d\theta$ and radial distance dr. We would then need to set up a two-dimensional integration. However, let's do something easier.



Every charge element in the ring contributes to the potential at *P*.

Figure 24-21 A plastic disk of radius R, charged on its top surface to a uniform surface charge density σ. We wish to find the potential V at point P on the central axis of the disk.

In Fig. 24-21, consider a differential element consisting of a flat ring of radius R' and radial width dR'. Its charge has magnitude

$$dq = \sigma(2\pi R')(dR'),$$

in which $(2\pi R')(dR')$ is the upper surface area of the ring. All parts of this charged element are the same distance r from point P on the disk's axis. With the aid of Fig. 24-21, we can use Eq. 24-31 to write the contribution of this ring to the electric potential at P as

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi R')(dR')}{\sqrt{z^2+R'^2}}. \tag{24-36}$$

We find the net potential at P by adding (via integration) the contributions of all the rings from $R' = 0$ to $R' = R$:

$$V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{R'\,dR'}{\sqrt{z^2+R'^2}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-z\right). \tag{24-37}$$

Note that the variable in the second integral of Eq. 24-37 is R' and not z, which remains constant while the integration over the surface of the disk is carried out. (Note also that, in evaluating the integral, we have assumed that $z \geq 0$.)

SAMPLE PROBLEM 24.11

Finding potential of charged disk

A plastic disk of radius $R = 64.0$ cm is charged on one side with a uniform surface charge density $\sigma = 7.73$ fC/m^2, and then three quadrants of the disk are removed. The remaining quadrant is shown in Fig. 24-22. With $V = 0$ at infinity, what is the potential due to the remaining quadrant at point P, which is on the central axis of the original disk at distance D = 25.9 cm from the original center?

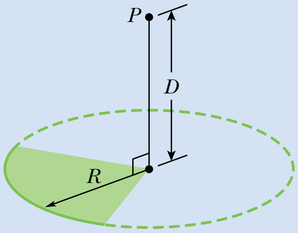


Figure 24-22 A plastic disk of radius R charged on one side with a uniform surface charge density σ.

KEY IDEA

Since the disk is uniformly charged, when the full disk is present each quadrant contributes equally to the electric potential at P.

Expressing potential: Electrical potential is a scalar quantity. The potential at P due to a single quadrant is one-fourth the potential due to the entire disk. We first find an expression for the potential at P due to the entire disk. To do so, consider a ring of charge with radius r and (infinitesimal) width dr. Its area is $2\pi r{\cdot}dr$ and it contains charge $dq = 2\pi\sigma r dr$. All the charge in it is at a distance $\sqrt{r^2+D^2}$ from P, so the potential it produces at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\sigma r dr}{\sqrt{r^2+D^2}} = \frac{\sigma r dr}{2\varepsilon_0\sqrt{r^2+D^2}}.$$

Calculation: Integrating over r, the total potential at P is

$$V = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r dr}{\sqrt{r^2+D^2}}$$

$$= \frac{\sigma}{2\varepsilon_0}\sqrt{r^2+D^2}\,\Big|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+D^2}-D\right].$$

Therefore, the potential V_{sq} at P due to a single quadrant is

$$V_{sq}=\frac{V}{4}=\frac{\sigma}{8\varepsilon_0}\left[\sqrt{R^2+D^2}-D\right]$$

$$=\frac{\left(7.73\times10^{-15}\,\mathrm{C/m^2}\right)}{8\left(8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}\right)}\left[\sqrt{(0.640\ \mathrm{m})^2+(0.259\ \mathrm{m})^2}-0.259\ \mathrm{m}\right]$$

$$=4.71\times10^{-5}\ \mathrm{V}.$$

Learn: Consider the limit $D\gg R$. The potential becomes

$$V_{sq}=\frac{\sigma}{8\varepsilon_0}\left[\sqrt{R^2+D^2}-D\right]\approx\frac{\sigma}{8\varepsilon_0}\left[D\left(1+\frac{1}{2}\frac{R^2}{D^2}+\cdots\right)-D\right]$$

$$=\frac{\sigma}{8\varepsilon_0}\frac{R^2}{2D}=\frac{\pi R^2\sigma/4}{4\pi\varepsilon_0 D}=\frac{q_{sq}}{4\pi\varepsilon_0 D}$$

where $q_{sq}=\pi R^2\sigma/4$ is the charge on the quadrant. In this limit, we see that the potential resembles that due to a point charge q_{sq}.

SAMPLE PROBLEM 24.12

Potential due to infinite sheet of charge

Find the potential due to an infinite sheet of charge at a distance x from it. Assume that the potential is zero at a distance a from it.

KEY IDEA

Although the infinite sheet can be treated as the special case of a disk, yet there are points that need be discussed. Let us do this by the formula we derived (Eq. 23-19), because the electric field of an infinite disk is known and is simple to handle.

Here we can use the formula we found before

$$\Delta V=-\int\vec{E}\cdot d\vec{x}=-\int_a^x\frac{\sigma}{2\varepsilon_0}\cdot dx.$$

Here, we have assumed the potential to be zero at a distance of a from the sheet.

$$V_x=\frac{\sigma(a-x)}{2\varepsilon_0}.$$

We can note here that we cannot take the potential to be zero at infinity. If we do so, the potential will be infinite. This is because the electric field is also extending to infinity.

We can also note that a is just a arbitrary number we chose according to our convenience. It can have any fixed value and still it will not affect our calculations because in any situation, we want to find the change in potential energy. For that, we need to find only the change in potential. It will be the same for every value of constant a.

Potential Due to Spherical Shell

The potential due to a spherical shell can be determined by finding out the potential at different points in space by dividing the shell into rings and then integrating. However, the integration process is lengthy and a different approach can be used. The potential at different points in space can be found by integrating the expression for the electric field. We will first find the potential outside the shell because the reference point happens to be outside the shell (at infinity).

$$V_{\text{out}}=-\int_\infty^r\vec{E}\cdot d\vec{l}=-\int_\infty^r\frac{q}{4\pi\varepsilon_0 r^2}dr=\frac{q}{4\pi\varepsilon_0 r}. \tag{24-38}$$

Although we are integrating form infinity to r, it is observed that $d\vec{l}$ is taken in outward direction by convention (Fig. 24-23). So, angle between it and electric field is zero.

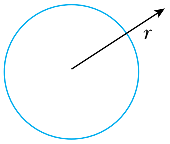


Figure 24-23 Finding the potential due to uniformly charged spherical shell at (a) point outside the shell and (b) at a point inside the shell.

Inside the shell, electric field is zero. This does not mean that potential is zero too. Recall your integration lessons. If the electric field is zero inside, the potential will be constant. But how to figure out the value of the constant! Now we do not have the flexibility of assigning any potential to the inside points. That we can do only with the reference point and that is already done. There is a way out. We know the potential at the surface of the shell which happens to be common to both the regions.

$$\int_{V_R}^{V_{\text{in}}} dV = -\int_R^r \vec{E}\cdot d\vec{l} = -\int_R^r 0\ dr = 0.$$

$$V_{\text{in}} = V_R = \frac{q}{4\pi\varepsilon_0 R}. \tag{24-39}$$

The potential due to a uniform spherical shell is plotted in Fig. 24-24.

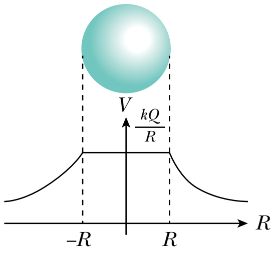


Figure 24-24 A graph showing variation of potential due to a uniformly charged spherical shell at different points on its diameter extended. Note the discontinuity on the slope of V at R, this would mean that electric field is defined at all the points, but is discontinuous at the surface.

Potential Due to Uniformly Charged Sphere

We will now determine the potential due to a uniformly charged sphere of radius R and charge per unit volume ρ at different points in space. We have learnt that on application of Gauss' law to spherical surface, Eq. 23-17 gave the electric field due to a sphere.

For a point outside the sphere, we can integrate the electric field outside to obtain an expression for the electric potential outside (Fig. 24-25).

$$V = -\int_\infty^r \vec{E}\cdot d\vec{l} = -\int_\infty^r \frac{\rho R^3}{3\varepsilon_0 x^2}dx = \frac{\rho R^3}{3\varepsilon_0 r}.$$

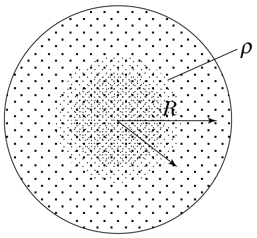


Figure 24-25 The dots represent a spherically symmetric distribution of charge of radius R, whose volume charge density ρ is a constant.

For finding the potential at an inside point, we integrate as before. Keep in mind that the lower limit in this integration cannot be infinity as infinity does not lie inside the sphere. But lower limit can be assumed to be R where potential is not zero but a known value.

$$\int_{V_R}^{V_{\text{in}}} dV = -\int_R^r \vec{E}\cdot d\vec{l} = -\int_R^r \frac{\rho x}{3\varepsilon_0}dx = \frac{\rho(R^2-r^2)}{6\varepsilon_0}$$

$$V_R = \frac{\rho R^2}{3\varepsilon_0} \Rightarrow V_r = \frac{\rho(3R^2-r^2)}{6\varepsilon_0}.$$

This potential is plotted in Fig. 24-26.

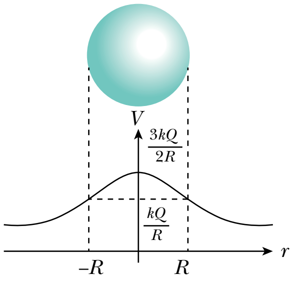


Figure 24-26 The graph showing variation of potential due to a uniformly charged sphere at points on its diameter extended.

It is important to remember that this method can be applied only when we know the electric field at all the points along the line that we are going to integrate. For example, we cannot find the electric potential due to a circular are at its center by this formula. But if we divide the arc in small elements and integrate, we can easily determine the potential at the center of the arc as

$$V = \frac{kq}{R}.$$

SAMPLE PROBLEM 24.13

Potential at the surface of the outer shell

Consider two concentric shells shown in Fig. 24-27, one of charge q and radius R and another outer shell of charge $-q$ and radius $2R$. Find the potential at the surface of the outer shell and at the center.

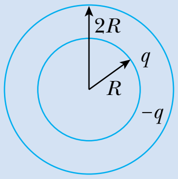


Figure 24-27 Two concentric spherical shells.

KEY IDEA

Remember that both the shells will contribute to the potential at any point. Center is a point which is inside both the shells. It can be said that net potential at any point is

$$V = V_1 + V_2,$$

where V_1 and V_2 are potential due to inner and outer shells, respectively.

Determining potential: The potential at the center due to the shell is given by Eq. 24-39.

$$V_1 = \frac{kq}{R},\quad V_2 = \frac{-kq}{2R}.$$

Therefore,

$$V = \frac{kq}{R} + \frac{-kq}{2R} = \frac{kq}{2R}.$$

The surface of the outer shell is a point lying outside both the shells. So, at that point the potential will not be constant but will be given by the equation

$$V = \frac{kq}{2R} + \frac{-kq}{2R} = 0.$$

Thus, we find that the potential everywhere out of the system of shells is zero. This can be explained on the basis of Gauss' law, the net charge in any Gaussian surface drawn outside the shells is zero. So, the electric field outside is zero. According to Eq. 24-19, the potential outside the shell is constant. But at an outside point, that is infinitely, the electric potential is already defined to be zero. So, the potential at all the points outside must be zero.

SAMPLE PROBLEM 24.14

Field and potential inside a charged spherical shell

In Fig. 24-28, a thin conducting spherical shell of radius $R = 2.00$ cm has a uniformly spread surface charge of $q = -5.00 \times 10^{-15}$ C. What are the electric potential and magnitude of the electric field on (a) the surface and at (b) $r = 2.00\,R$ and (c) $r = 0.500\,R$?

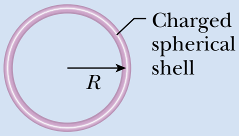


Figure 24-28 Thin conducting spherical shell of radius R.

KEY IDEA

For points on or outside a uniform spherical distribution of charge, the potential and field are the same as if the charge were concentrated as a particle at the center of the actual distribution. Thus, the potential is

$$V = k\frac{q}{r}. \tag{24-40}$$

where r is the radial distance and $k = 1/4\pi\varepsilon_0$, and the field has the magnitude

$$E = k\frac{|q|}{r^2}. \tag{24-41}$$

From Gauss' law, we know that in the empty hollow of the shell, the electric field is zero because an interior Gaussian sphere would enclose no charge and thus have no electric flux through it. A change in potential between two points is equal to the negative of the integration of the electric field along a path between those points:

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}, \tag{24-42}$$

On the surface: From Eqs. 24-40 and 24-41, the potential and field magnitude at $r = R$ are

$$E = (8.99\times10^9\ \mathrm{N\cdot m^2/C^2})\frac{|5.00\times10^{-15}\mathrm{C}|}{(0.0200\ \mathrm{m})^2}$$

$$= 0.112\ \mathrm{V/m} \qquad \text{(Answer)}$$

$$V = (8.99\times10^9\ \mathrm{N\cdot m^2/C^2})\frac{5.00\times10^{-15}\mathrm{C}}{0.0200\ \mathrm{m}}$$

$$= 2.25\times10^{-3}\ \mathrm{V} = 2.25\ \mathrm{mV} \qquad \text{(Answer)}$$

Outside the surface: From those same two equations, with $r = 2.00\ R = 0.0400$ m, we have

$$E = (8.99\times10^9\ \mathrm{N\cdot m^2/C^2})\frac{|5.00\times10^{-15}\mathrm{C}|}{(0.0400\ \mathrm{m})^2}$$

$$= 2.81\times10^{-2}\ \mathrm{V/m} \qquad \text{(Answer)}$$

$$V = (8.99\times10^9\mathrm{N\cdot m^2/C^2})\frac{5.00\times10^{-15}\mathrm{C}}{0.0400\ \mathrm{m}}$$

$$= 1.12\times10^{-3}\ \mathrm{V} = 1.12\ \mathrm{mV} \qquad \text{(Answer)}$$

Inside the empty shell: Because the hollow contains no charge, Gauss' law tells us that for any point inside the hollow

$$E = 0. \qquad \text{(Answer)}$$

To find the potential in the hollow, we use Eq. 24-42, integrating from any interior point (call it point *i*) to the surface (call it point *f*). Substituting $E = 0$, Eq. 24-42 simply boils down to

$$V_f - V_i = -\int_i^f 0\ d\vec{s}$$

$$2.25\times10^{-3}\ \mathrm{V} - V_i = 0$$

$$V_i = 2.25\times10^{-3}\ \mathrm{V} = 2.25\ \mathrm{mV} \qquad \text{(Answer)}$$

In words, the interior potential is the same as the surface potential because there is no internal electric field inside the empty hollow.

SAMPLE PROBLEM 24.15

Point charge projected from a large distance towards center of fixed charged ring

Assume that a point charge q is projected from a large distance toward the center of a fixed charged ring along its axis. What should be its initial velocity so that the charge may reach the other side of infinity?

KEY IDEA

Here, we can say that the charge will be repelled till it reaches the center of the ring. After that, it will experience repulsive force again, but in the opposite direction. So, if we are able to give the charge sufficient velocity so that it may reach the center of the ring, then it will cross the ring and reach the other side without any extra velocity.

We will use the law of conservation of energy to solve this problem.

Calculation: Assuming the velocity to be zero at the center of the ring,

$$U_i + K_i = U_f + K_f,$$

$$U = qV = \frac{kqQ}{\sqrt{a^2+x^2}},$$

$$0 + \frac{1}{2}mv^2 = \frac{kqQ}{\sqrt{a^2+0^2}} + 0$$

$$v = \sqrt{\frac{2kqQ}{ma}}.$$

Learn: In practice, the initial velocity should be slightly greater than this velocity so that the charged particle may cross the center.

PROBLEM-SOLVING TACTICS

Tactic 4: Signs of Trouble with Electric Potential When you calculate the potential V at some point P due to a line of charge or any other continuous charge configuration, the signs can cause you trouble. Here is a generic guide to sort out the signs.

If the charge is negative, should the symbols dq and λ represent negative quantities, or should you explicitly show the signs, using $-dq$ and $-\lambda$? You can do either as long as you remember what your notation means, so that when you get to the final step, you can correctly interpret the sign of V.

Another approach, which can be used when the entire charge distribution is of a single sign, is to let the symbols dq and λ represent magnitudes only. The result of the calculation will give you the magnitude of V at P. Then add a sign to V based on the sign of the charge. (If the zero potential is at infinity, positive charge gives a positive potential and negative charge gives a negative potential.)

If you happen to reverse the limits on the integral used to calculate a potential, you will obtain a negative value for V. The magnitude will be correct, but discard the minus sign. Then determine the proper sign for V from the sign of the charge. As an example, we would have obtained a minus sign in Eq. 24-35 if we had reversed the limits in the integral above that equation. We would then have discarded that minus sign and noted that the potential is positive because the charge producing it is positive.

24.8 | CALCULATING THE FIELD FROM THE POTENTIAL

Key Concepts

- The component $\vec{E}$ of in any direction is the negative of the rate at which the potential changes with distance in that direction:

$$E_s = -\frac{\partial V}{\partial s}.$$

- The x, y, and z components of $\vec{E}$ may be found from

$$E_x = -\frac{\partial V}{\partial x};\quad E_y = -\frac{\partial V}{\partial y};\quad E_z = -\frac{\partial V}{\partial z}.$$

When $\vec{E}$ is uniform, all this reduces to

$$E = -\frac{\Delta V}{\Delta s},$$

where s is perpendicular to the equipotential surfaces.

- The electric field is zero parallel to an equipotential surface.

In Section 24.4, you saw how to find the potential at a point f if you know the electric field along a path from a reference point to point f. In this module, we propose to go the other way—that is, to find the electric field when we know the potential. As Fig. 24-7 shows, solving this problem graphically is easy: If we know the potential V at all points near an assembly of charges, we can draw in a family of equipotential surfaces. The electric field lines, sketched perpendicular to those surfaces, reveal the variation of $\vec{E}$. What we are seeking here is the mathematical equivalent of this graphical procedure.

Figure 24-29 shows cross sections of a family of closely spaced equipotential surfaces, the potential difference between each pair of adjacent surfaces being dV. As the figure suggests, the field $\vec{E}$ at any point P is perpendicular to the equipotential surface through P.

Suppose that a positive test charge q_0 moves through a displacement $d\vec{s}$ from one equipotential surface to the adjacent surface. From Eq. 24-6, we see that the work the electric field does on the test charge during the move is $-q_0\,dV$. From Eq. 24-18 and Fig. 24-29, we see that the work done by the electric field may also be written as the scalar product $(q_0\vec{E})\cdot d\vec{s}$, or $q_0E(\cos\theta)\,ds$. Equating these two expressions for the work yields

$$-q_0 dV = q_0 E(\cos\theta)\,ds, \tag{24-43}$$

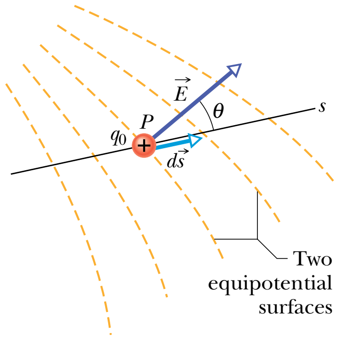


Figure 24-29 A test charge q_0 moves a distance $d\vec{s}$ from one equipotential surface to another. (The separation between the surfaces has been exaggerated for clarity.) The displacement $d\vec{s}$ makes an angle θ with the direction of the electric field $\vec{E}$.

or
$$E\cos\,\theta = -\frac{dV}{ds}. \tag{24-44}$$

Since E cos θ is the component of $\vec{E}$ in the direction of $d\vec{s}$, Eq. 24-44 becomes

$$E_s = -\frac{\partial V}{\partial s}. \tag{24-45}$$

We have added a subscript to E and switched to the partial derivative symbols to emphasize that Eq. 24-43 involves only the variation of V along a specified axis (here called the s axis) and only the component of $\vec{E}$ along that axis. In words, Eq. 24-45 (which is essentially the reverse operation of Eq. 24-20) states:

> The component of $\vec{E}$ in any direction is the negative of the rate at which the electric potential changes with distance in that direction.

If we take the s axis to be, in turn, the x, y, and z axes, we find that the x, y, and z components of $\vec{E}$ at any point are

$$E_x = -\frac{\partial V}{\partial x};\qquad E_y = -\frac{\partial V}{\partial y};\qquad E_z = -\frac{\partial V}{\partial z}. \tag{24-46}$$

Thus, if we know V for all points in the region around a charge distribution—that is, if we know the function $V(x, y, z)$—we can find the components of $\vec{E}$, and thus $\vec{E}$ itself, at any point by taking partial derivatives.

For the simple situation in which the electric field $\vec{E}$ is uniform, Eq. 24-45 becomes

$$E = -\frac{\Delta V}{\Delta s}, \tag{24-47}$$

where s is perpendicular to the equipotential surfaces. The component of the electric field is zero in any direction parallel to the equipotential surfaces because there is no change in potential along the surfaces.

The electric field will always point from high potential to low potential consider the following equation.

$$E_x = -\frac{dV}{dx}$$

If potential increases in x direction, dV/dx will be positive and electric field will be negative. This implies that electric field points in negative derection from higher potential to lower potential. The electric field is away from the sheet of charge, so from the equation we can say that potential decreases as x increases.

A very important conclusion can be drawn from this. The electric field will always point from high potential to low potential. We will show how in the following equation:

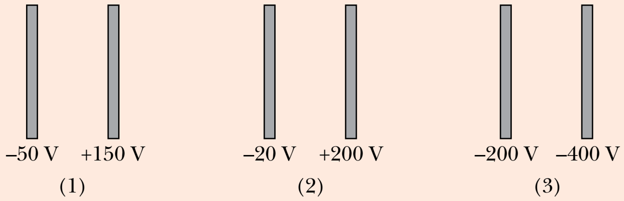
$$E_x = -\frac{dV}{dx}$$

If potential increases in x direction, dV/dx will be positive and the electric field will be negative. In other words, electric field points in negative direction from higher potential to lower potential.

The electric field is away from the sheet of charge. But from the expression for the potential, we can say that the potential decreases as x increases. So, the electric field points from higher potential to lower potential.

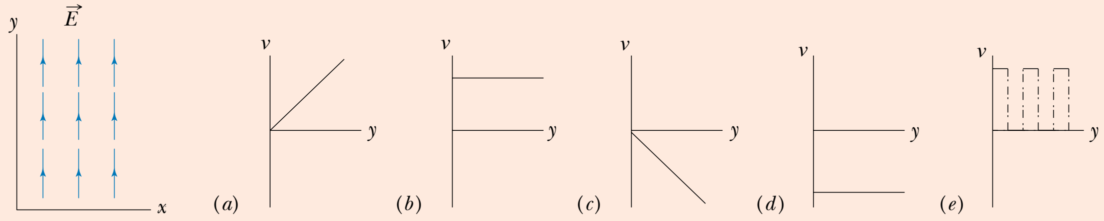
CHECKPOINT 5

The figure shows three pairs of parallel plates with the same separation, and the electric potential of each plate. The electric field between the plates is uniform and perpendicular to the plates. (a) Rank the pairs according to the magnitude of the electric field between the plates, greatest first. (b) For which pair is the electric field pointing rightward? (c) If an electron is released midway between the third pair of plates, does it remain there, move rightward at constant speed, move leftward at constant speed, accelerate rightward, or accelerate leftward?



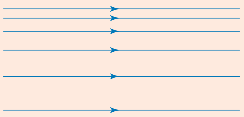
CHECKPOINT 6

Which potential graph describes the electric field shown here?



CHECKPOINT 7

Is the electrostatic field pattern shown in the figure possible? (*Hint*: Assume a rectangular loop in the field and find the work done by the electric field on a test charge in moving along that loop. Did you arrive at a contradiction?)



SAMPLE PROBLEM 24.16

Finding the field from the potential

The electric potential at any point on the central axis of a uniformly charged disk is given by Eq. 24-37,

$$V = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-z\right).$$

Starting with this expression, derive an expression for the electric field at any point on the axis of the disk.

KEY IDEA

We want the electric field $\vec{E}$ as a function of distance z along the axis of the disk. For any value of z, the direction of $\vec{E}$ must be along that axis because the disk has circular symmetry about that axis. Thus, we want the component E_z of $\vec{E}$ in the direction of z. This component is the negative of the rate at which the electric potential changes with distance z.

Calculation: Thus, from the last of Eq. 24-46, we can write

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\frac{d}{dz}\left(\sqrt{z^2+R^2}-z\right)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right). \qquad \text{(Answer)}$$

This is the same expression that we derived in Section 22.9 by integration, using Coulomb's law.

SAMPLE PROBLEM 24.17

Acceleration of charge of given potential

The potential at a point (x, y) is expressed as $V = 4x + 3y$. A charge 1 μC and mass 1 mg is released from (1, 1) from rest. (a) Find the acceleration of the charge.

KEY IDEA

The potential at a given point can be differentiated to obtain the electric field. This electric field can then be used to find the acceleration of the charge.

Calculation:

$$E_x = -\frac{\partial V}{\partial x} \quad \text{and} \quad E_y = -\frac{\partial V}{\partial y}.$$

$$\vec{E} = -4\hat{i} - 3\hat{j}$$

$$\vec{a} = \frac{q\vec{E}}{m} = \frac{(-4\hat{i} - 3\hat{j}) \times 10^{-6}}{10^{-6}} = -4\hat{i} - 3\hat{j}$$

(b) Find the time when it crosses x axis.

KEY IDEA

Since the acceleration is constant, we can apply the equations of motion.

Calculation: According to the equation of motion:

$$S = ut + \frac{1}{2}at^2.$$

When the charge crosses x axis, its y coordinate is 0. So, displacement in y direction is $0 - 1 = -1$ m. Hence,

$$-1 = \frac{3}{2}t^2 \Rightarrow t = \sqrt{\frac{2}{3}}\ \text{s}.$$

Learn: The charged particle would move in a straight line.

24.9 | POTENTIAL DUE TO AN ELECTRIC DIPOLE

Key Concept

- At a distance r from an electric dipole with dipole moment magnitude $p = qd$, the electric potential of the dipole is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

  for $r \geq d$; the angle θ lies between the dipole moment vector and a line extending from the dipole midpoint to the point of measurement.

Before trying to understand the electric field of a dipole, we need to define some new mathematics which will come handy here.

Polar Coordinates

The polar coordinate (r, q) of a point P are defined in Fig. 24-30; r is the distance from the origin (the magnitude of the position vector), (the angle down from the z axis) is called the **polar angle**. The relation to Cartesian coordinates (x, y, z) can be read from Fig. 24-30 as

$$x = r\sin, z = r\cos\theta \tag{24-48}$$

Figure 24-30 also shows two-unit vectors $\hat{\text{r}}$ and $\hat{\theta}$ pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{\text{i}}$ and $\hat{\text{j}}$, and any vector A can be expressed in terms of those in the usual way):

$$A = A_r\hat{\text{r}} + A_\theta\hat{\theta} \tag{24-49}$$

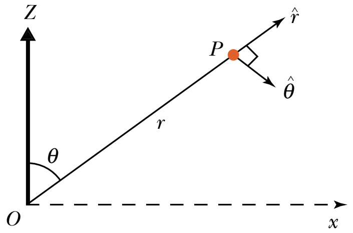


Figure 24-30 Polar coordinates of a general point in xz plane. For simplicity, we ignore the y axis.

A_r and A_θ are the radial and polar components of A.

However, there is something very important one must be aware of: $\hat{\text{r}}$ and $\hat{\theta}$ are associated with a particular point P, and they change direction as P moves around. For example, $\hat{\text{r}}$ always points radially outward, but "radially outward" can be the x direction, the y direction, or any other direction, depending on where you are. In Fig. 24-31, $\vec{A} = \hat{y}$ and $\vec{B} = -\hat{y}$, and yet both of them would be written as $\hat{r}$ in polar coordinates. In particular, do not combine the spherical components of vectors associated with different points (in Fig. 24-31, $\vec{A} + \vec{B} = 0$, not $2\hat{r}$, and $A \cdot B = -1$, not $+1$). Beware of differentiating a vector that is expressed in polar coordinates, as the unit vectors themselves are functions of position $\partial\hat{r}/\partial\theta = \hat{\theta}$, for example). And do not take $\hat{r}$ and $\hat{\theta}$ outside an integral, as we did with $\hat{i}$ and $\hat{j}$.

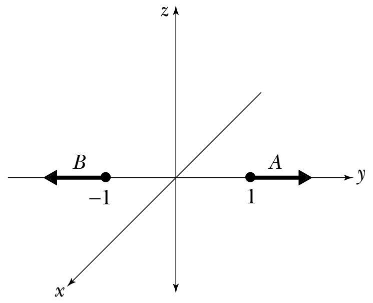


Figure 24-31 Vector A and B in polar coordinates.

An infinitesimal displacement in the $\hat{r}$ direction is simply dr (Fig. 24-32a), just as an infinitesimal element of length in the x direction is dx:

$$dl_r = dr. \tag{24-50}$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (Fig. 24-32b) is not just $d\theta$ (that's an angle—it does not even have the right units for a length), but rather $r\,d\theta$.

$$dl_\theta = r\theta \tag{24-51}$$

Thus, the general infinitesimal displacement dl is

$$dl = dr\hat{r} + rd\theta\hat{\theta}. \tag{24-52}$$

This plays the role (in line integrals, for example) that $dl = dx\hat{i} + dy\hat{j}$ played in Cartesian coordinates.

Now let us apply Eq. 24-29 to an electric dipole to find the potential at an arbitrary point P in Fig. 24-33a. At P, the positively charged particle (at distance $r_{(+)}$) sets up potential $V_{(+)}$ and the negatively charged particle (at distance $r_{(-)}$) sets up potential $V_{(-)}$. Then the net potential at P is given by Eq. 24-29 as

$$V = \sum_{i=1}^{2} V_i = V_{(+)} + V_{(-)} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_{(+)}} + \frac{-q}{r_{(-)}}\right) = \frac{q}{4\pi\varepsilon_0}\frac{r_{(-)} - r_{(+)}}{r_{(-)}r_{(+)}}. \tag{24-53}$$

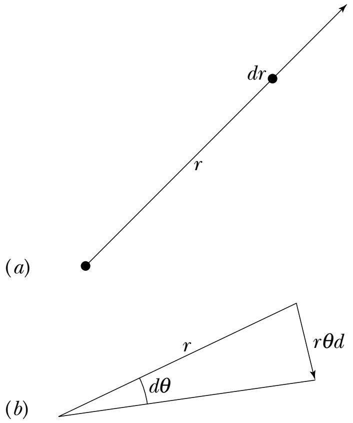


Figure 24-32 (a) dr represents a change in polar coordinate r; (b) $d\theta$ represents a change in polar coordinate θ.

Naturally occurring dipoles—such as those possessed by many molecules—are quite small; so we are usually interested only in points that are relatively far from the dipole, such that $r \geq d$, where d is the distance between the charges and r is the distance from the dipole's midpoint to P. In that case, we can approximate the two lines to P as

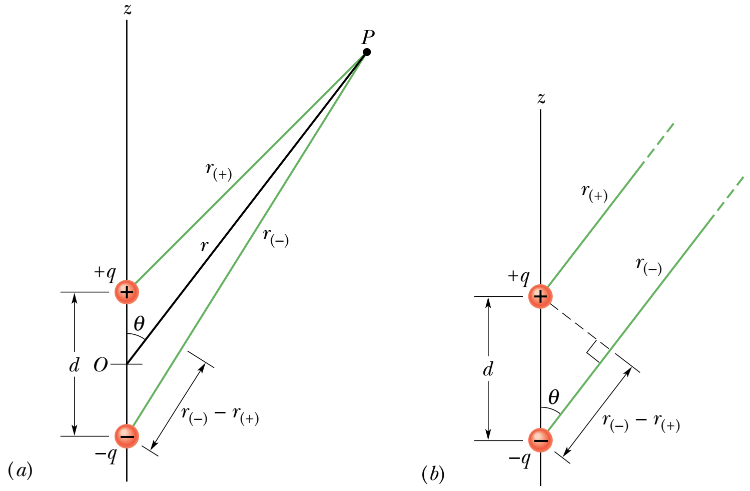


Figure 24-33 (a) Point P is a distance r from the midpoint O of a dipole. The line OP makes an angle θ with the dipole axis. (b) If P is far from the dipole, the lines of lengths $r_{(+)}$ and $r_{(-)}$ are approximately parallel to the line of length r, and the dashed black line is approximately perpendicular to the line of length $r_{(-)}$.

being parallel and their length difference as being the leg of a right triangle with hypotenuse d (Fig. 24-33b). Also, that difference is so small that the product of the lengths is approximately r^2. Thus,

$$r_{(-)} - r_{(+)} \approx d\cos\theta \quad \text{and} \quad r_{(-)}r_{(+)} \approx r^2.$$

If we substitute these quantities into Eq. 24-53, we can approximate V to be

$$V = \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2},$$

where θ is measured from the dipole axis as shown in Fig. 24-33a. We can now write V as

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} \quad \text{(electric dipole)}, \tag{24-54}$$

in which p (= qd) is the magnitude of the electric dipole moment $\vec{p}$ defined in Section 22.9. The vector $\vec{p}$ is directed along the dipole axis, from the negative to the positive charge. (Thus, θ is measured from the direction of $\vec{p}$.) We use this vector to report the orientation of an electric dipole.

If the location of the points and dipole is known in terms of their x, y, z coordinates, it is caseis to use the formula of potential in vector form as follows:

$$V = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \tag{24-55}$$

CHECKPOINT 8

Suppose that three points are set at equal (large) distances r from the center of the dipole in Fig. 24-33: Point a is on the dipole axis above the positive charge, point b is on the axis below the negative charge, and point c is on a perpendicular bisector through the line connecting the two charges. Rank the points according to the electric potential of the dipole there, greatest (most positive) first.

Electric Field Due to a Dipole

We have learnt in Section 24.4 that the electric field can be obtained if we know the potential $V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}$. However, the relationship is derived in terms of x and y coordinates, whereas in case of electric dipole, we have the expressionial in terms of polar coordinates, r and θ. Figure 24-34 shows the electric field of a dipole kept along z-axis. Note that the resultant electric field makes an angle of $q + f$ with the z axis.

To get the electric field from the potential, we take the derivative of the potential in radial and polar directions.

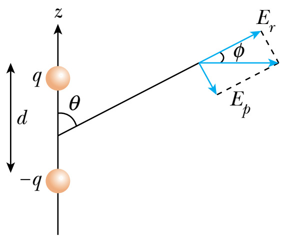


Figure 24-34 Electric field of a dipole kept along z axis.

$$E_r = \frac{-\partial V}{\partial r}.$$

For displacement in the tangential direction, the radius will be the same but the displacement will be $r\,d\theta$:

$$E_p = \frac{-\partial V}{r\partial\theta},$$

$$V = \frac{kp\cos\theta}{r^2}.$$

By differentiating the above expressions, we get

$$E_r = \frac{2kp\cos\theta}{r^3}, \quad E_p = \frac{kp\sin\theta}{r^3}. \tag{24-56}$$

Therefore,

$$E_T = \sqrt{E_r^2 + E_p^2} = \frac{kp}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{kp}{r^3}\sqrt{1+3\cos^2\theta}. \tag{24-57}$$

The angle made by resultant with the radial direction is

$$\tan\phi = \frac{E_p}{E_r} = \frac{\sin\theta}{2\cos\theta} = \frac{\tan\theta}{2}. \tag{24-58}$$

SAMPLE PROBLEM 24.18

Electric field due to a dipole at a point on dipole axis

Starting from Eq. 24-54 derive the expression for the electric field due to a dipole at a point on the dipole axis.

KEY IDEA

The component of the electric field $\vec{E}$ in any direction is the negative of the rate at which potential changes with distance in that direction. From Eq. 24-54, the electric potential of a dipole at a point a distance r away is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

where p is the magnitude of the dipole moment $\vec{p}$ and θ is the angle between $\vec{p}$ and the position vector of the point. The potential at infinity is taken to be zero.

Analysis: On the dipole axis $\theta = 0$ or π, so $|\cos\theta| = 1$. Therefore, magnitude of the electric field is

$$|E(r)| = \left|\frac{\partial V}{\partial r}\right| = \frac{p}{4\pi\varepsilon_0}\left|\frac{d}{dr}\left(\frac{1}{r^2}\right)\right| = \frac{p}{2\pi\varepsilon_0 r^3}.$$

Learn: Take the z axis to be the dipole axis. For $r = z > 0$ $(\theta = 0)$, $E = p/2\pi\varepsilon_0 z^3$. On the other hand, for $r = -z < 0$ $(\theta = \pi)$, $E = -p/2\pi\varepsilon_0 z^3$.

SAMPLE PROBLEM 24.19

Resultant potential and electric field of dipoles

A dipole of dipole moment p is placed at origin along x axis. Another dipole of dipole moment also p is kept at (0, 1, 0) along y axis. Find the resultant potential and electric field at (1, 0, 0).

KEY IDEA

We have studied that the electric field and potential both obey the law of superposition. We can find the electric field and the potential due to each of the dipole at the point concerned and then add them. The caution to be exercised is that while finding the electric field/potential due to a given dipole, we need to consider the center of that dipole as the origin.

Calculation: Due to the first dipole at the origin, the potential is

$$V_1 = \frac{kp\cos\theta}{r^2} = \frac{kp}{1^2} = kp$$

Due to the second dipole at $y = 1$, the potential is

$$V_2 = \frac{kp\cos\theta}{r^2} = \frac{kp}{(\sqrt{2})^2}\cos 135° = -\frac{kp}{2\sqrt{2}}$$

Therefore,

$$V = V_1 + V_2 = kp - \frac{kp\sqrt{2}}{4}. \qquad \text{(Answer)}$$

Similarly, the electric field in the radial direction due to the first dipole is

$$E_{r1} = \frac{2kp\cos 0}{r^3}\hat{\text{i}} = 2kp\hat{\text{i}}.$$

Here, the radial direction is also the x direction as can be seen from Fig. 24-35.

$$E_{p1} = \frac{2kp\sin 0}{r^3}\hat{\text{j}} = 0\hat{\text{j}}.$$

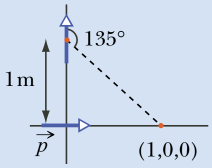


Figure 24-35 Finding the electric field and potential due to two dipole (*a*) at origin and (*b*) at (0, 1, 0). Note that $r = 1$ m and $\theta = 0°$ for first dipole and $r = 2$ m and $\theta = 135°$ for the second.

Here, the polar direction is the positive y direction.

The magnitude of the electric field in the radial direction due to the second dipole is

$$-\cos 45\hat{\mathrm{i}} - \sin 45\hat{\mathrm{j}} \qquad E_{r2} = \frac{2kp}{2\sqrt{2}}\cos 135° = -\frac{kp}{2}.$$

Here, the radial direction is shown in Fig. 24-36. A unit vector in the radial direction will be $\cos 45\hat{\mathrm{i}} - \sin 45\hat{\mathrm{j}}$. So,

$$E_{r2} = -\frac{kp}{2}\left(\frac{\hat{\mathrm{i}}}{\sqrt{2}} - \frac{\hat{\mathrm{j}}}{\sqrt{2}}\right).$$

Similarly, the electric field in the polar direction will be given by

$$\vec{E}_{p2} = \frac{kp\sin 135°}{(\sqrt{2})^3}\left(-\frac{\hat{\mathrm{i}}}{\sqrt{2}} - \frac{\hat{\mathrm{j}}}{\sqrt{2}}\right).$$

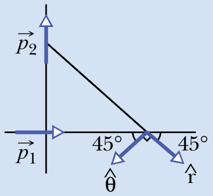


Figure 24-36 Radial and polar direction for dipole $\vec{p}_2$.

Because a unit vector in polar direction will be $-\cos 45\hat{\mathrm{i}} - \sin 45\hat{\mathrm{j}}$, so we have

$$\vec{E} = 2kp\hat{\mathrm{i}} - \frac{kp}{2\sqrt{2}}\hat{\mathrm{i}} + \frac{kp}{2\sqrt{2}}\hat{\mathrm{j}} - \frac{kp}{4\sqrt{2}}\hat{\mathrm{i}} - \frac{kp}{4\sqrt{2}}\hat{\mathrm{j}}$$

$$= \left(\frac{kp(8\sqrt{2}-3)\hat{\mathrm{i}}}{4\sqrt{2}} + \frac{kp\hat{\mathrm{j}}}{4\sqrt{2}}\right). \qquad \text{(Answer)}$$

Force on a Dipole Kept in a Nonuniform Electric Field

The vector sum of the forces exerted on dipoles in *nonuniform* fields is not zero. Consider, for example, the situation illustrated in Fig. 24-37*a*. A dipole, with its dipole moment $\vec{p}$ aligned with the x axis is placed in the nonuniform electric field of a particle carrying a charge Q on the x axis. Because the negative end of the dipole is closer to Q than the positive end, the magnitude of the attractive force on the negative end is larger than the repulsive force on the positive end. Consequently, the vector sum of the forces exerted by the nonuniform field of Q on the dipole is nonzero and the dipole is attracted toward the charged particle.

How does this attraction vary with the distance d? To answer this question, we can write down an expression for the vector sum of the forces exerted on the two ends of the dipole and examine how this sum varies with the distance d. But a simpler method would be to calculate the force exerted by the electric field of the dipole on Q using the formula for the electric field of a dipole. This force and the vector sum of the forces exerted by Q on the dipole form an action–reaction pair and so their magnitudes are the same (Fig. 24-37*b*). Equation 24-55 tells us that the magnitude of the electric field of the dipole along the dipole axis is

$$E_r = \frac{2kp}{r^3}$$

and so, the magnitude of the force on Q is

$$F_{dQ} = E_{dQ} = 2k\frac{p}{d^3}Q = k\frac{2pQ}{d^3}$$

The force exerted by Q on the dipole, being equal in magnitude to, is thus

$$F_{dQ} = F_{Q^+} + F_{Q^-} = k\frac{2pQ}{d^3}$$

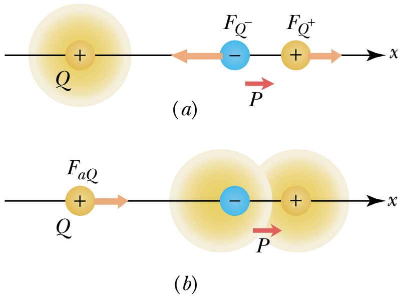


Figure 24-37 (*a*) The vector sum of the electric forces exerted by a charged particle on a dipole is nonzero. (*b*) The force exerted by the charged particle on the dipole and that exerted by the dipole on the charged particle form an action–reaction pair.

Many a times we come across situations where the total charge of a system is zero, but they are made up of more than two charges. In such cases, we may be required to make pairs of charges and find the dipole moment of each pair. Then we can add the dipole moments as vectors to obtain the total dipole moment.

SAMPLE PROBLEM 24.20

Dipole moment of combination of three charges

Three charges Q, Q, and $-2Q$ are located on the vertices of an equilateral triangle of side a. Find the dipole moment of the combination (Fig. 24-38).

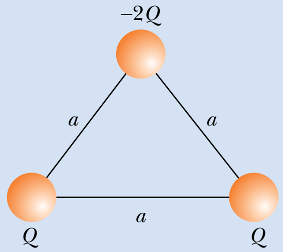


Figure 24-38 Finding dipole moment of three charges.

KEY IDEA

In the situation shown in Fig. 24-39, we can divide the charge $-2Q$ into two parts: $-Q$ and $-Q$. This makes two dipoles of moment Qa at an angle of 60° to each other.

Calculations: As can be easily seen their resultant should be in the y direction.

$$\vec{p} = 2Qa\cos 30°\hat{\mathrm{j}} = \sqrt{3}Qa\hat{\mathrm{j}}$$

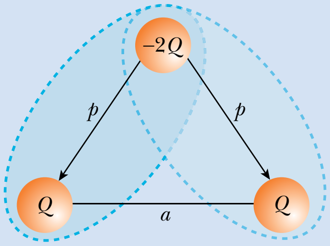


Figure 24-39 Making pairs of equal and opposite charges.

24.10 | ELECTRIC POTENTIAL ENERGY OF A SYSTEM OF CHARGED PARTICLES

Key Concept

- The electric potential energy of a system of charged particles is equal to the work needed to assemble the system with the particles initially at rest and infinitely distant from each other. For two particles at separation r,

$$U = W = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r}.$$

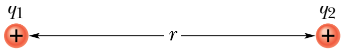


Figure 24-40 Two charges held a fixed distance r apart.

In this module we are going to calculate the potential energy of a system of two charged particles and then briefly discuss how to expand the result to a system of more than two particles. Our starting point is to examine the work we must do (as an external agent) to bring together two charged particles that are initially infinitely far apart and that end up near each other and stationary. If the two particles have the same sign of charge, we must fight against their mutual repulsion. Our work is then positive and results in a positive potential energy for the final two-particle system. If, instead, the two particles have opposite signs of charge, our job is easy because of the mutual attraction of the particles. Our work is then negative and results in a negative potential energy for the system.

Let's follow this procedure to build the two-particle system in Fig. 24-40, where particle 1 (with positive charge q_1) and particle 2 (with positive charge q_2) have separation r. Although both particles are positively charged, our result will apply also to situations where they are both negatively charged or have different signs.

We start with particle 2 fixed in place and particle 1 infinitely far away, with an initial potential energy U_i for the two-particle system. Next we bring particle 1 to its final position, and then the system's potential energy is U_f. Our work changes the system's potential energy by $\Delta U = U_f - U_i$.

With Eq. 24-4 ($\Delta U = q(V_f - V_i)$), we can relate ΔU to the change in potential through which we move particle 1:

$$U_f - U_i = q_1(V_f - V_i). \tag{24-59}$$

Let's evaluate these terms. The initial potential energy is $U_i = 0$ because the particles are in the reference configuration (as discussed in Section 24.2). The two potentials in Eq. 24-53 are due to particle 2 and are given by Eq. 24-28:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r}. \tag{24-60}$$

This tells us that when particle 1 is initially at distance $r = \infty$, the potential at its location is $V_i = 0$. When we move it to the final position at distance r, the potential at its location is

$$V_f = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r}. \tag{24-61}$$

Substituting these results into Eq. 24-58 and dropping the subscript f, we find that the final configuration has a potential energy of

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \quad \text{(two-particle system).} \tag{24-62}$$

Equation 24-61 includes the signs of the two charges. If the two charges have the same sign, U is positive. If they have opposite signs, U is negative.

If we next bring in a third particle, with charge q_3, we repeat our calculation, starting with particle 3 at an infinite distance and then bringing it to a final position at distance r_{31} from particle 1 and distance r_{32} from particle 2. At the final position, the potential V_f at the location of particle 3 is the algebraic sum of the potential V_1 due to particle 1 and the potential V_2 of particle 2. When we work out the algebra, we find that

The total potential energy of a system of particles is the sum of the potential energies for every pair of particles in the system.

This result applies to a system for any given number of particles.

Now that we have an expression for the potential energy of a system of particles, we can apply the principle of the conservation of energy to the system as expressed in Eq. 24-12. For example, if the system consists of many particles, we might consider the kinetic energy (and the associated *escape speed*) required of one of the particles to escape from the rest of the particles.

SAMPLE PROBLEM 24.21

Potential energy of a system of three charged particles

Figure 24-41 shows three charged particles held in fixed positions by forces that are not shown. What is the electric potential energy U of this system of charges? Assume that $d = 12$ cm and that

$$q_1 = +q, \quad q_2 = -4q, \quad \text{and} \quad q_3 = +2q,$$

in which $q = 150$ nC.

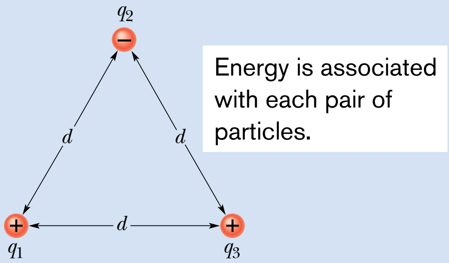


Figure 24-41 Three charges are fixed at the vertices of an equilateral triangle. What is the electric potential energy of the system?

KEY IDEA

The potential energy U of the system is equal to the work we must do to assemble the system, bringing in each charge from an infinite distance.

Calculations: Let's mentally build the system of Fig. 24-41, starting with one of the charges, say q_1, in place and the others at infinity. Then we bring another one, say q_2, in from infinity and put it in place. From Eq. 24-61 with d substituted for r, the potential energy U_{12} associated with the pair of charges q_1 and q_2 is

$$U_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{d}.$$

We then bring the last charge q_3 in from infinity and put it in place. The work that we must do in this last step is equal to the sum of the work we must do to bring q_3 near q_1 and the work we must do to bring it near q_2. From Eq. 24-61, with d substituted for r, that sum is

$$W_{13} + W_{23} = U_{13} + U_{23} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{d} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{d}.$$

The total potential energy U of the three-charge system is the sum of the potential energies associated with the three pairs of charges. This sum (which is actually independent of the order in which the charges are brought together) is

$$\begin{aligned} U &= U_{12} + U_{13} + U_{23} \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{(+q)(-4q)}{d} + \frac{(+q)(+2q)}{d} + \frac{(-4q)(+2q)}{d}\right) \\ &= \frac{10q^2}{4\pi\varepsilon_0 d} \\ &= -\frac{(8.99\times10^9\,\mathrm{N\cdot m^2/C^2})(10)(150\times10^{-9}\,\mathrm{C})^2}{0.12\,\mathrm{m}} \\ &= -1.7\times10^{-2}\,\mathrm{J} = -17\,\mathrm{mJ}. \qquad \text{(Answer)} \end{aligned}$$

The negative potential energy means that negative work would have to be done to assemble this structure, starting with the three charges infinitely separated and at rest. Put another way, an external agent would have to do 17 mJ of positive work to disassemble the structure completely, ending with the three charges infinitely far apart.

The lesson here is this: If you are given an assembly of charged particles, you can find the potential energy of the assembly by finding the potential energy of every possible pair of the particles and then summing the results.

SAMPLE PROBLEM 24.22

Conservation of mechanical energy with electric potential energy

An alpha particle (two protons, two neutrons) moves into a stationary gold atom (79 protons, 118 neutrons), passing through the electron region that surrounds the gold nucleus like a shell and headed directly toward the nucleus (Fig. 24-42). The alpha particle slows until it momentarily stops when its center is at radial distance $r = 9.23$ fm from the nuclear center. Then it moves back along its incoming path. (Because the gold nucleus is much more massive than the alpha particle, we can assume the gold nucleus does not move.) What was the kinetic energy K_i of the alpha particle when it was initially far away (hence external to the gold atom)? Assume that the only force acting between the alpha particle and the gold nucleus is the (electrostatic) Coulomb force and treat each as a single charged particle.

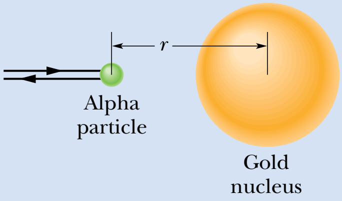


Figure 24-42 An alpha particle, traveling head-on toward the center of a gold nucleus, comes to a momentary stop (at which time all its kinetic energy has been transferred to electric potential energy) and then reverses its path.

KEY IDEA

During the entire process, the mechanical energy of the *alpha particle + gold atom* system is conserved.

Reasoning: When the alpha particle is outside the atom, the system's initial electric potential energy U_i is zero because the atom has an equal number of electrons and protons, which produce a *net* electric field of zero. However, once the alpha particle passes through the electron region surrounding the nucleus on its way to the nucleus, the electric field due to the electrons goes to zero. The reason is that the electrons act like a closed spherical shell of uniform negative charge and, as discussed in Section 23.4, such a shell produces zero electric field in the space it encloses. The alpha particle still experiences the electric field of the protons in the nucleus, which produces a repulsive force on the protons within the alpha particle.

As the incoming alpha particle is slowed by this repulsive force, its kinetic energy is transferred to electric potential energy of the system. The transfer is complete when the alpha particle momentarily stops and the kinetic energy is $K_f = 0$.

Calculations: The principle of conservation of mechanical energy tells us that

$$K_i + U_i = K_f + K_f. \tag{24-63}$$

We know two values: $U_i = 0$ and $K_f = 0$. We also know that the potential energy U_f at the stopping point is given by the right side of Eq. 24-61, with $q_1 = 2e$, $q_2 = 79e$ (in which e is the elementary charge, 1.60×10^{-19} C), and $r = 9.23$ fm. Thus, we can rewrite Eq. 24-63 as

$$K_i = \frac{1}{4\pi\varepsilon_0}\frac{(2e)(79e)}{9.23\text{ fm}}$$

$$= \frac{(8.99\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)(158)(1.60\times10^{-19}\,\text{C})^2}{9.23\times10^{-15}\text{ m}}$$

$$= 3.94\times10^{-12}\,\text{J} = 24.6\text{ MeV}. \qquad \text{(Answer)}$$

24.11 | POTENTIAL OF A CHARGED ISOLATED CONDUCTOR

Key Concepts

- An excess charge placed on a conductor will, in the equilibrium state, be located entirely on the outer surface of the conductor.
- The entire conductor, including interior points, is at a uniform potential.
- If an isolated conductor is placed in an external electric field, then at every internal point, the electric field due to the conduction electrons cancels the external electric field that otherwise would have been there.
- Also, the net electric field at every point on the surface is perpendicular to the surface.

In Section 23.3, we concluded that $\vec{E} = 0$ for all points inside an isolated conductor. We then used Gauss' law to prove that an excess charge placed on an isolated conductor lies entirely on its surface. (This is true even if the conductor has an empty internal cavity.) Here we use the first of these facts to prove an extension of the second:

> An excess charge placed on an isolated conductor will distribute itself on the surface of that conductor so that all points of the conductor—whether on the surface or inside—come to the same potential. This is true even if the conductor has an internal cavity and even if that cavity contains a net charge.

Our proof follows directly from Eq. 24-21, which is

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}.$$

Since $\vec{E} = 0$ for all points within a conductor, it follows directly that $V_f = V_i$ for all possible pairs of points i and f in the conductor.

Figure 24-43*a* is a plot of potential against radial distance r from the center for an isolated spherical conducting shell of 1.0 m radius, having a charge of 1.0 μC. For points outside the shell, we can calculate $V(r)$ from Eq. 24-28 because the charge q behaves for such external points as if it were concentrated at the center of the shell. That equation holds right up to the surface of the shell. Now let us push a small test charge through the shell—assuming a small hole exists—to its center. No extra work is needed to do this because no net electric force acts on the test charge once it is inside the shell. Thus, the potential at all points inside the shell has the same value as that on the surface, as Fig. 24-43*a* shows.

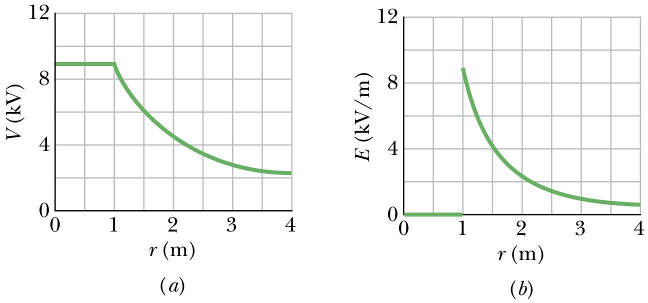


Figure 24-43 (*a*) A plot of $V(r)$ both inside and outside a charged spherical shell of radius 1.0 m. (*b*) A plot of $E(r)$ for the same shell.

Figure 24-43*b* shows the variation of electric field with radial distance for the same shell. Note that $E = 0$ everywhere inside the shell. The curves of Fig. 24-43*b* can be derived from the curve of Fig. 24-43*a* by differentiating with respect to r, using Eq. 24-45 (recall that the derivative of any constant is zero). The curve of Fig. 24-43*a* can be derived from the curves of Fig. 24-43*b* by integrating with respect to r, using Eq. 24-21.

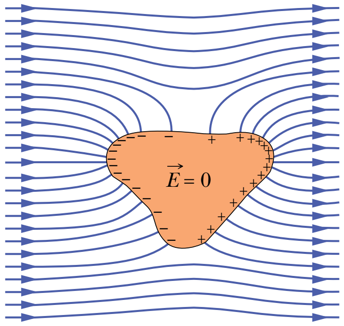


Figure 24-44 An uncharged conductor is suspended in an external electric field. The free electrons in the conductor distribute themselves on the surface as shown, so as to reduce the net electric field inside the conductor to zero and make the net field at the surface perpendicular to the surface.

Isolated Conductor in an External Electric Field

If an isolated conductor is placed in an *external electric field,* as in Fig. 24-44, all points of the conductor still come to a single potential regardless of whether the conductor has an excess charge. The free conduction electrons distribute themselves on the surface in such a way that the electric field they produce at interior points cancels the external electric field that would otherwise be there. Furthermore, the electron distribution causes the net electric field at all points on the surface to be perpendicular to the surface. If the conductor in Fig. 24-44 could be somehow removed, leaving the surface charges frozen in place, the internal and external electric field would remain absolutely unchanged.

SAMPLE PROBLEM 24.23

Finding potential on spherical conducting shell due to point charge kept outside the shell

Find the potential at point A due to (*a*) charge Q; (*b*) charge on the outer surface of the conducting shell; and (*c*) total potential at A (Fig. 24-45).

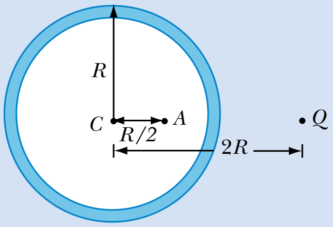


Figure 24-45 A point charge is kept outside a spherical conducting shell.

KEY IDEA

It is easy to find the potential due to a point charge. But the charge on the surface of the conducting shell is nonuniform and its potential can be tricky. However, we can use the fact that the entire conductor is an equipotential surface.

Calculation: The first part, potential due to Q is easy. It can be seen that

$$V_Q = \frac{q}{6\pi\varepsilon_0 R}.$$

There are induced charges on outer surface of the shell. As discussed before, they are nonuniformly distributed.

So, we cannot calculate the potential due to a nonuniform shell at A. But let us try to calculate the potential due to this shell at C, the center of the shell. We cannot use the formula for the uniformly charged shell, but the center is equidistant from all the induced charges at the surface. Since potential is a scalar, V at the center due to the induced charges will be

$$V_{C\text{ in}} = \frac{\sum q_{\text{in}}}{4\pi\varepsilon_0 R}.$$

Since the shell is initially uncharged, the total charge on it will be zero. So, the potential at C due to induced charges will be zero. Therefore, the total potential at C will be

$$V_{CQ} + V_{C\text{ in}} = \frac{Q}{8\pi\varepsilon_0 R} + 0.$$

As we discussed before, the potential at all the points within the cavity should be the same. So, $V_A = V_C$.

From this, we can say that

$$V_A = \frac{Q}{8\pi\varepsilon_0 R}$$

$$V_A = V_{AQ} + V_{A\text{ in}} = \frac{Q}{8\pi\varepsilon_0 R}$$

$$V_{A\text{ in}} = \frac{-Q}{24\pi\varepsilon_0 R}.$$

Learn: We can see why the potential at A is negative. The induced charges near Q will be negative; and equal and opposite charges will be induced on the other side of the conductor. They will be farther off, producing a net negative potential at A.

A Corollary: Corona Discharge

If some charge is given to a conductor, the charge on the conductor surface is not evenly distributed. The charge per unit area is maximum at those points where the radius of curvature is minimum. To understand this, consider two conducting spheres of radii r and R kept at a large distance from each other. They are joined by a conducting wire as shown in Fig. 24-46.

If $l \gg r$ and R then effect of one sphere on the potential of another will be negligible. After we have joined the two with conducting wire, the charge will be redistributed so that both the bodies acquire the same potential. Since the bodies are at a large distance, we can safely assume that the charge will be distributed evenly on the surface. So, to find the potential, we can use the formula of the potential due to a uniformly charged shell:

$$\frac{q'}{4\pi\varepsilon_0 r} = \frac{q''}{4\pi\varepsilon_0 R} = c \quad \text{(say)},$$

$$q' = \frac{rq''}{R},$$

$$\sigma_1 = \frac{q'}{4\pi r^2} = \frac{c\varepsilon_0}{r},$$

$$\sigma_2 = \frac{q''}{4\pi R^2} = \frac{c\varepsilon_0}{R}.$$

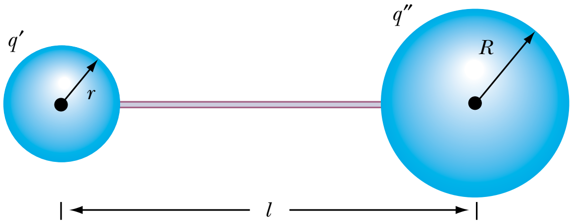


Figure 24-46 Two conductors are joined by a thin wire to bring them to the same potential. Once they reach the same potential, the spheres will remain at the same potential even if the wire is removed.

From this, we can note that:

The charge per unit area is inversely proportional to the radius of curvature of the surface, that is,

$$\sigma \propto \frac{1}{r}.$$

For conductors of a more intricate shape, the charge density distribution is much more complex. After charging the conductor shown in Fig. 24-47 and touching its lateral surface *a*, concave surface *b,* and the pointed region *c* with an electroscope, we find that the surface charge density for a conductor of an arbitrary shape is different at different regions of the surface. The minimum value corresponds to the concave surface, while the maximum value is observed on protruding tips. It should be recalled once again that although the surface of such a conductor is an equipotential surface, the density of the distributed charge can be quite nonuniform.

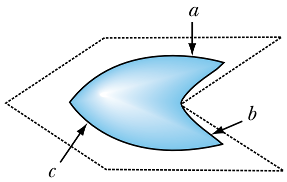
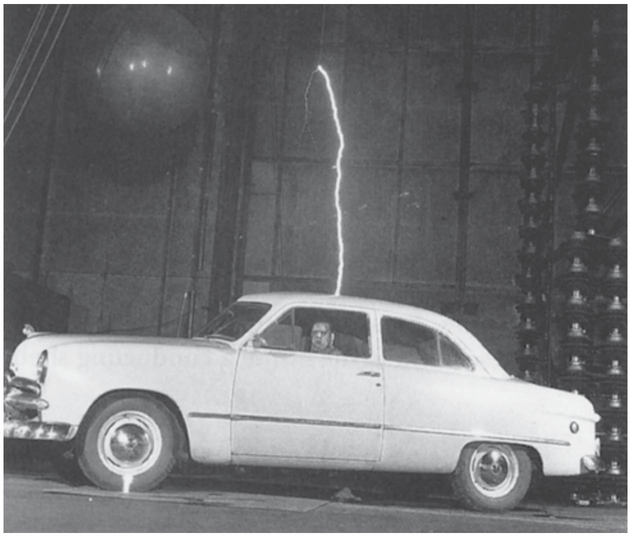


Figure 24-47 The distribution of the surface charge density over a conductor of a complex shape. If we imagine that the conductor is enveloped by a layer whose thickness is proportional to the surface charge density, the layer will have the shape of the figure shown by the dashed line.

Spark Discharge from a Charged Conductor

On nonspherical conductors, a surface charge does not distribute itself uniformly over the surface of the conductor. At sharp points or sharp edges, the surface charge density—and thus the external electric field, which is proportional to it—may reach very high values. The air around such sharp points or edges may become ionized, producing the corona discharge that golfers and mountaineers see on the tips of bushes, golf clubs, and rock hammers when thunderstorms threaten. Such corona discharges, like hair that stands on end, are often the precursors of lightning strikes. In such circumstances, it is wise to enclose yourself in a cavity inside a conducting shell, where the electric field is guaranteed to be zero. A car (unless it is a convertible or made with a plastic body) is almost ideal (Fig. 24-48).

If the distance between the charges is not small, the potential at any point will be due to both the conductors. We will also be required to take the effect of the induced charges into account. Surface charge density on a conductor can also be different for the same radii of curvature when there is a charge in the vicinity. For example, consider the situation shown in Fig. 24-49. Here, the surface charge density at different points of the sphere will not be the same even though *R* for all points is the same.



Courtesy Westinghouse Electric Corporation

Figure 24-48 A large spark jumps to a car's body and then exits by moving across the insulating left front tire (note the flash there), leaving the person inside unharmed.

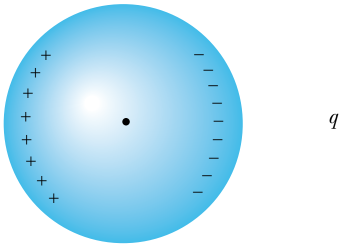


Figure 24-49 The radius of curvature of the surface is constant, yet the electric field at all the points near the surface is not the same.

SAMPLE PROBLEM 24.24

Potential due to charges on three concentric shells

Find the (a) potential of the shells in the situation shown in Fig. 24-50 and (b) the charge distribution on each of the shells after the inner shell and the outermost shells are connected by a conducting wire.

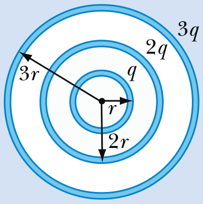


Figure 24-50 Three concentric spherical conducting shells are given charges, as shown.

KEY IDEA

Here, the problem is simplified because the shells are concentric. So, the charge distribution on them will be uniform. Now we use the formula for the potential of a uniformly charged shell, obtained in Chapter 23.

Calculations:

(a)
$$V_A = \frac{q}{4\pi\varepsilon_0 r} + \frac{2q}{2\times 4\pi\varepsilon_0 r} + \frac{3q}{3\times 4\pi\varepsilon_0 r} = \frac{3q}{4\pi\varepsilon_0 r}.$$

$$V_B = \frac{q}{8\pi\varepsilon_0 r} + \frac{2q}{2\times 4\pi\varepsilon_0 r} + \frac{3q}{3\times 4\pi\varepsilon_0 r}$$

$$= \frac{2q}{4\pi\varepsilon_0 r} + \frac{q}{2\times 4\pi\varepsilon_0 r} = \frac{5q}{8\pi\varepsilon_0 r}.$$

$$V_C = \frac{6q}{3\times 4\pi\varepsilon_0 r} = \frac{q}{2\pi\varepsilon_0 r}.$$

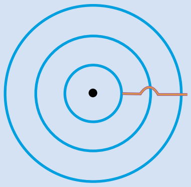


Figure 24-51 The charge redistributed between the inner and outer shells so that they come to the same potential.

(b) Once the shells are connected by a wire, the charge will be redistributed between the inner and outer shells (Fig. 24-51) so that they come to the same potential, that is,

$$V_A = V_C.$$

Assume that the charges on inner shell and outer shell are q' and q'', respectively. Then,

$$V_A = \frac{q'}{4\pi\varepsilon_0 r} + \frac{2q}{2\times 4\pi\varepsilon_0 r} + \frac{3q''}{3\times 4\pi\varepsilon_0 r}.$$

$$V_C = \frac{q'}{3\times 4\pi\varepsilon_0 r} + \frac{2q}{3\times 4\pi\varepsilon_0 r} + \frac{3q''}{3\times 4\pi\varepsilon_0 r}.$$

By charge conservation, we have

$$q' + q'' = 4q.$$

$$q' = -\frac{q}{2}.$$

$$q'' = -\frac{9q}{2}.$$

Learn: Charge on the middle shell will not change because it is not connected. Note that although there will be two pairs of charges on each shell, one on its inner surface and another on its outer surface, both of them have been clubbed together. Why?

24.12 | EARTHING

Here is an example of how conduction can eliminate excess charge on an object. If you rub a copper rod with wool, charge is transferred from the wool to the rod. However, if you are holding the rod while also touching a water tap, you cannot charge the rod in spite of the transfer. The reason is that you, the rod, and the water tap are all conductors connected, via the plumbing, to Earth's surface, which is a huge conductor. As the excess charges put on the rod by the wool repel one another, they move away from one another by moving first through the rod, then through you, and then through the water tap and plumbing to reach Earth's surface, where they can spread out.

Thus, in setting up a pathway of conductors between an object and Earth's surface, we are said to *ground* the object, and in neutralizing the object (by eliminating an unbalanced positive or negative charge), we are said to *discharge* the object. If instead of holding the copper rod in your hand, you hold it by an insulating handle, you eliminate the conducting path to Earth, and the rod can then be charged by rubbing (the charge remains on the rod), as long as you do not touch it directly with your hand.

Earth can be assumed to be a large spherical conductor. Since its radius is very large any change in amount of charge on it will not produce any substantial change in its potential:

$$V = \frac{Q}{6.4 \times 10^6 \times 4\pi\varepsilon_0} \simeq 0.$$

Thus, we can assume the potential of Earth to be zero always. If any conductor is connected to Earth, charges will flow from Earth or to the Earth so that the potential of that conductor will also become zero. Remember, it is said that the potential will become zero, not the charge! Actually, the charge on Earth at any moment is very large. But we are talking about change in it. Potential is assumed to be 0, taking earth as reference point as shown in Fig. 24-52.

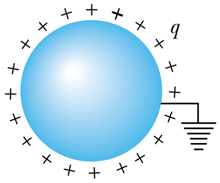


Figure 24-52 Earthing is shown with this sign.

SAMPLE PROBLEM 24.25

Charge and potential on a conductor that is earthed

Assume that two spherical concentric conducting shells of radius r and R are kept as shown in Fig. 24-53. The inner shell is given a charge Q and the outer shell is earthed. Find the charge supplied by the Earth and the potential of the inner shell.

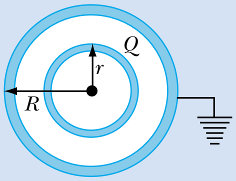


Figure 24-53 A charged spherical conducting shell is enclosed by another conducting shell and then the outer shell is connected to the Earth.

KEY IDEA

Let q be the charge given by the Earth. Note that when the body is connected to Earth, its charge will not be conserved. Some amount of charge can flow from the Earth or flow to the Earth to make its potential zero. The body which is not connected to the Earth will retain its original charge.

Calculation: Potential at surface of outer shell:

$$\frac{Q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 R} = 0.$$

So, $q = -Q$. So, the Earth has to supply a charge of $-Q$ to make the potential of the outer shell zero. But the potential of inner shell need not be zero.

$$V_{\text{in}} = \frac{Q}{4\pi\varepsilon_0 r} - \frac{Q}{4\pi\varepsilon_0 R}.$$

This potential will be positive.

Learn: Another interesting point that we can note here is that since the shells are conducting, the charge $-Q$ will reside on the inner surface of the outer shell. So, the charge on the outer surface of the shell will be zero!

Can you guess why! Let us recall that the joining of the outer shell to the Earth brings both the Earth and the shell at the same potential. So, the electric field between the Earth and the outer shell will be zero. This is possible only if the charge on the outer surface of shell is zero. The charge inside the cavity cannot produce any electric field outside because electric field lines cannot penetrate a conductor.

Electric Field of the Earth

Experiments show that an electroscope connected to a probe shows noticeable divergence even if there are no charged bodies near it. The larger the divergence of its leaves, the higher the point above the surface of the Earth. This means that there is a potential difference between points in the atmosphere at different altitudes, that is, there exists an electric field around the Earth's surface. The variation of potential with height is different in different seasons and for different regions. Near the surface, its approximate value is about 130 V/m. As we rise above the surface, this field rapidly attenuates, and at an altitude of 1 km its strength amounts to only 40 V/m, while at an altitude of 10 km it becomes negligibly weak. Thus, we always live and work in a considerable electric field.

Experimental investigation of this field and appropriate calculations show that the Earth as a whole has a negative charge whose mean value is estimated at 0.5 million coulombs. This charge remains practically unchanged due to some processes in the Earth's atmosphere and beyond it (in space), which have not been studied completely so far.

A natural question arises: if a constant negative charge exists on the surface of the Earth, then where are the corresponding positive charges? Where do the electric field lines terminating at the Earth's surface start? It can be easily seen that these positive charges cannot be located very far from the Earth, for example, on the Moon, stars, or planets. Otherwise, the field near the Earth would have the same configuration as the field of an isolated ball. The strength of this field would decrease in inverse proportion to the square of the distance, from the center of the Earth (and not from the Earth's surface). However, the Earth's radius is equal to about 6400 km, and therefore the change in the distance by several kilometers or tens of kilometers from the center of the Earth would have caused only a negligible change in the electric field strength. However, experiments show that, as was mentioned earlier, the strength of the Earth's electric field decreases very rapidly as we move away from it. This points to the fact that the positive charge corresponding to the negative charge of the Earth is located at a not very large altitude above its surface. Indeed, a layer of positively charged (ionized) molecules was observed at an altitude of a few tens of kilometers above the Earth. The positive bulk charge of this "cloud" compensates the negative charge of the Earth. The lines of the electric field of the Earth stretch from this layer to the surface of the Earth.

Pollination and Electrostatics

The ability of a bee to transport pollen from flower to flower depends on two features. (1) Bees become charged while flying through the air. (2) The anther of a flower (Fig. 24-54*a*) is electrically isolated from the ground but the stigma is electrically connected to the ground. In other words, it is earthed. When a bee hovers near an anther, the electric field due to the charge on the bee induces charge on a neutral pollen grain, making the closer side slightly more negative than the farther side (Fig. 24-54*b*). The charges on the two sides are equal but their distances from the bee are not, and the attractive force on the closer side is slightly larger than the repulsive force on the farther side. As a result, the pollen grain is pulled to the bee, where it clings to hairs during the bee's flight to the next flower.

When the bee happens to come close to a stigma in the next flower, the charge on the bee and induced charge on the grain bring conduction electrons up to the tip of the stigma (Fig. 24-54*c*) because the stigma is electrically connected to the ground. Those electrons attract the closer side of the grain and repel the farther side. If the grain is close enough to the stigma, the net force causes the grain to jump to the stigma, starting the fertilization of the flower.

Agricultural engineers now mimic this process by spraying plants with charged pollen grains, so that the grains collect on the stigma rather than fall uselessly to the ground.

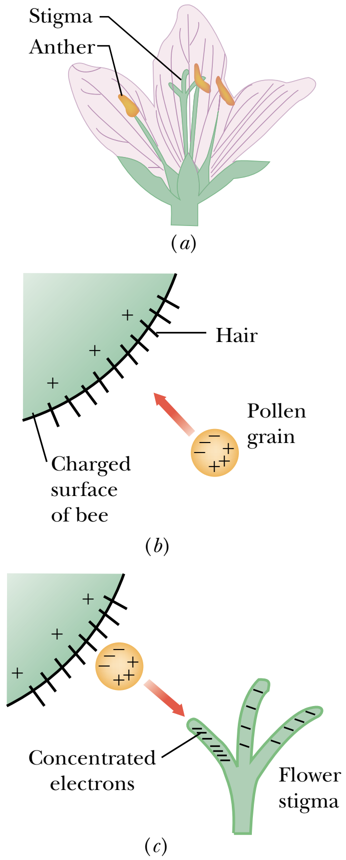


Figure 24-54 (*a*) The anther and stigma portions of a flower. (*b*) A pollen grain near an anther has induced charge due to a bee. (*c*) Electrons collect at a stigma tip, attracting the grain.

24.13 | ELECTROSTATIC SELF-ENERGY

Key Concept

- The self energy of a uniformly charged spherical shell is

$$U = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

Self-energy of a system of charge distribution is the work required to be done by an external agent in building the charge configuration to the present state from a state of infinite separation.

$$W_{\text{ext}} + W_E = 0$$

$$W_{\text{ext}} = -W_E = -(-\Delta U) = \Delta U.$$

In this situation (Fig. 24-55),

$$W_{\text{ext}} = U_f - U_i = \frac{q_1 q_2}{4\pi\varepsilon_0 r}.$$

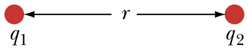


Figure 24-55 Two charges at a distance r from each other have an electrostatic potential energy which gets converted into potential energy as soon as they are released to move apart.

In the case of a continuous charge distribution, this is the energy of interaction of all parts of the charge distribution with each other.

To find it, let us consider a small part of the charge distribution having charge dq. Find the potential due to all the charges at dq's location (since dq is a small charge this will be equal to the potential at that point):

$$dU = Vdq.$$

dU is the potential energy of interaction of dq with rest of the charges:

$$U = \frac{1}{2}\int Vdq.$$

We divided the entire value by 2 because each of the pair will be covered twice.

We now calculate the self-energy of a uniformly charged spherical shell shown in Fig. 24-56.

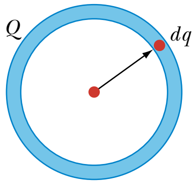


Figure 24-56 A uniformly charged shell has some self-energy due to interaction of different parts of the charged shell. Here, we are considering the shell to be made of a large number of charges dq.

$$V = \frac{kQ}{R} \quad \text{(due to all other charges; } dq \text{ is not considered).}$$

$$U = \frac{1}{2}\int Vdq = \frac{1}{2}\int_0^Q \frac{Qdq}{4\pi\varepsilon_0 R} = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

SAMPLE PROBLEM 24.26

Self-energy of a uniformly charged sphere

Find the self-energy of a uniformly charged sphere of radius R as shown in Fig. 24-57. Assume that the charge density is ρ.

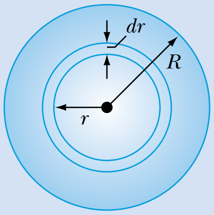


Figure 24-57 A uniformly charged sphere.

KEY IDEA

The charged sphere can be divided into small shells as shown in Fig. 24-57. We can find the potential due to rest of the charges at the location of the shell. This will give us interaction energy of the shell with the rest of the sphere. This energy can then be integrated to find the total energy. Here, we have to find the energy of interaction of different parts of the shell with each other.

Calculation: Considering the expression for potential of a uniformly charged sphere, we have

$$V = \frac{\rho}{6\varepsilon_0}(3R^2 - x^2).$$

The charge on a spherical shell can be found by multiplying the charge density with the volume of the shell.

$$dq = \rho \times \frac{4}{3}\pi \times 3r^2 dr.$$

$$U = \frac{1}{2}\int \frac{\rho}{6\varepsilon_0}(3R^2 - r^2) \times \rho \times 4\pi r^2 dr$$

$$= \frac{4\pi\rho^2}{6\times 2\varepsilon_0}\int_0^R (3R^2 r^2 dr - r^4 dr)$$

$$= \frac{4\pi\rho^2 R^5}{15\varepsilon_0}$$

$$= \frac{3}{20\pi\varepsilon_0}\frac{Q^2}{R}.$$

Learn: Do not commit mistake. This is a uniformly charged sphere, it cannot be a conductor.

SAMPLE PROBLEM 24.27

Electrostatic energy of two charged conducting shells, with and without earthing

(a) Find the total electrostatic energy of the configuration shown in Fig. 24-58

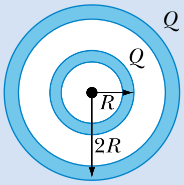


Figure 24-58 Finding the total electrostatic energy of two charged conducting shells.

Here, we find that there are three shells of charges. In the innermost shell, charges on the inner surface of the outer shell form one shell and charges on the outer shell also form a shell. Each of these shells will have a self-energy and an interaction energy with the other two. Since there are three shells, there will be three pairs of interaction energy as seen in Fig. 24-59. So, in total, there will be six terms in the electrostatic potential energy expression.

Calculation:

$$U_{\text{self}} = \frac{Q^2}{2\times 4\pi\varepsilon_0 R} + \frac{(-Q)^2}{2\times 2\times 4\pi\varepsilon_0 R} + \frac{4Q^2}{2\times 2\times 4\pi\varepsilon_0 R}.$$

$$U_{AB} = Q\times\frac{-Q}{2\times 4\pi\varepsilon_0 R}.$$

$$U_{AC} = Q\times\frac{2Q}{2\times 4\pi\varepsilon_0 R}.$$

$$U_{BC} = -Q\times\frac{2Q}{2\times 4\pi\varepsilon_0 R}.$$

$$U = \frac{5Q^2}{16\pi\varepsilon_0 R}.$$

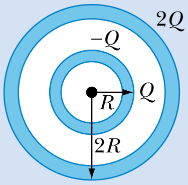


Figure 24-59 Distribution of charges on the surface of a conductor.

Learn: Note that the interaction energies of outer shell with charge Q and with inner surface of the outer shell cancel each other. This is expected. Since there is no electric field due to the outer shell at a point inside, its interaction energy with inner shells will also be zero.

(b) If the inner shell is earthed, find the new charges on each surface and the total electrostatic energy of the configuration.

KEY IDEA

When the inner shell is earthed, its potential becomes zero. Let us assume that its charge becomes q as seen in Fig. 24-60.

Calculation:

$$V = \frac{Q}{2\times 4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 R} = 0.$$

Learn: Here the potential due to the inner surface and the outer surface have been combined because the outer shell is thin, and both the surfaces have the same radii. So, charge on the inner shell will be $-Q/2$. The facing surface will get a charge of $Q/2$, and charge on outer surface will be $Q/2$.

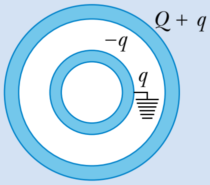


Figure 24-60 The charge distribution after the inner shell is earthed.

REVIEW AND SUMMARY

Electric Potential The electric potential V at a point P in the electric field of a charged object is

$$V = \frac{-W_\infty}{q_0} = \frac{U}{q_0}, \qquad (24\text{-}2)$$

where W_∞ is the work that would be done by the electric force on a positive test charge were it brought from an infinite distance to P, and U is the potential energy that would then be stored in the test charge–object system.

Electric Potential Energy If a particle with charge q is placed at a point where the electric potential of a charged object is V, the electric potential energy U of the particle–object system is

$$U = qV. \tag{24-3}$$

If the particle moves through a potential difference ΔV, the change in the electric potential energy is

$$\Delta U = q\,\Delta V = q(V_f - V_i). \tag{24-4}$$

Mechanical Energy If a particle moves through a change ΔV in electric potential without an applied force acting on it, applying the conservation of mechanical energy gives the change in kinetic energy as

$$\Delta K = -q\,\Delta V. \tag{24-11}$$

If, instead, an applied force acts on the particle, doing work W_{app}, the change in kinetic energy is

$$\Delta K = -q\,\Delta V + W_{\text{app}}. \tag{24-13}$$

In the special case when $\Delta K = 0$, the work of an applied force involves only the motion of the particle through a potential difference:

$$W_{\text{app}} = q\,\Delta V \quad (\text{for } K_i = K_f). \tag{24-14}$$

Equipotential Surfaces The points on an **equipotential surface** all have the same electric potential. The work done on a test charge in moving it from one such surface to another is independent of the locations of the initial and final points on these surfaces and of the path that joins the points. The electric field $\vec{E}$ is always directed perpendicularly to corresponding equipotential surfaces.

Finding V from $\vec{E}$ The electric potential difference between two points i and f is

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}, \tag{24-20}$$

where the integral is taken over any path connecting the points. If the integration is difficult along any particular path, we can choose a different path along which the integration might be easier. If we choose $V_i = 0$,we have, for the potential at a particular point,

$$V = -\int_i^f \vec{E}\cdot d\vec{s}. \tag{24-21}$$

In the special case of a uniform field of magnitude E, the potential change between two adjacent (parallel) equipotential lines separated by distance Δx is

$$\Delta V = -E\,\Delta x. \tag{24-23}$$

Potential Due to a Charged Particle The electric potential due to a single charged particle at a distance r from that particle is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \tag{24-28}$$

where V has the same sign as q. The potential due to a collection of charged particles is

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}. \tag{24-29}$$

Potential Due to a Continuous Charge Distribution For a continuous distribution of charge, Eq. 24-29 becomes

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}, \tag{24-32}$$

in which the integral is taken over the entire distribution.

Calculating $\vec{E}$ from V The component of $\vec{E}$ in any direction is the negative of the rate at which the potential changes with distance in that direction:

$$E_s = \frac{\partial V}{\partial s}. \tag{24-45}$$

The x, y, and z components of $\vec{E}$ may be found from

$$E_x = -\frac{\partial V}{\partial x};\quad E_y = -\frac{\partial V}{\partial y};\quad E_z = -\frac{\partial V}{\partial z}. \tag{24-46}$$

When $\vec{E}$ is uniform, Eq. 24-42 reduces to

$$E = -\frac{\Delta V}{\Delta s}, \tag{24-47}$$

where s is perpendicular to the equipotential surfaces.

Potential Due to an Electric Dipole At a distance r from an electric dipole with dipole moment magnitude $p = qd$, the electric potential of the dipole is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} \tag{24-54}$$

for $r \gg d$; the angle θ is defined in Fig. 24-17.

Electric Potential Energy of a System of Charged Particles The electric potential energy of a system of charged particles is equal to the work needed to assemble the system with the particles initially at rest and infinitely distant from each other. For two particles at separation r,

$$U = W = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}. \tag{24-62}$$

Potential of a Charged Conductor An excess charge placed on a conductor will, in the equilibrium state, be located entirely on the outer surface of the conductor. The charge will distribute itself so that the following occur: (1) The entire conductor, including interior points, is at a uniform potential. (2) At every internal point, the electric field due to the charge cancels the external electric field that otherwise would have been there. (3) The net electric field at every point on the surface is perpendicular to the surface.

PROBLEMS

1. A charged, conducting sphere of radius 5.5 cm sets up a potential of 75 V at a radial distance of 2.2 m (with $V = 0$ set at infinity). (a) What is the potential on the sphere's surface? (b) What is the surface charge density?

2. The thin plastic rod shown in Fig. 24-61 has length L = 24.0 cm and a nonuniform linear charge density $\lambda = cx$, where $c = 28.9$ pC/m². With $V = 0$ at infinity, find the electric potential at point P_1 on the axis, at distance $d = 3.00$ cm from one end.

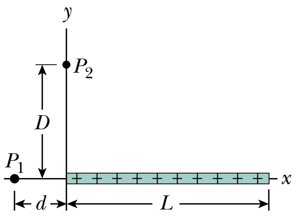


Figure 24-61 Problems 2, 3, and 4.

3. Figure 24-61 shows a thin plastic rod of length $L = 13.5$ cm and uniform charge 43.6 fC. (a) In terms of distance d, find an expression for the electric potential at point P_1. (b) Next, substitute variable x for d and find an expression for the magnitude of the component E_x of the electric field at P_1. (c) What is the direction of E_x relative to the positive direction of the x axis? (d) What is the value of E_x at P_1 for $x = d = 6.60$ cm? (e) From the symmetry in Fig. 24-61, determine E_y at P_1.

4. The thin plastic rod of length L = 12.0 cm in Fig. 24-61 has a nonuniform linear charge density $\lambda = cx$, where c = 49.9 pC/m². (a) With $V = 0$ at infinity, find the electric potential at point P_2 on the y axis at $y = D = 3.56$ cm. (b) Find the electric field component E_y at P_2. (c) Why cannot the field component E_x at P_2 be found using the result of (a)?

5. A plastic rod has been bent into a circle of radius $R = 8.20$ cm. It has a charge $Q_1 = +7.07$ pC uniformly distributed along onequarter of its circumference and a charge $Q_2 = -6Q_1$ uniformly distributed along the rest of the circumference (Fig. 24-62). With $V = 0$ at infinity, what is the electric potential at (a) the center C of the circle and (b) point P, on the central axis of the circle at distance $D = 2.05$ cm from the center?

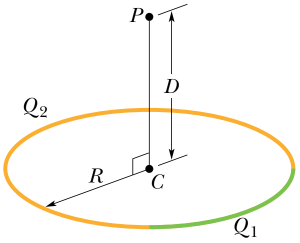


Figure 24-62 Problem 5.

6. A thin, spherical, conducting shell of radius R is mounted on an isolating support and charged to a potential of −170 V. An electron is then fired directly toward the center of the shell, from point P at distance r from the center of the shell ($r \gg R$). What initial speed v_0 is needed for the electron to just reach the shell before reversing direction?

7. An electron is placed in an xy plane where the electric potential depends on x and y as shown, for the coordinate axes, in Fig. 24-63 (the potential does not depend on z). The scale of the vertical axis is set by $V_s = 1000$ V. In unit-vector notation, what is the electric force on the electron?

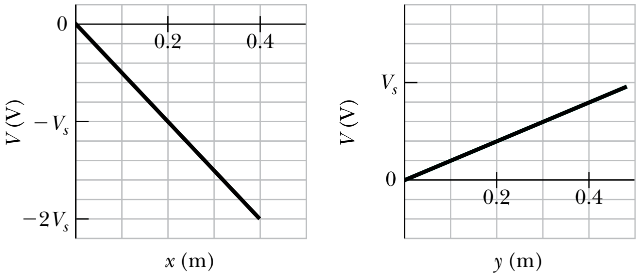


Figure 24-63 Problem 7.

8. The electric potential V in the space between two flat parallel plates 1 and 2 is given (in volts) by $V = 1500x^2$, where x (in meters) is the perpendicular distance from plate 1. At $x = 1.8$ cm, (a) what is the magnitude of the electric field and (b) is the field directed toward or away from plate 1?

9. A particle of charge q is fixed at point P, and a second particle of mass m and the same charge q is initially held a distance r_1 from P. The second particle is then released. Determine its momentum magnitude when it is a distance r_2 from P. Let $q = 3.1\ \mu$C, $m = 20$ mg, $r_1 = 0.90$ mm, and $r_2 = 1.5$ mm.

10. If a lightning discharge lasts 1.4 ms and carries a current of 5.0×10^4 A through a potential difference of 2.4×10^9 V, what is the change in the energy of the charge that is transferred by the discharge?

11. What is the magnitude of the electric field at the point $(-1.00\hat{i} - 2.00\hat{j} + 4.00\hat{k})$ m if the electric potential in the region is given by $V = 2.00xyz^2$, where V is in volts and coordinates x, y, and z are in meters?

12. Two electrons are fixed 4.0 cm apart. Another electron is shot from infinity and stops midway between the two. What is its initial speed?

13. (a) What is the electric potential energy of two electrons separated by 3.00 nm? (b) If the separation increases, does the potential energy increase or decrease?

14. In the rectangle of Fig. 24-64, the sides have lengths 5.0 cm and 15 cm, $q_1 = -5.0\ \mu$C, and $q_2 = +2.0\ \mu$C. With $V = 0$ at infinity, what is the electric potential at (a) corner A and (b) corner B? (c) How much work is required to move a charge $q_3 = -2.0\ \mu$C from B to A along a diagonal of the rectangle? (d) Does this work increase or decrease the electric potential energy of the three-charge system? Is more, less, or the same work required if q_3 is moved along a path that is (e) inside the rectangle but not on a diagonal and (f) outside the rectangle?

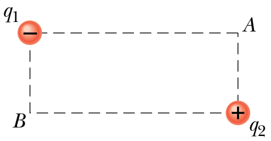


Figure 24-64 Problem 14.

15. If a proton moves through a potential difference of 4.5 kV, what is the magnitude of the change in the proton's potential energy expressed in the unit electron-volt?

16. Identical 50 μC charges are fixed on an x axis at $x = \pm 2.0$ m. A particle of charge $q = -15\ \mu$C is then released from rest at a point on the positive part of the y axis. Due to the symmetry of the situation, the particle moves along the y axis and has kinetic energy 1.2 J as it passes through the point $x = 0$, $y = 4.0$ m. (a) What is the kinetic energy of the particle as it passes through the origin? (b) At what negative value of y will the particle momentarily stop?

17. In Fig. 24-65*a*, a particle of elementary charge $+e$ is initially at coordinate $z = 20$ nm on the dipole axis (here a z axis) through an electric dipole, on the positive side of the dipole. (The origin of z is at the center of the dipole.) The particle is then moved along a circular path around the dipole center until it is at coordinate $z = -20$ nm, on the negative side of the dipole axis. Figure 24-37*b* gives the work W_a done by the force moving the particle versus the angle θ that locates the particle relative to the positive direction of the z axis. The scale of the vertical axis is set by $W_{as} = 2.0 \times 10^{-30}$ J. What is the magnitude of the dipole moment?

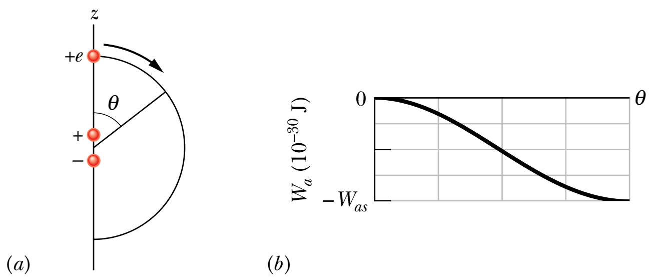


Figure 24-65 Problem 17.

18. A particle of charge +7.5 μC is released from rest at the point $x = 60$ cm on an x axis. The particle begins to move due to the presence of a charge Q that remains fixed at the origin. What is the kinetic energy of the particle at the instant it has moved 50 cm if (a) $Q = +20$ μC and (b) $Q = -20$ μC?

19. When an electron moves from A to B along an electric field line in Fig. 24-66, the electric field does 4.78×10^{-19} J of work on it. What are the electric potential differences (a) $V_B - V_A$, (b) $V_C - V_A$, and (c) $V_C - V_B$?

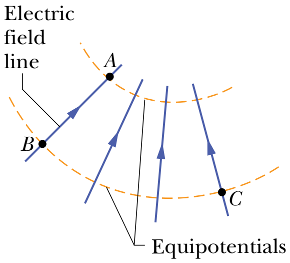


Figure 24-66 Problem 19.

20. A positron (charge $+e$, mass equal to the electron mass) is moving at 1.5×10^7 m/s in the positive direction of an x axis when, at $x = 0$, it encounters an electric field directed along the x axis. The electric potential V associated with the field is given in Fig. 24-67. The scale of the vertical axis is set by $V_s = 500.0$ V. (a) Does the positron emerge from the field at $x = 0$ (which means its motion is reversed) or at $x = 0.50$ m (which means its motion is not reversed)? (b) What is its speed when it emerges?

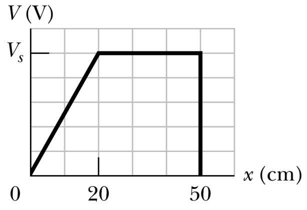


Figure 24-67 Problem 20.

21. Suppose N electrons can be placed in either of two configurations. In configuration 1, they are all placed on the circumference of a narrow ring of radius R and are uniformly distributed so that the distance between adjacent electrons is the same everywhere. In configuration 2, $N - 1$ electrons are uniformly distributed on the ring and one electron is placed in the center of the ring. (a) What is the smallest value of N for which the second configuration is less energetic than the first? (b) For that value of N, consider any one circumference electron—call it e_0. How many other circumference electrons are closer to e_0 than the central electron is?

22. The smiling face of Fig. 24-68 consists of three items:

(1) a thin rod of charge −3.0 μC that forms a full circle of radius 6.0 cm;

(2) a second thin rod of charge 1.0 μC that forms a circular arc of radius 4.0 cm, subtending an angle of 90° about the center of the full circle;

(3) an electric dipole with a dipole moment that is perpendicular to a radial line and has a magnitude of 1.28×10^{-21} C·m.

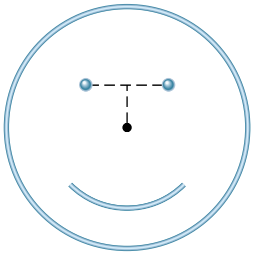


Figure 24-68 Problem 22.

What is the net electric potential at the center?

23. The particles shown in Fig. 24-69 each have charge magnitude $q = 5.00$ pC and were initially infinitely far apart. To form the square with edge length $a = 64.0$ cm, (a) how much work must be done by an external agent, (b) how work must be done by the electric forces, and (c) what is the potential energy of the system?

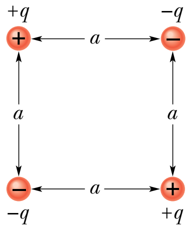


Figure 24-69 Problem 23.

24. Two uniformly charged, infinite, nonconducting planes are parallel to a yz plane and positioned at $x = -50$ cm and $x = +50$ cm. The charge densities on the planes are -50 nC/m² and $+25$ nC/m², respectively. What is the magnitude of the potential difference between the origin and the point on the x axis at $x = +100$ cm? (*Hint:* Use Gauss' law.)

25. *Proton in a well.* Figure 24-70 shows electric potential V along an x axis. The scale of the vertical axis is set by $V_s = 10.0$ V. A proton is to be released at $x = 3.5$ cm with initial kinetic energy 5.00 eV. (a) If it is initially moving in the negative direction of the axis, does it reach a turning point (if so, what is the x coordinate of that point) or does it escape from the plotted region (if so, what is its speed at $x = 0$)? (b) If it is initially moving in the positive direction of the axis, does it reach a turning point (if so, what is the x coordinate of that point) or does it escape from the plotted region (if so, what is its speed at $x = 6.0$ cm)? What are the (c) magnitude F and (d) direction (positive or negative direction of the x axis) of the electric force on the proton if the proton moves just to the left of $x = 3.0$ cm? What are (e) F and (f) the direction if the proton moves just to the right of $x = 5.0$ cm?

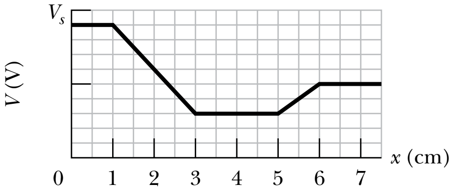


Figure 24-70 Problem 25.

26. A charge of 70 A·h (ampere-hours) moves through a potential difference of 25 V. What are (a) the charge in coulombs and (b) the magnitude of the change in the potential energy of the charge?

27. Figure 24-71 shows a rectangular array of charged particles fixed in place, with distance $a = 35.0$ cm and the charges shown as integer multiples of $q_1 = 3.40$ pC and $q_2 = 6.00$ pC. With $V = 0$ at infinity, what is the net electric potential at the rectangle's center? (*Hint:* Thoughtful examination of the arrangement can reduce the calculation.)

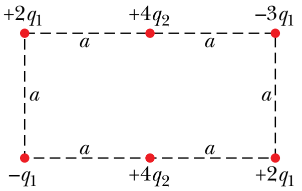


Figure 24-71 Problem 27.

28. An infinite nonconducting sheet with a uniform surface charge density sets up parallel equipotential surfaces. Any pair of surfaces differing by 25.0 V are separated by 8.80 mm. (a) What is the magnitude of the surface charge density? (b) If an electron is released near the sheet, does it tend to move from higher to lower potential or vice versa?

29. A hollow metal sphere has a potential of +300 V with respect to ground (defined to be at $V = 0$) and a charge of 5.0×10^{-9} C. Find the electric potential at the center of the sphere.

30. Two metal spheres, each of radius 3.0 cm, have a center-to-center separation of 2.0 m. Sphere 1 has charge $+1.0 \times 10^{-8}$ C; sphere 2 has charge -8.0×10^{-8} C. Assume that the separation is large enough for us to say that the charge on each sphere is uniformly distributed (the spheres do not affect each other). With $V = 0$ at infinity, calculate (a) the potential at the point halfway between the centers and the potential on the surface of (b) sphere 1 and (c) sphere 2.

31. A charge of −9.0 nC is uniformly distributed around a thin plastic ring lying in a yz plane with the ring center at the origin. A −3.0 pC particle is located on the x axis at $x = 3.0$ m. For a ring radius of 1.5 m, how much work must an external force do on the particle to move it to the origin?

32. The ammonia molecule NH_3 has a permanent electric dipole moment equal to 1.47 D, where 1 D = 1 debye unit = 3.34×10^{-30} C·m. Calculate the electric potential due to an ammonia molecule at a point 103 nm away along the axis of the dipole. (Set $V = 0$ at infinity.)

33. The electric field between two large, parallel, metal plates is approximately uniform, especially away from the edges where there can be some fringing. Suppose the plate separation is 8.00 cm. If the electric force on an electron placed in the uniform field has a magnitude of 7.90×10^{-16} N, (a) what is the potential difference between the plates and (b) is the force directed toward the plate with the higher potential or the lower potential?

34. A nonconducting sphere has radius $R = 2.31$ cm and uniformly distributed charge $q = +3.50$ fC. Take the electric potential at the sphere's center to be $V_0 = 0$. What is V at radial distance (a) $r = 1.45$ cm and (b) $r = R$. (c) If, instead, $V_0 = 0$ at infinity, what is V at $r = R$?

35. Figure 24-72 shows a thin rod with a uniform charge density of 1.00 μC/m. Evaluate the electric potential at point P if $d = D = L/4.00$. Assume that the potential is zero at infinity.

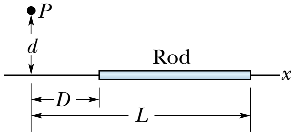


Figure 24-72 Problem 35.

36. What is the excess charge on a conducting sphere of radius $r = 0.35$ m if the potential of the sphere is 1500 V and $V = 0$ at infinity?

37. Two large parallel metal plates are 1.5 cm apart and have charges of equal magnitudes but opposite signs on their facing surfaces. Take the potential of the negative plate to be zero. If the potential halfway between the plates is then +10.0 V, what is the electric field in the region between the plates?

38. The electric field in a region of space has the components $E_y = E_z = 0$ and $E_x = (4.00 \text{ N/C})x^2$. Point A is on the y axis at $y = 3.00$ m, and point B is on the x axis at $x = 4.00$ m. What is the potential difference $V_B - V_A$?

39. In Fig. 24-73, a plastic rod having a uniformly distributed charge $Q = -28.9$ pC has been bent into a circular arc of radius $R = 3.71$ cm and central angle $\phi = 120°$. With $V = 0$ at infinity, what is the electric potential at P, the center of curvature of the rod?

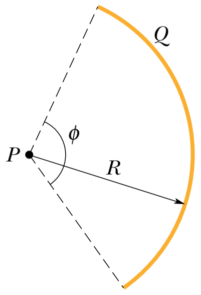


Figure 24-73 Problem 39.

40. Consider a particle with charge $q = 3.0$ nC, point A at distance $d_1 = 2.0$ m from q, and point B at distance $d_2 = 1.0$ m. (a) If A and B are diametrically opposite each other, as in Fig. 24-74a, what is the electric potential difference $V_A - V_B$? (b) What is that electric potential difference if A and B are located as in Fig. 24-74b?

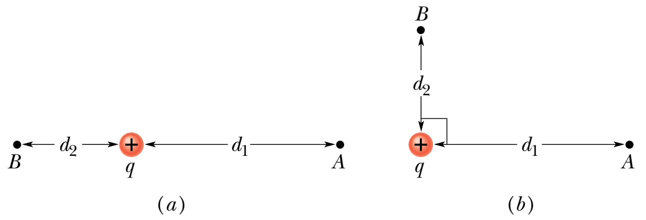
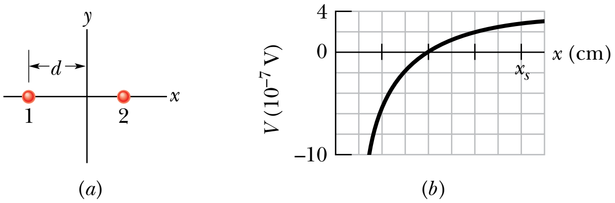


Figure 24-74 Problem 40.

41. The electric potential at points in an xy plane is given by $V = (2.00\ \text{V/m}^2)x^2 - (3.00\ \text{V/m}^2)y^2$. What are (a) the magnitude and (b) angle (relative to $+x$) of the electric field at the point (4.00 m, 2.00 m)?

42. Two charged particles are shown in Fig. 24-75*a*. Particle 1, with charge q_1, is fixed in place at distance d. Particle 2, with charge q_2, can be moved along the x axis. Figure 24-75*b* gives the net electric potential V at the origin due to the two particles as a function of the x coordinate of particle 2. The scale of the x axis is set by $x_s = 16.0$ cm. The plot has an asymptote of $V = 5.92 \times 10^{-7}$ V as $x \to \infty$. What is q_2 in terms of e?



(*a*) (*b*)

Figure 24-75 Problem 42.

43. A ball of mass 1 kg and charge 1 μC is directly projected from infinity toward another stationary ball of mass 2 kg and same charge as the first. The initial velocity of projection is 10 m/s. Find the distance of closest separation and their velocities at the closest separation (Fig. 24-76).

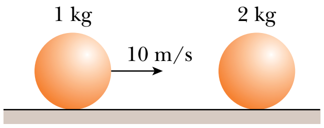


Figure 24-76 Problem 43.

44. Sphere 1 with radius R_1 has positive charge q. Sphere 2 with radius $2.00R_1$ is far from sphere 1 and initially uncharged. After the separated spheres are connected with a wire thin enough to retain only negligible charge, (a) is potential V_1 of sphere 1 greater than, less than, or equal to potential V_2 of sphere 2? What fraction of q ends up on (b) sphere 1 and (c) sphere 2? (d) What is the ratio σ_1/σ_2 of the surface charge densities of the spheres?

45. Two metal spheres, each of radius 3.0 cm, have a center-to-center separation of 2.0 m. Sphere 1 has charge $+1.0 \times 10^{-8}$ C; sphere 2 has charge -3.0×10^{-8} C. Assume that the separation is large enough for us to assume that the charge on each sphere is uniformly distributed (the spheres do not affect each other). With $V = 0$ at infinity, calculate (a) the potential at the point halfway between the centers and the potential on the surface of (b) sphere 1 and (c) sphere 2.

46. Thousand small mercury drops each of the radius r and charge q coalesce together to form one spherical drop. What is the ratio of the potential of the bigger drop to that of the smaller one?

47. A point object with a charge $+Q$ is placed at the center of a conducting shell of inner radius R, outer radius $2R$, and a charge of $-4Q$. A thin-walled conducting shell of radius $3R$ and a charge of $+4Q$ is concentric with the point object and the first shell. Defining $V = 0$ at infinity, find all the distances from the center at which the electric potential is zero.

48. On the middle one of three thin concentric spherical metal shells of radius R there is electric charge of magnitude Q. The inner spherical shell of radius $R/2$ and the outer spherical shell of radius $3R/2$ are earthed. (a) What is the amount of the electric charge on the earthed spherical shells? (b) Plot the electric field strength versus the distance from the center.

49. If a thin spherical shell of radius r and charge q is kept inside a thick spherical shell of inner radius $2r$, outer radius $3r$, and total charge $2q$. (a) Find the charge distribution on each surface. (b) Find the self-energy of each charge distribution. (c) Find the total electrostatic energy stored in the system.

50. Two isolated metallic solid spheres of radii R and $2R$ are charged such that both have the same charge Q. The spheres are located far away from each other and connected by a thin conducting wire. Find the heat dissipated in the wire.

51. A point charge, $q_1 = +2\ \mu$C, is placed at the origin of coordinates. A second, $q_2 = -3\ \mu$C, is placed on the x axis at $x = 100$ cm (Fig. 24-77). At what point (or points) on the x axis will the potential be zero?

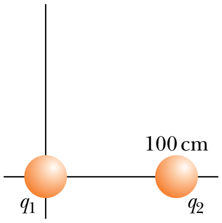


Figure 24-77 Two charges are kept on x axis.

PRACTICE QUESTIONS

Single Correct Choice Type

1. Which one of the following statements best explains why it is possible to define an electrostatic potential in a region of space that contains an electrostatic field?

(a) Work must be done to bring two positive charges closer together

(b) Like charges repel one another and unlike charges attract one another

(c) A positive charge will gain kinetic energy as it approaches a negative charge

(d) The work required to bring two charges together is independent of the path taken

2. Two positive point charges are separated by a distance R. If the distance between the charges is reduced to $R/2$, what happens to the total electric potential energy of the system?

(a) It is doubled

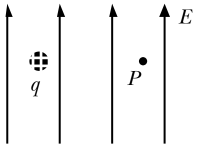
(b) It is reduced to one-half of its original value

(c) It remains the same
(d) It is reduced to one-fourth of its original value

3. Two point charges are located at two of the vertices of a right triangle. If a third charge $-2q$ is brought from infinity and placed at the third vertex, what will its electric potential energy be? Use the following values: $a = 0.15$ m; $b = 0.45$ m, and $q = 2.0 \times 10^{-5}$ C.

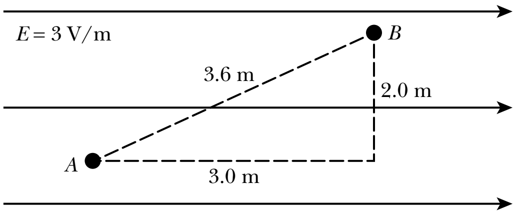
(a) -17 J **(b)** $+8.5$ J
(c) -12 J **(d)** $+14$ J

4. A charge $q = -4.0\ \mu$C is moved 0.25 m horizontally to point ***P*** in a region where an electric field is 150 V/m and directed vertically as shown in the figure. What is the change in the electric potential energy of the charge?



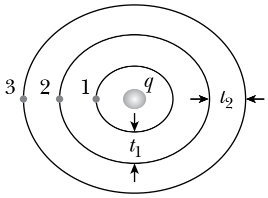
(a) -2.4×10^{-3} J **(b)** -1.5×10^{-4} J
(c) zero joules **(d)** $+1.5 \times 10^{-4}$ J

5. A $+1.0\ \mu$C point charge is moved from point A to B in the uniform electric field as shown in the figure. Which one of the following statements is necessarily true concerning the potential energy of the point charge?



(a) It increases by 6.0×10^{-6} J
(b) It decreases by 6.0×10^{-6} J
(c) It decreases by 9.0×10^{-6} J
(d) It increases by 10.8×10^{-6} J

6. The given figure shows three spherical and equipotential surfaces 1, 2, and 3 around a point charge q. The potential difference $V_1 - V_2 = V_2 - V_3$. If t_1 and t_2 be the distance between them, then



(a) Their radii are in AP
(b) Their radii are in GP
(c) The inverse of their radii are in AP
(d) The inverse of their radii are in GP

7. In total, 10 charges (5 of them are $+q$ each and other 5 are $-q$ each) are placed randomly on the circumference of a circle. The radius of the circle is R. The electric potential at the center of this circle due to these charges will be

(a) 0
(b) $10\ Kq/R$
(c) $5\ Kq/R$
(d) Cannot be calculated unless the positions of the charges on the circle are specified.

8. The potential at a point x (measured in μm) due to some charges situated on the x axis is given by $V(x) = 20/(x^2 - 4)$ volts. The electric field E at $x = 4\ \mu$m is given by

(a) 5/3 volt/μm and in the negative x direction
(b) 5/3 volt/μm and in the positive x direction
(c) 10/9 volt/μm and in the negative x direction
(d) 10/9 volt/μm and in the positive x direction

9. A particle has a charge of $+1.5\mu$C and moves from point A to point B, a distance of 0.20 m. The particle experiences a constant electric force, and its motion is along the line of action of the force. The difference between the particle's electric potential energy at A and B is $\text{EPE}_A - \text{EPE}_B = +9.0 \times 10^{-4}$ J. Find the magnitude and direction of the electric force that acts on the particle.

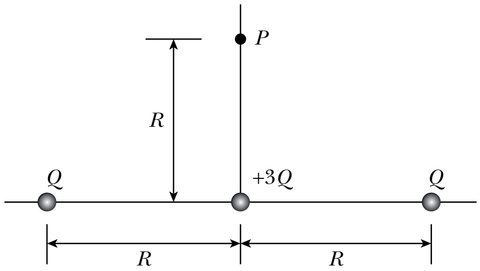
(a) 3.0×10^{-3} N, from A toward B
(b) 4.5×10^{-3} N, from A toward B
(c) 3.0×10^{-3} N, from B toward A
(d) 4.5×10^{-3} N, from B toward A

10. Electric field on the axis of a small electric dipole at a distance r is $\vec{E}_1$ and at a distance of $2r$ on a line of perpendicular bisector is $\vec{E}_2$. Then

(a) $\vec{E}_2 = -\dfrac{\vec{E}_1}{8}$ **(b)** $\vec{E}_2 = -\dfrac{\vec{E}_1}{16}$

(c) $\vec{E}_2 = -\dfrac{\vec{E}_1}{4}$ **(d)** $\vec{E}_2 = -\dfrac{\vec{E}_1}{10}$

11. Three point charges $-Q$, $-Q$, and $+3Q$ are arranged along a line as shown in the following figure.



What is the electric potential at the point P?

(a) $+kQ/R$ **(b)** $-1.6\,kQ/R$
(c) $-2k\,Q/R$ **(d)** $+1.6\,kQ/R$

12. A charge is located at the center of sphere A (radius $R_A = 0.0010$ m), which is in the center of sphere B (radius $R_B = 0.0012$ m). Spheres A and B are both equipotential surfaces. What is the ratio V_A/V_B of the potentials of these surfaces?

(a) 0.42 **(b)** 1.2
(c) 0.83 **(d)** 1.4

13. Point A is located 0.25 m away from a charge of -2.1×10^{-9} C. Point B is located 0.50 m away from the charge. What is the electric potential difference $V_B - V_A$ between these two points?

(a) 19 V **(b)** 26 V
(c) 38 V **(d)** 76 V

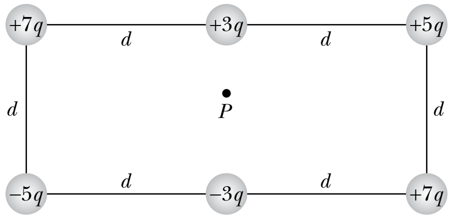
14. The electrostatic potential inside a charged spherical ball is given by $\phi = \alpha r^2 + b$ where r is the distance from the center; a, b are constants. Then the charge density inside ball is

(a) $-6a\varepsilon_0 r$ **(b)** $-24\pi a\varepsilon_0 r$
(c) $-6a\varepsilon_0$ **(d)** $-24\pi a\varepsilon_0 r$

15. Identical point charges of +1.7 μC are fixed to diagonally opposite corners of a square. A third charge is then fixed at the center of the square, such that it causes the potentials at the empty corners to change signs without changing magnitudes. Find the sign and magnitude of the third charge.

(a) −4.8 μC **(b)** −3.4 μC
(c) +5.9 μC **(d)** +1.7 μC

16. The given figure shows six point charges arranged in a rectangle. The value of q is 9.0μC, and the distance d is 0.13 m. Find the total electric potential at location P, which is at the center of the rectangle.



(a) Zero **(b)** $+8.7 \times 10^6$ V
(c) $+7.8 \times 10^6$ V **(d)** -6.1×10^6 V

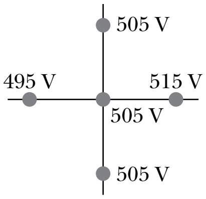
17. A positive charge $+q_1$ is located to the left of a negative charge $-q_2$. On a line passing through the two charges, there are two places where the total potential is zero. The first place is between the charges and is 4.00 cm to the left of the negative charge. The second place is 7.00 cm to the right of the negative charge. What is the distance between the charges?

(a) 0.637 m **(b)** 0.773 m
(c) 0.442 m **(d)** 0.187 m

18. One particle has a mass of 3.0×10^{-3} kg and a charge of +8.00μC. A second particle has a mass of 6.0×10^{-3} kg and the same charge. The two particles are initially held in place and then released. The particles fly apart, and when the separation between them is 0.100 m, the speed of the 3.00×10^{-3} kg particle is 125 m/s. Find the initial separation between the particles.

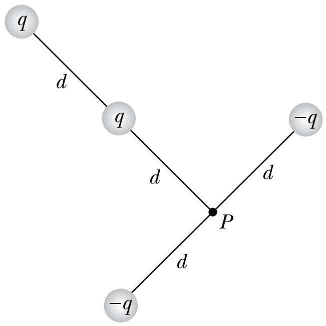
(a) 6.63×10^{-2} m **(b)** 8.83×10^{-2} m
(c) 2.06×10^{-2} m **(d)** 1.41×10^{-2} m

19. The given figure shows the potential at five points on a set of axes. Each of the four outer points is 6.0×10^{-3} m from the point at the origin. From the data shown, find the magnitude and direction of the electric field in the vicinity of the origin.



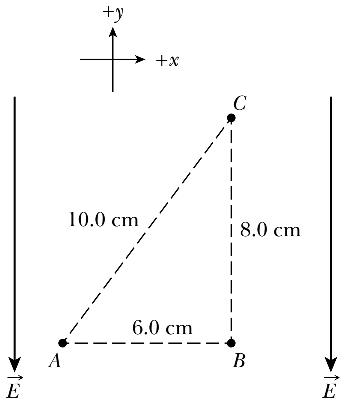
(a) 3.4×10^3 V/m, to the right
(b) 1.7×10^3 V/m, to the right
(c) 3.4×10^3 V/m, to the left
(d) 1.7×10^3 V/m, to the left

20. The given figure shows four point charges. The value of q is 2.0 μC, and the distance d is 0.96 m. Find the total potential at the location P. Assume that the potential of a point charge is zero at infinity.



(a) -9.4×10^3 V **(c)** -4.7×10^3 V
(b) -2.8×10^4 V **(d)** -7.1×10^3 V

21. The given figure shows a uniform electric field that points in the negative y direction; the magnitude of the field is 3600 N/C. Determine the electric potential difference (a) $V_B - V_A$ between points A and B, (b) $V_C - V_B$ between points B and C, and (c) $V_A - V_C$ between points C and A.



(a) −290 V **(b)** zero V
(c) −220 V **(d)** +220 V

22. Two particles each have a mass of 6.0×10^{-3} kg. One has a charge of $+5.0 \times 10^{-6}$ C, and the other has a charge of -5.0×10^{-6} C. They are initially held at rest at a distance of 0.80 m apart. Both are then released and accelerate toward each other. How fast is each particle moving when the separation between them is one-third its initial value?

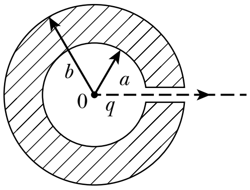
(a) 7.3 m/s **(b)** 3.9 m/s
(c) 9.7 m/s **(d)** 5.4 m/s

23. A large solid sphere with uniformly distributed positive charge has a smooth narrow tunnel through its center. A small particle with negative charge, initially at rest far from the sphere, approaches it along the line of the tunnel, reaches its surface with a speed v, and passes through the tunnel. What will be its speed at the center of the tunnel?

(a) Zero **(b)** v

(c) $\sqrt{2}\,v$ **(d)** $\sqrt{1.5}\,v$

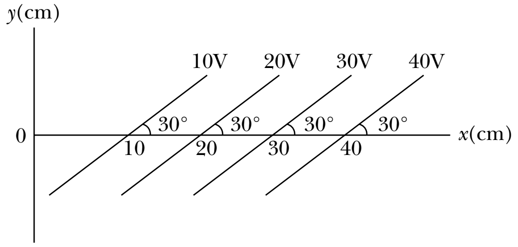
24. A point charge q is located at the centre O of a spherical uncharged conducting layer provided with a small orifice (see figure). The inside and outside radii of the layer are equal to a and b respectively. What amount of work has to be performed to slowly transfer the charge q from the point O through the orifice and into infinity?



(a) $\dfrac{q^2}{8\pi\varepsilon_0}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$ **(b)** $\dfrac{q^2}{4\varepsilon_0}\left(\dfrac{1}{b}-\dfrac{1}{a}\right)$

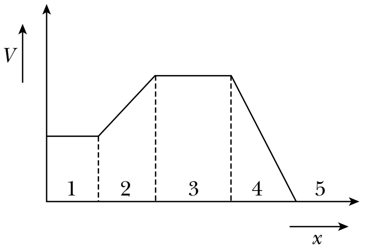
(c) $\dfrac{q^2}{8\pi\varepsilon_0}\left(\dfrac{1}{b}-\dfrac{1}{a}\right)$ **(d)** $\dfrac{q^2}{4\pi\varepsilon_0}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$

25. An equipotential surfaces is shown in the following figure. Find the magnitude and the direction of the electric field?



(a) 200 V/m making an angle 60° with the positive x-axis
(b) 200 V/m making an angle 120° with the positive x-axis
(c) 200 V/m making an angle 120° with the negative x-axis
(d) 100 V/m making an angle 60° with the x-axis

26. The figure below gives the electric potential V as a function of distance through five regions on x-axis. Which of the following is true for the electric field E in these regions?



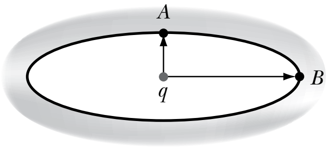
(a) $E_1 > E_2 > E_3 > E_4 > E_5$
(b) $E_1 = E_3 = E_5$ and $E_2 < E_4$
(c) $E_2 = E_4 = E_5$ and $E_1 < E_3$
(d) $E_1 < E_2 < E_3 < E_4 < E_5$

27. A solid spherical conducting object has a nonzero charge on it. Which of the following statements is most appropriate?
(a) Electric potential is zero throughout the volume but the electric field is nonzero constant
(b) Electric field is zero throughout the volume but the electric potential is nonzero constant
(c) Electric potential as well as field are zero throughout the volume
(d) Electric potential as well as field are nonzero constant throughout the volume

28. Two conducting concentric spherical shells are given positive charges. Choose the correct outcome.
(a) Outer shell will be at higher potential
(b) Inner shell will be at higher potential
(c) Outer will always be at higher potential irrespective of the sign of charges given to two spheres
(d) No prediction can be made using the given data

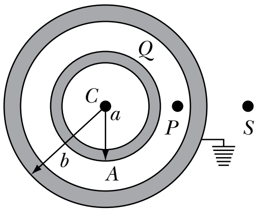
29. Which of the following statements about solid conductors in electrostatics is (are) true?
(I) The electric field inside the conductor is always zero
(II) The electric potential inside the conductor is always zero
(III) Any net charge is on the surface
(a) I only (b) III only
(c) II and III only (d) I and III only

30. An ellipsoidal cavity is carved within a perfect conductor. A positive charge q is placed at the center of the cavity. The points A and B are on the cavity surface as shown in the following figure.



(a) Electric field near A in the cavity = electric field near B in the cavity
(b) Charge density at A = charge density at B
(c) Potential at A = potential at B
(d) None of these

31. Consider the arrangement of two concentric conducting shells with radii a and b, respectively, as shown in the following figure. The inner shell carries a charge Q. The outer shell is grounded



(a) Potential at C equals zero
(b) Potential at S equals zero
(c) Potential at P equals zero
(d) None

32. A very large sphere having charge Q uniformly distributed on the surface is compressed uniformly till its radius reduces to R. The work done by electric forces in this process is

(a) $\dfrac{Q^2}{8\pi\varepsilon_0 R}$ **(b)** $-\dfrac{Q^2}{8\pi\varepsilon_0 R}$

(c) ∞ **(d)** 0

More than One Correct Choice Type

33. When a negative charge moves in a direction opposite to the direction of an electric field,

(a) The field does work on the charge
(b) The charge does work on the field
(c) The charge gains potential energy
(d) The charge loses potential energy

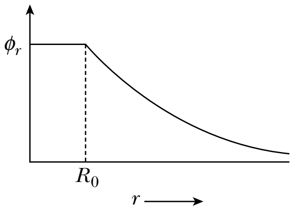
34. Which statement about a system of point charges that are fixed in space is necessarily true? Assuming electrostatic potential energy at infinity to be zero.

(a) If the electrostatic potential energy of the system is negative, net positive work by an external agent is required to take the charges in the system back to infinity.
(b) If the electrostatic potential energy of the system is zero, all charges in the configuration cannot have same sign.
(c) If the electrostatic potential energy of the system is negative, net positive work by an external agent was required to assemble the system of charges.
(d) If the electrostatic potential energy of the system is negative, then there is no electric force anywhere in space on any other charged particle not part of the system.

35. The electrostatic potential (ϕ_r) of a spherical symmetric system, kept at origin, is shown in the following figure, and given as

$$\phi_r = \frac{q}{4\pi\varepsilon_0 r}(r \ge R_0)$$

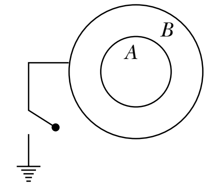
$$\phi_r = \frac{q}{4\pi\varepsilon_0 R_0}(r \le R_0)$$



Which of the following option(s) is/are correct?

(a) For spherical region ($r \le R_0$), total electrostatic energy stored is zero.
(b) Within $r = 2R_0$ total charge is q.
(c) There will be no charge anywhere except at $r = R_0$.
(d) Electric field is discontinuous at $r = R_0$.

36. Two concentric shells have radii R and $2R$, charges q_A and q_B and potentials $2V$ and $(\frac{3}{2})$ V, respectively, as shown in the following figure. Now, shell B is earthed and let charges on them become q'_A and q'_B. Then:



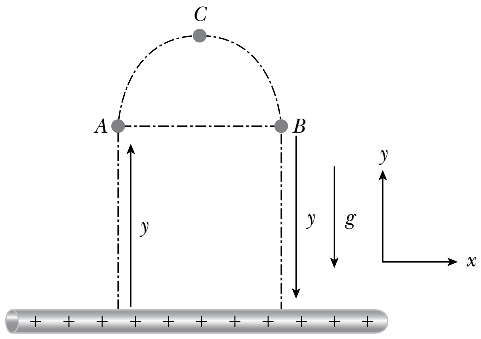
(a) $\frac{q_A}{q_B} = \frac{1}{2}$

(b) $\frac{|q'_A|}{|q'_B|} = 1$

(c) potential of A after earthing becomes $(\frac{3}{2})$ V

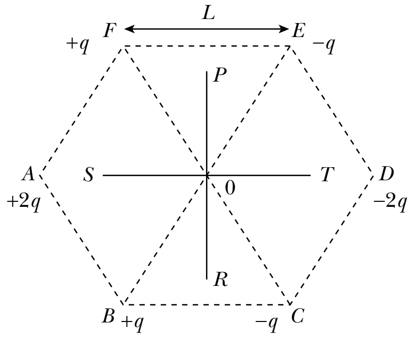
(d) potential difference between A and B after earthing becomes $\frac{V}{2}$

37. An infinitely long uniformly charged non conducting rod is placed along x axis. A test charge is slowly taken from point A to B in the xy plane along a semi-circular path $A \to C \to B$ as shown in the following figure, then which alternative (s) may be correct?



(a) Work done by external agent is zero
(b) The work done by external agent is path independent
(c) $W_{A \to C} = -W_{B \to C}$
(d) At point C, the force applied by external agent is maximum

38. Six point charges are kept at the vertices of a regular hexagon of side L and centre O, as shown in the following figure. Given that $K = \frac{1}{4\pi\varepsilon_0}\frac{q}{L^2}$, which of the following statement(s) is (are) correct?

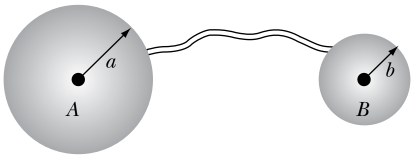


(a) The electric field at O is $6K$ along OD
(b) The potential at O is zero
(c) The potential at all points on the line PR is same
(d) The potential at all points on the line ST is same

39. A uniformly charged solid sphere of radius R has potential V_0 (measured with respect to ∞) on its surface. For this sphere the equipotential surface with potentials $\frac{3V_0}{2}, \frac{5V_0}{4}, \frac{3V_0}{4}$ and $\frac{V_0}{4}$ have radius R_1, R_2, R_3 and R_4, respectively. Then

(a) $R_1 \ne 0$ and $(R_2 - R_1) > (R_4 - R_3)$
(b) $R_1 = 0$ and $R_2 < (R_4 - R_3)$
(c) $2R < R_4$
(d) $R_1 = 0$ and $R_2 > (R_4 - R_3)$

40. Two conducting spheres, A of radius a and B of radius b, are shown in the given figure. Both are positively charged and isolated from their surroundings. They had been connected by a conducting wire but the wire has been removed. Assuming the potential an infinite distance away is zero, which of the following statements about the spheres are true:



- **(a)** Sphere A has the larger charge
- **(b)** Sphere B has the larger charge
- **(c)** Both spheres have the same potential
- **(d)** Both spheres have the same charge

41. A conductor A is given a charge of amount $+Q$ and then placed inside an uncharged deep metal can B, without touching it.

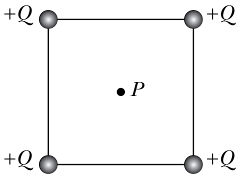
- **(a)** The potential of A does not change when it is placed inside B
- **(b)** If B is earthed, $+Q$ amount of charge flows from it into the Earth
- **(c)** If B is earthed, the potential of A is reduced
- **(d)** If B is earthed, the potential of A and B both becomes zero

42. A sphere of radius R is charged uniformly with a volume charge density. It is disassembled and reassembled in two identical spheres kept far from each other.

- **(a)** Positive work is done by external agent
- **(b)** Potential energy of system decreases
- **(c)** Potential energy of system increases
- **(d)** Work is done by the system

Linked Comprehension

Paragraph for Questions 43—45: Four point charges are individually brought from infinity and placed at the corners of a square as shown in the given figure. Each charge has the identical value $+Q$. The length of the diagonal of the square is $2a$.



43. The first two charges are brought from infinity and placed at adjacent corners. What is the electric potential energy of these two charges?

(a) $\dfrac{kQ^2}{a\sqrt{2}}$ **(b)** $\dfrac{kQ}{a\sqrt{2}}$

(c) $\dfrac{2kQ}{a}$ **(d)** $\dfrac{kQ^2}{2a}$

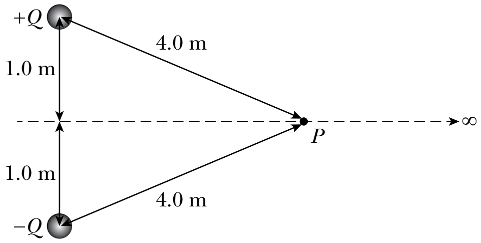
44. What is the magnitude of the electric field at P, the center of the square?

(a) kQ/a^2 **(b)** zero
(c) $2kQ/a^2$ **(d)** $kQ/4a^2$

45. What is the electric potential at P, the center of the square?

(a) kQ/a **(b)** $2kQ/a$
(c) $4kQ/a$ **(d)** $kQ/4a$

Paragraph for Questions 46 and 47: Two charges of opposite sign and equal magnitude $Q = 2.0$ C are held 2.0 m apart as shown in the following figure.



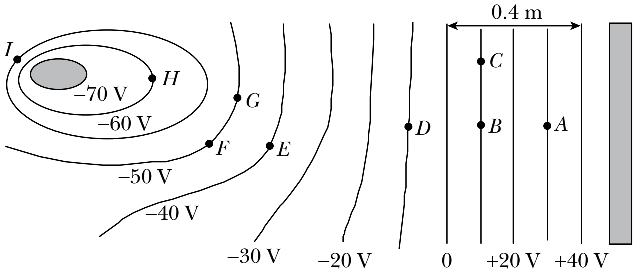
46. Determine the magnitude of the electric field at the point P.

- **(a)** 2.2×10^9 V/m
- **(b)** 5.6×10^8 V/m
- **(c)** 4.4×10^8 V/m
- **(d)** 2.8×10^8 V/m

47. How much work is required to move a 1.0 C charge from infinity to the point P?

(a) 0 J **(b)** 4.5×10^9 J
(c) 2.2×10^9 J **(d)** infinity

Paragraph for Questions 48—52: The given figure shows cross sections of equipotential surfaces between two charged conductors that are shown in solid grey. Various points on the equipotential surfaces near the conductors are labeled A, B, C, ..., I.



48. At which of the labeled points will an electron have the greatest potential energy?

(a) A **(b)** G
(c) D **(d)** H

49. What is the direction of the electric field at B?

- **(a)** toward A
- **(b)** toward C
- **(c)** toward D
- **(d)** into the page

50. How much work is required to move a –1.0 μC charge from A to E?
- **(a)** $+3.0 \times 10^{-5}$ J
- **(b)** $+7.0 \times 10^{-5}$ J
- **(c)** -4.0×10^{-5} J
- **(d)** -7.0×10^{-5} J

51. A positive point charge is placed at F. Complete the following statement: When it is released,
- **(a)** no force will be exerted on it.
- **(b)** a force will cause it to move toward E.
- **(c)** a force will cause it to move toward G.
- **(d)** a force will cause it to move away from E.

52. A point charge gains 50 μJ of electric potential energy when it is moved from point D to point G. Determine the magnitude of the charge.
- **(a)** 1.0 μC
- **(b)** 1.3 μC
- **(c)** 25 μC
- **(d)** 50 μC

Paragraph for Questions 53 and 54: An isolated system consists of two conducting spheres A and B. Sphere A has five times the radius of sphere B. Initially, the spheres are given equal amounts of positive charge and are isolated from each other. The two spheres are then connected by a conducting wire.

Note: The potential of a sphere of radius R that carries a charge Q is $V = kQ/R$, if the potential at infinity is zero.

53. Which one of the following statements is true after the spheres are connected by the wire?
- **(a)** The electric potential of A is 1/25 as large as that of B.
- **(b)** The electric potential of A equals that of B.
- **(c)** The electric potential of A is 25 times larger than that of B.
- **(d)** The electric potential of A is 1/5 as large as that of B.

54. Determine the ratio of the charge on sphere A to that on sphere B, q_A/q_B, after the spheres are connected by the wire.
- **(a)** 1
- **(b)** 1/5
- **(c)** 5
- **(d)** 25

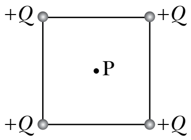
Matrix-Match

55. Match the appropriate Column I with Column II.

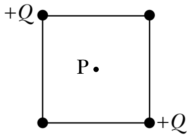
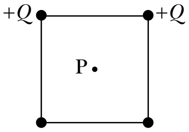
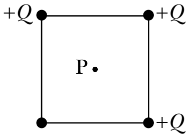
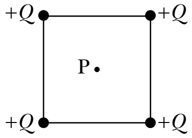
Column I	Column II
(A) Same dimension of mass	**(P)** electric charge
(B) Negative dimension of mass	**(Q)** electric potential
(C) Zero dimension of mass	**(R)** electric capacity
(D) Dimension of length is ± D	**(S)** energy density

Directions for Questions 56 and 57: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

56. Four point charges are individually brought from infinity and placed at the corners of a square. Each charge has the identical value $+Q$. The length of the diagonal of the square is $2a$.



In the given table there are four values each of electric field (Column I), potential difference (Column II) and configuration of charges on four corners of square (Column III).

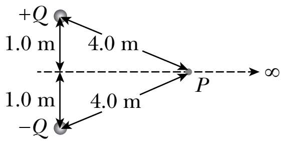
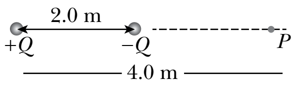
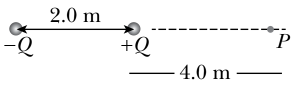
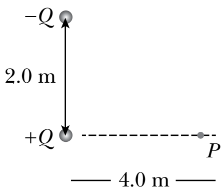
Column I	Column II	Column III
(I) $E = \dfrac{2kQ}{a^2}$	**(i)** $V = \dfrac{2\sqrt{2}kQ}{a}$	**(J)** 
(II) $E = 0$	**(ii)** $V = 0$	**(K)** 
(III) $E = \dfrac{2\sqrt{2}kQ}{a^2}$	**(iii)** $V = \dfrac{3\sqrt{2}kQ}{a}$	**(L)** 
(IV) $E = \dfrac{kQ}{2a^2}$	**(iv)** $V = \dfrac{4\sqrt{2}kQ}{a}$	**(M)** 

(1) The combination for minimum electric field and minimum electric potential energy is
- **(a)** (II) (i) (J)
- **(b)** (IV) (iii) (K)
- **(c)** (I) (ii) (M)
- **(d)** (III) (iv) (L)

(2) The combination for maximum electric field is
- **(a)** (II) (i) (J)
- **(b)** (IV) (ii) (L)
- **(c)** (I) (iv) (M)
- **(d)** (III) (iii) (K)

(3) The combination for minimum electric field and maximum electric potential energy is
- **(a)** (II) (iv) (M)
- **(b)** (IV) (iii) (K)
- **(c)** (I) (ii) (J)
- **(d)** (III) (i) (L)

57. Two charges of opposite sign and equal magnitude Q = 2.0 C are held 2.0 m apart. And there is a point P at 4.0 cm from $+Q$ charge. In the given table there are four values each of electric field (Column I), potential difference (Column II) and configuration of charges (Column III).

Column I	Column II	Column III
(I) $E = 1.27 \times 10^9$ V/m	**(i)** $V = 1.5 \times 10^9$ V	**(J)** 
(II) $E = 0$ V/m	**(ii)** $V = 4.75 \times 10^8$ V	**(K)** 
(III) $E = 6.25 \times 10^8$ V/m	**(iii)** $V = -1.5 \times 10^9$ V	**(L)** 
(IV) $E = 5.6 \times 10^8$ V/m	**(iv)** $V = 0$ V	**(M)** 

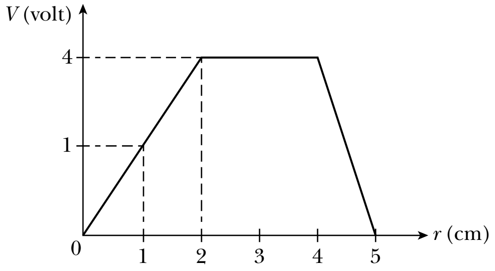
(1) The combination for maximum electric field is
(a) (I) (ii) (M) **(b)** (III) (iv) (K)
(c) (IV) (iii) (J) **(d)** (II) (i) (L)

(2) The combination for maximum electric potential is
(a) (I) (ii) (M) **(b)** (III) (iv) (K)
(c) (IV) (iii) (J) **(d)** (II) (i) (L)

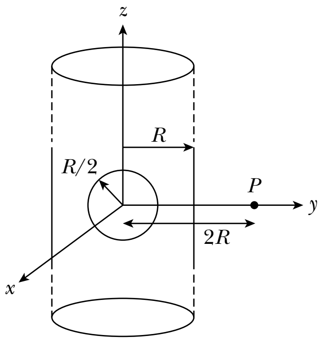
(3) The combination for minimum electric potential is
(a) (I) (iii) (M) **(b)** (II) (ii) (K)
(c) (III) (i) (L) **(d)** (IV) (iv) (J)

Integer Type

58. The following figure shows the variation of potential V with distance r from fixed point. At $r = 2$ cm, what is the magnitude of the electric field?



59. An infinitely long solid cylinder of radius R has a uniform volume charge density ρ. It has a spherical cavity of radius $R/2$ with its center on the axis of the cylinder, as shown in the following figure. The magnitude of the electric field at the point P, which is at a distance $2R$ from the axis of the cylinder, is given by the expression $23\rho R/16k\varepsilon_0$. The value of k is



ANSWER KEY

Checkpoints

1. (a) negative; (b) positive; (c) increase; (d) higher

2. (a) positive; (b) higher

3. (a) rightward; (b) 1, 2, 3, 5: positive; 4, negative; (c) 3, then 1, 2, and 5 tie, then 4

4. all tie

5. (a) 2, then 1 and 3 tie; (b) 3; (c) accelerated leftward

6. (c)

7. Not possible

8. a, c (zero), b

Problems

1. (a) 3.0 kV; (b) 0.48 μC/m^2

2. 45.2 mV

3. (a) $(2.90\times10^{-3}\,\text{V})\ln\left(1+\dfrac{0.135\text{ m}}{d}\right)$; (b) $\dfrac{(3.92\times10^{-4}\text{ N}\cdot\text{m}^2/\text{C})}{x(x+0.135\text{ m})}$; (c) 180°; (d) 0.0296 N/C; (e) 0

4. (a) 4.02×10^{-2} V; (b) 0.321 N/C

5. (a) –3.88 V; (b) –3.76 V

6. 7.73×10^{6} m/s

7. $(-8.0\times10^{-16}\text{N})\hat{\text{i}} + (3.2\times10^{-16}\text{N})\hat{\text{j}}$

8. (a) 54 V/m; (b) toward

9. 3.9×10^{-2} kg·m/s

10. 1.7×10^{11} J

11. 78.4 N/C

12. 0.22 km/s

13. (a) 7.67×10^{-20} J; (b) decreases

14. (a) $+6.0\times10^{4}$ V; (b) -7.8×10^{5} V; (c) –1.7 J; (d) decrease; (e) same; (f) same

15. 4.5 keV

16. (a) 4.9 J; (b) –7.2 m

17. 2.8×10^{-37} C·m

18. (a) 1.0 J; (b) 11 J

19. (a) 2.99 V; (b) 2.99 V; (c) 0

20. (a) From $x = 0.50$ m; (b) 1.5×10^{7} m/s

21. (a) 12; (b) 2

22. -2.2×10^{5} V

23. (a) –0.908 pJ; (b) 0.908 pJ; (c) –0.908 pJ

24. 2.8×10^{3} V

25. (a) turning point at $x = 1.3$ cm; (b) 2.4×10^{4} m/s; (c) $+4.8\times10^{-17}$ N; (d) $+x$ direction; (e) 3.2×10^{-17} N; (f) $-x$ direction

26. (a) 2.5×10^{5} C; (b) 6.3×10^{6} J

27. 2.47 V

28. (a) 5.03×10^{-8} C/m^2; (b) lower to higher

29. 300 V

30. (a) -6.3×10^{2} V; (b) 2.6 kV; (c) –24 kV

31. 8.9×10^{-11} J

32. 4.16 μV

33. (a) 395 V; (b) lower potential to higher potential

34. (a) –0.268 mV; (b) –0.681 mV; (c) +1.36 mV

35. 1.09×10^{4} V

36. 5.8×10^{-8} C

37. 1.33×10^{3} V/m

38. –85.3 V

39. –7.0 V

40. (a) –13 V; (b) –13 V

41. (a) 20.0 V/m; (b) 143°

42. $-33e$

43. 3.70×10^{3} m; $\dfrac{10}{3}$ m/s

44. (a) equal; (b) 0.333; (c) 0.667; (d) 2.00

45. (a) -1.8×10^{2} V; (b) 2.9 kV; (c) –8.9 kV

46. 100

47. $\dfrac{9R}{4}, \dfrac{6R}{7}$

48. (a) $\dfrac{-Q}{4}$; (b) $\dfrac{-3Q}{4}$

49. (a) $0, q, -q, 3q$; (b) $0, \dfrac{kq^2}{2R}, \dfrac{kq^2}{4R}, \dfrac{3kq^2}{2R}$; (c) $\dfrac{7q^2}{16\pi\varepsilon_0 R}$

50. $\dfrac{kQ^2}{12R}$

51. 40 cm, –200 cm

Practice Questions

Single Correct Choice Type

1. (d)
2. (a)
3. (a)
4. (c)
5. (c)
6. (c)
7. (a)
8. (d)
9. (b)
10. (b)
11. (d)
12. (b)
13. (c)
14. (c)
15. (a)
16. (c)
17. (d)
18. (d)
19. (d)
20. (a)
21. (b)
22. (c)
23. (d)
24. (a)
25. (b)
26. (b)
27. (b)
28. (b)
29. (d)
30. (c)
31. (b)
32. (b)

More than One Correct Choice Type

33. (a), (d)
34. (a), (b)
35. (a), (b), (c), (d)
36. (a), (b)
37. (a), (b)
38. (a), (b), (c)
39. (b), (c)
40. (a), (c)
41. (b), (c)
42. (b), (d)

Linked Comprehension

43. (a)
44. (b)
45. (c)
46. (b)
47. (a)
48. (d)
49. (c)
50. (b)
51. (d)
52. (b)
53. (b)
54. (c)

Matrix-Match

55. (a) → (r), (s); (b) → (s); (c) → (p), (q); (d) → (r)
56. (1) → (a); (2) → (d); (3) → (a)
57. (1) → (a); (2) → (a); (3) → (d)

Integer Type

58. 2
59. 6

CHAPTER 25

Capacitance

Contents

25.1 What is Physics?
25.2 Capacitance
25.3 Calculating the Capacitance
25.4 Capacitors in Parallel and in Series
25.5 Energy Stored in an Electric Field
25.6 Capacitor with a Dielectric
25.7 Dielectrics: An Atomic View
25.8 Dielectrics and Gauss' Law

25.1 | WHAT IS PHYSICS?

One goal of physics is to provide the basic science for practical devices designed by engineers. The focus of this chapter is on one extremely common example—the capacitor, a device in which electrical energy can be stored. For example, the batteries in a camera store energy in the photoflash unit by charging a capacitor. The batteries can supply energy at only a modest rate, too slowly for the photoflash unit to emit a flash of light. However, once the capacitor is charged, it can supply energy at a much greater rate when the photoflash unit is triggered—enough energy to allow the unit to emit a burst of bright light.

The physics of capacitors can be generalized to other devices and to any situation involving electric fields. For example, Earth's atmospheric electric field is modeled by meteorologists as being produced by a huge spherical capacitor that partially discharges via lightning. The charge that skis collect as they slide along snow can be modeled as being stored in a capacitor that frequently discharges as sparks (which can be seen by nighttime skiers on dry snow).

The first step in our discussion of capacitors is to determine how much charge can be stored. This "how much" is called capacitance.

25.2 | CAPACITANCE

Key Concepts

- A capacitor consists of two isolated conductors (the plates) with charges $+q$ and $-q$. Its capacitance C is defined from

$$q = CV,$$

  where V is the potential difference between the plates.
- When a circuit with a battery, an open switch, and an uncharged capacitor is completed by closing the switch, conduction electrons shift, leaving the capacitor plates with opposite charges.

Paul Silvermann/Fundamental Photographs

Figure 25-1 An assortment of capacitors.

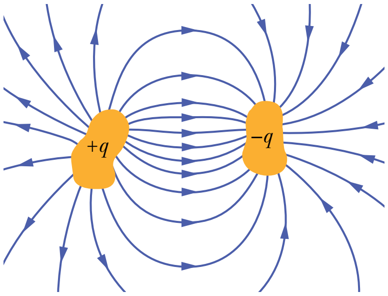


Figure 25-2 Two conductors, isolated electrically from each other and from their surroundings, form a *capacitor*. When the capacitor is charged, the charges on the conductors, or *plates* as they are called, have the same magnitude q but opposite signs.

Figure 25-1 shows some of the many sizes and shapes of capacitors. Figure 25-2 shows the basic elements of *any* capacitor—two isolated conductors of any shape. No matter what their geometry, flat or not, we call these conductors *plates*.

Figure 25-3*a* shows a less general but more conventional arrangement, called a *parallel-plate capacitor*, consisting of two parallel conducting plates of area A separated by a distance d. The symbol we use to represent a capacitor (⊣⊢) is based on the structure of a parallel-plate capacitor but is used for capacitors of all geometries. We assume for the time being that no material medium (such as glass or plastic) is present in the region between the plates. In Section 25.5, we shall remove this restriction.

When a capacitor is *charged,* its plates have charges of equal magnitudes but opposite signs: $+q$ and $-q$. However, we refer to the *charge of a capacitor* as being q, the absolute value of these charges on the plates. (Note that q is not the net charge on the capacitor, which is zero.)

Because the plates are conductors, they are equipotential surfaces; all points on a plate are at the same electric potential. Moreover, there is a potential difference between the two plates. For historical reasons, we represent the absolute value of this potential difference with V rather than with the ΔV we used in previous notation.

The charge q and the potential difference V for a capacitor are proportional to each other; that is,

$$q = CV. \tag{25-1}$$

The proportionality constant C is called the **capacitance** of the capacitor. Its value depends only on the geometry of the plates and *not* on their charge or potential difference. The capacitance is a measure of how much charge must be put on the plates to produce a certain potential difference between them: The *greater the capacitance, the more charge is required*.

The SI unit of capacitance that follows from Eq. 25-1 is the coulomb per volt. This unit occurs so often that it is given a special name, the *farad* (F):

$$1 \text{ farad} = 1 \text{ F} = 1 \text{ coulomb per volt} = 1 \text{ C/V}. \tag{25-2}$$

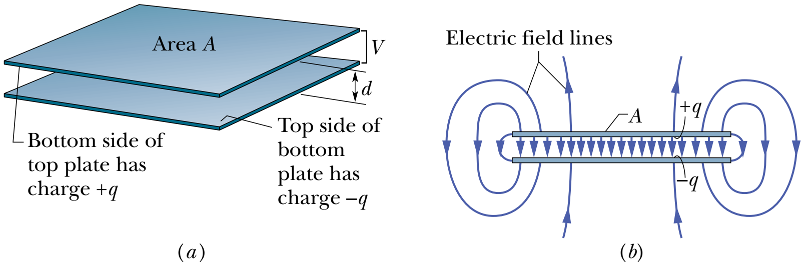

Area A
V
d
Bottom side of top plate has charge +q
Top side of bottom plate has charge −q
(a)
Electric field lines
A
+q
−q
(b)

Figure 25-3 (*a*) A parallel-plate capacitor, made up of two plates of area A separated by a distance d. The charges on the facing plate surfaces have the same magnitude q but opposite signs. (*b*) As the field lines show, the electric field due to the charged plates is uniform in the central region between the plates. The field is not uniform at the edges of the plates, as indicated by the "fringing" of the field lines there.

As you will see, the farad is a very large unit. Submultiples of the farad, such as the microfarad ($1\ \mu F = 10^{-6}$ F) and the picofarad ($1\ pF = 10^{-12}$ F), are more convenient units in practice.

Charging a Capacitor

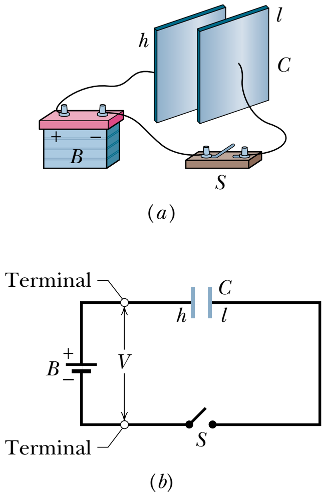


Figure 25-4 (*a*) Battery B, switch S, and plates *h* and *l* of capacitor C, connected in a circuit. (*b*) A schematic diagram with the *circuit elements* represented by their symbols.

One way to charge a capacitor is to place it in an electric circuit with a battery. An *electric circuit* is a path through which charge can flow. A *battery* is a device that maintains a certain potential difference between its *terminals* (points at which charge can enter or leave the battery) by means of internal electrochemical reactions in which electric forces can move internal charge.

In Fig. 25-4*a*, a battery *B*, a switch *S*, an uncharged capacitor *C*, and interconnecting wires form a circuit. The same circuit is shown in the *schematic diagram* of Fig. 25-4*b*, in which the symbols for a battery, a switch, and a capacitor represent those devices. The battery maintains potential difference *V* between its terminals. The terminal of higher potential is labeled + and is often called the *positive* terminal; the terminal of lower potential is labeled − and is often called the *negative* terminal.

The circuit shown in Figs. 25-4*a* and *b* is said to be *incomplete* because switch *S* is *open*; that is, the switch does not electrically connect the wires attached to it. When the switch is *closed*, electrically connecting those wires, the circuit is complete and charge can then flow through the switch and the wires. We know that the charge that can flow through a conductor, such as a wire, is that of electrons. When the circuit of Fig. 25-4 is completed, electrons are driven through the wires by an electric field that the battery sets up in the wires. The field drives electrons from capacitor plate *h* to the positive terminal of the battery; thus, plate *h*, losing electrons, becomes positively charged. The field drives just as many electrons from the negative terminal of the battery to capacitor plate *l*; thus, plate *l*, gaining electrons, becomes negatively charged *just as much* as plate *h*, losing electrons, becomes positively charged.

Initially, when the plates are uncharged, the potential difference between them is zero. As the plates become oppositely charged, that potential difference increases until it equals the potential difference *V* between the terminals of the battery. Then plate *h* and the positive terminal of the battery are at the same potential, and there is no longer an electric field in the wire between them. Similarly, plate *l* and the negative terminal reach the same potential, and there is then no electric field in the wire between them. Thus, with the field zero, there is no further drive of electrons. The capacitor is then said to be *fully charged*, with a potential difference *V* and charge *q* that are related by Eq. 25-1.

In this book we assume that during the charging of a capacitor and afterward, charge cannot pass from one plate to the other across the gap separating them. Also, we assume that a capacitor can retain (or *store*) charge indefinitely, until it is put into a circuit where it can be *discharged*.

CHECKPOINT 1

Does the capacitance *C* of a capacitor increase, decrease, or remain the same (a) when the charge *q* on it is doubled and (b) when the potential difference *V* across it is tripled?

PROBLEM-SOLVING TACTIC

Tactic 1: The Symbol V and Potential Difference In previous chapters, the symbol *V* represents an electric potential at a point or along an equipotential surface. However, in matters concerning electrical devices, *V* often represents a *potential difference* between two points or two equipotential surfaces. Equation 25-1 is an example of this second use of the symbol. In Section 25.3, you will see a mixture of the two meanings of *V*. There and in later chapters, you need to be alert as to the intent of this symbol.

You will also be seeing, in this book and elsewhere, a variety of phrases regarding potential difference. A potential difference or a "potential" or a "voltage" may be *applied* to a device, or it may be *across* a device. A capacitor can be charged to a potential difference, as in "a capacitor is charged to 12 V." Also, a battery can be characterized by the potential difference across it, as in "a 12 V battery." Always keep in mind what is meant by such phrases: There is a potential difference between two points, such as two points in a circuit or at the terminals of a device such as a battery.

25.3 | CALCULATING THE CAPACITANCE

Key Concepts

- We generally determine the capacitance of a particular capacitor configuration by (1) assuming a charge q to have been placed on the plates, (2) finding the electric field $\vec{E}$ due to this charge, (3) evaluating the potential difference V between the plates, and (4) calculating C from $q = CV$. Some results are the following:
- A parallel-plate capacitor with flat parallel plates of area A and spacing d has capacitance

$$C = \frac{\varepsilon_0 A}{d}.$$

- A cylindrical capacitor (two long coaxial cylinders) of length L and radii a and b has capacitance

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(\frac{b}{a})}.$$

- A spherical capacitor with concentric spherical plates of radii a and b has capacitance

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a}.$$

- An isolated sphere of radius R has capacitance

$$C = 4\pi\varepsilon_0 R.$$

Our goal here is to calculate the capacitance of a capacitor once we know its geometry. Because we shall consider a number of different geometries, it seems wise to develop a general plan to simplify the work. In brief our plan is as follows: (1) Assume a charge q on the plates; (2) calculate the electric field $\vec{E}$ between the plates in terms of this charge, using Gauss' law; (3) knowing $\vec{E}$, calculate the potential difference V between the plates from Eq. 24-18; (4) calculate C from Eq. 25-1.

Before we start, we can simplify the calculation of both the electric field and the potential difference by making certain assumptions. We discuss each in turn.

Calculating the Electric Field

To relate the electric field $\vec{E}$ between the plates of a capacitor to the charge q on either plate, we shall use Gauss' law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q. \tag{25-3}$$

Here q is the charge enclosed by a Gaussian surface and $\oint \vec{E} \cdot d\vec{A}$ is the net electric flux through that surface. In all cases that we shall consider, the Gaussian surface will be such that whenever there is an electric flux through it, $\vec{E}$ will have a uniform magnitude E and the vectors $\vec{E}$ and $d\vec{A}$ will be parallel. Equation 25-3 then reduces to

$$q = \varepsilon_0 EA \quad \text{(special case of Eq. 25-3)}, \tag{25-4}$$

in which A is the area of that part of the Gaussian surface through which there is a flux. For convenience, we shall always draw the Gaussian surface in such a way that it completely encloses the charge on the positive plate; see Fig. 25-5 for an example.

We use Gauss' law to relate q and E. Then we integrate the E to get the potential difference.

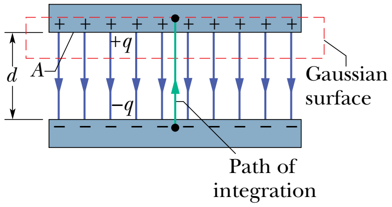


Figure 25-5 A charged parallel-plate capacitor. A Gaussian surface encloses the charge on the positive plate. The integration of Eq. 25-6 is taken along a path extending directly from the negative plate to the positive plate.

Calculating the Potential Difference

In the notation of Chapter 24 (Eq. 24-18), the potential difference between the plates of a capacitor is related to the field $\vec{E}$ by

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}, \tag{25-5}$$

in which the integral is to be evaluated along any path that starts on one plate and ends on the other. We shall always choose a path that follows an electric field line, from the negative plate to the positive plate. For this path, the vectors $\vec{E}$ and $d\vec{s}$ will have opposite directions; so the dot product $\vec{E} \cdot d\vec{s}$ will be equal to $-E\,ds$. Thus, the right side of Eq. 25-5 will then be positive. Letting V represent the difference $V_f - V_i$, we can then recast Eq. 25-5 as

$$V = \int_-^+ E\,ds \quad \text{(special case of Eq. 25-5)}, \tag{25-6}$$

in which the $-$ and $+$ remind us that our path of integration starts on the negative plate and ends on the positive plate.

We are now ready to apply Eqs. 25-4 and 25-6 to some particular cases.

A Parallel-Plate Capacitor

We assume, as Fig. 25-5 suggests, that the plates of our parallel-plate capacitor are so large and so close together that we can neglect the fringing of the electric field at the edges of the plates, taking $\vec{E}$ to be constant throughout the region between the plates.

We draw a Gaussian surface that encloses just the charge q on the positive plate, as in Fig. 25-5. From Eq. 25-4 we can then write

$$q = \varepsilon_0 EA, \tag{25-7}$$

where A is the area of the plate.

Equation 25-6 yields

$$V = \int_-^+ E\,ds = E\int_0^d ds = Ed. \tag{25-8}$$

In Eq. 25-8, E can be placed outside the integral because it is a constant; the second integral then is simply the plate separation d.

If we now substitute q from Eq. 25-7 and V from Eq. 25-8 into the relation $q = CV$ (Eq. 25-1), we find

$$C = \frac{\varepsilon_0 A}{d} \quad \text{(parallel-plate capacitor)}. \tag{25-9}$$

Thus, the capacitance does indeed depend only on geometrical factors—namely, the plate area A and the plate separation d. Note that C increases as we increase area A or decrease separation d.

As an aside, we point out that Eq. 25-9 suggests one of our reasons for writing the electrostatic constant in Coulomb's law in the form $1/4\pi\varepsilon_0$. If we had not done so, Eq. 25-9—which is used more often in engineering practice than Coulomb's law—would have been less simple in form. We note further that Eq. 25-9 permits us to express the permittivity constant ε_0 in a unit more appropriate for use in problems involving capacitors; namely,

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} = 8.85\ \text{pF/m}. \tag{25-10}$$

We have previously expressed this constant as

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2. \tag{25-11}$$

A Cylindrical Capacitor

Figure 25-6 shows, in cross section, a cylindrical capacitor of length L formed by two coaxial cylinders of radii a and b. We assume that $L \gg b$ so that we can neglect the fringing of the electric field that occurs at the ends of the cylinders. Each plate contains a charge of magnitude q.

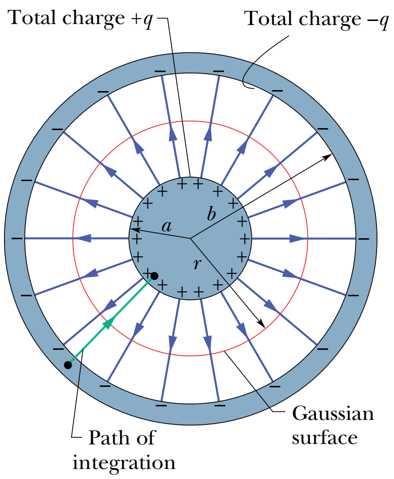


Figure 25-6 A cross section of a long cylindrical capacitor, showing a cylindrical Gaussian surface of radius r (that encloses the positive plate) and the radial path of integration along which Eq. 25-6 is to be applied. This figure also serves to illustrate a spherical capacitor in a cross section through its center.

As a Gaussian surface, we choose a cylinder of length L and radius r, closed by end caps and placed as is shown in Fig. 25-6. It is coaxial with the cylinders and encloses the central cylinder and thus also the charge q on that cylinder. Equation 25-4 then relates that charge and the field magnitude E as

$$q = \varepsilon_0 EA = \varepsilon_0 E(2\pi rL),$$

in which $2\pi rL$ is the area of the curved part of the Gaussian surface. There is no flux through the end caps. Solving for E yields

$$E = \frac{q}{2\pi\varepsilon_0 Lr}. \tag{25-12}$$

Substitution of this result into Eq. 25-6 yields

$$V = \int_-^+ E\,ds = -\frac{q}{2\pi\varepsilon_0 L}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right), \tag{25-13}$$

where we have used the fact that here $ds = -dr$ (we integrated radially inward).

From the relation $C = q/V$, we then have

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)} \quad \text{(cylindrical capacitor)}. \tag{25-14}$$

We see that the capacitance of a cylindrical capacitor, like that of a parallel-plate capacitor, depends only on geometrical factors, in this case the length L and the two radii b and a.

A Spherical Capacitor

Figure 25-6 can also serve as a central cross section of a capacitor that consists of two concentric spherical shells, of radii a and b. As a Gaussian surface we draw a sphere of radius r concentric with the two shells; then Eq. 25-4 yields

$$q = \varepsilon_0 EA = \varepsilon_0 E(4\pi r^2),$$

in which $4\pi r^2$ is the area of the spherical Gaussian surface. We solve this equation for E, obtaining

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, \tag{25-15}$$

which we recognize as the expression for the electric field due to a uniform spherical charge distribution (Eq. 23-12).

If we substitute this expression into Eq. 25-6, we find

$$V = \int_-^+ E\,ds = -\frac{q}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{q}{4\pi\varepsilon_0}\frac{b-a}{ab}, \tag{25-16}$$

where again we have substituted $-dr$ for ds. If we now substitute Eq. 25-16 into Eq. 25-1 and solve for C, we find

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a} \quad \text{(spherical capacitor)}. \tag{25-17}$$

An Isolated Sphere

We can assign a capacitance to a *single* isolated spherical conductor of radius R by assuming that the "missing plate" is a conducting sphere of infinite radius. After all, the field lines that leave the surface of a positively charged isolated conductor must end somewhere; the walls of the room in which the conductor is housed can serve effectively as our sphere of infinite radius.

To find the capacitance of the conductor, we first rewrite Eq. 25-17 as

$$C = 4\pi\varepsilon_0 \frac{a}{1-(a/b)}.$$

If we then let $b \to \infty$ and substitute R for a, we find

$$C = 4\pi\varepsilon_0 R \quad \text{(isolated sphere).} \tag{25-18}$$

Note that this formula and the others we have derived for capacitance (Eqs. 25-9, 25-14, and 25-17) involve the constant ε_0 multiplied by a quantity that has the dimensions of a length.

CHECKPOINT 2

For capacitors charged by the same battery, does the charge stored by the capacitor increase, decrease, or remain the same in each of the following situations? (a) The plate separation of a parallel-plate capacitor is increased. (b) The radius of the inner cylinder of a cylindrical capacitor is increased. (c) The radius of the outer spherical shell of a spherical capacitor is increased.

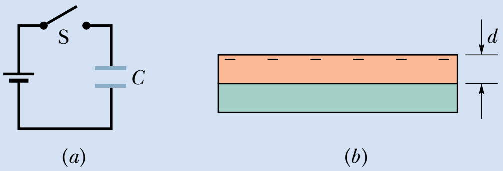
SAMPLE PROBLEM 25.01

Charging the plates in a parallel-plate capacitor

In Fig. 25-7*a*, switch S is closed to connect the uncharged capacitor of capacitance $C = 0.25\ \mu\text{F}$ to the battery of potential difference $V = 12$ V. The lower capacitor plate has thickness $L = 0.50$ cm and face area $A = 2.0 \times 10^{-4}\ \text{m}^2$, and it consists of copper, in which the density of conduction electrons is $n = 8.49 \times 10^{28}$ electrons/m^3. From what depth d within the plate (Fig. 25-7*b*) must electrons move to the plate face as the capacitor becomes charged?

KEY IDEA

The charge collected on the plate is related to the capacitance and the potential difference across the capacitor by Eq. 25-1 ($q = CV$).



(*a*) (*b*)

Figure 25-7 (*a*) A battery and capacitor circuit. (*b*) The lower capacitor plate.

Calculations: Because the lower plate is connected to the negative terminal of the battery, conduction electrons move up to the face of the plate. From Eq. 25-1, the total charge magnitude that collects there is

$$q = CV = (0.25 \times 10^{-6}\ \text{F})(12\ \text{V})$$
$$= 3.0 \times 10^{-6}\ \text{C}.$$

Dividing this result by e gives us the number N of conduction electrons that come up to the face:

$$N = \frac{q}{e} = \frac{3.0\times10^{-6}\ \text{C}}{1.602\times10^{-19}\ \text{C}}$$
$$= 1.873\times10^{13}\ \text{electrons}.$$

These electrons come from a volume that is the product of the face area A and the depth d we seek. Thus, from the density of conduction electrons (number per volume), we can write

$$n = \frac{N}{Ad},$$

or

$$d=\frac{N}{An}=\frac{1.873\times10^{13}\ \text{electrons}}{(2.0\times10^{-4}\ \text{m}^2)(8.49\times10^{28}\ \text{electrons/m}^3)}$$

$$=1.1\times10^{-12}\ \text{m}=1.1\ \text{pm}. \qquad \text{(Answer)}$$

We commonly say that electrons move from the battery to the negative face but, actually, the battery sets up an electric field in the wires and plate such that electrons very close to the plate face move up to the negative face.

25.4 | CAPACITORS IN PARALLEL AND IN SERIES

Key Concepts

- The equivalent capacitances C_{eq} of combinations of individual capacitors connected in parallel and in series can be found from

$$C_{\text{eq}}=\sum_{j=1}^{n}C_j \quad (n \text{ capacitors in parallel})$$

and

$$\frac{1}{C_{\text{eq}}}=\sum_{j=1}^{n}\frac{1}{C_j} \quad (n \text{ capacitors in series}).$$

Equivalent capacitances can be used to calculate the capacitances of more complicated series–parallel combinations.

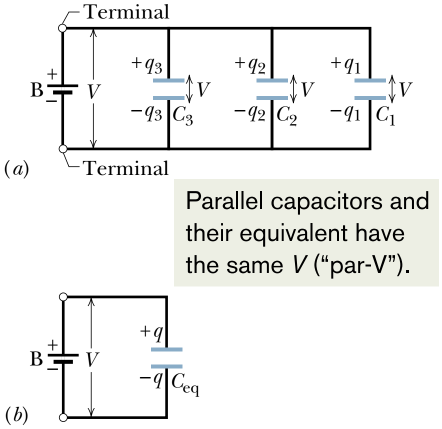
When there is a combination of capacitors in a circuit, we can sometimes replace that combination with an **equivalent capacitor**—that is, a single capacitor that has the same capacitance as the actual combination of capacitors. With such a replacement, we can simplify the circuit, affording easier solutions for unknown quantities of the circuit. Here we discuss two basic combinations of capacitors that allow such a replacement.

Capacitors in Parallel

Figure 25-8*a* shows an electric circuit in which three capacitors are connected *in parallel* to battery B. This description has little to do with how the capacitor plates are drawn. Rather, "in parallel" means that the capacitors are directly wired together at one plate and directly wired together at the other plate, and that the same potential difference V is applied across the two groups of wired-together plates. Thus, each capacitor has the same potential difference V, which produces charge on the capacitor. (In Fig. 25-8*a*, the applied potential V is maintained by the battery.) In general:

> When a potential difference V is applied across several capacitors connected in parallel, that potential difference V is applied across each capacitor. The total charge q stored on the capacitors is the sum of the charges stored on all the capacitors.

When we analyze a circuit of capacitors in parallel, we can simplify it with this mental replacement:



Parallel capacitors and their equivalent have the same V ("par-V").

Figure 25-8 (*a*) Three capacitors connected in parallel to battery B. The battery maintains potential difference V across its terminals and thus across *each* capacitor. (*b*) The equivalent capacitor, with capacitance C_{eq}, replaces the parallel combination.

> Capacitors connected in parallel can be replaced with an equivalent capacitor that has the same *total* charge q and the same potential difference V as the actual capacitors.

(You might remember this result with the nonsense word "par-V," which is close to "party," to mean "capacitors in parallel have the same V.") Figure 25-8*b* shows the equivalent capacitor (with equivalent capacitance C_{eq}) that has replaced the three capacitors (with actual capacitances C_1, C_2, and C_3) of Fig. 25-8*a*.

To derive an expression for C_{eq} in Fig. 25-8*b*, we first use Eq. 25-1 to find the charge on each actual capacitor:

$$q_1 = C_1V, q_2 = C_2V, \quad \text{and} \quad q_3 = C_3V.$$

The total charge on the parallel combination of Fig. 25-8*a* is then

$$q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V.$$

The equivalent capacitance, with the same total charge q and applied potential difference V as the combination, is then

$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3,$$

a result that we can easily extend to any number n of capacitors, as

$$C_{eq} = \sum_{j=1}^{n} C_j \quad (n \text{ capacitors in parallel}). \tag{25-19}$$

Thus, to find the equivalent capacitance of a parallel combination, we simply add the individual capacitances.

Capacitors in Series

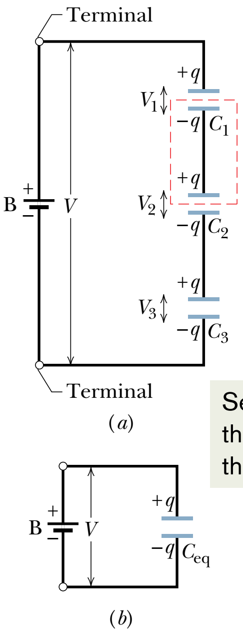
Figure 25-9*a* shows three capacitors connected *in series* to battery B. This description has little to do with how the capacitors are drawn. Rather, "in series" means that the capacitors are wired serially, one after the other, and that a potential difference V is applied across the two ends of the series. (In Fig. 25-9*a*, this potential difference V is maintained by battery B.) The potential differences that then exist across the capacitors in the series produce identical charges q on them.

> When a potential difference V is applied across several capacitors connected in series, the capacitors have identical charge q. The sum of the potential differences across all the capacitors is equal to the applied potential difference V.

We can explain how the capacitors end up with identical charge by following a *chain reaction* of events, in which the charging of each capacitor causes the charging of the next capacitor. We start with capacitor 3 and work upward to capacitor 1. When the battery is first connected to the series of capacitors, it produces charge $-q$ on the bottom plate of capacitor 3. That charge then repels negative charge from the top plate of capacitor 3 (leaving it with charge $+q$). The repelled negative charge moves to the bottom plate of capacitor 2 (giving it charge $-q$). That charge on the bottom plate of capacitor 2 then repels negative charge from the top plate of capacitor 2 (leaving it with charge $+q$) to the bottom plate of capacitor 1 (giving it charge $-q$). Finally, the charge on the bottom plate of capacitor 1 helps move negative charge from the top plate of capacitor 1 to the battery, leaving that top plate with charge $+q$.

Here are two important points about capacitors in series:

1. When charge is shifted from one capacitor to another in a series of capacitors, it can move along only one route, such as from capacitor 3 to capacitor 2 in Fig. 25-9*a*. If there are additional routes, the capacitors are not in series.



Series capacitors and their equivalent have the same q ("seri-q").

Figure 25-9 (*a*) Three capacitors connected in series to battery B. The battery maintains potential difference V between the top and bottom plates of the series combination. (*b*) The equivalent capacitor, with capacitance C_{eq}, replaces the series combination.

2. The battery directly produces charges on only the two plates to which it is connected (the bottom plate of capacitor 3 and the top plate of capacitor 1 in Fig. 25-9*a*). Charges that are produced on the other plates are due merely to the shifting of charge already there. For example, in Fig. 25-9*a*, the part of the circuit enclosed by dashed lines is electrically isolated from the rest of the circuit. Thus, its charge can only be redistributed.

When we analyze a circuit of capacitors in series, we can simplify it with this mental replacement:

> Capacitors that are connected in series can be replaced with an equivalent capacitor that has the same charge q and the same *total* potential difference V as the actual series capacitors.

Obviously the charges on connected plates are equal and opposite if they are initially uncharged. (You might remember this with the nonsense word "seri-q" to mean "capacitors in series have the same q.") Figure 25-9*b* shows the equivalent capacitor (with equivalent capacitance C_{eq}) that has replaced the three actual capacitors (with actual capacitances C_1, C_2, and C_3) of Fig. 25-9*a*.

To derive an expression for C_{eq} in Fig. 25-9*b*, we first use Eq. 25-1 to find the potential difference of each actual capacitor:

$$V_1 = \frac{q}{C_1}, \quad V_2 = \frac{q}{C_2}, \quad \text{and} \quad V_3 = \frac{q}{C_3}.$$

The total potential difference V due to the battery is the sum

$$V = V_1 + V_2 + V_3 = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right).$$

The equivalent capacitance is then

$$C_{eq} = \frac{q}{V} = \frac{1}{1/C_1 + 1/C_2 + 1/C_3},$$

or

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

We can easily extend this to any number n of capacitors as

$$\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j} \quad (n \text{ capacitors in series}). \tag{25-20}$$

Using Eq. 25-20 you can show that the equivalent capacitance of a series of capacitances is always *less* than the least capacitance in the series.

CHECKPOINT 3

A battery of potential V stores charge q on a combination of two identical capacitors. What are the potential difference across and the charge on either capacitor if the capacitors are (a) in parallel and (b) in series?

SAMPLE PROBLEM 25.02

Capacitors in parallel and in series

(a) Find the equivalent capacitance for the combination of capacitances shown in Fig. 25-10*a*, across which potential difference V is applied. Assume

$$C_1 = 12.0\ \mu\text{F},\ C_2 = 5.30\ \mu\text{F},\quad \text{and}\quad C_3 = 4.50\ \mu\text{F}.$$

KEY IDEA

Any capacitors connected in series can be replaced with their equivalent capacitor, and any capacitors connected in parallel can be replaced with their equivalent capacitor. Therefore, we should first check whether any of the capacitors in Fig. 25-10*a* are in parallel or series.

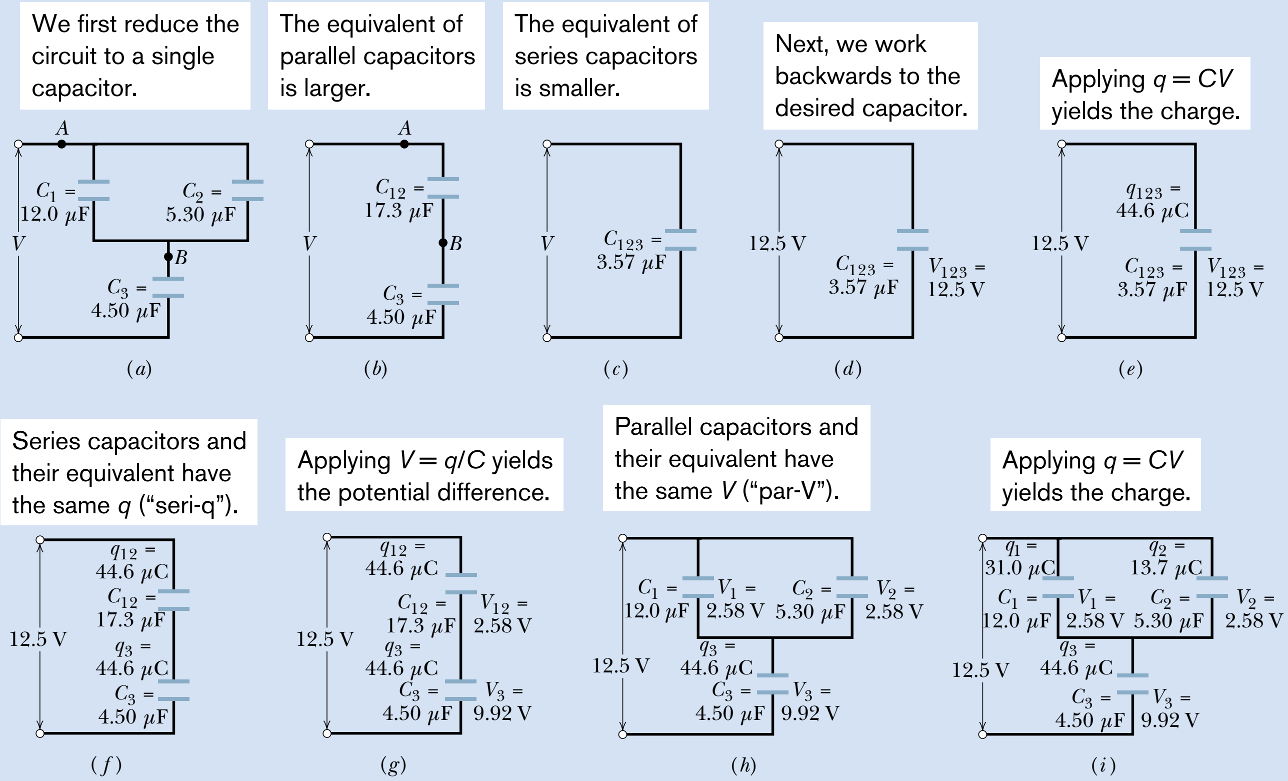
Finding equivalent capacitance: Capacitors 1 and 3 are connected one after the other, but are they in series? No. The potential V that is applied to the capacitors produces charge on the bottom plate of capacitor 3. That charge causes charge to shift from the top plate of capacitor 3. However, note that the shifting charge can move to the bottom plates of both capacitor 1 and capacitor 2. Because there is more than one route for the shifting charge, capacitor 3 is not in series with capacitor 1 (or capacitor 2). Any time you think you might have two capacitors in series, apply this check about the shifting charge.

Are capacitor 1 and capacitor 2 in parallel? Yes. Their top plates are directly wired together and their bottom plates are directly wired together, and electric potential is applied between the top-plate pair and the bottom-plate pair. Thus, capacitor 1 and capacitor 2 are in parallel, and Eq. 25-19 tells us that their equivalent capacitance C_{12} is

$$C_{12} = C_1 + C_2 = 12.0\ \mu\text{F} + 5.30\ \mu\text{F} = 17.3\ \mu\text{F}.$$

In Fig. 25-10*b*, we have replaced capacitors 1 and 2 with their equivalent capacitor, called capacitor 12 (say "one two" and not "twelve"). (The connections at points A and B are exactly the same in Figs. 25-10*a* and *b*.)

Is capacitor 12 in series with capacitor 3? Again applying the test for series capacitances, we see that the charge that shifts from the top plate of capacitor 3 must entirely


We first reduce the circuit to a single capacitor.

The equivalent of parallel capacitors is larger.

The equivalent of series capacitors is smaller.

Next, we work backwards to the desired capacitor.

Applying $q = CV$ yields the charge.

Series capacitors and their equivalent have the same q ("seri-q").

Applying $V = q/C$ yields the potential difference.

Parallel capacitors and their equivalent have the same V ("par-V").

Applying $q = CV$ yields the charge.

Figure 25-10 (*a*)–(*d*) Three capacitors are reduced to one equivalent capacitor. (*e*)–(*i*) Working backwards to get the charges.

go to the bottom plate of capacitor 12. Thus, capacitor 12 and capacitor 3 are in series, and we can replace them with their equivalent C_{123} ("one two three"), as shown in Fig. 25-10*c*. From Eq. 25-20, we have

$$\frac{1}{C_{123}} = \frac{1}{C_{12}} + \frac{1}{C_3} = \frac{1}{17.3\ \mu\text{F}} + \frac{1}{4.50\ \mu\text{F}} = 0.280\ \mu\text{F}^{-1},$$

from which

$$C_{123} = \frac{1}{0.280\ \mu\text{F}^{-1}} = 3.57\ \mu\text{F}. \qquad \text{(Answer)}$$

(b) The potential difference applied to the input terminals in Fig. 25-10*a* is $V = 12.5$ V. What is the charge on C_1?

KEY IDEAS

We now need to work backwards from the equivalent capacitance to get the charge on a particular capacitor. We have two techniques for such "backwards work": (1) Seri-q: Series capacitors have the same charge as their equivalent capacitor. (2) Par-V: Parallel capacitors have the same potential difference as their equivalent capacitor.

Working backwards: To get the charge q_1 on capacitor 1, we work backwards to that capacitor, starting with the equivalent capacitor 123. Because the given potential difference V (= 12.5 V) is applied across the actual combination of three capacitors in Fig. 25-10*a*, it is also applied across C_{123} in Figs. 25-10*d* and *e*. Thus, Eq. 25-1 ($q = CV$) gives us

$$q_{123} = C_{123}V = (3.57\ \mu\text{F})(12.5\ \text{V}) = 44.6\ \mu\text{C}.$$

The series capacitors 12 and 3 in Fig. 25-10*b* each have the same charge as their equivalent capacitor 123 (Fig. 25-10*f*). Thus, capacitor 12 has charge $q_{12} = q_{123} = 44.6\ \mu$C. From Eq. 25-1 and Fig. 25-10*g*, the potential difference across capacitor 12 must be

$$V_{12} = \frac{q_{12}}{C_{12}} = \frac{44.6\ \mu\text{C}}{17.3\ \mu\text{F}} = 2.58\ \text{V}.$$

The parallel capacitors 1 and 2 each have the same potential difference as their equivalent capacitor 12 (Fig. 25-10*h*). Thus, capacitor 1 has potential difference $V_1 = V_{12} = 2.58$ V, and, from Eq. 25-1 and Fig. 25-10*i*, the charge on capacitor 1 must be

$$q_1 = C_1V_1 = (12.0\ \mu\text{F})(2.58\ \text{V}) = 31.0\ \mu\text{C}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 25.03

One capacitor charging up another capacitor

Capacitor 1, with $C_1 = 3.55\ \mu$F, is charged to a potential difference $V_0 = 6.30$ V, using a 6.30 V battery. The battery is then removed, and the capacitor is connected as in Fig. 25-11 to an uncharged capacitor 2, with $C_2 = 8.95\ \mu$F. When switch S is closed, charge flows between the capacitors. Find the charge on each capacitor when equilibrium is reached.

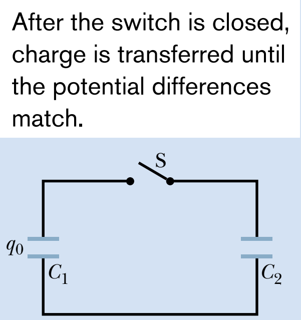

After the switch is closed, charge is transferred until the potential differences match.
S
q_0
C_1
C_2

Figure 25-11 A potential difference V_0 is applied to capacitor 1 and the charging battery is removed. Switch S is then closed so that the charge on capacitor 1 is shared with capacitor 2.

KEY IDEAS

The situation here differs from the previous example because here an applied electric potential is *not* maintained across a combination of capacitors by a battery or some other source. Here, just after switch S is closed, the only applied electric potential is that of capacitor 1 on capacitor 2, and that potential is decreasing. Thus, the capacitors in Fig. 25-11 are not connected *in series*; and although they are drawn parallel, in this situation they are not *in parallel*.

As the electric potential across capacitor 1 decreases, that across capacitor 2 increases. Equilibrium is reached when the two potentials are equal because, with no potential difference between connected plates of the capacitors, there is no electric field within the connecting wires to move conduction electrons. The initial charge on capacitor 1 is then shared between the two capacitors.

Calculations: Initially, when capacitor 1 is connected to the battery, the charge it acquires is, from Eq. 25-1,

$$q_0 = C_1V_0 = (3.55 \times 10^{-6}\ \text{F})(6.30\ \text{V})$$
$$= 22.365 \times 10^{-6}\ \text{C}.$$

When switch S in Fig. 25-11 is closed and capacitor 1 begins to charge capacitor 2, the electric potential and charge on capacitor 1 decrease and those on capacitor 2 increase until

$$V_1 = V_2 \quad \text{(equilibrium).}$$

From Eq. 25-1, we can rewrite this as

$$\frac{q_1}{C_1} = \frac{q_2}{C_2} \quad \text{(equilibrium).}$$

Because the total charge cannot magically change, the total after the transfer must be

$$q_1 + q_2 = q_0 \quad \text{(charge conservation);}$$

thus $$q_2 = q_0 - q_1.$$

We can now rewrite the second equilibrium equation as

$$\frac{q_1}{C_1} = \frac{q_0 - q_1}{C_2}.$$

Solving this for q_1 and substituting given data, we find

$$q_1 = 6.35\ \mu\text{C}. \qquad \text{(Answer)}$$

The rest of the initial charge ($q_0 = 22.365\ \mu$C) must be on capacitor 2:

$$q_2 = 16.0\ \mu\text{C}. \qquad \text{(Answer)}$$

PROBLEM-SOLVING TACTICS

Tactic 2: Multiple-Capacitor Circuits Let us review the procedure used in the solution of Sample Problem 25.02, in which several capacitors are connected to a battery. To find a single equivalent capacitance, we simplify the given arrangement of capacitances by replacing them, in steps, with equivalent capacitances, using Eq. 25-19 when we find capacitances in parallel and Eq. 25-20 when we find capacitances in series. Then, to find the charge stored by that single equivalent capacitance, we use Eq. 25-1 and the potential difference V applied by the battery.

That result tells us the net charge stored on the actual arrangement of capacitors. However, to find the charge on, or the potential difference across, any particular capacitor in the actual arrangement, we need to reverse our steps of simplification. With each reversed step, we use these two rules: When capacitances are in parallel, they have the same potential difference as their equivalent capacitance, and we use Eq. 25-1 to find the charge on each capacitance; when they are in series, they have the same charge as their equivalent capacitance, and we use Eq. 25-1 to find the potential difference across each capacitance.

Tactic 3: Batteries and Capacitors A battery maintains a certain potential difference across its terminals. Thus, when capacitor 1 of Sample Problem 25.03 is connected to the 6.30 V battery, charge flows between the capacitor and the battery until the capacitor has the same potential difference across it as the battery. A capacitor differs from a battery in that a capacitor lacks the internal electrochemical reactions needed to release charged particles (electrons) from internal atoms and molecules. Thus, when the charged capacitor 1 of Sample Problem 25.03 is disconnected from the battery and then connected to the uncharged capacitor 2 with switch S closed, the potential difference across capacitor 1 is not maintained. The quantity that *is* maintained is the charge q_0 of the two-capacitor system; that is, charge obeys a conservation law, *not* electric potential.

Capacitors in Complex Circuits

For more complicated circuits, we are going to introduce two methods. You are free to choose any one of them as per your convenience. We will formalize them in Chapter 26.

Let us consider a circuit consisting of three capacitors as shown in Fig. 25-12. Two of them C_1 and C_2 are already charged as shown and the third one C_3 has just now been connected. This arrangement is a little deceptive. The capacitors seem to be in series, but they are not. The charges on them are not equal.

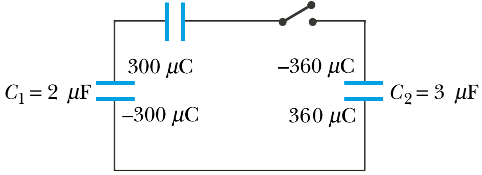


Figure. 25-12 Three capacitors connected to charge the capacitor C_3.

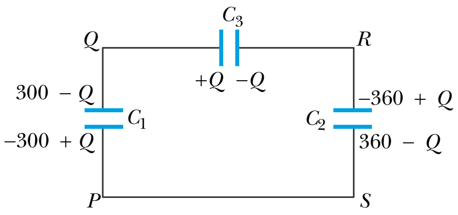


Figure 25-13 Steady-state charges on C_1, C_2, and C_3.

Method 1: Let us assume that the left plate of a C_3 gets a charge of $+Q$ in the steady state. Then its right plate will acquire a charge of $-Q$. By the law of conservation of charge, we can say that the upper plate of C_2 will get an additional charge of $+Q$. So, its lower plate charge will reduce by Q. This plate acquires this charge from the lower plate of C_1. Thus, in steady state, the plate charges will look the way shown in Fig. 25-13.

Now we know that if we start from point P and traverse around the loop $PQRS$, by the conservative nature of electric field

$$\oint \vec{E}\cdot d\vec{l} = 0.$$

So, if we denote V_1, V_2 and V_3 as the potential drops across C_1, C_2 and C_3, respectively, then conservative nature of electric field gives

$$V_1 + V_2 + V_3 = 0.$$

In this situation, it implies that

$$\frac{300-Q}{C_1} - \frac{Q}{C_3} + \frac{360-Q}{C_2} = 0.$$

Note that potential increases when we move from negative plate to positive plate and vice versa. Solving the above equation, we get

$$Q = 180\ \mu\text{C}.$$

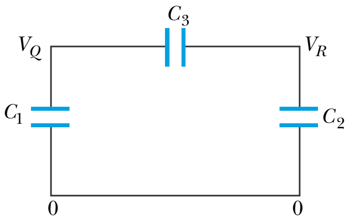


Figure 25-14 Potentials for the circuit of Fig. 26-13.

Method 2: Here we will use the law of conservation of charges. At all the three nodes P, Q, and R, the charge before the connection should be equal to the charge after the connection. Thus, the charge at node P before connection was 360 – 360 = 60 μC, at Q it was 300 μC, at R it was –360 μC.

Let us assume that point P is at zero potential. (Recall that: Any point can be taken to be reference at zero potential.) Assume that potential of Q in steady state is V_Q and that of R in steady state is V_R as seen in Fig. 25-14.

The total charge at Q is now

$$C_1(V_Q - 0) + C_3(V_Q - V_R) \quad \text{(at upper plate of } C_1 \text{ and left plate of } C_3\text{)}$$

$$2(V_Q - 0) + 1.5(V_Q - V_R) = 300.$$

The total charge at R is now

$$C_3(V_R - V_Q) + C_2(V_R - 0) \quad \text{(at right plate of } C_3 \text{ and upper plate of } C_2\text{)}$$

$$1.5(V_R - V_Q) + 3V_R = -360.$$

Solving these two equations, we get $V_Q = 60$ V and $V_R = -60$ V.

So, the charge on C_3 in steady state is $Q = 1.5[60 - (60)] = 180\ \mu$C, as we found before.

25.5 | ENERGY STORED IN AN ELECTRIC FIELD

Key Concepts

- The electric potential energy U of a charged capacitor,

$$U = \frac{q^2}{2C} = \frac{1}{2}CV^2,$$

is equal to the work required to charge the capacitor. This energy can be associated with the capacitor's electric field $\vec{E}$.

- Every electric field, in a capacitor or from any other source, has an associated stored energy. In vacuum, the energy density u (potential energy per unit volume) in a field of magnitude E is

$$u = \frac{1}{2}\varepsilon_0 E^2.$$

Work must be done by an external agent to charge a capacitor. We can imagine doing the work ourselves by transferring electrons from one plate to the other, one by one. As the charges build, so does the electric field between the plates, which opposes the continued transfer. So, greater amounts of work are required. Actually, a battery does all this for us, at the expense of its stored chemical energy. We visualize the work as being stored as electric potential energy U in the electric field between the plates. You can recover this energy at will, by discharging the capacitor in a circuit.

Suppose that, at a given instant, a charge q' has been transferred from one plate of a capacitor to the other. The potential difference V' between the plates at that instant will be q'/C. If an extra increment of charge dq' is then transferred, the increment of work required will be, from Eq. 24-6,

$$dW = V'\,dq' = \frac{q'}{C}\,dq'.$$

The work required to bring the total capacitor charge up to a final value q is

$$W = \int dW = \frac{1}{C}\int_0^q q'\,dq' = \frac{q^2}{2C}.$$

This work is stored as potential energy U in the capacitor, so that

$$U = \frac{q^2}{2C} \quad \text{(potential energy)}. \tag{25-21}$$

From Eq. 25-1, we can also write this as

$$U = \frac{1}{2}CV^2 \quad \text{(potential energy)}. \tag{25-22}$$

Equations 25-21 and 25-22 hold no matter what the geometry of the capacitor is.

To gain some physical insight into energy storage, consider two parallel-plate capacitors that are identical except that capacitor 1 has twice the plate separation of capacitor 2. Then capacitor 1 has twice the volume between its plates and also, from Eq. 25-9, half the capacitance of capacitor 2. Equation 25-4 tells us that if both capacitors have the same charge q, the electric fields between their plates are identical. And Eq. 25-21 tells us that capacitor 1 has twice the stored potential energy of capacitor 2. Thus, of two otherwise identical capacitors with the same charge and same electric field, the one with twice the volume between its plates has twice the stored potential energy. Arguments like this tend to verify our earlier assumption:

> The potential energy of a charged capacitor may be viewed as being stored in the electric field between its plates.

Explosions in Airborne Dust

We have discussed earlier that making contact with certain materials, such as clothing, carpets, and even playground slides, can leave you with a significant electrical potential. You might become painfully aware of that potential if a spark leaps between you and a grounded object, such as a faucet. In many industries involving the production and transport of powder, such as in the cosmetic and food industries, such a spark can be disastrous. Although the powder in bulk may not burn at all, when individual powder grains are airborne and thus surrounded by oxygen, they can burn so fiercely that a cloud of the grains burns as an explosion. Safety engineers cannot eliminate all possible sources of sparks in the powder industries. Instead, they attempt to keep the amount of energy available in the sparks below the threshold value U_t ($\approx$ 150 mJ) typically required to ignite airborne grains.

Suppose a person becomes charged by contact with various surfaces as he walks through an airborne powder. We can roughly model the person as a spherical capacitor of radius $R = 1.8$ m. From Eq. 25-18 ($C = 4\pi\varepsilon_0 R$) and Eq. 25-22 ($U = 1/2CV^2$), we see that the energy of the capacitor is

$$U = \frac{1}{2}(4\pi\varepsilon_0 R)V^2.$$

From this we see that the threshold energy corresponds to a potential of

$$V = \sqrt{\frac{2U_t}{4\pi\varepsilon_0 R}} = \sqrt{\frac{2(150\times10^{-3}\text{ J})}{4\pi(8.85\times10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2)(1.8\text{ m})}}$$
$$= 3.9\times10^4\text{ V}.$$

Safety engineers attempt to keep the potential of the personnel below this level by “bleeding” off the charge through, say, a conducting floor.

Energy Density

In a parallel-plate capacitor, neglecting fringing, the electric field has the same value at all points between the plates. Thus, the **energy density** u—that is, the potential energy per unit volume between the plates—should also be uniform. We can find u by dividing the total potential energy by the volume Ad of the space between the plates. Using Eq. 25-22, we obtain

$$u = \frac{U}{Ad} = \frac{CV^2}{2Ad}. \qquad (25\text{-}23)$$

With Eq. 25-9 ($C = \varepsilon_0 A/d$), this result becomes

$$u = \frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2. \qquad (25\text{-}24)$$

However, from Eq. 24-42 ($E = -\Delta V/\Delta s$), V/d equals the electric field magnitude E; so

$$u = \frac{1}{2}\varepsilon_0 E^2 \quad \text{(energy density)}. \qquad (25\text{-}25)$$

Although we derived this result for the special case of an electric field of a parallel-plate capacitor, it holds for any electric field. If an electric field $\vec{E}$ exists at any point in space, that site has an electric potential energy with a density (amount per unit volume) given by Eq. 25-25.

SAMPLE PROBLEM 25.04

Potential energy and energy density of an electric field

An isolated conducting sphere whose radius R is 6.85 cm has a charge $q = 1.25$ nC.

(a) How much potential energy is stored in the electric field of this charged conductor?

KEY IDEAS

(1) An isolated sphere has capacitance given by Eq. 25-18 ($C = 4\pi\varepsilon_0 R$). (2) The energy U stored in a capacitor depends on the capacitor's charge q and capacitance C according to Eq. 25-21 ($U = q^2/2C$).

Calculation: Substituting $C = 4\pi\varepsilon_0 R$ into Eq. 25-21 gives us

$$U = \frac{q^2}{2C} = \frac{q^2}{8\pi\varepsilon_0 R}$$
$$= \frac{(1.25\times10^{-9}\text{ C})^2}{(8\pi)(8.85\times10^{-12}\text{ F/m})(0.0685\text{ m})}$$
$$= 1.03\times10^{-7}\text{ J} = 103\text{ nJ}. \qquad \text{(Answer)}$$

(b) What is the energy density at the surface of the sphere?

KEY IDEA

The density u of the energy stored in an electric field depends on the magnitude E of the field, according to Eq. 25-25 ($u = 1/2\varepsilon_0 E^2$).

Calculations: Here we must first find E at the surface of the sphere, as given by Eq. 23-15:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}.$$

The energy density is then

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{q^2}{32\pi^2\varepsilon_0 R^4}$$

$$= \frac{(1.25\times10^{-9}\ \text{C})^2}{(32\pi^2)(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(0.0685\ \text{m})^4}$$

$$= 2.54\times10^{-5}\ \text{J/m}^3 = 25.4\ \mu\text{J/m}^3. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 25.05

Energy stored in the system being charged by a battery

A capacitor of capacitance C is charged by connecting it to a battery of emf ξ. The capacitor is now disconnected and reconnected to the battery with the polarity reversed. Calculate the heat dissipated in the connecting wires (Fig. 25-15).

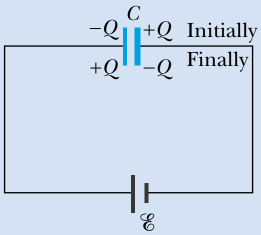


Figure 25-15 A capacitor being charged by a battery.

KEY IDEAS

The work done by the external agent on the system goes into two parts: (a) in increasing the potential energy of the system and (b) in heat. Here the battery is the agent which drives the charges around in the circuit. Obviously, it is doing work at the expense of its internal chemical energy. Although a part of it goes in the form of potential energy of the charges on the capacitor, a part of it goes as heat too. To find the heat dissipated in the circuit, we need to invoke the first law of thermodynamics.

Calculations: When the capacitor is connected to the battery, a charge $Q = C\xi$ appears on one plate and $-Q$ on the other. When the polarity is reversed, a charge $-Q$ appears on the first plate and $+Q$ on the second. A charge $2Q$, therefore, passes through the battery from the negative terminal to the positive terminal. The battery does a work equal to

$$W = (2Q)\xi = 2C\xi^2$$

in the process. The potential energy stored in the capacitor is the same in the two cases. Thus, the work done by the battery appears as heat in the connecting wires. The heat produced is, therefore, $2C\xi^2$. (Answer)

One might wonder that where did this heat energy go. Its is not mentioned in the problem that the connecting wires had some resistance, which would imply that the wires get heated up as the charge passes through them. The answer is that even if the wires were superconductors, some amount of energy will definitely be dissipated because the charges would get accelerated in the process and according to Maxwell's equations, accelerating charges would always radiate some energy. The surprising part is that no matter what the resistance of the circuit is, the energy dissipated would turn out to be the same; the only difference is that one process would be very quick and another rather slow.

25.6 | CAPACITOR WITH A DIELECTRIC

Key Concepts

- If the space between the plates of a capacitor is completely filled with a dielectric material, the capacitance C in vacuum (or, effectively, in air) is multiplied by the material's dielectric constant κ, which is a number greater than 1.
- In a region that is completely filled by a dielectric, all electrostatic equations containing the permittivity constant ε_0 must be modified by replacing ε_0 with $\kappa\varepsilon_0$.

Table 25-1 Some Properties of Dielectrics[a]

Material	Dielectric Constant κ	Dielectric Strength (kV/mm)
Air (1 atm)	1.00054	3
Polystyrene	2.6	24
Paper	3.5	16
Transformer oil	4.5	
Pyrex	4.7	14
Ruby mica	5.4	
Porcelain	6.5	
Silicon	12	
Germanium	16	
Ethanol	25	
Water (20°C)	80.4	
Water (25°C)	78.5	
Titania ceramic	130	
Strontium titanate	310	8

For a vacuum, $\kappa =$ unity.

[a] Measured at room temperature, except for the water.

If you fill the space between the plates of a capacitor with a *dielectric*, which is an insulating material such as mineral oil or plastic, what happens to the capacitance? Michael Faraday—to whom the whole concept of capacitance is largely due and for whom the SI unit of capacitance is named—first looked into this matter in 1837. Using simple equipment much like that shown in Fig. 25-16, he found that the capacitance *increased* by a numerical factor κ, which he called the **dielectric constant** of the insulating material. Table 25-1 shows some dielectric materials and their dielectric constants. The dielectric constant of a vacuum is unity by definition. Because air is mostly empty space, its measured dielectric constant is only slightly greater than unity. Even common paper can significantly increase the capacitance of a capacitor, and some materials, such as strontium titanate, can increase the capacitance by more than two orders of magnitude.

Another effect of the introduction of a dielectric is to limit the potential difference that can be applied between the plates to a certain value V_{max}, called the *breakdown potential*. If this value is substantially exceeded, the dielectric material will break down and form a conducting path between the plates. Every dielectric material has a characteristic *dielectric strength*, which is the maximum value of the electric field that it can tolerate without breakdown. A few such values are listed in Table 25-1.

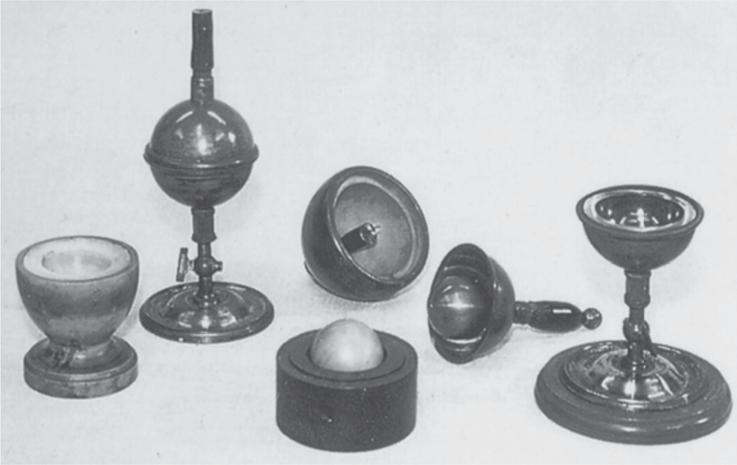
As we discussed just after Eq. 25-18, the capacitance of any capacitor can be written in the form

$$C = \varepsilon_0 \mathscr{L}, \tag{25-26}$$

in which $\mathscr{L}$ has the dimension of length. For example, $\mathscr{L} = A/d$ for a parallel-plate capacitor. Faraday's discovery was that, with a dielectric *completely* filling the space between the plates, Eq. 25-26 becomes

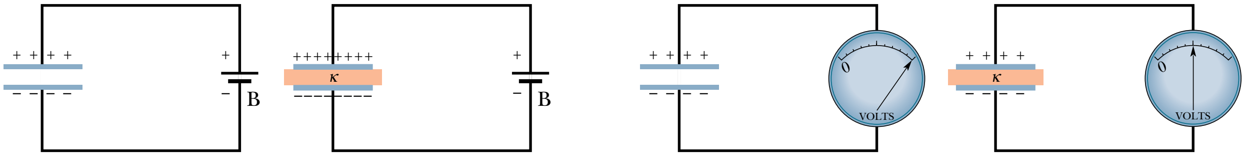
$$C = \kappa\varepsilon_0 \mathscr{L}, = \kappa C_{\text{air}}, \tag{25-27}$$

where C_{air} is the value of the capacitance with only air between the plates. For example, if we fill a capacitor with strontium titanate, with a dielectric constant of 310, we multiply the capacitance by 310.



The Royal Institute, England/Bridgeman Art Library/NY

Figure 25-16 The simple electrostatic apparatus used by Faraday. An assembled apparatus (second from left) forms a spherical capacitor consisting of a central brass ball and a concentric brass shell. Faraday placed dielectric materials in the space between the ball and the shell.



V = a constant

(a)

q = a constant

(b)

Figure 25-17 (a) If the potential difference between the plates of a capacitor is maintained, as by battery B, the effect of a dielectric is to increase the charge on the plates. (b) If the charge on the capacitor plates is maintained, as in this case, the effect of a dielectric is to reduce the potential difference between the plates. The scale shown is that of a *potentiometer*, a device used to measure potential difference (here, between the plates). A capacitor cannot discharge through a potentiometer.

Figure 25-17 provides some insight into Faraday's experiments. In Fig. 25-17a the battery ensures that the potential difference V between the plates will remain constant. When a dielectric slab is inserted between the plates, the charge q on the plates increases by a factor of κ; the additional charge is delivered to the capacitor plates by the battery. In Fig. 25-17b there is no battery, and therefore the charge q must remain constant when the dielectric slab is inserted; then the potential difference V between the plates decreases by a factor of κ. Both these observations are consistent (through the relation $q = CV$) with the increase in capacitance caused by the dielectric.

Comparison of Eqs. 25-26 and 25-27 suggests that the effect of a dielectric can be summed up in more general terms:

In a region completely filled by a dielectric material of dielectric constant κ, all electrostatic equations containing the permittivity constant ε_0 are to be modified by replacing ε_0 with $\kappa\varepsilon_0$.

Thus, the magnitude of the electric field produced by a point charge inside a dielectric is given by this modified form of Eq. 23-15:

$$E = \frac{1}{4\pi\kappa\varepsilon_0}\frac{q}{r^2}. \tag{25-28}$$

Also, the expression for the electric field just outside an isolated conductor immersed in a dielectric (see Eq. 23-11) becomes

$$E = \frac{\sigma}{\kappa\varepsilon_0}. \tag{25-29}$$

Because κ is always greater than unity, both these equations show that *for a fixed distribution of charges, the effect of a dielectric is to weaken the electric field* that would otherwise be present.

SAMPLE PROBLEM 25.06

Work and energy when a dielectric is inserted into a capacitor

A parallel-plate capacitor whose capacitance C is 13.5 pF is charged by a battery to a potential difference V = 12.5 V between its plates. The charging battery is now disconnected, and a porcelain slab (κ = 6.50) is slipped between the plates.

(a) What is the potential energy of the capacitor before the slab is inserted?

KEY IDEA

We can relate the potential energy U_i of the capacitor to the capacitance C and either the potential V (with Eq. 25-22) or the charge q (with Eq. 25-21):

$$U_i = \frac{1}{2}CV^2 = \frac{q^2}{2C}.$$

Calculation: Because we are given the initial potential V (= 12.5 V), we use Eq. 25-22 to find the initial stored energy:

$$U_i = \frac{1}{2}CV^2 = \frac{1}{2}(13.5\times10^{-12}\text{ F})(12.5\text{ V})^2$$
$$= 1.055\times10^{-9}\text{ J} = 1055\text{ pJ} \approx 1100\text{ pJ}. \quad \text{(Answer)}$$

(b) What is the potential energy of the capacitor–slab device after the slab is inserted?

KEY IDEA

Because the battery has been disconnected, the charge on the capacitor cannot change when the dielectric is inserted. However, the potential *does* change.

Calculations: Thus, we must now use Eq. 25-21 to write the final potential energy U_f, but now that the slab is within the capacitor, the capacitance is κC. We then have

$$U_f = \frac{q^2}{2\kappa C} = \frac{U_i}{\kappa} = \frac{1055\text{ pJ}}{6.50}$$
$$= 162\text{ pJ} \approx 160\text{ pJ}. \quad \text{(Answer)}$$

When the slab is introduced, the potential energy decreases by a factor of κ.

The "missing" energy, in principle, would be apparent to the person who introduced the slab. The capacitor would exert a tiny tug on the slab and would do work on it, in amount

$$W = U_i - U_f = (1055 - 162)\text{ pJ} = 893\text{ pJ}.$$

If the slab were allowed to slide between the plates with no restraint and if there were no friction, the slab would oscillate back and forth between the plates with a (constant) mechanical energy of 893 pJ, and this system energy would transfer back and forth between kinetic energy of the moving slab and potential energy stored in the electric field.

25.7 | DIELECTRICS: AN ATOMIC VIEW

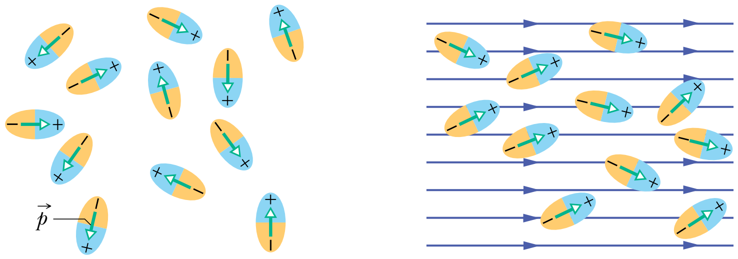
Key Concepts

- When a dielectric material is placed in an external electric field, it develops an internal electric field that is oriented opposite the external field, thus reducing the magnitude of the electric field inside the material.
- When a dielectric material is placed in a capacitor with a fixed amount of charge on the surface, the net electric field between the plates is decreased.

What happens, in atomic and molecular terms, when we put a dielectric in an electric field? There are two possibilities, depending on the type of molecule:

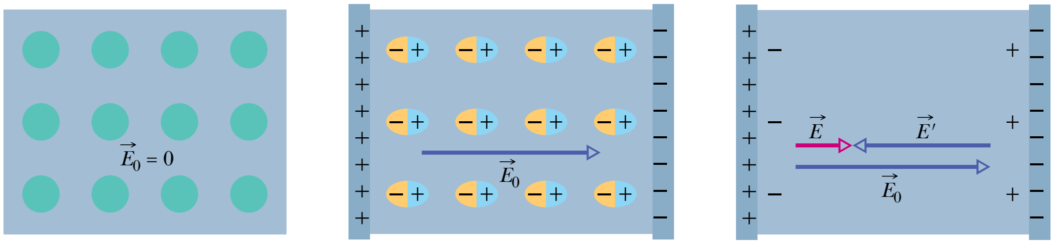
1. *Polar dielectrics.* The molecules of some dielectrics, like water, have permanent electric dipole moments. In such materials (called *polar dielectrics*), the electric dipoles tend to line up with an external electric field as in Fig. 25-18. Because the molecules are continuously jostling each other as a result of their random thermal motion, this alignment is not complete, but it becomes more complete as the magnitude of the applied field is increased (or as the temperature, and thus the jostling, are decreased). The alignment of the electric dipoles produces an electric field that is directed opposite the applied field and is smaller in magnitude.
2. *Nonpolar dielectrics*. Regardless of whether they have permanent electric dipole moments, molecules acquire dipole moments by induction when placed in an external electric field. This occurs because the external field tends to "stretch" the molecules, slightly separating the centers of negative and positive charge.

Figure 25-19*a* shows a nonpolar dielectric slab with no external electric field applied. In Fig. 25-19*b*, an electric field $\vec{E}_0$ is applied via a capacitor, whose plates are charged as shown. The result is a slight separation of the centers of the positive and negative charge distributions within the slab, producing positive charge on one face of the slab (due to the positive ends of dipoles there) and negative charge on the opposite face (due to the negative ends of dipoles there). The slab as a whole remains electrically neutral and—within the slab—there is no excess charge in any volume element.



(*a*) (*b*)

Figure 25-18 (*a*) Molecules with a permanent electric dipole moment, showing their random orientation in the absence of an external electric field. (*b*) An electric field is applied, producing partial alignment of the dipoles. Thermal agitation prevents complete alignment.


The initial electric field inside this nonpolar dielectric slab is zero.

The applied field aligns the atomic dipole moments.

The field of the aligned atoms is opposite the applied field.

(*a*) (*b*) (*c*)

Figure 25-19 (*a*) A nonpolar dielectric slab. The circles represent the electrically neutral atoms within the slab. (*b*) An electric field is applied via charged capacitor plates; the field slightly stretches the atoms, separating the centers of positive and negative charge. (*c*) The separation produces surface charges on the slab faces. These charges set up a field $\vec{E}'$, which opposes the applied field $\vec{E}_0$. The resultant field $\vec{E}$ inside the dielectric (the vector sum of $\vec{E}_0$ and $\vec{E}'$) has the same direction as $\vec{E}_0$ but a smaller magnitude.

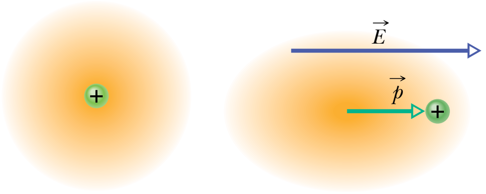
Figure 25-19*c* shows that the induced surface charges on the faces produce an electric field $\vec{E}'$ in the direction opposite that of the applied electric field $\vec{E}_0$. The resultant field $\vec{E}$ inside the dielectric (the vector sum of fields $\vec{E}_0$ and $\vec{E}'$) has the direction of $\vec{E}_0$ but is smaller in magnitude.

Both the field $\vec{E}'$ produced by the surface charges in Fig. 25-19*c* and the electric field produced by the permanent electric dipoles in Fig. 25-18 act in the same way—they oppose the applied field $\vec{E}$. Thus, the effect of both polar and nonpolar dielectrics is to weaken any applied field within them, as between the plates of a capacitor.

Induced Dipole Moment

Many molecules, such as water, have *permanent* electric dipole moments. In other molecules (called *nonpolar molecules*) and in every isolated atom, the centers of the positive and negative charges coincide (Fig. 25-20*a*) and thus no dipole moment is set up. However, if we place an atom or a nonpolar molecule in an external electric field, the field distorts the electron orbits and separates the centers of positive and negative charge (Fig. 25-20*b*). Because the

The electric field shifts the positive and negative charges, creating a dipole.



(*a*) (*b*)

Figure 25-20 (*a*) An atom showing the positively charged nucleus (circle) and the negatively charged electrons (cloud). The centers of positive and negative charge coincide. (*b*) If the atom is placed in an external electric field, the electron orbits are distorted so that the centers of positive and negative charge no longer coincide. An induced dipole moment $\vec{p}$ appears. The distortion is greatly exaggerated here.

electrons are negatively charged, they tend to be shifted in a direction opposite the field. This shift sets up a dipole moment $\vec{p}$ that points in the direction of the field. This dipole moment is said to be *induced* by the field and the atom or molecule is then said to be *polarized* by the field (i.e., it has a positive side and a negative side). When the field is removed, the induced dipole moment and the polarization disappear.

CHECKPOINT 4

In a cylindrical capacitor, if the potential difference is too great, dielectric breakdown will occur. Will such breakdown begin near the inner or the outer conductor? Why?

25.8 | DIELECTRICS AND GAUSS' LAW

Key Concepts

- Inserting a dielectric into a capacitor causes induced charge to appear on the faces of the dielectric and weakens the electric field between the plates.
- The induced charge is less than the free charge on the plates.
- When a dielectric is present, Gauss' law may be generalized to

$$\varepsilon_0 \oint \kappa \vec{E} \cdot d\vec{A} = q,$$

where q is the free charge. Any induced surface charge is accounted for by including the dielectric constant κ inside the integral.

In our discussion of Gauss' law in Chapter 23, we assumed that the charges existed in a vacuum. Here we shall see how to modify and generalize that law if dielectric materials, such as those listed in Table 25-1, are present. Figure 25-21 shows a parallel-plate capacitor of plate area A, both with and without a dielectric. We assume that the charge q on the plates is the same in both situations. Note that the field between the plates induces charges on the faces of the dielectric by one of the methods described in Section 25.6.

For the situation of Fig. 25-21a, without a dielectric, we can find the electric field $\vec{E}_0$ between the plates as we did in Fig. 25-5: We enclose the charge $+q$ on the top plate with a Gaussian surface and then apply Gauss' law. Letting E_0 represent the magnitude of the field, we find

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 EA = q, \tag{25-30}$$

or

$$E_0 = \frac{q}{\varepsilon_0 A}. \tag{25-31}$$

In Fig. 25-21b, with the dielectric in place, we can find the electric field between the plates (and within the dielectric) by using the same Gaussian surface. However, now the surface encloses two types of charge: It still

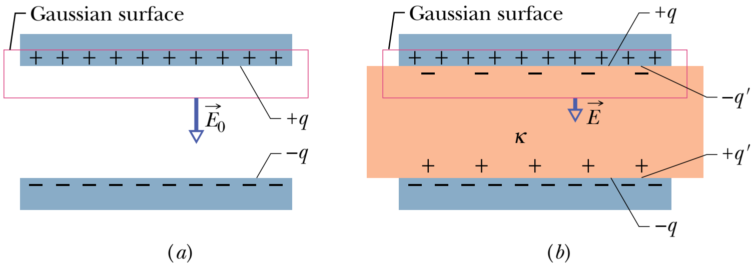


Figure 25-21 A parallel-plate capacitor (a) without and (b) with a dielectric slab inserted. The charge q on the plates is assumed to be the same in both cases.

encloses charge $+q$ on the top plate, but it now also encloses the induced charge $-q'$ on the top face of the dielectric. The charge on the conducting plate is said to be *free charge* because it can move if we change the electric potential of the plate; the induced charge on the surface of the dielectric is not free charge because it cannot move from that surface.

The net charge enclosed by the Gaussian surface in Fig. 25-21*b* is $q - q'$, so Gauss' law now gives

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 EA = q - q', \tag{25-32}$$

or

$$E = \frac{q - q'}{\varepsilon_0 A}. \tag{25-33}$$

The effect of the dielectric is to weaken the original field E_0 by a factor of κ; so we may write

$$E = \frac{E_0}{\kappa} = \frac{q}{\kappa \varepsilon_0 A}. \tag{25-34}$$

Comparison of Eqs. 25-33 and 25-34 shows that

$$q - q' = \frac{q}{\kappa}. \tag{25-35}$$

Equation 25-35 shows correctly that the magnitude q' of the induced surface charge is less than that of the free charge q and is zero if no dielectric is present (because then $\kappa = 1$ in Eq. 25-35).

By substituting for $q - q'$ from Eq. 25-35 in Eq. 25-32, we can write Gauss' law in the form

$$\varepsilon_0 \oint \kappa \vec{E} \cdot d\vec{A} = q \quad \text{(Gauss' law with dielectric).} \tag{25-36}$$

This equation, although derived for a parallel-plate capacitor, is true generally and is the most general form in which Gauss' law can be written. Note:

1. The flux integral now involves $\kappa\vec{E}$, not just $\vec{E}$. (The vector $\varepsilon_0 \kappa \vec{E}$ is sometimes called the *electric displacement* $\vec{D}$, so that Eq. 25-36 can be written in the form $\oint \vec{D} \cdot d\vec{A} = q$.)
2. The charge q enclosed by the Gaussian surface is now taken to be the *free charge only*. The induced surface charge is deliberately ignored on the right side of Eq. 25-36, having been taken fully into account by introducing the dielectric constant κ on the left side.
3. Equation 25-36 differs from Eq. 23-7, our original statement of Gauss' law, only in that ε_0 in the latter equation has been replaced by $\kappa\varepsilon_0$. We keep κ inside the integral of Eq. 25-36 to allow for cases in which κ is not constant over the entire Gaussian surface.

SAMPLE PROBLEM 25.07

Dielectric partially filling the gap in a capacitor

Figure 25-22 shows a parallel-plate capacitor of plate area A and plate separation d. A potential difference V_0 is applied between the plates by connecting a battery between them. The battery is then disconnected, and a dielectric slab of thickness b and dielectric constant κ is placed between the plates as shown. Assume $A = 115\ \text{cm}^2$, $d = 1.24$ cm, $V_0 = 85.5$ V, $b = 0.780$ cm, and $\kappa = 2.61$.

(a) What is the capacitance C_0 before the dielectric slab is inserted?

Calculation: From Eq. 25-9 we have

$$C_0 = \frac{\varepsilon_0 A}{d} = \frac{(8.85 \times 10^{-12}\ \text{F/m})(115 \times 10^{-4}\ \text{m}^2)}{1.24 \times 10^{-2}\ \text{m}}$$
$$= 8.21 \times 10^{-12}\ \text{F} = 8.21\ \text{pF}. \quad \text{(Answer)}$$

(b) What free charge appears on the plates?

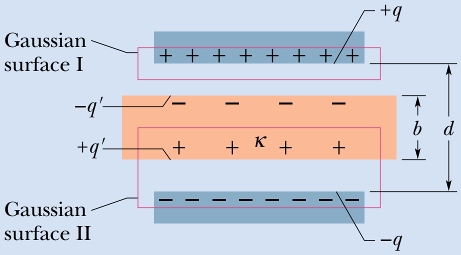


Figure 25-22 A parallel-plate capacitor containing a dielectric slab that only partially fills the space between the plates.

Calculation: From Eq. 25-1,

$$q = C_0V_0 = (8.21 \times 10^{-12}\ \text{F})(85.5\ \text{V})$$
$$= 7.02 \times 10^{-10}\ \text{C} = 702\ \text{pC}. \qquad \text{(Answer)}$$

Because the battery was disconnected before the slab was inserted, the free charge is unchanged.

(c) What is the electric field E_0 in the gaps between the plates and the dielectric slab?

KEY IDEA

We need to apply Gauss' law, in the form of Eq. 25-36, to Gaussian surface I in Fig. 25-22.

Calculations: That surface passes through the gap, and so it encloses *only* the free charge on the upper capacitor plate. Electric field pierces only the bottom of the Gaussian surface. Because there the area vector $d\vec{A}$ and the field vector $\vec{E}_0$ are both directed downward, the dot product in Eq. 25-36 becomes

$$\vec{E}_0 \cdot d\vec{A} = E_0\, dA \cos 0° = E_0\, dA.$$

Equation 25-36 then becomes

$$\varepsilon_0 \kappa E_0 \oint dA = q.$$

The integration now simply gives the surface area A of the plate. Thus, we obtain

$$\varepsilon_0 \kappa E_0 A = q,$$

or
$$E_0 = \frac{q}{\varepsilon_0 \kappa A}.$$

We must put $\kappa = 1$ here because Gaussian surface I does not pass through the dielectric. Thus, we have

$$E_0 = \frac{q}{\varepsilon_0 \kappa A} = \frac{7.02 \times 10^{-10}\ \text{C}}{(8.85 \times 10^{-12}\ \text{F/m})(1)(115 \times 10^{-4}\ \text{m}^2)}$$
$$= 6900\ \text{V/m} = 6.90\ \text{kV/m}. \qquad \text{(Answer)}$$

Note that the value of E_0 does not change when the slab is introduced because the amount of charge enclosed by Gaussian surface I in Fig. 25-22 does not change.

(d) What is the electric field E_1 in the dielectric slab?

KEY IDEA

Now we apply Gauss' law in the form of Eq. 25-36 to Gaussian surface II in Fig. 25-22.

Calculations: Only the free charge $-q$ is in Eq. 25-36, so

$$\varepsilon_0 \oint \kappa \vec{E}_1 \cdot d\vec{A} = -\varepsilon_0 \kappa E_1 A = -q. \qquad (25\text{-}37)$$

The first minus sign in this equation comes from the dot product $\vec{E}_1 \cdot d\vec{A}$ along the top of the Gaussian surface because now the field vector $\vec{E}_1$ is directed downward and the area vector $d\vec{A}$ (which, as always, points outward from the interior of a closed Gaussian surface) is directed upward. With 180° between the vectors, the dot product is negative. Now $\kappa = 2.61$. Thus, Eq. 25-37 gives us

$$E_1 = \frac{q}{\varepsilon_0 \kappa A} = \frac{E_0}{\kappa} = \frac{6.90\ \text{kV/m}}{2.61}$$
$$= 2.64\ \text{kV/m}. \qquad \text{(Answer)}$$

(e) What is the potential difference V between the plates after the slab has been introduced?

KEY IDEA

We find V by integrating along a straight line directly from the bottom plate to the top plate.

Calculation: Within the dielectric, the path length is b and the electric field is E_1. Within the two gaps above and below the dielectric, the total path length is $d - b$ and the electric field is E_0. Equation 25-6 then yields

$$V = \int_-^+ E\, ds = E_0(d-b) + E_1 b$$
$$= (6900\ \text{V/m})(0.0124\ \text{m} - 0.00780\ \text{m})$$
$$+ (2640\ \text{V/m})(0.00780\ \text{m})$$
$$= 52.3\ \text{V}. \qquad \text{(Answer)}$$

This is less than the original potential difference of 85.5 V.

(f) What is the capacitance with the slab in place?

KEY IDEA

The capacitance C is related to q and V via Eq. 25-1.

Calculation: Taking q from (b) and V from (e), we have

$$C = \frac{q}{V} = \frac{7.02 \times 10^{-10}\ \text{C}}{52.3\ \text{V}}$$
$$= 1.34 \times 10^{-11}\ \text{F} = 13.4\ \text{pF}. \qquad \text{(Answer)}$$

This is greater than the original capacitance of 8.21 pF.

Force on a Dielectric Slab Inside a Capacitor

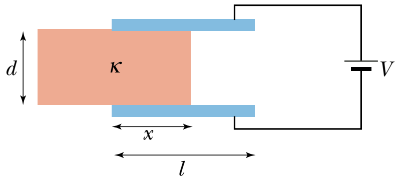


Figure 25-23 A dielectric slab with same dimensions as the inner dimensions of the capacitor, is partially inserted inside the capacitor.

Figure 25-23 shows a parallel-plate capacitor with plates of width w and length l. The separation between the plates is d. The plates are rigidly clamped and connected to a battery of emf V. A dielectric slab of thickness d and dielectric constant κ is slowly inserted between the plates. We wish to find the force exerted by the electric field on the dielectric slab.

Just as a conductor is attracted to an electric field, so it too is dielectric – and for essentially the same reason, the bound charge tends to accumulate near the free charge of the opposite sign. But the calculation of forces on dielectrics can be surprisingly tricky.

Consider, for example, the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor (Fig. 25-23). We have always pretended that the field is uniform inside a parallel-plate capacitor and zero outside. If this were literally true, there would be no net force on the dielectric at all, since the field everywhere would be perpendicular to the plates. However, there is in reality a fringing field around the edges, which for most purposes can be ignored but in this case is responsible for the whole effect. Indeed, the field could not terminate abruptly at the edge of the capacitor, for if it did the line integral of E around any closed loop in Fig. 25-23 would not be zero. It is this nonuniform fringing field that pulls the dielectric into the capacitor.

In order to use the principle of work correctly, the presence of fringing must be acknowledged. To this end, let all space be divided into six regions as shown in Fig. 25-24. Region I consists of the dielectric volume up to the left-hand edge of the plates, region II is the dielectric volume from this point up to where the field is uniform, and region III is the remainder of the dielectric volume. The other regions are in vacuum. Region IV consists of the volume of uniform field between the plates, region V continues from this point to the right-hand edge of the plates, and region VI is the rest of space.

Consider the special case where the left-hand edge of the dielectric extends beyond the end of the fringe field and its right-hand edge lies in the region of uniform field, as shown in Fig. 25-24. In this case, an infinitesimal displacement dx to the right does not change the field energy in regions I, II, V, and VI. Instead, the only effect is to increase the volume of region III by an amount $(wd)dx$ while decreasing that of region IV by the same amount. Since the field is given by $|E| = V/d$ in both regions, the resulting change in the total field energy is

$$\begin{aligned} dU &= \frac{1}{2}\kappa\varepsilon_0\left(\frac{V}{d}\right)^2 (wd)dx - \frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2 (wd)dx \\ &= \frac{(\kappa-1)\varepsilon_0 wV^2}{2d}dx. \end{aligned}$$

This can be in turn represented in terms of the capacitance: The plate area of the part with the dielectric is wx. Its capacitance is

$$C_1 = \frac{\kappa\varepsilon_0 wx}{d}.$$

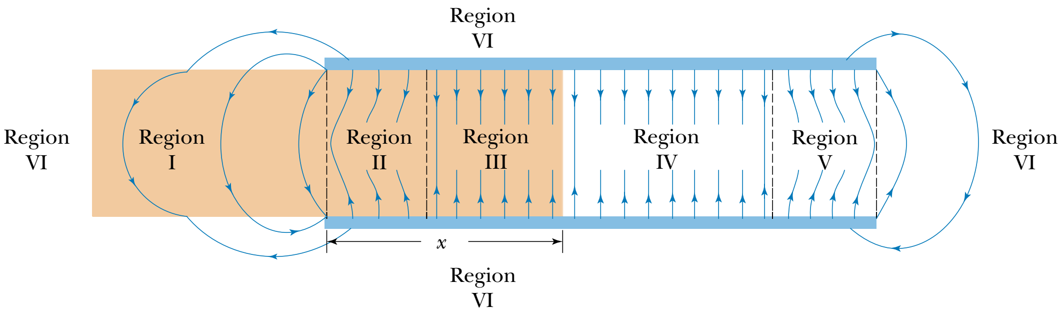


Figure 25-24 Division of all space into six regions for the purpose of evaluating the total field energy. Region I, II, and III are within the dielectric and regions IV, V, and VI are in vacuum.

Similarly, the capacitance of the part without the dielectric is

$$C_2 = \frac{\varepsilon_0 w(l-x)}{d}.$$

These two parts are connected in parallel. The capacitance of the system is, therefore,

$$C = C_1 + C_2 = \frac{\varepsilon_0 w}{d}[l + x(\kappa - 1)].$$

The energy of the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{\varepsilon_0 w V^2}{2d}[l + x(\kappa - 1)].$$

So, it follows that

$$dU = \frac{(\kappa - 1)\varepsilon_0 w V^2}{2d}dx.$$

Suppose, the electric field attracts the dielectric slab with a force F. An external force of magnitude equal to F should be applied in opposite direction so that the plate moves slowly (no acceleration). As the slab moves inside, the capacitance increases to $C + dC$. As the potential difference remains constant at V, the battery has to supply a further charge of

$$dQ = (dC)V$$

to the capacitor. The work done by the battery is, therefore,

$$dW_b = VdQ = (dC)V^2.$$

The external force F does a work

$$dW_e = (-Fdx)$$

during the displacement. The total work done on the capacitor is

$$dW_b + dW_e = (dC)V^2 - Fdx.$$

This should be equal to the increase dU in the stored energy. Thus,

$$\frac{1}{2}(dC)V^2 = (dC)V^2 - Fdx$$

$$F = \frac{1}{2}V^2\frac{dC}{dx} = \frac{\varepsilon_0 w V^2(\kappa - 1)}{2d}$$

Thus, the electric field attracts the dielectric into the capacitor with a force $\varepsilon_0 w V^2(\kappa - 1)/2d$ and this much force should be applied in the opposite direction to move it slowly.

REVIEW AND SUMMARY

Capacitor; Capacitance A **capacitor** consists of two isolated conductors (the *plates*) with charges $+q$ and $-q$. Its **capacitance** C is defined from

$$q = CV, \tag{25-1}$$

where V is the potential difference between the plates.

Determining Capacitance We generally determine the capacitance of a particular capacitor configuration by (1) assuming a charge q to have been placed on the plates, (2) finding the electric field $\vec{E}$ due to this charge, (3) evaluating the potential difference V, and (4) calculating C from Eq. 25-1. Some specific results are the following:

A *parallel-plate capacitor* with flat parallel plates of area A and spacing d has capacitance

$$C = \frac{\varepsilon_0 A}{d}. \tag{25-9}$$

A *cylindrical capacitor* (two long coaxial cylinders) of length L and radii a and b has capacitance

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)}. \tag{25-14}$$

A *spherical capacitor* with concentric spherical plates of radii a and b has capacitance

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a}. \tag{25-17}$$

An *isolated sphere* of radius R has capacitance

$$C = 4\pi\varepsilon_0 R. \tag{25-18}$$

Capacitors in Parallel and in Series The **equivalent capacitances** C_{eq} of combinations of individual capacitors connected in **parallel** and in **series** can be found from

$$C_{\text{eq}} = \sum_{j=1}^{n} C_j \quad (n \text{ capacitors in parallel}) \tag{25-19}$$

and

$$\frac{1}{C_{\text{eq}}} = \sum_{j=1}^{n} \frac{1}{C_j} \quad (n \text{ capacitors in series}). \tag{25-20}$$

Equivalent capacitances can be used to calculate the capacitances of more complicated series–parallel combinations.

Potential Energy and Energy Density The **electric potential energy** U of a charged capacitor,

$$U = \frac{q^2}{2C} = \frac{1}{2}CV^2, \tag{25-21, 25-22}$$

is equal to the work required to charge the capacitor. This energy can be associated with the capacitor's electric field $\vec{E}$. By extension we can associate stored energy with any electric field. In vacuum, the **energy density** u, or potential energy per unit volume, within an electric field of magnitude E is given by

$$u = \frac{1}{2}\varepsilon_0 E^2. \tag{25-25}$$

Capacitance with a Dielectric If the space between the plates of a capacitor is completely filled with a dielectric material, the capacitance C is increased by a factor κ, called the **dielectric constant**, which is characteristic of the material. In a region that is completely filled by a dielectric, all electrostatic equations containing ε_0 must be modified by replacing ε_0 with $\kappa\varepsilon_0$.

The effects of adding a dielectric can be understood physically in terms of the action of an electric field on the permanent or induced electric dipoles in the dielectric slab. The result is the formation of induced charges on the surfaces of the dielectric, which results in a weakening of the field within the dielectric for a given amount of free charge on the plates.

Gauss' Law with a Dielectric When a dielectric is present, Gauss' law may be generalized to

$$\varepsilon_0 \oint \kappa \vec{E} \cdot d\vec{A} = q. \tag{25-36}$$

Here q is the free charge; any induced surface charge is accounted for by including the dielectric constant κ inside the integral.

PROBLEMS

1. In Fig. 25-25, $C_1 = 10.0\ \mu\text{F}$, $C_2 = 5.0\ \mu\text{F}$, and $C_3 = 4.0\ \mu\text{F}$. What is the change in their equivalent capacitance if (a) capacitors 1 and 2 are interchanged and (separately) (b) capacitors 1 and 3 are interchanged?

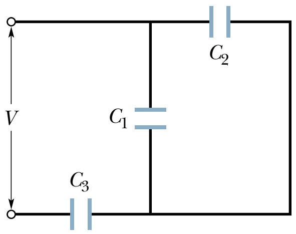


Figure 25-25 Problems 1, 2, and 3.

2. In Fig. 25-25, a potential difference $V = 75.0$ V is applied across a capacitor arrangement with capacitances $C_1 = 10.0\ \mu\text{F}$, $C_2 = 5.00\ \mu\text{F}$, and $C_3 = 15.0\ \mu\text{F}$. What are (a) charge q_3, (b) potential difference V_3, and (c) stored energy U_3 for capacitor 3, (d) q_1, (e) V_1, and (f) U_1 for capacitor 1, and (g) q_2, (h) V_2, and (i) U_2 for capacitor 2?

3. In Fig. 25-25, a potential difference of $V = 65.0$ V is applied across a capacitor arrangement with capacitances $C_1 = 10.0\ \mu\text{F}$, $C_2 = 5.00\ \mu\text{F}$, and $C_3 = 4.00\ \mu\text{F}$. If capacitor 3 undergoes electrical breakdown so that it becomes equivalent to conducting wire, then for capacitor 1 what are the increases in (a) charge, (b) potential difference, and (c) stored energy?

4. In Fig. 25-26, find the equivalent capacitance of the combination. Assume that C_1 is 10.0 μF, C_2 is 8.00 μF, and C_3 is 4.00 μF.

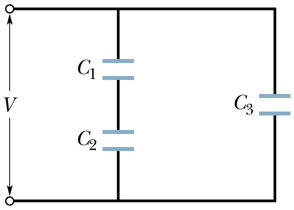


Figure 25-26 Problems 4 and 6.

5. What capacitance is required to store an energy of 10 kW·h at a potential difference of 1700 V?

6. In Fig. 25-26, a potential difference $V = 100$ V is applied across a capacitor arrangement with capacitances $C_1 = 10.0\ \mu$F, $C_2 = 5.00\ \mu$F, and $C_3 = 2.00\ \mu$F. What are (a) charge q_3, (b) potential difference V_3, and (c) stored energy U_3 for capacitor 3, (d) q_1, (e) V_1, and (f) U_1 for capacitor 1, and (g) q_2, (h) V_2, and (i) U_2 for capacitor 2?

7. A parallel-plate capacitor has plates of area 0.080 m^2 and a separation of 1.2 cm. A battery charges the plates to a potential difference of 120 V and is then disconnected. A dielectric slab of thickness 4.0 mm and dielectric constant 4.8 is then placed symmetrically between the plates. (a) What is the capacitance before the slab is inserted? (b) What is the capacitance with the slab in place? What is the free charge q (c) before and (d) after the slab is inserted? What is the magnitude of the electric field (e) in the space between the plates and dielectric and (f) in the dielectric itself? (g) With the slab in place, what is the potential difference across the plates? (h) How much external work is involved in inserting the slab?

8. Figure 25-27 displays a 16.0 V battery and 3 uncharged capacitors of capacitances $C_1 = 4.00\ \mu$F, $C_2 = 6.00\ \mu$F, and $C_3 = 3.00\ \mu$F. The switch is thrown to the left side until capacitor 1 is fully charged. Then the switch is thrown to the right. What is the final charge on (a) capacitor 1, (b) capacitor 2, and (c) capacitor 3?

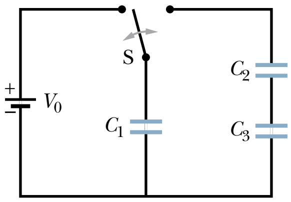


Figure 25-27 Problem 8.

9. In Fig. 25-28, the battery has potential difference $V = 14.0$ V, $C_2 = 3.00\ \mu$F, $C_4 = 4.00\ \mu$F, and all the capacitors are initially uncharged. When switch S is closed, a total charge of 12 μC passes through point a and a total charge of 8.0 μC passes through point b. What are (a) C_1 and (b) C_3?

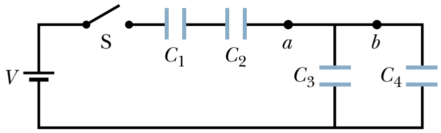


Figure 25-28 Problem 9.

10. A parallel-plate air-filled capacitor having area 40 cm^2 and plate spacing 1.0 mm is charged to a potential difference of 500 V. Find (a) the capacitance, (b) the magnitude of the charge on each plate, (c) the stored energy, (d) the electric field between the plates, and (e) the energy density between the plates.

11. A dielectric material is to fill the space in a capacitor. Initially, with only air in place, the capacitance is 8.0 pF. With the dielectric material in place, the capacitor should store 3.2 μJ at a maximum potential difference of 350.8 V. (a) What dielectric constant is required? (b) Of the materials in Table 25-1, which material should be used?

12. An air-filled parallel-plate capacitor has a capacitance of 2.1 pF. The separation of the plates is doubled, and wax is inserted between them. The new capacitance is 2.6 pF. Find the dielectric constant of the wax.

13. A 2.0 μF capacitor and a 4.0 μF capacitor are connected in parallel across a 300 V potential difference. (a) What is the total energy stored by them? (b) They are next connected in series across that potential difference. What is the ratio of the total energy stored by them in the parallel arrangement to that in the series arrangement?

14. In Fig. 25-29, how much charge is stored on the parallel-plate capacitors by the 10.0 V battery? One is filled with air, and the other is filled with a dielectric for which $\kappa = 3.00$; both capacitors have a plate area of 5.00×10^{-3} m^2 and a plate separation of 2.00 mm.

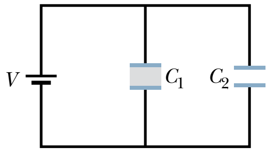


Figure 25-29 Problem 14.

15. Assume that a stationary electron is a point of charge. What is the energy density u of its electric field at radial distances (a) $r = 1.00$ mm, (b) $r = 1.00\ \mu$m, (c) $r = 1.00$ nm, (d) $r = 1.00$ pm, and (e) $r = 1.00$ fm? (f) What is u in the limit as $r \to 0$?

16. You are asked to construct a capacitor having a capacitance near 1 nF and a breakdown potential in excess of 10,000 V. You think of using the sides of a tall Pyrex drinking glass as a dielectric, lining the inside and outside curved surfaces with aluminum foil to act as the plates. The glass is 10 cm tall with an inner radius of 3.6 cm and an outer radius of 3.8 cm. What are the (a) capacitance and (b) breakdown potential of this capacitor?

17. The parallel plates in a capacitor, with a plate area of 8.50 cm^2 and an air-filled separation of 8.00 mm, are charged by a 16.0 V battery. They are then disconnected from the battery and pushed together (without discharge) to a separation of 3.00 mm. Neglecting fringing, find (a) the potential difference between the plates, (b) the initial stored energy, (c) the final stored energy, and (d) the (negative) work in pushing them together.

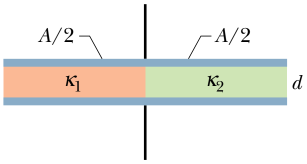


Figure 25-30 Problem 18.

18. Figure 25-30 shows a parallel plate capacitor with a plate area $A = 5.56$ cm^2 and separation $d = 5.56$ mm. The left half of the gap is filled with material of dielectric constant $\kappa_1 = 7.00$;

the right half is filled with material of dielectric constant $\kappa_2 = 10.0$. What is the capacitance?

19. In Fig. 25-31, $C_1 = 10.0\ \mu F$, $C_2 = 20.0\ \mu F$, and $C_3 = 5.00\ \mu F$. If no capacitor can withstand a potential difference of more than 100 V without failure, what are (a) the magnitude of the maximum potential difference that can exist between points A and B and (b) the maximum energy that can be stored in the three-capacitor arrangement?

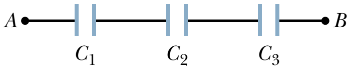


Figure 25-31 Problem 19.

20. A coaxial cable used in a transmission line has an inner radius of 0.10 mm and an outer radius of 0.40 mm. Calculate the capacitance per meter for the cable. Assume that the space between the conductors is filled with epoxy resin with dielectric constant 3.6.

21. Figure 25-32 shows a parallel-plate capacitor of plate area $A = 12.5\ \text{cm}^2$ and plate separation $2d$ = 7.12 mm. The left half of the gap is filled with material of dielectric constant κ_1 = 21.0; the top of the right half is filled with material of dielectric constant κ_2 = 42.0; the bottom of the right half is filled with material of dielectric constant $\kappa_3 = 58.0$. What is the capacitance?

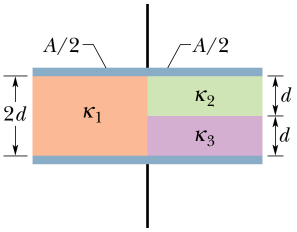


Figure 25-32 Problem 21.

22. A parallel-plate capacitor has square plates with edge length 8.20 cm and 1.30 mm separation. (a) Calculate the capacitance. (b) Find the charge for a potential difference of 120 V.

23. In Fig. 25-33, the battery has a potential difference of V = 12.0 V and the five capacitors each have a capacitance of 10.0 μF. What is the charge on (a) capacitor 1 and (b) capacitor 2?

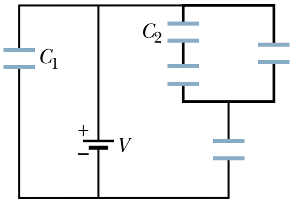


Figure 25-33 Problem 23.

24. A parallel-plate air-filled capacitor has a capacitance of 50 pF. (a) If each of its plates has an area of 0.30 m², what is the separation? (b) If the region between the plates is now filled with material having $\kappa = 5.6$, what is the capacitance?

25. Each of the uncharged capacitors in Fig. 25-34 has a capacitance of 25.0 μF. A potential difference of V = 750 V is established when the switch is closed. How many coulombs of charge then pass through meter A?

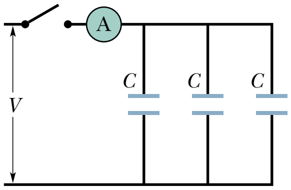


Figure 25-34 Problem 25.

26. Figure 25-35 shows a variable "air gap" capacitor for manual tuning. Alternate plates are connected together; one group of plates is fixed in position, and the other group is capable of rotation. Consider a capacitor of n = 8 plates of alternating polarity, each plate having area $A = 1.50\ \text{cm}^2$ and separated from adjacent plates by distance d = 3.40 mm. What is the maximum capacitance of the device?

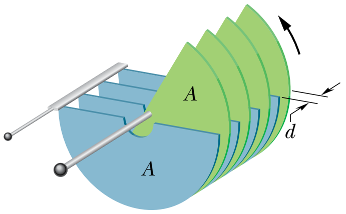


Figure 25-35 Problem 26.

27. Figure 25-36 shows a parallel-plate capacitor with a plate area $A = 7.89\ \text{cm}^2$ and plate separation d = 4.62 mm. The top half of the gap is filled with material of dielectric constant κ_1 = 11.0; the bottom half is filled with material of dielectric constant $\kappa_2 = 4.0$. What is the capacitance?

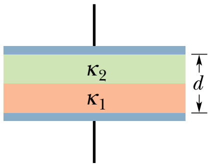


Figure 25-36 Problem 27.

28. The capacitor in Fig. 25-37 has a capacitance of 30 μF and is initially uncharged. The battery provides a potential difference of 120 V. After switch S is closed, how much charge will pass through it?

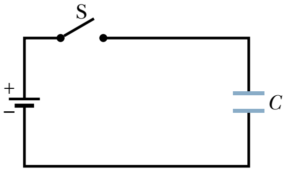


Figure 25-37 Problem 28.

29. Two parallel-plate capacitors, 8.0 μF each, are connected in parallel to a 10 V battery. One of the capacitors is then squeezed so that its plate separation is 50.0% of its initial value. Because of the squeezing, (a) how much additional charge is transferred to the capacitors by the battery and (b) what is the increase in the total charge stored on the capacitors?

30. In Fig. 25-38, a 20.0 V battery is connected across capacitors of capacitances $C_1 = C_6 = 6.00\ \mu F$ and $C_3 = C_5 = 2.00C_2 = 2.00C_4 = 4.00\ \mu F$. What are (a) the equivalent capacitance C_{eq} of the capacitors and (b) the charge stored by C_{eq}? What are (c) V_1 and (d) q_1 of capacitor 1,

(e) V_2 and (f) q_2 of capacitor 2, and (g) V_3 and (h) q_3 of capacitor 3?

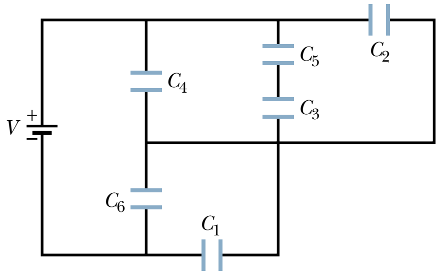


Figure 25-38 Problem 30.

31. A certain substance has a dielectric constant of 5.6 and a dielectric strength of 18 MV/m. If it is used as the dielectric material in a parallel-plate capacitor, what minimum area should the plates of the capacitor have to obtain a capacitance of 3.9×10^{-2} μF and to ensure that the capacitor will be able to withstand a potential difference of 4.0 kV?

32. For the arrangement of Fig. 25-22, suppose that the battery remains connected while the dielectric slab is being introduced. Calculate (a) the capacitance, (b) the charge on the capacitor plates, (c) the electric field in the gap, and (d) the electric field in the slab, after the slab is in place.

33. In Fig. 25-39, the capacitances are $C_1 = 1.0$ μF and $C_2 = 3.0$ μF, and both capacitors are charged to a potential difference of $V = 200$ V but with opposite polarity as shown. Switches S_1 and S_2 are now closed. (a) What is now the potential difference between points a and b? What now is the charge on capacitor (b) 1 and (c) 2?

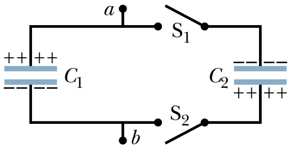


Figure 25-39 Problem 33.

34. Capacitor 3 in Fig. 25-40a is a variable capacitor (its capacitance C_3 can be varied). Figure 25-40b gives the electric potential V_1 across capacitor 1 versus C_3. The horizontal scale is set by $C_{3s} = 12.0$ μF. Electric potential V_1 approaches an asymptote of 8.0 V as $C_3 \to \infty$. What are (a) the electric potential V across the battery, (b) C_1, and (c) C_2?

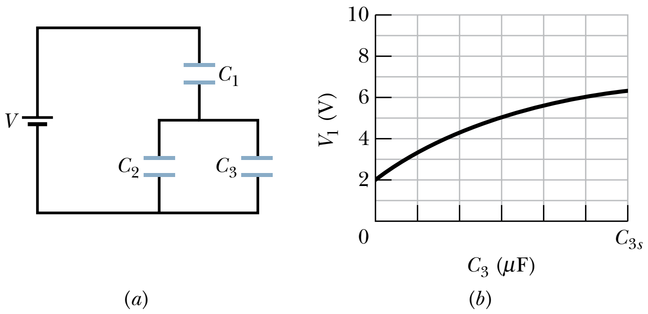


Figure 25-40 Problem 34.

35. Figure 25-41 shows a circuit section of four air-filled capacitors that is connected to a larger circuit. The graph below the section shows the electric potential $V(x)$ as a function of position x along the lower part of the section, through capacitor 4. Similarly, the graph above the section shows the electric potential $V(x)$ as a function of position x along the upper part of the section, through capacitors 1, 2, and 3. Capacitor 3 has a capacitance of 1.60 μF. What are the capacitances of (a) capacitor 1 and (b) capacitor 2?

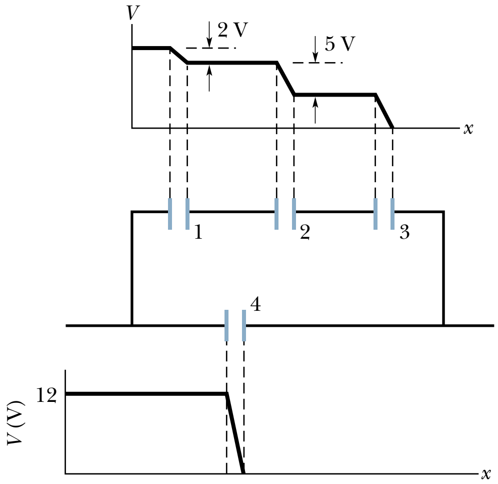


Figure 25-41 Problem 35.

36. A charged isolated metal sphere of diameter 15 cm has a potential of 6500 V relative to $V = 0$ at infinity. (a) Calculate the energy density in the electric field near the surface of the sphere. (b) If the diameter is decreased, does the energy density near the surface increase, decrease, or remain the same?

37. Two parallel plates of area 100 cm^2 are given charges of equal magnitudes 8.4×10^{-7} C but opposite signs. The electric field within the dielectric material filling the space between the plates is 1.4×10^6 V/m. (a) Calculate the dielectric constant of the material. (b) Determine the magnitude of the charge induced on each dielectric surface.

38. What is the capacitance of a drop that results when two mercury spheres, each of radius $R = 3.00$ mm, merge?

39. Plot 1 in Fig. 25-42a gives the charge q that can be stored on capacitor 1 versus the electric potential V set up across it. The vertical scale is set by $q_s = 16.0$ μC, and the horizontal scale is set by $V_s = 2.0$ V. Plots 2 and 3 are similar plots for capacitors 2 and 3, respectively. Figure 25-42b shows a circuit with those three capacitors and a 10.0 V battery. What is the charge stored on capacitor 2 in that circuit?

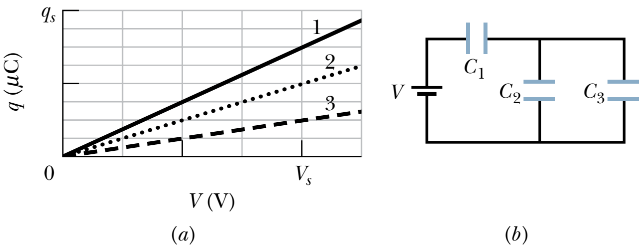


Figure 25-42 Problem 39.

40. Figure 25-43 shows a 24.0 V battery and four uncharged capacitors of capacitances $C_1 = 1.00\ \mu F$, $C_2 = 2.00\ \mu F$, $C_3 = 3.00\ \mu F$, and $C_4 = 4.00\ \mu F$. If only switch S_1 is closed, what is the charge on (a) capacitor 1, (b) capacitor 2, (c) capacitor 3, and (d) capacitor 4? If both switches are closed, what is the charge on (e) capacitor 1, (f) capacitor 2, (g) capacitor 3, and (h) capacitor 4?

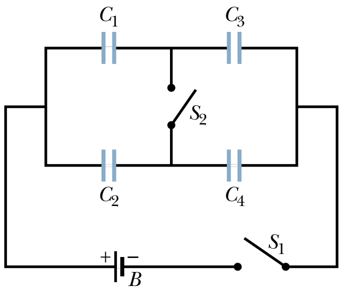


Figure 25-43 Problem 40.

41. How many 12.5 μF capacitors must be connected in parallel to store a charge of 33.0 mC with a potential of 110 V across the capacitors?

42. The space between two concentric conducting spherical shells of radii $b = 1.70$ cm and $a = 1.20$ cm is filled with a substance of dielectric constant $\kappa = 6.91$. A potential difference $V = 73.0$ V is applied across the inner and outer shells. Determine (a) the capacitance of the device, (b) the free charge q on the inner shell, and (c) the charge q' induced along the surface of the inner shell.

43. You have two flat metal plates, each of area 1.00 m^2, with which to construct a parallel-plate capacitor. (a) If the capacitance of the device is to be 2.00 F, what must be the separation between the plates? (b) Could this capacitor actually be constructed?

44. The two metal objects in Fig. 25-44 have net charges of +70 pC and −70 pC, which result in a 35 V potential difference between them. (a) What is the capacitance of the system? (b) If the charges are changed to +200 pC and −200 pC, what does the capacitance become? (c) What does the potential difference become?

Figure 25-44 Problem 44.

45. In Fig. 25-45, $V = 12$ V, $C_1 = 10\ \mu F$, and $C_2 = C_3 = 20\ \mu F$. Switch S is first thrown to the left side until capacitor 1 reaches equilibrium. Then the switch is thrown to the right. When equilibrium is again reached, how much charge is on capacitor 1?

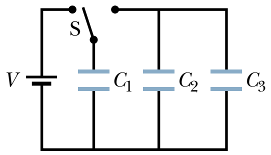


Figure 25-45 Problem 45.

46. A 100 pF capacitor is charged to a potential difference of 80.0 V, and the charging battery is disconnected. The capacitor is then connected in parallel with a second (initially uncharged) capacitor. If the potential difference across the first capacitor drops to 35.0 V, what is the capacitance of this second capacitor?

47. The plates of a spherical capacitor have radii 37.0 mm and 40.0 mm. (a) Calculate the capacitance. (b) What must be the plate area of a parallel-plate capacitor with the same plate separation and capacitance?

48. In Fig. 25-46, two parallel-plate capacitors (with air between the plates) are connected to a battery. Capacitor 1 has a plate area of 1.5 cm^2 and an electric field (between its plates) of magnitude 3500 V/m. Capacitor 2 has a plate area of 0.70 cm^2 and an electric field of magnitude 1500 V/m. (a) What is the total charge on the two capacitors? (b) If the first plate area is cut in half, does the total charge increase, decrease, or remain the same?

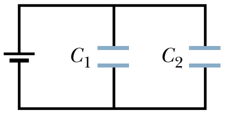
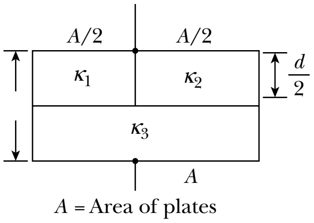


Figure 25-46 Problem 48.

PRACTICE QUESTIONS

Single Correct Choice Type

1. A parallel plate capacitor of area A, plate separation d and capacitance C is filled with three different dielectric materials having dielectric constants κ_1, κ_2 and κ_3 as shown in the following figure. If a single dielectric material is to be used to have the same capacitance C in this capacitor then its dielectric constant κ is given by



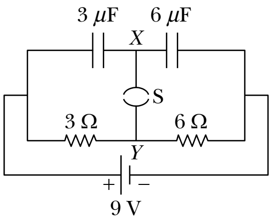
(a) $\dfrac{1}{\kappa} = \dfrac{1}{\kappa_1} + \dfrac{1}{\kappa_2} + \dfrac{1}{2\kappa_3}$

(b) $\dfrac{1}{\kappa} = \dfrac{1}{\kappa_1 + \kappa_2} + \dfrac{1}{2\kappa_3}$

(c) $\kappa = \dfrac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2} + 2\kappa_3$

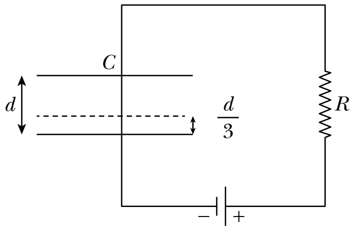
(d) $\kappa = \dfrac{\kappa_1\kappa_3}{\kappa_1 + \kappa_3} + \dfrac{\kappa_2\kappa_3}{\kappa_2 + \kappa_3}$

2. A circuit is connected as shown in the following figure with the switch S open. When the switch is closed, the total amount of charge that flows from Y to X is



(a) 0
(b) 54 μC
(c) 27 μC
(d) 81 μC

3. A parallel-plate capacitor C with plates of unit area and separation d is filled with a liquid of dielectric constant $\kappa = 2$. The level of liquid is $d/3$ initially. Suppose the liquid level decreases at a constant speed v, the time constant as a function of time t is



(a) $\dfrac{6\varepsilon_0 R}{5d + 3vt}$
(b) $\dfrac{(15d + 9vt)\varepsilon_0 R}{2d^2 - 3dvt - 9v^2t^2}$
(c) $\dfrac{6\varepsilon_0 R}{5d - 3vt}$
(d) $\dfrac{(15d - 9vt)\varepsilon_0 R}{2d^2 + 3dvt - 9v^2t^2}$

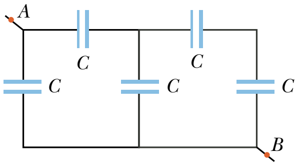
4. Capacitance (in F) of a spherical conductor with radius 1 m is
(a) 1.1×10^{-10}
(b) 10^{-6}
(c) 9×10^{-9}
(d) 10^{-3}

5. A sheet of aluminium foil of negligible thickness is introduced between the plates of a capacitor. The capacitance of the capacitor
(a) Decreases
(b) Remains unchanged
(c) Becomes infinite
(d) Increases

6. The plate areas and plate separations of five parallel-plate capacitors are
Capacitor 1: Area A_0, separation d_0
Capacitor 2: Area $2A_0$, separation $2d_0$
Capacitor 3: Area $2A_0$, separation $d_0/2$
Capacitor 4: Area $A_0/2$, separation $2d_0$
Capacitor 5: Area A_0, separation $d_0/2$
Rank these according to their capacitances, least to greatest.
(a) 1, 2, 3, 4, 5
(b) 5, 4, 3, 2, 1
(c) 5, 3, and 4 tie, then 1, 2
(d) 4, 1, and 2 tie, then 5, 3

7. Electric potential difference between the positive plate and halfway point between the two parallel plates of a fully charged capacitor is
(a) Zero
(b) One-half of the total potential difference between the two plates
(c) One-quarter of the total potential difference between the two plates
(d) None of the above

8. What is the equivalent capacitance of the system of capacitors between A and B as shown in the following figure.

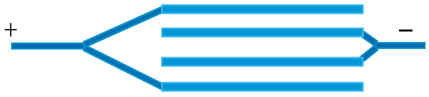


(a) 1 C
(b) 1.6 C
(c) C
(d) None of these

9. How many different capacitances can you produce using three capacitors if you have all three connected in the circuit each time?
(a) 5
(b) 6
(c) 7
(d) 8

10. Two capacitors with C_A greater than C_B and are connected in series with a battery. Which of the following is true?
(a) There is more charge stored on C_A
(b) There is more charge stored on C_B
(c) There is the same charge stored on each capacitor
(d) There is the same potential difference across both capacitors

11. Four metallic plates each with a surface area of one side A are placed at a distance d from each other as shown in the following figure. The capacitance of the system is

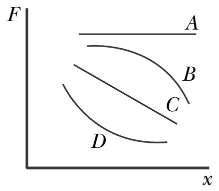


(a) $\dfrac{\varepsilon_0 A}{d}$
(b) $\dfrac{2\varepsilon_0 A}{d}$
(c) $\dfrac{3\varepsilon_0 A}{d}$
(d) $\dfrac{4\varepsilon_0 A}{d}$

12. Two capacitors of capacitances 2 μF and a 1 μF are connected in series and charged from a battery. They store charges P and Q, respectively. When disconnected and charged separately using the same battery, they have charges R and S, respectively. Then
(a) $R > S > Q = P$
(b) $P > Q > R = S$
(c) $R > P = Q > S$
(d) $R = P > S = Q$

13. You have five capacitors. To store maximum amount of energy when using a source with terminal voltage V.
(a) You need to connect them all in series
(b) You need to connect them all in parallel
(c) You need to connect three capacitors in parallel and the rest in series with that combination
(d) You need to connect two capacitors in parallel and the rest in series with that combination

14. Which of following graph in the following figure correctly represents the force between plates of an isolated charged parallel-plate capacitor with distance x between them.

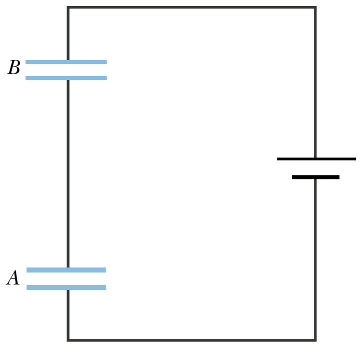


(a) A **(b)** B
(c) C **(d)** D

15. Consider an isolated capacitor made of two parallel metallic plates separated by a distance d. The top plate has a surface charge density of $-\sigma$. A slab of metal of thickness $l < d$ is inserted between the plates, not touching either plates. Upon insertion of the metal slab, the potential difference between the plates

(a) Increase **(b)** Decreases
(c) Remains the same **(d)** Become zero

16. Two identical capacitors are connected as shown in the following figure. If a dielectric slab is inserted in B, choose correct statement.

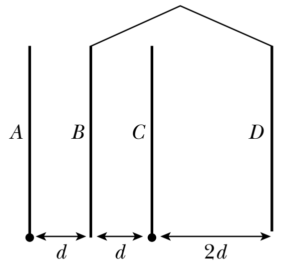


(a) Energy of both capacitors will increase
(b) Energy of both capacitors will decrease
(c) Energy of A will increase but B will decrease
(d) Energy of B will increase but A will decrease

17. A parallel plate capacitor is charged completely and then disconnected from the battery. If the separation between the plates is reduced by 50% and the space between the plates is filled with a dielectric slab of dielectric constant 10, then the potential difference between the plates

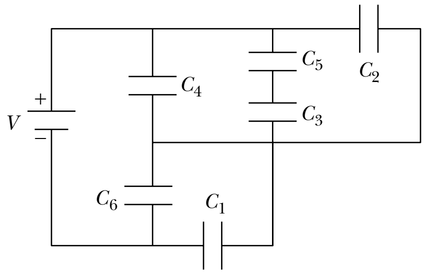
(a) Decreases by 95% **(b)** Increases by 95%
(c) Decreases by 50% **(d)** Increases by 50%

18. Four square plates, each with edge length a, are arranged as shown in the following figure. The equivalent capacitance between A and C is



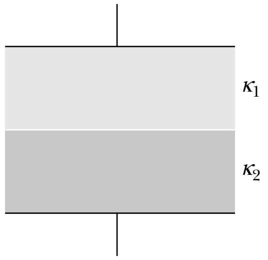
(a) $\dfrac{3\varepsilon_0 a^2}{5d}$ **(b)** $\dfrac{3\varepsilon_0 a^2}{2d}$
(c) $\dfrac{2\varepsilon_0 a^2}{3d}$ **(d)** $\dfrac{5\varepsilon_0 a^2}{3d}$

19. What is the equivalent capacitance of the capacitors in the following figure?
Given that: $C_1 = C_6 = 3\ \mu\text{F}$ and $C_3 = C_5 = 2C_2 = 2C_4 = 4\ \mu\text{F}$.



(a) 4.5 μF **(b)** 1.5 μF
(c) 2 μF **(d)** 3 μF

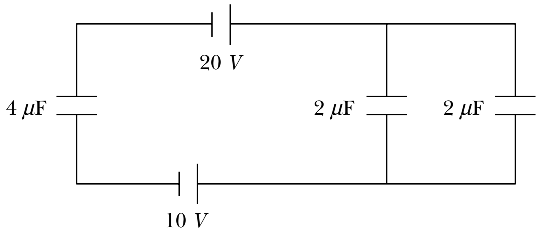
20. A parallel plate capacitor of plate area 2.00×10^{-2} m^2 is filled with two dielectric slabs, each with thickness 2.00 mm. One slab has dielectric constant 3.00, and the other, 4.00 as shown in the figure.



How much charge does the capacitor store when charged by a 7.00 V battery?

(a) 1.08 nC **(b)** 2.16 nC
(c) 0.84 nC **(d)** 1.68 nC

21. Charge stored in 4 μF capacitor in the circuit shown in following figure is:

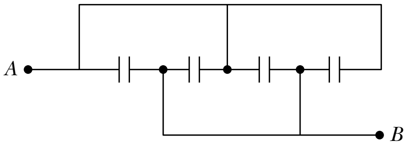


(a) 20 μC **(b)** 40 μC
(c) 10 μC **(d)** 120 μC

22. The plates of a parallel plate capacitor are charged to 100 V. Then a 4 mm thick dielectric slab is inserted between the plates and then to obtain the original potential difference, the distance between the system plates is increased by 2.0 mm. The dielectric constant of the slab is

(a) 5.4 **(b)** 2.2
(c) 2.0 **(d)** 2.6

23. Four capacitors, each of 8 μF, are joined as shown in the figure. The equivalent capacitance between the points A and B is

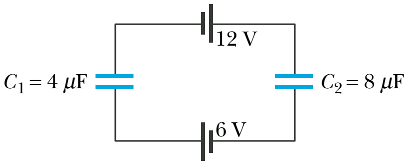


(a) 32 μF **(b)** 2 μF
(c) 8 μF **(d)** 16 μF

More than One Correct Choice Type

24. A dielectric slab of thickness d is inserted in a parallel plate capacitor whose negative plate is at $x = 0$ and positive plate is at $x = 3d$. The slab is equidistant from the plates. The capacitor is given some charge. As x goes from 0 to $3d$,
(a) The magnitude of the electric field remains the same
(b) The direction of the electric field remains the same
(c) The electric potential increases continuously
(d) The electric potential increases at first, then decreases and again increases

25. Consider the circuit shown in the following figure.



(a) The charge on C_2 is greater that on C_1
(b) The charge on C_1 and C_2 are the same
(c) The potential drops across C_1 and C_2 are the same
(d) The potential drops across C_1 is greater than that across C_2

26. You charge a parallel-plate capacitor, remove it from the battery, and prevent the wires connected to the plates from touching each other. When you pull the plates farther apart, which one of the following quantities increases?
(a) Electric field between the plates
(b) Capacitance
(c) Voltage between the plates
(d) Energy stored in the capacitor

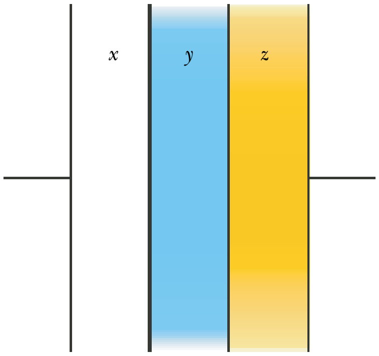
27. Two identical capacitors are charged individually to different potentials and are disconnected from the source. The capacitors are then connected in parallel to each other.
(a) Net charge is less than the sum of initial individual charges
(b) Net charge equals the sum of initial charge
(c) The net potential difference across them is different from the sum of the individual initial potential difference
(d) The net energy stored in the two capacitors is less than the sum of the initial individual energies

28. The top plate of a charged plane capacitor is fixed, whereas the bottom one is kept in equilibrium by the gravitational and electrostatic forces.
(a) When the capacitor is isolated, the plate is in stable equilibrium
(b) When the capacitor is connected to a battery, the plate is in unstable equilibrium
(c) When the capacitor is isolated, the plate is in neutral equilibrium
(d) When the capacitor is connected to a battery, the plate is in neutral equilibrium

29. A parallel-plate capacitor is charged by a battery. The battery is removed and a thick glass slab is inserted between the plates. Now,
(a) The capacity of the capacitor is increased
(b) The electrical energy stored in the capacitor is decreased
(c) The potential across the plates is decreased
(d) The electric field between the plates is decreased

30. A parallel-plate capacitor has parallel sheet of copper inserted between and parallel to the two plate, without touching the plates. The capacity of the capacitor after the introduction of the copper sheet is
(a) Minimum when the copper sheet touches one of the plates
(b) Maximum when the copper sheet is midway between the two plates
(c) Invariant for all positions of the sheet between the plates
(d) Greater than that before introducing the sheet

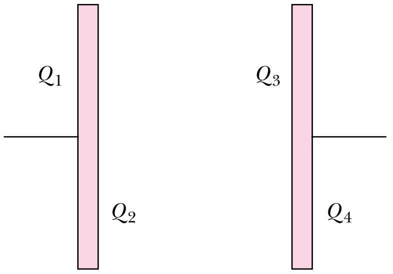
31. In the given figure a capacitor having three layers between its plates. Layer x is vacuum, y is conductor, and z is a dielectric. Which of the following change(s) will result in increase in capacitance?



(a) Replace x by conductor
(b) Replace y by dielectric
(c) Replace z by conductor
(d) Replace x by dielectric

32. A parallel-plate capacitor is charged from a cell and then isolated from it. The separation between the plates is now increased.
(a) The force of attraction between the plates will decrease
(b) The field in the region between the plates will not change
(c) The energy stored in the capacitor will increase
(d) The potential difference between the plates will decrease

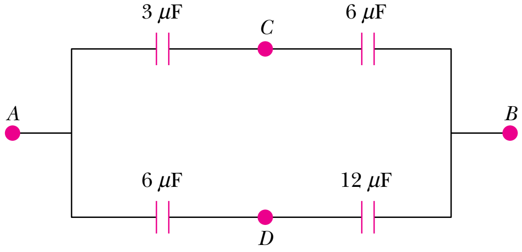
33. In an isolated parallel-plate capacitor of capacitance C, the four surfaces have charges Q_1, Q_2, Q_3, and Q_4, as shown in the following figure. The potential difference between the plates is



(a) $\dfrac{Q_1+Q_2}{C}$ **(b)** $\left|\dfrac{Q_2}{C}\right|$

(c) $\left|\dfrac{Q_3}{C}\right|$ **(d)** $\dfrac{1}{C}[(Q_1+Q_2)-(Q_3-Q_4)]$

34. In the circuit shown in the following figure, some potential difference is applied between A and B. If C is joined to D,



(a) No charge will flow between C and D
(b) Some charge will flow between C and D
(c) The equivalent capacitance between C and D will not change
(d) The equivalent capacitance between C and D will change

35. The capacitance of a parallel-plate capacitor is C_0 when the region between the plates has air. This region is now filled with a dielectric slab of dielectric constant κ. The capacitor is connected to a cell of emf ε and the slab is taken out.
(a) Charge $\varepsilon C_0(\kappa-1)$ flows through the cell
(b) Energy $\varepsilon^2 C_0(\kappa-1)$ is absorbed by the cell
(c) The energy stored in the capacitor is reduced by $\varepsilon^2 C_0(\kappa-1)$
(d) The external agent has to do $(1/2)\varepsilon^2 C_0(\kappa-1)$ amount of work to take the slab out

Linked Comprehension

Passage for Questions 36–38: Consider a parallel-plate capacitor originally with a charge q_0, capacitance C_0, and potential difference ΔV_0. There is an electrostatic force of magnitude F_0 between the plates, and the capacitor has a stored energy U_0. The terminals of the capacitor are connected to another capacitor of same capacitance and charge.

36. A dielectric slab with $\kappa_e > 1$ is inserted between the plates of the first capacitor. Which quantity decreases?
(a) q **(b)** C
(c) ΔV **(d)** F

37. What is the direction of the electrostatic force on the dielectric slab while it is being inserted?
(a) The force pulls the slab into the capacitor
(b) The force pushes the slab out of the capacitor
(c) There is no electrostatic force on the slab
(d) Can't be said

38. Later the dielectric slab is removed. While the slab is being removed
(a) Charge on second capacitor increases.
(b) Charge on second capacitor decreases.
(c) Charge on second capacitor is constant.
(d) Can't be said.

Paragraph for Questions 39–43: Two parallel-plate condensers A and B having capacities of 1 μF and 5 μF are charged separately to the same potential of 100 V. Now the positive plate of A is connected to the negative plate of B and the negative plate of A to the positive plate of B.

39. Common potential is

(a) $\dfrac{200}{5}$ V **(b)** $\dfrac{200}{3}$ V

(c) $\dfrac{200}{4}$ V **(d)** $\dfrac{200}{7}$ V

40. Energy stored in capacitor is
(a) 2×10^{-2} J **(b)** 3×10^{-2} J
(c) 4×10^{-4} J **(d)** 5×10^{-5} J

41. Final charge on capacitor B is
(a) 111 μF **(b)** 222 μF
(c) 444 μF **(d)** 333 μF

42. Final charge on capacitor A is
(a) 76.67 μC **(b)** 84.43
(c) 66.67 **(d)** 37.74 μC

43. Final energy stored in capacitor is
(a) 1.67×10^{-2} J **(b)** 6.71×10^{-2} J
(c) 1.2×10^{-2} J **(d)** 1.33×10^{-2} J

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

44.

Column I	Column II
(a) $(1/4\pi\varepsilon_0)q_1q_2/r^2$, where q_1 and q_2 are charges and r is distance	**(p)** (kilogram) (meter) (second)$^{-2}$
(b) qE, q is charge and E is electric field	**(q)** newton
(c) $(1/2)CV^2$, where C is capacitance and V is voltage	**(r)** (newton) (meter)^2 (coulomb)$^{-1}$
(d) q/ε_0, q is charge and ε_0 is absolute permittivity of free space	**(s)** joule

45.

Column I	Column II
(a) A parallel-plate capacitor is connected to a battery. A dielectric of dielectric constant κ is introduced in the capacitor	**(p)** charge on each plate remains the same.
(b) A parallel-plate capacitor is connected to a battery. Now, battery is disconnected and a dielectric of dielectric constant κ is introduced in the capacitor	**(q)** charge on each plate increases by κ.
(c) A conducting slab completely fills the space between the plate of a capacitor	**(r)** potential difference between the plates of the capacitor becomes infinite.
(d) The distance between the plates of a parallel-plate capacitor is increased	**(s)** the capacitance of the capacitor becomes infinite.

Directions for Questions 46 and 47: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

46. In the given table there are four values of capacitance (Column 1), electric field (Column 2) and potential (Column 3) of capacitors of different shapes and sizes formed by a combination of two conductors separated by an insulator.

Column 1	Column 2	Column 3
(I) $C = \frac{4\pi\varepsilon_0 ba}{(b-a)}$	**(i)** $E = \frac{\lambda}{2\pi\varepsilon_0 r}$	**(J)** $V = \frac{Q\ln(b/a)}{2\pi\varepsilon_0 L}$
(II) $C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{r_b}{r_a}\right)}$	**(ii)** $E = \frac{Q}{\varepsilon_0 A}$	**(K)** $V = \frac{Q(b-a)}{4\pi\varepsilon_0 ba}$
(III) $C = \frac{\varepsilon_0 A}{d}$	**(iii)** $E = \frac{Q}{2\pi\varepsilon_0 r^2}$	**(L)** $V_a - V_b = \int E\cdot dr$
(IV) $C = \frac{Q}{V}$	**(iv)** $E = \frac{\sigma}{\varepsilon_0}$	**(M)** $V = \frac{Qd}{\varepsilon_0 A}$

Here all variables have their usual meaning.

(1) What are the equations for C, E and V of a cylindrical plate capacitor?

(a) (III) (ii) (L)
(b) (II) (i) (J)
(c) (III) (iii) (J)
(d) (I) (i) (M)

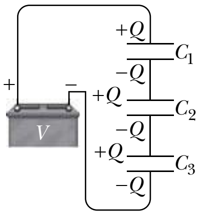
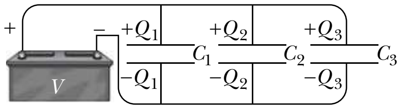
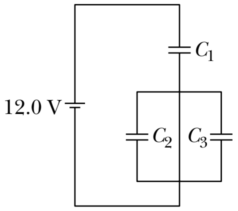
(2) Give the general formulas of C, E and V of capacitor?

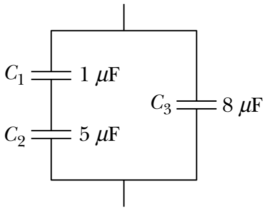
(a) (II) (ii) (L)
(b) (II) (iii) (L)
(c) (IV) (iv) (L)
(d) (I) (i) (K)

(3) What are the equations for C, E and V of a spherical plate capacitor?

(a) (I) (i) (J) **(b)** (I) (i) (M)
(c) (III) (iii) (J) **(d)** (I) (iii) (K)

47. Capacitors can be arranged in two simple and common types of connections, known as series and parallel. In the given table, column 1 shows voltages of different arrangements of capacitors, column 2 shows different arrangement of capacitors and column 3 shows charges of different arrangements of capacitors

Column 1	Column 2	Column 3
(I) All have the same voltage V across their plates.	**(i)**  +Q C1 −Q +Q C2 −Q +Q C3 −Q + − V	**(J)** Each capacitor may store a different charge.
(II) Voltage may vary across their plates.	**(ii)**  + − +Q1 C1 −Q1 +Q2 C2 −Q2 +Q3 C3 −Q3 V	**(K)** Each of the capacitors acquires an identical charge Q.
(III) In mixed combination the series capacitors have different voltages and parallel capacitors have same voltages.	**(iii)** 	**(L)** Each capacitor may store a different charge in first parallel connection, then each of the capacitors acquires an identical charge Q in series connection.

Column 1	Column 2	Column 3
(IV) In mixed combination the series capacitors have same voltages and parallel capacitors have same voltages.	**(iv)** 	**(M)** Each of the capacitors acquires an identical charge Q in first parallel connection, then each capacitor may store a different charge in series connection.

(1) What are the conditions when capacitor is in the parallel network?

(a) (I) (ii) (J) **(b)** (IV) (ii) (M)
(c) (II) (ii) (J) **(d)** (I) (iv) (K)

(2) What are the conditions when capacitors is in mixed combination, that is, parallel and series combination respectively?

(a) (III) (iv) (K) **(b)** (IV) (i) (M)
(c) (III) (iii) (L) **(d)** (I) (i) (M)

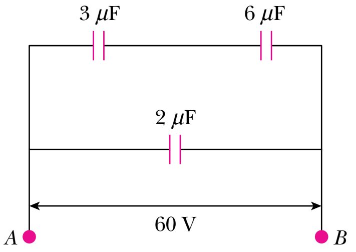
(3) What are the conditions when capacitor is in the series network?

(a) (III) (i) (L) **(b)** (IV) (i) (J)
(c) (II) (i) (K) **(d)** (II) (iii) (M)

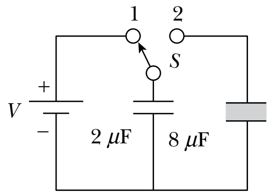
Integer Type

48. P and Q are two capacitors with capacitance 10 μF and 20 μF, respectively, which are connected in series with a battery of value 12 V. Find the ratio between the charges of P and Q.

49. In the circuit shown in the following figure, the potential difference across the 3 μF capacitor is



50. A 2 μF capacitor is charged as shown in the figure. The percentage of its stored energy dissipated after the switch S is turned to position 2 is



ANSWER KEY

Checkpoints

1. (a) same; (b) same **2.** (a) decreases; (b) increases; (c) decreases **3.** (a) V, $q/2$; (b) $V/2$, q

4. Inner conductor as electric filed strength will reach its breakdown value there first.

Problems

1. (a) unchanged; (b) 1.58 μF

2. (a) 5.625×10^{-4} C ≈ 563 C; (b) 37.5 V; (c) 10.5 mJ; (d) 37.5×10^{-4} C; (e) 37.5 V; (f) 7.03 mJ; (g) 1.88×10^{-4} C; (h) 37.5 V; (i) 3.52 mJ

3. (a) 5.13×10^{-4} C; (b) 51.3 V; (c) 20.1 mJ **4.** 8.44 μF **5.** 25 F

6. (a) 2.00×10^{-4} C; (b) 100 V; (c) 1.00×10^{-2} J; (d) 3.33×10^{-4} C; (e) 33.3 V; (f) 5.55×10^{-3} J; (g) 3.33×10^{-4} C; (h) 66.7 V; (i) 1.11×10^{-2} J

7. (a) 59 pF; (b) 80 pF; (c) 7.1 nC; (d) 7.1 nC; (e) 10 kV/m; (f) 2.1 kV/m; (g) 88 V; (h) −0.11 μJ

8. (a) 42.7 μC; (b) 21.3 μC; (c) 21.3 μC **9.** (a) 1.5 μF; (b) 2.0 μF

10. (a) 10 pF; (b) 18 nC; (c) 4.4 μJ; (d) 5.0×10^5 V/m; (e) 1.1 J/m^3 **11.** (a) 6.5; (b) porcelain **12.** 2.5

13. (a) 0.27 J; (b) 4.5 **14.** 8.85×10^{-10} C

15. (a) 9.16×10^{-18} J/m^3; (b) 9.16×10^{-6} J/m^3; (c) 9.16×10^{6} J/m^3; (d) 9.16×10^{18} J/m^3; (e) 9.16×10^{30} J/m^3; (f) for $r \to 0$, the energy density $u \to \infty$.

16. (a) 0.48 nF; (b) 28 kV

17. (a) 6.00 V; (b) 1.20×10^{-10} J; (c) 4.51×10^{-11} J; (d) -7.52×10^{-11} J.

18. 7.52×10^{-12} F

19. (a) 175 V; (b) 4.375×10^{-2} J $\approx$ 44 mJ

20. 1.44×10^{-10} F/m $\approx$ 0.14 nF/m

21. 5.42×10^{-11} F

22. (a) 45.8 pF; (b) 5.49 nC

23. (a) 1.20×10^{-4} C; (b) 2.40×10^{-5} C

24. (a) 5.3×10^{-2} m; (b) 2.8×10^{2} pF

25. 0.0563 C

26. 2.73×10^{-12} F

27. 8.87×10^{-12} F

28. 3.6×10^{-3} C

29. (a) 8.0 μF; (b) 80 μC

30. (a) 4.00 μF; (b) 8.00×10^{-5} C; (c) 6 67 V; (d) 4.00×10^{-5} C; (e) 13.3 V; (f) 2.67×10^{-5} C; (g) 6.67 V; (h) 2.67×10^{-5} C

31. 0.17 m^2

32. (a) 13.4 pF; (b) 1.15 nC; (c) 1.13×10^{4} N/C; (d) 4.33×10^{3} N/C

33. (a) 100 V; (b) 1.0×10^{-4} C; (c) 3.0×10^{-4} C

34. (a) 8.0 V; (b) 4.5 μF; (c) 1.5 μF

35. (a) 4.0 μF; (b) 1.6 μF

36. (a) 0.333 J/m^3; (b) decreasing the diameter, or the radius, will increase u

37. (a) 6.8; (b) 7.2×10^{-7} C

38. 4.20×10^{-13} F

39. 20 μC

40. (a) 18.0 μC; (b) 3.20 μC; (c) 18.0 μC; (d) 3.20 μC; (e) 16.8 μC; (f) 33.6 μC; (g) 21.6 μC; (h) 28.8 μC

41. 24

42. (a) 31.4 pF; (b) 2.29 nC; (c) 1.96 nC

43. (a) 4.43×10^{-12} m; (b) Since d is much less than the size of an atom ($\sim 10^{-10}$ m), this capacitor cannot be constructed

44. (a) 2.0 pF; (b) 2.0 pF; (c) 1.0×10^{2} V.

45. 24 μC

46. 129 pF

47. (a) 59.4 pF; (b) 186 cm^2

48. (a) 5.6×10^{-12} C; (b) decreasing area decreases the charge q

Practice Questions

Single Correct Choice Type

1. (b)
2. (c)
3. (a)
4. (a)
5. (b)
6. (d)
7. (b)
8. (b)
9. (d)
10. (c)
11. (b)
12. (a)
13. (b)
14. (a)
15. (b)
16. (c)
17. (a)
18. (a)
19. (d)
20. (a)
21. (a)
22. (c)
23. (a)

More than One Correct Choice Type

24. (b), (c)
25. (b), (c)
26. (c), (d)
27. (b), (c), (d)
28. (b), (c)
29. (a), (b), (c), (d)
30. (c), (d)
31. (a), (c), (d)
32. (b), (c)
33. (b), (c)
34. (a), (c)
35. (a), (b), (d)

Linked Comprehension

36. (c)
37. (a)
38. (a)
39. (b)
40. (b)
41. (d)
42. (c)
43. (d)

Matrix-Match

44. (a) → (p), (q); (b) → (p), (q); (c) → (s); (d) → (r)

45. (a) → (q); (b) → (p); (c) → (r), (s); (d) → (p)

46. (1) → (b); (2) → (c); (3) → (d)

47. (1) → (a); (2) → (c); (3) → (c)

Integer Type

48. 1
49. 40
50. 80

CHAPTER

26

Current and Resistance

Contents

26.1 What is Physics?
26.2 Electric Current
26.3 Current Density
26.4 Resistance and Resistivity
26.5 Ohm's Law
26.6 A Microscopic View of Ohm's Law
26.7 Power in Electric Circuits
26.8 Semiconductors
26.9 Superconductors

26.1 | WHAT IS PHYSICS?

In the last four chapters we discussed electrostatics—the physics of stationary charges. In this and the next chapter, we discuss the physics of **electric currents**—that is, charges in motion.

Examples of electric currents abound and involve many professions. Meteorologists are concerned with lightning and with the less dramatic slow flow of charge through the atmosphere. Biologists, physiologists, and engineers working in medical technology are concerned with the nerve currents that control muscles and especially with how those currents can be reestablished after spinal cord injuries. Electrical engineers are concerned with countless electrical systems, such as power systems, lightning protection systems, information storage systems, and music systems. Space engineers monitor and study the flow of charged particles from our Sun because that flow can wipe out telecommunication systems in orbit and even power transmission systems on the ground. In addition to such scholarly work, almost every aspect of daily life now depends on information carried by electric currents, from stock trades to ATM transfers and from video entertainment to social networking.

In this chapter we discuss the basic physics of electric currents and why they can be established in some materials but not in others. We begin with the meaning of electric current.

26.2 | ELECTRIC CURRENT

Key Concepts

- An electric current i in a conductor is defined by

$$i = \frac{dq}{dt},$$

where dq is the amount of positive charge that passes in time dt.

- By convention, the direction of electric current is taken as the direction in which positive charge carriers would move even though (normally) only conduction electrons can move.

Although an electric current is a stream of moving charges, not all moving charges constitute an electric current. If there is to be an electric current through a given surface, there must be a net flow of charge through that surface. Two examples clarify our meaning.

1. The free electrons (conduction electrons) in an isolated length of copper wire are in random motion at speeds of the order of 10^6 m/s. If you pass a hypothetical plane through such a wire, conduction electrons pass through it *in both directions* at the rate of many billions per second—but there is *no net transport* of charge and thus *no current* through the wire. However, if you connect the ends of the wire to a battery, you slightly bias the flow in one direction, with the result that there now is a net transport of charge and thus an electric current through the wire.
2. The flow of water through a garden hose represents the directed flow of positive charge (the protons in the water molecules) at a rate of perhaps several million coulombs per second. There is no net transport of charge, however, because there is a parallel flow of negative charge (the electrons in the water molecules) of exactly the same amount moving in exactly the same direction.

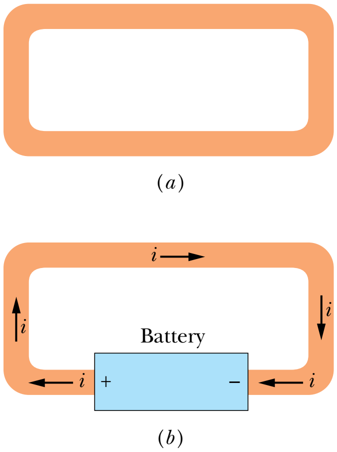


Figure 26-1 (*a*) A loop of copper in electrostatic equilibrium. The entire loop is at a single potential, and the electric field is zero at all points inside the copper. (*b*) Adding a battery imposes an electric potential difference between the ends of the loop that are connected to the terminals of the battery. The battery thus produces an electric field within the loop, from terminal to terminal, and the field causes charges to move around the loop. This movement of charges is a current *i*.

In this chapter we restrict ourselves largely to the study—within the framework of classical physics—of *steady* currents of *conduction electrons* moving through *metallic conductors* such as copper wires.

As Fig. 26-1*a* reminds us, any isolated conducting loop—regardless of whether it has an excess charge—is all at the same potential. No electric field can exist within it or along its surface. Although conduction electrons are available, no net electric force acts on them and thus there is no current.

If, as in Fig. 26-1*b*, we insert a battery in the loop, the conducting loop is no longer at a single potential. Electric fields act inside the material making up the loop, exerting forces on the conduction electrons, causing them to move and thus establishing a current. After a very short time, the electron flow reaches a constant value and the current is in its *steady state* (it does not vary with time). Steady current means that the amount of charge entering an end of the wire at any time is equal to the amount of charge leaving the wire at the other end. In other words, no charge is accumulating at any point.

The current may change with time, but its value is the same at every cross section. If the current out of any cross section is more than the incoming current, then some negative charge will build up at the cross section. First, this would mean that there is source or sink of charge which does not exist in nature. Even if there are such points where we hypothetically assume that the charges are stored, then this negative charge built up at the cross section will decrease the outgoing current (by decelerating the positive charge flow) and increase the incoming current (by accelerating the positive charge inflow). The net effect is that the current will be equal in a very short time. This mechanism of nature to keep the current constant is so refined that even if the current is changing the direction with time (as in AC), still the current at all the cross sections at any given time will have the same value.

The current is the same in any cross section.

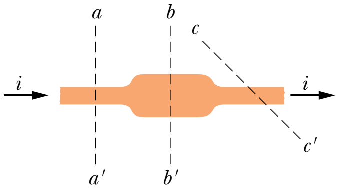


Figure 26-2 The current *i* through the conductor has the same value at planes *aa*′, *bb*′, and *cc*′.

Figure 26-2 shows a section of a conductor, part of a conducting loop in which current has been established. If charge dq passes through a hypothetical plane (such as aa') in time dt, then the current i through that plane is defined as

$$i = \frac{dq}{dt} \quad \text{(definition of current).} \tag{26-1}$$

We can find the charge that passes through the plane in a time interval extending from 0 to t by integration:

$$q = \int dq = \int_0^t i\,dt, \tag{26-2}$$

in which the current i may vary with time.

Under steady-state conditions, the current is the same for planes aa', bb', and cc' and indeed for all planes that pass completely through the conductor, no matter what their location or orientation. This follows from the fact that charge is conserved. Under the steady-state conditions assumed here, an electron must pass through plane aa' for every electron that passes through plane cc'. In the same way, if we have a steady flow of water through a garden hose, a drop of water must leave the nozzle for every drop that enters the hose at the other end. The amount of water in the hose is a conserved quantity.

The SI unit for current is the coulomb per second, or the ampere (A), which is an SI base unit:

$$1 \text{ ampere} = 1 \text{ A} = 1 \text{ coulomb per second} = 1 \text{ C/s}.$$

The formal definition of the ampere is discussed in Chapter 29.

Current, as defined by Eq. 26-1, is a scalar because both charge and time in that equation are scalars. Yet, as in Fig. 26-1*b*, we often represent a current with an arrow to indicate that charge is moving. Such arrows are not vectors, however, and they do not require vector addition. Figure 26-3*a* shows a conductor with current i_0 splitting at a junction into two branches. Because charge is conserved, the magnitudes of the currents in the branches must add to yield the magnitude of the current in the original conductor, so that

$$i_0 = i_1 + i_2. \tag{26-3}$$

As Fig. 26-3*b* suggests, bending or reorienting the wires in space does not change the validity of Eq. 26-3. Current arrows show only a direction (or sense) of flow along a conductor, not a direction in space.

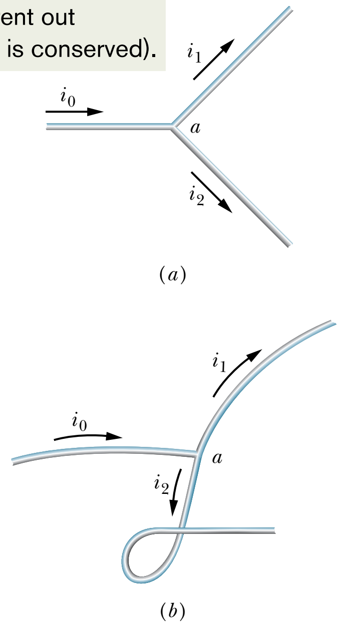

The current into the junction must equal the current out (charge is conserved).

Figure 26-3 The relation $i_0 = i_1 + i_2$ is true at junction a no matter what the orientation in space of the three wires. Currents are scalars, not vectors.

The Directions of Currents

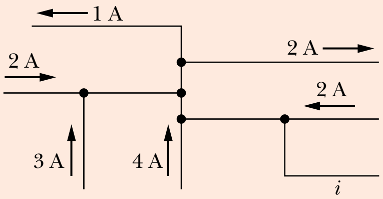
In Fig. 26-1*b* we drew the current arrows in the direction in which positively charged particles would be forced to move through the loop by the electric field. Such positive *charge carriers*, as they are often called, would move away from the positive battery terminal and toward the negative terminal. Actually, the charge carriers in the copper loop of Fig. 26-1*b* are electrons and thus are negatively charged. The electric field forces them to move in the direction opposite the current arrows, from the negative terminal to the positive terminal. For historical reasons, however, we use the following convention:

A current arrow is drawn in the direction in which positive charge carriers would move, even if the actual charge carriers are negative and move in the opposite direction.

We can use this convention because in *most* situations, the assumed motion of positive charge carriers in one direction has the same effect as the actual motion of negative charge carriers in the opposite direction. (When the effect is not the same, we shall drop the convention and describe the actual motion.)

CHECKPOINT 1

The figure here shows a portion of a circuit. What are the magnitude and direction of the current i in the lower right-hand wire?



SAMPLE PROBLEM 26.01

Current is the rate at which charge passes a point

Water flows through a garden hose at a volume flow rate dV/dt of 450 cm^3/s. What is the current of negative charge?

KEY IDEAS

The current i of negative charge is due to the electrons in the water molecules moving through the hose. The current is the rate at which that negative charge passes through any plane that cuts completely across the hose.

Calculations: We can write the current in terms of the number of molecules that pass through such a plane per second as

$$i = \begin{pmatrix}\text{charge}\\\text{per}\\\text{electron}\end{pmatrix}\begin{pmatrix}\text{electrons}\\\text{per}\\\text{molecule}\end{pmatrix}\begin{pmatrix}\text{molecules}\\\text{per}\\\text{second}\end{pmatrix}$$

or

$$i = (e)(10)\frac{dN}{dt}.$$

We substitute 10 electrons per molecule because a water (H_2O) molecule contains 8 electrons in the single oxygen atom and 1 electron in each of the two hydrogen atoms.

We can express the rate dN/dt in terms of the given volume flow rate dV/dt by first writing

$$\begin{pmatrix}\text{molecules}\\\text{per}\\\text{second}\end{pmatrix} = \begin{pmatrix}\text{molecules}\\\text{per}\\\text{mole}\end{pmatrix}\begin{pmatrix}\text{moles}\\\text{per unit}\\\text{mass}\end{pmatrix} \times \begin{pmatrix}\text{mass}\\\text{per unit}\\\text{volume}\end{pmatrix}\begin{pmatrix}\text{volume}\\\text{per}\\\text{second}\end{pmatrix}.$$

"Molecules per mole" is Avogadro's number N_A. "Moles per unit mass" is the inverse of the mass per mole, which is the molar mass M of water. "Mass per unit volume" is the (mass) density ρ_{mass} of water. The volume per second is the volume flow rate dV/dt. Thus, we have

$$\frac{dN}{dt} = N_A\left(\frac{1}{M}\right)\rho_{\text{mass}}\left(\frac{dV}{dt}\right) = \frac{N_A\rho_{\text{mass}}}{M}\frac{dV}{dt}.$$

Substituting this into the equation for i, we find

$$i = 10eN_A M^{-1}\rho_{\text{mass}}\frac{dV}{dt}.$$

We know that Avogadro's number N_A is 6.02×10^{23} molecules/mol, or 6.02×10^{23} mol^{-1}, and from Table 15-1 we know that the density of water ρ_{mass} under normal conditions is 1000 kg/m^3. We can get the molar mass of water from the molar masses listed in Appendix F (in grams per mole): We add the molar mass of oxygen (16 g/mol) to twice the molar mass of hydrogen (1 g/mol), obtaining 18 g/mol = 0.018 kg/mol. So, the current of negative charge due to the electrons in the water is

$$\begin{aligned} i &= (10)(1.6\times10^{-19}\ \text{C})(6.02\times10^{23}\ \text{mol}^{-1})\\ &\quad\times(0.018\ \text{kg/mol})^{-1}(1000\ \text{kg/m}^3)(450\times10^{-6}\ \text{m}^3/\text{s})\\ &= 2.41\times10^{7}\ \text{C/s} = 2.41\times10^{7}\ \text{A}\\ &= 24.1\ \text{MA}. \qquad \text{(Answer)}\end{aligned}$$

This current of negative charge is exactly compensated by a current of positive charge associated with the nuclei of the three atoms that make up the water molecule. Thus, there is no net flow of charge through the hose.

26.3 | CURRENT DENSITY

Key Concepts

- Current i (a scalar quantity) is related to current density $\vec{J}$ (a vector quantity) by

$$i = \int \vec{J}\cdot d\vec{A},$$

  where $d\vec{A}$ is a vector perpendicular to a surface element of area dA and the integral is taken over any surface cutting across the conductor. The current density $\vec{J}$ has the same direction as the velocity of the moving charges if they are positive and the opposite direction if they are negative.
- When an electric field $\vec{E}$ is established in a conductor, the charge carriers (assumed positive) acquire a drift speed v_d in the direction of $\vec{E}$.
- The drift velocity $\vec{v}_d$ is related to the current density by

$$\vec{J} = (ne)\vec{v}_d,$$

  where ne is the carrier charge density.

Sometimes we are interested in the current i in a particular conductor. At other times we take a localized view and study the flow of charge through a cross section of the conductor at a particular point. To describe this flow, we can use the **current density** $\vec{J}$, which has the same direction as the velocity of the moving charges if they are positive and the opposite direction if they are negative. For each element of the cross section, the magnitude J is equal to the current per unit area through that element. We can write the amount of current through the element as $\vec{J}\cdot d\vec{A}$, where $d\vec{A}$ is the area vector of the element, perpendicular to the element. The total current through the surface is then

$$i = \int \vec{J}\cdot d\vec{A}. \tag{26-4}$$

If the current is uniform across the surface and parallel to $d\vec{A}$, then $\vec{J}$ is also uniform and parallel to $d\vec{A}$. Then Eq. 26-4 becomes

$$i = \int J\,dA = J\int dA = JA,$$

so

$$J = \frac{i}{A}, \tag{26-5}$$

where A is the total area of the surface. From Eq. 26-4 or 26-5 we see that the SI unit for current density is the ampere per square meter (A/m^2).

In Chapter 22 we saw that we can represent an electric field with electric field lines. Figure 26-4 shows how current density can be represented with a similar set of lines, which we can call *streamlines*. The current, which is toward the right in Fig. 26-4, makes a transition from the wider conductor at the left to the narrower conductor at the right. Because charge is conserved during the transition, the amount of charge and thus the amount of current cannot change. However, the current density does change—it is greater in the narrower conductor. The spacing of the streamlines suggests this increase in current density; streamlines that are closer together imply greater current density.

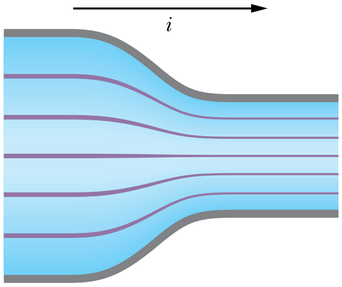


Figure 26-4 Streamlines representing current density in the flow of charge through a constricted conductor.

Drift Speed

When a conductor does not have a current through it, its conduction electrons move randomly, with no net motion in any direction. When the conductor does have a current through it, these electrons actually still move randomly, but now they tend to *drift* with a **drift speed** v_d in the direction opposite that of the applied electric field that causes the current. The drift speed is tiny compared with the speeds in the random motion. For example, in the copper conductors of household wiring, electron drift speeds are perhaps 10^{-5} or 10^{-4} m/s, whereas the random-motion speeds are around 10^6 m/s.

We can use Fig. 26-5 to relate the drift speed v_d of the conduction electrons in a current through a wire to the magnitude J of the current density in the wire. For convenience, Fig. 26-5 shows the equivalent drift of *positive* charge carriers in the direction of the applied electric field $\vec{E}$. Let us assume that these charge carriers all move with the same drift speed v_d and that the current density J is uniform across the wire's cross-sectional area A. The number of charge carriers in a length L of the wire is nAL, where n is the number of carriers per unit volume. The total charge of the carriers in the length L, each with charge e, is then

$$q = (nAL)e.$$

Because the carriers all move along the wire with speed v_d, this total charge moves through any cross section of the wire in the time interval

$$t = \frac{L}{v_d}.$$

Current is said to be due to positive charges that are propelled by the electric field.

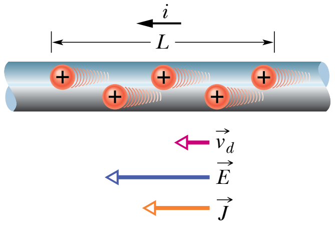


Figure 26-5 Positive charge carriers drift at speed v_d in the direction of the applied electric field $\vec{E}$. By convention, the direction of the current density $\vec{J}$ and the sense of the current arrow are drawn in that same direction.

Equation 26-1 tells us that the current i is the time rate of transfer of charge across a cross section, so here we have

$$i = \frac{q}{t} = \frac{nALe}{L/v_d} = nAev_d. \tag{26-6}$$

Solving for v_d and recalling Eq. 26-5 ($J = i/A$), we obtain

$$v_d = \frac{i}{nAe} = \frac{J}{ne}$$

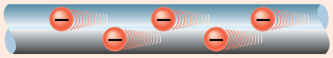
or, extended to vector form,

$$\vec{J} = (ne)\vec{v}_d. \tag{26-7}$$

Here the product ne, whose SI unit is the coulomb per cubic meter (C/m^3), is the *carrier charge density*. For positive carriers, ne is positive and Eq. 26-7 predicts that $\vec{J}$ and $\vec{v}_d$ have the same direction. For negative carriers, ne is negative and $\vec{J}$ and $\vec{v}_d$ have opposite directions.

CHECKPOINT 2

The figure shows conduction electrons moving leftward in a wire. Are the following leftward or rightward: (a) the current i, (b) the current density $\vec{J}$, (c) the electric field $\vec{E}$ in the wire?



SAMPLE PROBLEM 26.02

Current density, uniform and nonuniform

(a) The current density in a cylindrical wire of radius $R = 2.0$ mm is uniform across a cross section of the wire and is $J = 2.0 \times 10^5$ A/m^2. What is the current through the outer portion of the wire between radial distances $R/2$ and R (Fig. 26-6*a*)?

KEY IDEA

Because the current density is uniform across the cross section, the current density J, the current i, and the cross-sectional area A are related by Eq. 26-5 ($J = i/A$).

Calculations: We want only the current through a reduced cross-sectional area A' of the wire (rather than the entire area), where

$$A' = \pi R^2 - \pi\left(\frac{R}{2}\right)^2 = \pi\left(\frac{3R^2}{4}\right)$$

$$= \frac{3\pi}{4}(0.0020\ \text{m})^2 = 9.424 \times 10^{-6}\ \text{m}^2.$$

So, we rewrite Eq. 26-5 as

$$i = JA'$$

and then substitute the data to find

$$i = (2.0 \times 10^5\ \text{A/m}^2)(9.424 \times 10^{-6}\ \text{m}^2)$$

$$= 1.9\ \text{A}. \qquad \text{(Answer)}$$

(b) Suppose, instead, that the current density through a cross section varies with radial distance r as $J = ar^2$, in which $a = 3.0 \times 10^{11}$ A/m^4 and r is in meters. What now is the current through the same outer portion of the wire?

KEY IDEA

Because the current density is not uniform across a cross section of the wire, we must resort to Eq. 26-4 ($i = \int \vec{J} \cdot d\vec{A}$) and integrate the current density over the portion of the wire from $r = R/2$ to $r = R$.

Calculations: The current density vector $\vec{J}$ (along the wire's length) and the differential area vector $d\vec{A}$

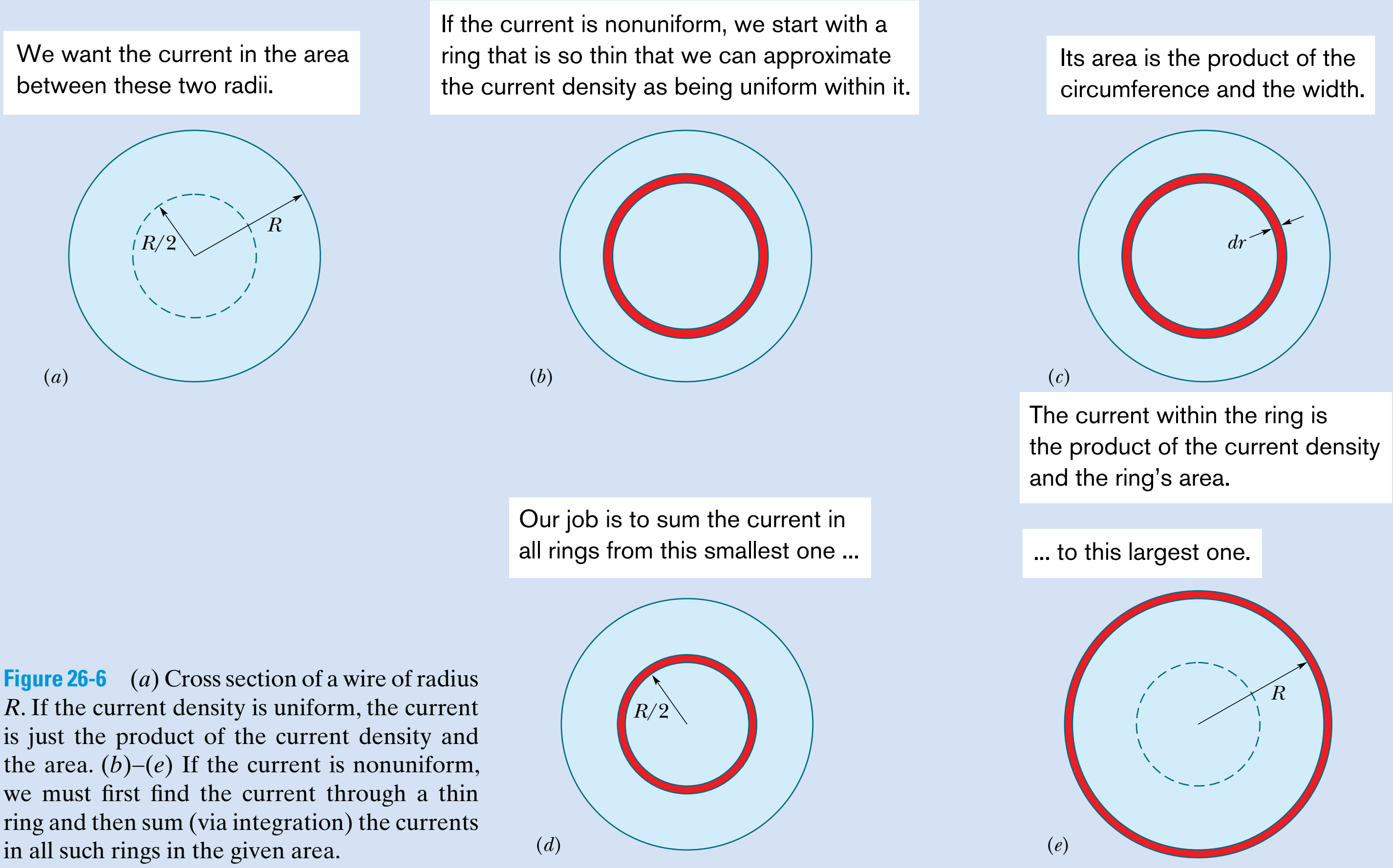
(perpendicular to a cross section of the wire) have the same direction. Thus,

$$\vec{J}\cdot d\vec{A} = J\,dA\cos 0 = J\,dA.$$

We need to replace the differential area dA with something we can actually integrate between the limits $r = R/2$ and $r = R$. The simplest replacement (because J is given as a function of r) is the area $2\pi r\,dr$ of a thin ring of circumference $2\pi r$ and width dr (Fig. 26-6*b*). We can then integrate with r as the variable of integration. Equation 26-4 then gives us

$$\begin{aligned} i &= \int \vec{J}\cdot d\vec{A} = \int J\,dA \\ &= \int_{R/2}^{R} ar^2\,2\pi r\,dr = 2\pi a\int_{R/2}^{R} r^3\,dr \\ &= 2\pi a\left[\frac{r^4}{4}\right]_{R/2}^{R} = \frac{\pi a}{2}\left[R^4 - \frac{R^4}{16}\right] = \frac{15}{32}\pi a R^4 \\ &= \frac{15}{32}\pi(3.0\times10^{11}\ \text{A/m}^4)(0.0020\ \text{m})^4 = 7.1\ \text{A}. \end{aligned}$$

(Answer)


We want the current in the area between these two radii.

If the current is nonuniform, we start with a ring that is so thin that we can approximate the current density as being uniform within it.

Its area is the product of the circumference and the width.

The current within the ring is the product of the current density and the ring's area.

Our job is to sum the current in all rings from this smallest one ...

... to this largest one.

Figure 26-6 (*a*) Cross section of a wire of radius *R*. If the current density is uniform, the current is just the product of the current density and the area. (*b*)–(*e*) If the current is nonuniform, we must first find the current through a thin ring and then sum (via integration) the currents in all such rings in the given area.

SAMPLE PROBLEM 26.03

In a current, the conduction electrons move very slowly

What is the drift speed of the conduction electrons in a copper wire with radius $r = 900\ \mu\text{m}$ when it has a uniform current $i = 17$ mA? Assume that each copper atom contributes one conduction electron to the current and that the current density is uniform across the wire's cross section.

KEY IDEAS

(1) The drift speed v_d is related to the current density $\vec{J}$ and the number n of conduction electrons per unit volume according to Eq. 26-7, which we can write as $J = nev_d$.

(2) Because the current density is uniform, its magnitude J is related to the given current i and wire size by Eq. 26-5 ($J = i/A$, where A is the cross-sectional area of the wire). (3) Because we assume one conduction electron per atom, the number n of conduction electrons per unit volume is the same as the number of atoms per unit volume.

Calculations: Let us start with the third idea by writing

$$n = \begin{pmatrix}\text{atoms}\\ \text{per unit}\\ \text{volume}\end{pmatrix} = \begin{pmatrix}\text{atoms}\\ \text{per}\\ \text{mole}\end{pmatrix}\begin{pmatrix}\text{moles}\\ \text{per unit}\\ \text{mass}\end{pmatrix}\begin{pmatrix}\text{mass}\\ \text{per unit}\\ \text{volume}\end{pmatrix}.$$

The number of atoms per mole is just Avogadro's number N_A ($= 6.02 \times 10^{23}\ \text{mol}^{-1}$). Moles per unit mass is the inverse of the mass per mole, which here is the molar mass M of copper. The mass per unit volume is the (mass) density ρ_{mass} of copper. Thus,

$$n = N_A\left(\frac{1}{M}\right)\rho_{\text{mass}} = \frac{N_A\rho_{\text{mass}}}{M}.$$

Taking copper's molar mass M and density ρ_{mass} from Appendix F, we then have (with some conversions of units)

$$\begin{aligned} n &= \frac{(6.02\times10^{23}\ \text{mol}^{-1})(8.96\times10^{3}\ \text{kg/m}^3)}{63.54\times10^{-3}\ \text{kg/mol}} \\ &= 8.49\times10^{28}\ \text{electrons/m}^3 \end{aligned}$$

or

$$n = 8.49 \times 10^{28}\ \text{m}^{-3}.$$

Next let us combine the first two key ideas by writing

$$\frac{i}{A} = nev_d.$$

Substituting for A with πr^2 ($= 2.54 \times 10^{-6}\ \text{m}^2$) and solving for v_d, we then find

$$\begin{aligned} v_d &= \frac{i}{ne(\pi r^2)} \\ &= \frac{17\times10^{-3}\ \text{A}}{(8.49\times10^{28}\ \text{m}^{-3})(1.6\times10^{-19}\ \text{C})(2.54\times10^{-6}\ \text{m}^2)} \\ &= 4.9\times10^{-7}\ \text{m/s}, \qquad \text{(Answer)} \end{aligned}$$

which is only 1.8 mm/h, slower than a sluggish snail.

Lights are fast: You may well ask: "If the electrons drift so slowly, why do the room lights turn on so quickly when I throw the switch?" Confusion on this point results from not distinguishing between the drift speed of the electrons and the speed at which *changes* in the electric field configuration travel along wires. This latter speed is nearly that of light; electrons everywhere in the wire begin drifting almost at once, including into the lightbulbs. Similarly, when you open the valve on your garden hose with the hose full of water, a pressure wave travels along the hose at the speed of sound in water. The speed at which the water itself moves through the hose—measured perhaps with a dye marker—is much slower.

26.4 | RESISTANCE AND RESISTIVITY

Key Concepts

- The resistance R of a conductor is defined as

$$R = \frac{V}{i},$$

where V is the potential difference across the conductor and i is the current.

- The resistivity ρ and conductivity σ of a material are related by

$$\rho = \frac{1}{\sigma} = \frac{E}{J},$$

where E is the magnitude of the applied electric field and J is the magnitude of the current density.

- The electric field and current density are related to the resistivity by

$$\vec{E} = \rho\vec{J}.$$

- The resistance R of a conducting wire of length L and uniform cross section is

$$R = \rho\frac{L}{A},$$

where A is the cross-sectional area.

- The resistivity ρ for most materials changes with temperature. For many materials, including metals, the relation between ρ and temperature T is approximated by the equation

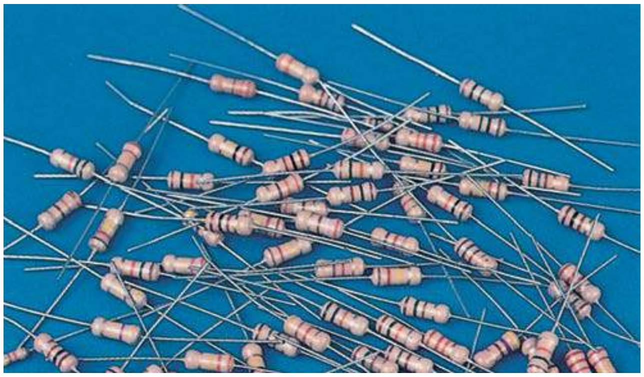
$$\rho - \rho_0 = \rho_0\alpha(T - T_0).$$

Here T_0 is a reference temperature, ρ_0 is the resistivity at T_0, and α is the temperature coefficient of resistivity for the material.

If we apply the same potential difference between the ends of geometrically similar rods of copper and of glass, very different currents result. The characteristic of the conductor that enters here is its electrical **resistance**. We determine the resistance between any two points of a conductor by applying a potential difference V between those points and measuring the current i that results.

The resistance R is then

$$R = \frac{V}{i} \quad \text{(definition of } R\text{).} \tag{26-8}$$



The Image Works

Figure 26-7 An assortment of resistors. The circular bands are color-coding marks that identify the value of the resistance.

The SI unit for resistance that follows from Eq. 26-8 is the volt per ampere. This combination occurs so often that we give it a special name, the **ohm** (symbol Ω); that is,

$$1 \text{ ohm} = 1\ \Omega = 1 \text{ volt per ampere}$$

$$= 1 \text{ V/A}. \tag{26-9}$$

A conductor whose function in a circuit is to provide a specified resistance is called a **resistor** (see Fig. 26-7). In a circuit diagram, we represent a resistor and a resistance with the symbol -⋀⋀⋀-. If we write Eq. 26-8 as

$$i = \frac{V}{R},$$

we see that, for a given V, the greater the resistance, the smaller the current.

The resistance of a conductor depends on the manner in which the potential difference is applied to it. Figure 26-8, for example, shows a given potential difference applied in two different ways to the same conductor. As the current density streamlines suggest, the currents in the two cases—hence the measured resistances—will be different. Unless otherwise stated, we shall assume that any given potential difference is applied as in Fig. 26-8*b*.

As we have done several times in other connections, we often wish to take a general view and deal not with particular objects but with materials. Here we do so by focusing not on the potential difference V across a particular resistor but on the electric field $\vec{E}$ at a point in a resistive material. Instead of dealing with the current i through the resistor, we deal with the current density $\vec{J}$ at the point in question. Instead of the resistance R of an object, we deal with the **resistivity** ρ of the *material:*

$$\rho = \frac{E}{J} \quad \text{(definition of } \rho\text{).} \tag{26-10}$$

(Compare this equation with Eq. 26-8.)

If we combine the SI units of E and J according to Eq. 26-10, we get, for the unit of ρ, the ohm-meter ($\Omega \cdot \text{m}$):

$$\frac{\text{unit } (E)}{\text{unit } (J)} = \frac{\text{V/m}}{\text{A/m}^2} = \frac{\text{V}}{\text{A}}\ \text{m} = \Omega \cdot \text{m}.$$

(Do not confuse the *ohm-meter,* the unit of resistivity, with the *ohmmeter,* which is an instrument that measures resistance.) Table 26-1 lists the resistivities of some materials.

We can write Eq. 26-10 in vector form as

$$\vec{E} = \rho \vec{J}. \tag{26-11}$$

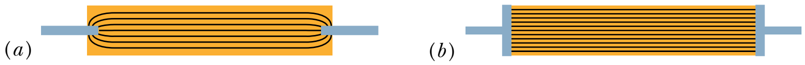


Figure 26-8 Two ways of applying a potential difference to a conducting rod. The gray connectors are assumed to have negligible resistance. When they are arranged as in (*a*) in a small region at each rod end, the measured resistance is larger than when they are arranged as in (*b*) to cover the entire rod end.

Equations 26-10 and 26-11 hold only for *isotropic* materials—materials whose electrical properties are the same in all directions.

We often speak of the **conductivity** σ of a material. This is simply the reciprocal of its resistivity, so

$$\sigma = \frac{1}{\rho} \quad \text{(definition of } \sigma\text{)}. \tag{26-12}$$

The SI unit of conductivity is the reciprocal ohm-meter, $(\Omega \cdot \text{m})^{-1}$. The unit name mhos per meter is sometimes used (mho is ohm backwards). The definition of σ allows us to write Eq. 26-11 in the alternative form

$$\vec{J} = \sigma\vec{E}. \tag{26-13}$$

Calculating Resistance from Resistivity

We have just made an important distinction:

> Resistance is a property of an object. Resistivity is a property of a material.

If we know the resistivity of a substance such as copper, we can calculate the resistance of a length of wire made of that substance. Let A be the cross-sectional area of the wire, let L be its length, and let a potential difference V exist between its ends (Fig. 26-9). If the streamlines representing the current density are uniform throughout the wire, the electric field and the current density will be constant for all points within the wire and, from Eqs. 24-45 and 26-5, will have the values

$$E = V/L \quad \text{and} \quad J = i/A. \tag{26-14}$$

Table 26-1 Resistivities of Some Materials at Room Temperature (20°C)

Material	Resistivity, ρ ($\Omega \cdot$ m)	Temperature Coefficient of Resistivity α (K^{-1})
Typical Metals		
Silver	1.62×10^{-8}	4.1×10^{-3}
Copper	1.69×10^{-8}	4.3×10^{-3}
Gold	2.35×10^{-8}	4.0×10^{-3}
Aluminum	2.75×10^{-8}	4.4×10^{-3}
Manganin[a]	4.82×10^{-8}	0.002×10^{-3}
Tungsten	5.25×10^{-8}	4.5×10^{-3}
Iron	9.68×10^{-8}	6.5×10^{-3}
Platinum	10.6×10^{-8}	3.9×10^{-3}
Typical Semiconductors		
Silicon, pure	2.5×10^{3}	-70×10^{-3}
Silicon, *n*-type[b]	8.7×10^{-4}	
Silicon, *p*-type[c]	2.8×10^{-3}	
Typical Insulators		
Glass	$10^{10} - 10^{14}$	
Fused quartz	$\sim 10^{16}$	

[a] An alloy specifically designed to have a small value of α.

[b] Pure silicon doped with phosphorus impurities to a charge carrier density of 10^{23} m^{-3}.

[c] Pure silicon doped with aluminum impurities to a charge carrier density of 10^{23} m^{-3}.

We can then combine Eqs. 26-10 and 26-14 to write

$$\rho = \frac{E}{J} = \frac{V/L}{i/A}. \tag{26-15}$$

However, V/i is the resistance R, which allows us to recast Eq. 26-15 as

$$R = \rho\frac{L}{A}. \tag{26-16}$$

Equation 26-16 can be applied only to a homogeneous isotropic conductor of uniform cross section, with the potential difference applied as in Fig. 26-8*b*.

The macroscopic quantities V, i, and R are of greatest interest when we are making electrical measurements on specific conductors. They are the quantities that we read directly on meters. We turn to the microscopic quantities E, J, and ρ when we are interested in the fundamental electrical properties of materials.

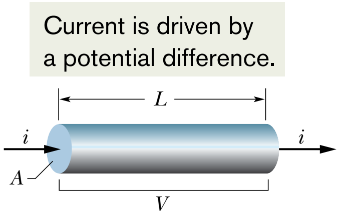
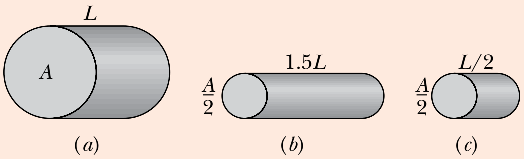

Current is driven by a potential difference.

Figure 26-9 A potential difference V is applied between the ends of a wire of length L and cross section A, establishing a current i.

CHECKPOINT 3

The figure here shows three cylindrical copper conductors along with their face areas and lengths. Rank them according to the current through them, greatest first, when the same potential difference V is placed across their lengths.



Variation with Temperature

The values of most physical properties vary with temperature, and resistivity is no exception. Figure 26-10, for example, shows the variation of this property for copper over a wide temperature range. The relation between temperature and resistivity for copper—and for metals in general—is fairly linear over a rather broad temperature range. For such linear relations we can write an empirical approximation that is good enough for most engineering purposes:

$$\rho - \rho_0 = \rho_0 \alpha (T - T_0). \tag{26-17}$$

Here T_0 is a selected reference temperature and ρ_0 is the resistivity at that temperature. Usually $T_0 = 293$ K (room temperature), for which $\rho_0 = 1.69 \times 10^{-8}\ \Omega \cdot \text{m}$ for copper.

Because temperature enters Eq. 26-17 only as a difference, it does not matter whether you use the Celsius or Kelvin scale in that equation because the sizes of degrees on these scales are identical. The quantity α in Eq. 26-17, called the *temperature coefficient of resistivity*, is chosen so that the equation gives good agreement with experiment for temperatures in the chosen range. Some values of α for metals are listed in Table 26-1.

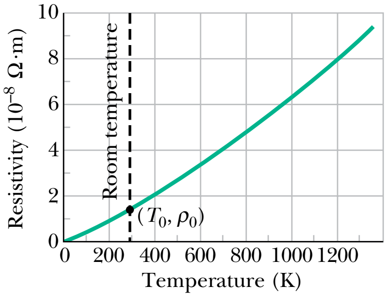


Figure 26-10 The resistivity of copper as a function of temperature. The dot on the curve marks a convenient reference point at temperature $T_0 = 293$ K and resistivity $\rho_0 = 1.69 \times 10^{-8}\ \Omega \cdot \text{m}$.

Resistivity can depend on temperature.

SAMPLE PROBLEM 26.04

A material has resistivity, a block of the material has resistance

A rectangular block of iron has dimensions 1.2 cm × 1.2 cm × 15 cm. A potential difference is to be applied to the block between parallel sides and in such a way that those sides are equipotential surfaces (as in Fig. 26-8*b*). What is the resistance of the block if the two parallel sides are (1) the square ends (with dimensions 1.2 cm × 1.2 cm) and (2) two rectangular sides (with dimensions 1.2 cm × 15 cm)?

KEY IDEAS

The resistance R of an object depends on how the electric potential is applied to the object. In particular, it depends on the ratio L/A, according to Eq. 26-16 ($R = \rho L/A$), where A is the area of the surfaces to which the potential difference is applied and L is the distance between those surfaces.

Calculations: For arrangement 1, we have $L = 15$ cm $= 0.15$ m and

$$A = (1.2\ \text{cm})^2 = 1.44 \times 10^{-4}\ \text{m}^2.$$

Substituting into Eq. 26-16 with the resistivity ρ from Table 26-1, we then find that for arrangement 1,

$$R = \frac{\rho L}{A} = \frac{(9.68 \times 10^{-8}\ \Omega \cdot \text{m})(0.15\ \text{m})}{1.44 \times 10^{-4}\ \text{m}^2}$$
$$= 1.0 \times 10^{-4}\ \Omega = 100\ \mu\Omega. \qquad \text{(Answer)}$$

Similarly, for arrangement 2, with distance $L = 1.2$ cm and area $A = (1.2\ \text{cm})(15\ \text{cm})$, we obtain

$$R = \frac{\rho L}{A} = \frac{(9.68 \times 10^{-8}\ \Omega \cdot \text{m})(1.2 \times 10^{-2}\ \text{m})}{1.80 \times 10^{-3}\ \text{m}^2}$$
$$= 6.5 \times 10^{-7}\ \Omega = 0.65\ \mu\Omega. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 26.05

Swimming during a storm

Figure 26-11 shows a swimmer at distance $D = 38.0$ m from a lightning strike to the water, with current $I = 78$ kA. The water has resistivity 30 Ω · m, the width of the swimmer along a radial line from the strike is 0.70 m, and his resistance across that width is 4.00 kΩ. Assume that the current spreads through the water over a hemisphere centered on the strike point. What is the current through the swimmer?

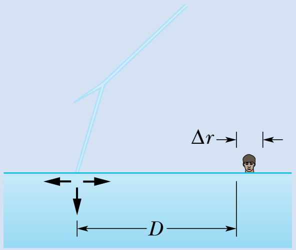


Figure 26-11 Swimmer at a distance of 38 m from where lightning strikes water.

KEY IDEAS

Since the current spreads uniformly over the hemisphere, the current density at any given radius r from the striking point is $J = I/2\pi r^2$. From Eq. 26-10, the magnitude of the electric field at a radial distance r is

$$E = \rho_w J = \frac{\rho_w I}{2\pi r^2},$$

where $\rho_w = 30\ \Omega\cdot\text{m}$ is the resistivity of water. The potential difference between a point at radial distance D and a point at $D + \Delta r$ is

$$E = -\int_D^{D+\Delta r} E\,dr = -\int_D^{D+\Delta r} \frac{\rho_w I}{2\pi r^2}\,dr$$

$$= -\frac{\rho_w I}{2\pi r}\left(\frac{1}{D+\Delta r} - \frac{1}{D}\right) = -\frac{\rho_w I}{2\pi r}\,\frac{\Delta r}{D(D+\Delta r)},$$

which implies that the current across the swimmer is

$$i = \frac{|\Delta V|}{R} = \frac{\rho_w I}{2\pi r}\,\frac{\Delta r}{D(D+\Delta r)}.$$

Calculations: Substituting the values given, we obtain

$$i = \frac{(30.0\ \Omega\cdot\text{m})(7.80\times10^4\ \text{A})}{2\pi(4.00\times10^3\Omega)}\,\frac{0.70\ \text{m}}{(38.0\ \text{m})(38.0\ \text{m}+0.70\ \text{m})}$$

$$= 4.43\times10^{-2}\ \text{A} \qquad \text{(Answer)}$$

26.5 | OHM'S LAW

Key Concepts

- A given device (conductor, resistor, or any other electrical device) obeys Ohm's law if its resistance $R\ (= V/i)$ is independent of the applied potential difference V.
- A given material obeys Ohm's law if its resistivity $\rho\ (= E/J)$ is independent of the magnitude and direction of the applied electric field $\vec{E}$.

As we just discussed, a resistor is a conductor with a specified resistance. It has that same resistance no matter what the magnitude and direction (*polarity*) of the applied potential difference are. Other conducting devices, however, might have resistances that change with the applied potential difference.

Figure 26-12*a* shows how to distinguish such devices. A potential difference V is applied across the device being tested, and the resulting current i through the device is measured as V is varied in both magnitude and polarity.

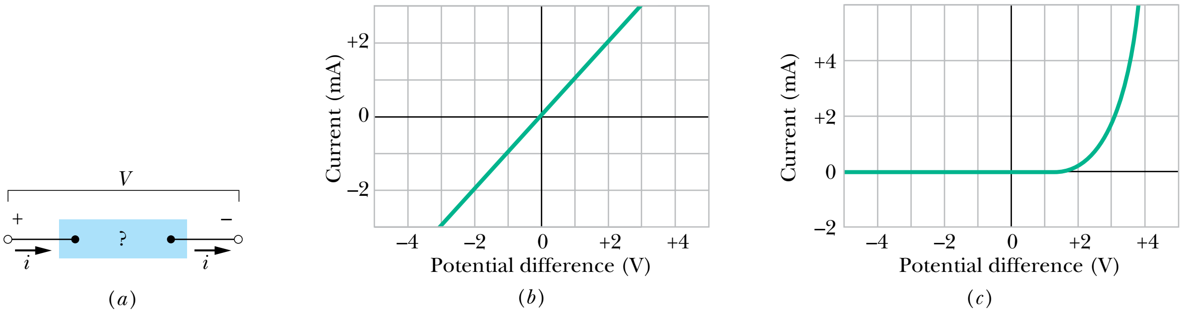


Figure 26-12 (*a*) A potential difference V is applied to the terminals of a device, establishing a current i. (*b*) A plot of current i versus applied potential difference V when the device is a 1000 Ω resistor. (*c*) A plot when the device is a semiconducting *pn* junction diode.

The polarity of V is arbitrarily taken to be positive when the left terminal of the device is at a higher potential than the right terminal. The direction of the resulting current (from left to right) is arbitrarily assigned a plus sign. The reverse polarity of V (with the right terminal at a higher potential) is then negative; the current it causes is assigned a minus sign.

Figure 26-12*b* is a plot of i versus V for one device. This plot is a straight line passing through the origin, so the ratio i/V (which is the slope of the straight line) is the same for all values of V. This means that the resistance $R = V/i$ of the device is independent of the magnitude and polarity of the applied potential difference V.

Figure 26-12*c* is a plot for another conducting device. Current can exist in this device only when the polarity of V is positive and the applied potential difference is more than about 1.5 V. When current does exist, the relation between i and V is not linear; it depends on the value of the applied potential difference V.

We distinguish between the two types of device by saying that one obeys Ohm's law and the other does not.

Ohm's law is an assertion that the current through a device is *always* directly proportional to the potential difference applied to the device.

(This assertion is correct only in certain situations; still, for historical reasons, the term "law" is used.) The device of Fig. 26-12*b*—which turns out to be a 1000 Ω resistor—obeys Ohm's law. The device of Fig. 26-12*c*—which is called a *pn* junction diode—does not.

A conducting device obeys Ohm's law when the resistance of the device is independent of the magnitude and polarity of the applied potential difference.

It is often contended that $V = iR$ is a statement of Ohm's law. That is not true! This equation is the defining equation for resistance, and it applies to all conducting devices, whether they obey Ohm's law or not. If we measure the potential difference V across, and the current i through, any device, even a *pn* junction diode, we can find its resistance *at that value of* V as $R = V/i$. The essence of Ohm's law, however, is that a plot of i versus V is linear; that is, R is independent of V. We can generalize this for conducting materials by using Eq. 26-11 ($\vec{E} = \rho\vec{J}$):

A conducting material obeys Ohm's law when the resistivity of the material is independent of the magnitude and direction of the applied electric field.

All homogeneous materials, whether they are conductors like copper or semiconductors like pure silicon or silicon containing special impurities, obey Ohm's law within some range of values of the electric field. If the field is too strong, however, there are departures from Ohm's law in all cases.

CHECKPOINT 4

The following table gives the current i (in amperes) through two devices for several values of potential difference V (in volts). From these data, determine which device does not obey Ohm's law.

Device 1		Device 2	
V	i	V	i
2.00	4.50	2.00	1.50
3.00	6.75	3.00	2.20
4.00	9.00	4.00	2.80

26.6 | A MICROSCOPIC VIEW OF OHM'S LAW

Key Concepts

- The assumption that the conduction electrons in a metal are free to move like the molecules in a gas leads to an expression for the resistivity of a metal:

$$\rho = \frac{m}{e^2 n\tau}.$$

Here n is the number of free electrons per unit volume and τ is the mean time between the collisions of an electron with the atoms of the metal.

- Metals obey Ohm's law because the mean free time τ is approximately independent of the magnitude E of any electric field applied to a metal.

To find out *why* particular materials obey Ohm's law, we must look into the details of the conduction process at the atomic level. Here we consider only conduction in metals, such as copper. We base our analysis on the *free-electron model,* in which we assume that the conduction electrons in the metal are free to move throughout the volume of a sample, like the molecules of a gas in a closed container. We also assume that the electrons collide not with one another but only with atoms of the metal.

According to classical physics, the electrons should have a Maxwellian speed distribution somewhat like that of the molecules in a gas, and thus the average electron speed should depend on the temperature. The motions of electrons are, however, governed not by the laws of classical physics but by those of quantum physics. As it turns out, an assumption that is much closer to the quantum reality is that conduction electrons in a metal move with a single effective speed v_{eff}, and this speed is essentially independent of the temperature. For copper, $v_{\text{eff}} \approx 1.6 \times 10^6$ m/s.

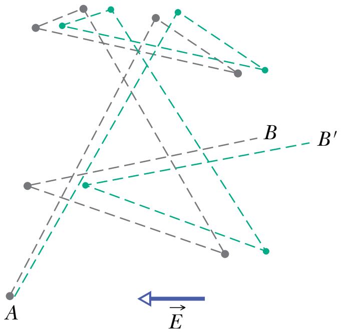


Figure 26-13 The gray lines show an electron moving from A to B, making six collisions en route. The green lines show what the electron's path might be in the presence of an applied electric field $\vec{E}$. Note the steady drift in the direction of $-\vec{E}$. (Actually, the green lines should be slightly curved, to represent the parabolic paths followed by the electrons between collisions, under the influence of an electric field.)

When we apply an electric field to a metal sample, the electrons modify their random motions slightly and drift very slowly—in a direction opposite that of the field—with an average drift speed v_d. The drift speed in a typical metallic conductor is about 5×10^{-7} m/s, less than the effective speed (1.6×10^6 m/s) by many orders of magnitude. Figure 26-13 suggests the relation between these two speeds. The gray lines show a possible random path for an electron in the absence of an applied field; the electron proceeds from A to B, making six collisions along the way. The green lines show how the same events *might* occur when an electric field $\vec{E}$ is applied. We see that the electron drifts steadily to the right, ending at B' rather than at B. Figure 26-13 was drawn with the assumption that $v_d \approx 0.02 v_{\text{eff}}$. However, because the actual value is more like $v_d \approx (10^{-13}) v_{\text{eff}}$, the drift displayed in the figure is greatly exaggerated.

The motion of conduction electrons in an electric field $\vec{E}$ is thus a combination of the motion due to random collisions and that due to $\vec{E}$. When we consider all the free electrons, their random motions average to zero and make no contribution to the drift speed. Thus, the drift speed is due only to the effect of the electric field on the electrons.

If an electron of mass m is placed in an electric field of magnitude E, the electron will experience an acceleration given by Newton's second law:

$$a = \frac{F}{m} = \frac{eE}{m}. \qquad (26\text{-}18)$$

After a typical collision, each electron will—so to speak—completely lose its memory of its previous drift velocity, starting fresh and moving off in a random direction. In the average time τ between collisions, the average electron will acquire a drift speed of $v_d = a\tau$. Moreover, if we measure the drift speeds of all the electrons at any instant, we will find that their average drift speed is also $a\tau$. Thus, at any instant, on average, the electrons will have drift speed $v_d = a\tau$. Then Eq. 26-18 gives us

$$v_d = a\tau = \frac{eE\tau}{m}. \qquad (26\text{-}19)$$

Combining this result with Eq. 26-7 ($\vec{J} = ne\vec{v}_d$), in magnitude form, yields

$$v_d = \frac{J}{ne} = \frac{eE\tau}{m}, \tag{26-20}$$

which we can write as

$$E = \left(\frac{m}{e^2 n\tau}\right) J. \tag{26-21}$$

Comparing this with Eq. 26-11 ($\vec{E} = \rho\vec{J}$), in magnitude form, leads to

$$\rho = \frac{m}{e^2 n\tau}. \tag{26-22}$$

Equation 26-22 may be taken as a statement that metals obey Ohm's law if we can show that, for metals, their resistivity ρ is a constant, independent of the strength of the applied electric field $\vec{E}$. Let's consider the quantities in Eq. 26-22. We can reasonably assume that n, the number of conduction electrons per volume, is independent of the field, and m and e are constants. Thus, we only need to convince ourselves that τ, the average time (or *mean free time*) between collisions, is a constant, independent of the strength of the applied electric field. Indeed, τ can be considered to be a constant because the drift speed v_d caused by the field is so much smaller than the effective speed v_{eff} that the electron speed—and thus τ—is hardly affected by the field. Thus, because the right side of Eq. 26-22 is independent of the field magnitude, metals obey Ohm's law.

The field inside circuit wires can be shown to be due not only to charges in and on the battery but also to those on the surfaces of the circuit elements.

Figure 26-14 shows the pattern of the electric field and a schematic charge distribution on a simple circuit consisting of a battery connected to a resistive wire of constant cross section and composition. You might be wondering what makes the electrons in the wire realize the electric field of the battery. The answer is simple. In the steady state there is a gradient of surface charge density on the surface of the resistive wire, which is a major contributor to the field inside the wire (charges in and on the battery also contribute). These charges produce an electric field which keeps on moving the electrons despite their being subjected to the resistive forces with the resistance. Thus, when there is a bend in a wire, charge accumulates at the bend at the right places to change the direction of the electric field. Note that the battery does not supply any charge. The net charge on the wires is still zero. It is just redistributed to produce the desired effect.

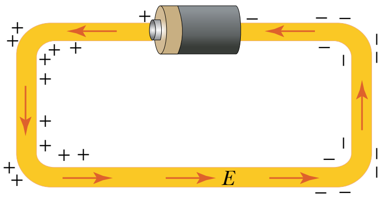


Figure 26-14 The steady-state pattern of the electric field and a schematic surface charge distribution for a simple circuit in the steady state. The surface charge is shown at sample locations along the wire to depict a charge gradient.

Consideration of surface charge can also help us understand the function of a resistor in a circuit. Let us consider a resistor made of carbon, and for simplicity we will give it the same diameter as the conducting wires in our circuit (Fig. 26-15). Carbon is a poor conductor, but not quite a good insulator–its resistivity is about $3 \times 10^{-5}\ \Omega \cdot \text{m}$ about 2000 times larger than that of copper. Because the resistor and the wires have the same cross-sectional area, the current density is the same in both. Using our previous current density for the 1 A current in the 1 mm radius wire, we can find the electric field in the resistor

$$E = \rho J = \frac{3\times 10^{-5}\,\Omega\cdot\text{m}}{3\times 10^{-6}\ \text{A/m}^2} \approx 10\ \text{V/m},$$

giving a field about 2000 times larger than the field in the copper wires. (Can you now see why the potential drop in the wires is negligible compared with the potential drop in the resistor?) This large electric field is necessary to force the electrons through the resistor.

As shown in Fig. 26-15, charges building up on the end of the wire (similar to the charges on a capacitor) are responsible for producing this large electric field. It does not take the whole lot; in fact this electric field can be produced by only about 1000 electrons at the interface.

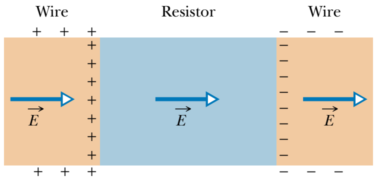


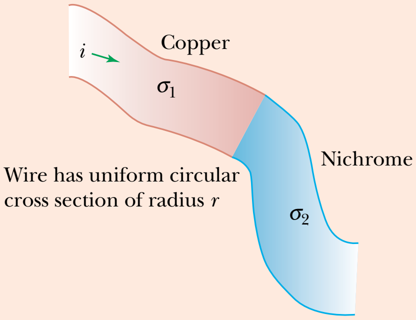
Figure 26-15 Details of surface changes near a resistor. The build-up charge on the wire ends produces the large electric field in the resistor.

What we have described here is a wonderful, self-regulating system. The battery provides the initial "burst" of current to the circuit, and almost instantly the charge finds its way to the locations where it guides the steady current and prevents further build-up of charge on the surface of the wires. This equilibrium is maintained as long as the battery continues to pump charge around the circuit.

Do you know that the Gauss' law is also applicable in the electric circuits?

CHECKPOINT 5

Can you see how the two wires having different conductivities connected in a circuit will have different values of the electric field inside them? What will be the sign of excess charge built up on the interface between two cylindrical conductors as shown in the figure?



SAMPLE PROBLEM 26.06

Mean free time and mean free distance

(a) What is the mean free time τ between collisions for the conduction electrons in copper?

KEY IDEAS

The mean free time τ of copper is approximately constant, and in particular does not depend on any electric field that might be applied to a sample of the copper. Thus, we need not consider any particular value of applied electric field. However, because the resistivity ρ displayed by copper under an electric field depends on τ, we can find the mean free time τ from Eq. 26-22 ($\rho = m/e^2 n\tau$).

Calculations: That equation gives us

$$\tau = \frac{m}{ne^2\rho}. \tag{26-23}$$

The number of conduction electrons per unit volume in copper is $8.49 \times 10^{28}\ \text{m}^{-3}$. We take the value of ρ from Table 26-1. The denominator then becomes

$$(8.49 \times 10^{28}\ \text{m}^{-3})(1.6 \times 10^{-19}\ \text{C})^2(1.69 \times 10^{-8}\ \Omega \cdot \text{m})$$
$$= 3.67 \times 10^{-17}\ \text{C}^2 \cdot \Omega/\text{m}^2 = 3.67 \times 10^{-17}\ \text{kg/s},$$

where we converted units as

$$\frac{\text{C}^2 \cdot \Omega}{\text{m}^2} = \frac{\text{C}^2 \cdot \text{V}}{\text{m}^2 \cdot \text{A}} = \frac{\text{C}^2 \cdot \text{J/C}}{\text{m}^2 \cdot \text{C/s}} = \frac{\text{kg} \cdot \text{m}^2/\text{s}^2}{\text{m}^2/\text{s}} = \frac{\text{kg}}{\text{s}}.$$

Using these results and substituting for the electron mass m, we then have

$$\tau = \frac{9.1 \times 10^{-31}\,\text{kg}}{3.67 \times 10^{-17}\,\text{kg/s}} = 2.5 \times 10^{-14}\,\text{s}. \qquad \text{(Answer)}$$

(b) The mean free path λ of the conduction electrons in a conductor is the average distance traveled by an electron between collisions. What is λ for the conduction electrons in copper, assuming that their effective speed v_{eff} is 1.6×10^6 m/s?

KEY IDEA

The distance d any particle travels in a certain time t at a constant speed v is $d = vt$.

Calculation: For the electrons in copper, this gives us

$$\lambda = v_{\text{eff}}\tau \tag{26-24}$$
$$= (1.6 \times 10^6\ \text{m/s})(2.5 \times 10^{-14}\ \text{s})$$
$$= 4.0 \times 10^{-8}\ \text{m} = 40\ \text{nm}. \qquad \text{(Answer)}$$

This is about 150 times the distance between nearest-neighbor atoms in a copper lattice. Thus, on the average, each conduction electron passes many copper atoms before finally hitting one.

26.7 | POWER IN ELECTRIC CIRCUITS

Key Concepts

- The power P, or rate of energy transfer, in an electrical device across which a potential difference V is maintained is

$$P = iV.$$

- If the device is a resistor, the power can also be written as

$$P = i^2R = \frac{V^2}{R}.$$

- In a resistor, electric potential energy is converted to internal thermal energy via collisions between charge carriers and atoms.

Figure 26-16 shows a circuit consisting of a battery B that is connected by wires, which we assume have negligible resistance, to an unspecified conducting device. The device might be a resistor, a storage battery (a rechargeable battery), a motor, or some other electrical device. The battery maintains a potential difference of magnitude V across its own terminals and thus (because of the wires) across the terminals of the unspecified device, with a greater potential at terminal a of the device than at terminal b.

Because there is an external conducting path between the two terminals of the battery, and because the potential differences set up by the battery are maintained, a steady current i is produced in the circuit, directed from terminal a to terminal b. The amount of charge dq that moves between those terminals in time interval dt is equal to $i\,dt$. This charge dq moves through a decrease in potential of magnitude V, and thus its electric potential energy decreases in magnitude by the amount

$$dU = dq\,V = i\,dt\,V. \tag{26-25}$$

The battery at the left supplies energy to the conduction electrons that form the current.

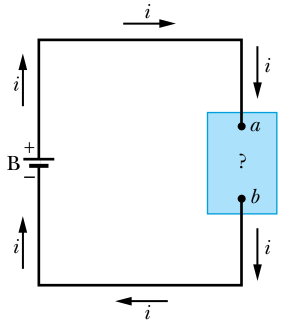


Figure 26-16 A battery B sets up a current i in a circuit containing an unspecified conducting device.

The principle of conservation of energy tells us that the decrease in electric potential energy from a to b is accompanied by a transfer of energy to some other form. The power P associated with that transfer is the rate of transfer dU/dt, which is given by Eq. 26-25 as

$$P = iV \quad \text{(rate of electrical energy transfer).} \tag{26-26}$$

Moreover, this power P is also the rate at which energy is transferred from the battery to the unspecified device. If that device is a motor connected to a mechanical load, the energy is transferred as work done on the load. If the device is a storage battery that is being charged, the energy is transferred to stored chemical energy in the storage battery. If the device is a resistor, the energy is transferred to internal thermal energy, tending to increase the resistor's temperature.

The unit of power that follows from Eq. 26-26 is the volt-ampere (V · A). We can write it as

$$1\ \text{V}\cdot\text{A} = \left(1\frac{\text{J}}{\text{C}}\right)\left(1\frac{\text{C}}{\text{s}}\right) = 1\frac{\text{J}}{\text{s}} = 1\ \text{W}.$$

As an electron moves through a resistor at constant drift speed, its average kinetic energy remains constant and its lost electric potential energy appears as thermal energy in the resistor and the surroundings. On a microscopic scale this energy transfer is due to collisions between the electron and the molecules of the resistor, which leads to an increase in the temperature of the resistor lattice. The mechanical energy thus transferred to thermal energy is *dissipated* (lost) because the transfer cannot be reversed.

For a resistor or some other device with resistance R, we can combine Eqs. 26-8 ($R = V/i$) and 26-26 to obtain, for the rate of electrical energy dissipation due to a resistance, either

$$P = i^2R \quad \text{(resistive dissipation)} \tag{26-27}$$

or

$$P = \frac{V^2}{R} \quad \text{(resistive dissipation).} \tag{26-28}$$

Caution: We must be careful to distinguish these two equations from Eq. 26-26: $P = iV$ applies to electrical energy transfers of all kinds; $P = i^2R$ and $P = V^2/R$ apply only to the transfer of electric potential energy to thermal energy in a device with resistance.

CHECKPOINT 6

A potential difference V is connected across a device with resistance R, causing current i through the device. Rank the following variations according to the change in the rate at which electrical energy is converted to thermal energy due to the resistance, greatest change first: (a) V is doubled with R unchanged, (b) i is doubled with R unchanged, (c) R is doubled with V unchanged, (d) R is doubled with i unchanged.

SAMPLE PROBLEM 26.07

Rate of energy dissipation in a wire carrying current

You are given a length of uniform heating wire made of a nickel–chromium–iron alloy called Nichrome; it has a resistance R of 72 Ω. At what rate is energy dissipated in each of the following situations? (1) A potential difference of 120 V is applied across the full length of the wire. (2) The wire is cut in half, and a potential difference of 120 V is applied across the length of each half.

KEY IDEA

Current in a resistive material produces a transfer of mechanical energy to thermal energy; the rate of transfer (dissipation) is given by Eqs. 26-26 to 26-28.

Calculations: Because we know the potential V and resistance R, we use Eq. 26-28, which yields, for situation 1,

$$P = \frac{V^2}{R} = \frac{(120\text{ V})^2}{72\ \Omega} = 200\text{ W}. \qquad \text{(Answer)}$$

In situation 2, the resistance of each half of the wire is (72 Ω)/2, or 36 Ω. Thus, the dissipation rate for each half is

$$P' = \frac{(120\text{ V})^2}{36\ \Omega} = 400\text{ W},$$

and that for the two halves is

$$P = 2P' = 800\text{ W}. \qquad \text{(Answer)}$$

This is four times the dissipation rate of the full length of wire. Thus, you might conclude that you could buy a heating coil, cut it in half, and reconnect it to obtain four times the heat output. Why is this unwise? (What would happen to the amount of current in the coil?)

26.8 | SEMICONDUCTORS

Key Concept

◆ Semiconductors are materials that have few conduction electrons but can become conductors when they are doped with other atoms that contribute charge carriers.

Semiconducting devices are at the heart of the microelectronic revolution that ushered in the information age. Table 26-2 compares the properties of silicon—a typical semiconductor—and copper—a typical metallic conductor. We see that silicon has many fewer charge carriers, a much higher resistivity, and a temperature coefficient of resistivity that is both large and negative. Thus, although the resistivity of copper increases with increasing temperature, that of pure silicon decreases.

Table 26-2 Some Electrical Properties of Copper and Silicon

Property	Copper	Silicon
Type of material	Metal	Semiconductor
Charge carrier density, m^{-3}	8.49×10^{28}	1×10^{16}
Resistivity, $\Omega \cdot \text{m}$	1.69×10^{-8}	2.5×10^{3}
Temperature coefficient of resistivity, K^{-1}	$+4.3 \times 10^{-3}$	-70×10^{-3}

Pure silicon has such a high resistivity that it is effectively an insulator and thus not of much direct use in microelectronic circuits. However, its resistivity can be greatly reduced in a controlled way by adding minute amounts of specific "impurity" atoms in a process called *doping.* Table 26-1 gives typical values of resistivity for silicon before and after doping with two different impurities.

We can roughly explain the differences in resistivity (and thus in conductivity) between semiconductors, insulators, and metallic conductors in terms of the energies of their electrons. (We need quantum physics to explain in more detail.) In a metallic conductor such as copper wire, most of the electrons are firmly locked in place within the atoms; much energy would be required to free them so they could move and participate in an electric current. However, there are also some electrons that, roughly speaking, are only loosely held in place and that require only little energy to become free. Thermal energy can supply that energy, as can an electric field applied across the conductor. The field would not only free these loosely held electrons but would also propel them along the wire; thus, the field would drive a current through the conductor.

In an insulator, significantly greater energy is required to free electrons so they can move through the material. Thermal energy cannot supply enough energy, and neither can any reasonable electric field applied to the insulator. Thus, no electrons are available to move through the insulator, and hence no current occurs even with an applied electric field.

A semiconductor is like an insulator *except* that the energy required to free some electrons is not quite so great. More important, doping can supply electrons or positive charge carriers that are very loosely held within the material and thus are easy to get moving. Moreover, by controlling the doping of a semiconductor, we can control the density of charge carriers that can participate in a current and thereby can control some of its electrical properties. Most semiconducting devices, such as transistors and junction diodes, are fabricated by the selective doping of different regions of the silicon with impurity atoms of different kinds.

Let us now look again at Eq. 26-22 for the resistivity of a conductor:

$$\rho = \frac{m}{e^2 n\tau}, \tag{26-29}$$

where n is the number of charge carriers per unit volume and τ is the mean time between collisions of the charge carriers. The equation also applies to semiconductors. Let's consider how n and τ change as the temperature is increased.

In a conductor, n is large but very nearly constant with any change in temperature. The increase of resistivity with temperature for metals (Fig. 26-10) is due to an increase in the collision rate of the charge carriers, which shows up in Eq. 26-29 as a decrease in τ, the mean time between collisions.

In a semiconductor, n is small but increases very rapidly with temperature as the increased thermal agitation makes more charge carriers available. This causes a *decrease* of resistivity with increasing temperature, as indicated by the negative temperature coefficient of resistivity for silicon in Table 26-2. The same increase in collision rate that we noted for metals also occurs for semiconductors, but its effect is swamped by the rapid increase in the number of charge carriers.

26.9 | SUPERCONDUCTORS

Key Concept

- Superconductors are materials that lose all electrical resistance. Most such materials require very low temperatures, but some become superconducting at temperatures as high as room temperature.

In 1911, Dutch physicist Kamerlingh Onnes discovered that the resistivity of mercury absolutely disappears at temperatures below about 4 K (Fig. 26-17). This phenomenon of **superconductivity** is of vast potential importance in technology because it means that charge can flow through a superconducting conductor without losing its energy to thermal energy. Currents created in a superconducting ring, for example, have persisted for several years without loss; the electrons making up the current require a force and a source of energy at start-up time but not thereafter.

Prior to 1986, the technological development of superconductivity was throttled by the cost of producing the extremely low temperatures required to achieve the effect. In 1986, however, new ceramic materials were discovered

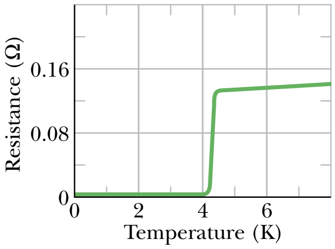


Figure 26-17 The resistance of mercury drops to zero at a temperature of about 4 K.

that become superconducting at considerably higher (and thus cheaper to produce) temperatures. Practical application of superconducting devices at room temperature may eventually become commonplace.

Superconductivity is a phenomenon much different from conductivity. In fact, the best of the normal conductors, such as silver and copper, cannot become superconducting at any temperature, and the new ceramic superconductors are actually good insulators when they are not at low enough temperatures to be in a superconducting state.

One explanation for superconductivity is that the electrons that make up the current move in coordinated pairs. One of the electrons in a pair may electrically distort the molecular structure of the superconducting material as it moves through, creating nearby a short-lived concentration of positive charge. The other electron in the pair may then be attracted toward this positive charge. According to the theory, such coordination between electrons would prevent them from colliding with the molecules of the material and thus would eliminate electrical resistance. The theory worked well to explain the pre-1986, lower temperature superconductors, but new theories appear to be needed for the newer, higher temperature superconductors.

REVIEW AND SUMMARY

Current An **electric current** i in a conductor is defined by

$$i = \frac{dq}{dt}. \tag{26-1}$$

Here dq is the amount of (positive) charge that passes in time dt through a hypothetical surface that cuts across the conductor. By convention, the direction of electric current is taken as the direction in which positive charge carriers would move. The SI unit of electric current is the **ampere** (A): 1 A = 1 C/s.

Current Density Current (a scalar) is related to **current density** $\vec{J}$ (a vector) by

$$i = \int \vec{J} \cdot d\vec{A}, \tag{26-4}$$

where $d\vec{A}$ is a vector perpendicular to a surface element of area dA and the integral is taken over any surface cutting across the conductor. $\vec{J}$ has the same direction as the velocity of the moving charges if they are positive and the opposite direction if they are negative.

Drift Speed of the Charge Carriers When an electric field $\vec{E}$ is established in a conductor, the charge carriers (assumed positive) acquire a **drift speed** v_d in the direction of $\vec{E}$; the velocity $\vec{v}_d$ is related to the current density by

$$\vec{J} = (ne)\vec{v}_d, \tag{26-7}$$

where ne is the *carrier charge density.*

Resistance of a Conductor The **resistance** R of a conductor is defined as

$$R = \frac{V}{i} \quad \text{(definition of } R\text{)}, \tag{26-8}$$

where V is the potential difference across the conductor and i is the current. The SI unit of resistance is the **ohm** (Ω): 1 Ω = 1 V/A.

Similar equations define the **resistivity** ρ and **conductivity** σ of a material:

$$\rho = \frac{1}{\sigma} = \frac{E}{J} \quad \text{(definitions of } \rho \text{ and } \sigma\text{)}, \qquad \text{(26-12, 26-10)}$$

where E is the magnitude of the applied electric field. The SI unit of resistivity is the ohm-meter (Ω·m). Equation 26-10 corresponds to the vector equation

$$\vec{E} = \rho\vec{J}. \tag{26-11}$$

The resistance R of a conducting wire of length L and uniform cross section is

$$R = \rho\frac{L}{A}, \tag{26-16}$$

where A is the cross-sectional area.

Change of ρ with Temperature The resistivity ρ for most materials changes with temperature. For many materials, including metals, the relation between ρ and temperature T is approximated by the equation

$$\rho - \rho_0 = \rho_0\alpha(T - T_0). \tag{26-17}$$

Here T_0 is a reference temperature, ρ_0 is the resistivity at T_0, and α is the temperature coefficient of resistivity for the material.

Ohm's Law A given device (conductor, resistor, or any other electrical device) obeys *Ohm's law* if its resistance R, defined by Eq. 26-8 as V/i, is independent of the applied potential difference V. A given *material* obeys Ohm's law if its resistivity, defined by Eq. 26-10, is independent of the magnitude and direction of the applied electric field $\vec{E}$.

Resistivity of a Metal By assuming that the conduction electrons in a metal are free to move like the molecules of a gas, it is possible to derive an expression for the resistivity of a metal:

$$\rho = \frac{m}{e^2 n \tau}. \tag{26-22}$$

Here n is the number of free electrons per unit volume and τ is the mean time between the collisions of an electron with the atoms of the metal. We can explain why metals obey Ohm's law by pointing out that τ is essentially independent of the magnitude E of any electric field applied to a metal.

Power The power P, or rate of energy transfer, in an electrical device across which a potential difference V is maintained is

$$P = iV \quad \text{(rate of electrical energy transfer)}. \tag{26-26}$$

Resistive Dissipation If the device is a resistor, we can write Eq. 26-26 as

$$P = i^2 R = \frac{V^2}{R} \quad \text{(resistive dissipation)}. \qquad \text{(26-27, 26-28)}$$

In a resistor, electric potential energy is converted to internal thermal energy via collisions between charge carriers and atoms.

Semiconductors *Semiconductors* are materials that have few conduction electrons but can become conductors when they are *doped* with other atoms that contribute charge carriers.

Superconductors *Superconductors* are materials that lose all electrical resistance at low temperatures. Some materials are superconducting at surprisingly high temperatures.

PROBLEMS

1. A wire 8.00 m long and 6.00 mm in diameter has a resistance of 30.0 mΩ. A potential difference of 23.0 V is applied between the ends. (a) What is the current in the wire? (b) What is the magnitude of the current density? (c) Calculate the resistivity of the wire material.

2. In Fig. 26-18*a*, a 9.00 V battery is connected to a resistive strip that consists of three sections with the same cross-sectional areas but different conductivities. Figure 26-18*b* gives the electric potential $V(x)$ versus position x along the strip. The horizontal scale is set by $x_s = 8.00$ mm. Section 3 has conductivity $4.00 \times 10^7\ (\Omega\cdot\text{m})^{-1}$. What is the conductivity of section (a) 1 and (b) 2?

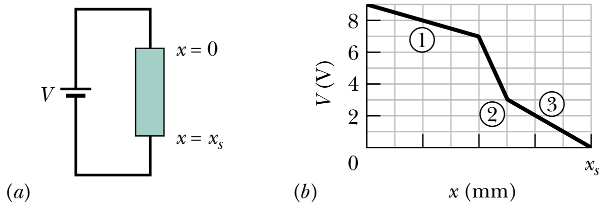


Figure 26-18 Problem 2.

3. In Fig. 26-19, current is set up through a truncated right circular cone of resistivity 731 Ω·m, left radius $a = 1.70$ mm, right radius $b = 2.30$ mm, and length $L = 3.50$ cm. Assume that the current density is uniform across any cross section taken perpendicular to the length. What is the resistance of the cone?

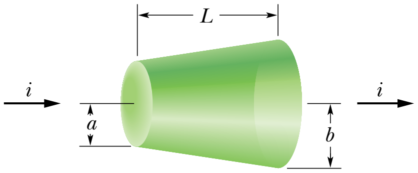


Figure 26-19 Problem 3.

4. *Kiting during a storm.* The legend that Benjamin Franklin flew a kite as a storm approached is only a legend—he was neither stupid nor suicidal. Suppose a kite string of radius 2.00 mm extends directly upward by 1.80 km and is coated with a 0.500 mm layer of water having resistivity 150 Ω·m. If the potential difference between the two ends of the string is 213 MV, what is the current through the water layer? The danger is not this current but the chance that the string draws a lightning strike, which can have a current as large as 500 000 A (way beyond just being lethal).

5. Earth's lower atmosphere contains negative and positive ions that are produced by radioactive elements in the soil and cosmic rays from space. In a certain region, the atmospheric electric field strength is 120 V/m and the field is directed vertically down. This field causes singly charged positive ions, at a density of 640 cm^{-3}, to drift downward and singly charged negative ions, at a density of 550 cm^{-3}, to drift upward (Fig. 26-20). The measured conductivity of the air in that region is $2.70 \times 10^{-14}\ (\Omega\cdot\text{m})^{-1}$. Calculate (a) the magnitude of the current density and (b) the ion drift speed, assumed to be the same for positive and negative ions.

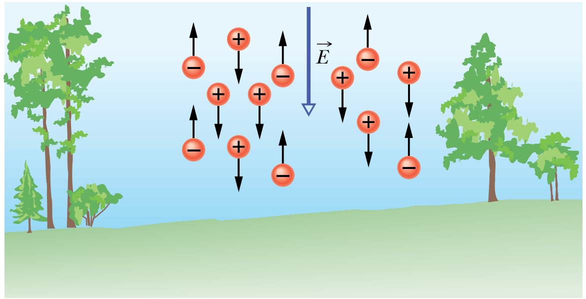


Figure 26-20 Problem 5.

6. A potential difference of 6.00 nV is set up across a 1.50 cm length of copper wire that has a radius of 2.00 mm. How much charge drifts through a cross section in 4.70 ms?

7. Figure 26-21 shows wire section 1 of diameter $D_1 = 4.00R$ and wire section 2 of diameter $D_2 = 1.75R$, connected by a tapered section. The wire is copper and carries a current. Assume that the current is uniformly distributed across any cross-sectional area through the wire's width. The electric potential change V along the length $L = 2.00$ m shown in section 2 is 10.0 μV. The number of charge carriers per unit volume is 8.49×10^{28} m^{-3}. What is the drift speed of the conduction electrons in section 1?

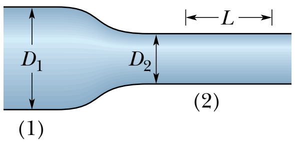


Figure 26-21 Problem 7.

8. When 230 V is applied across a wire that is 14.1 m long and has a 0.30 mm radius, the magnitude of the current density is 1.98×10^8 A/m^2. Find the resistivity of the wire.

9. In Fig. 26-22*a*, a 15 Ω resistor is connected to a battery. Figure 26-22*b* shows the increase of thermal energy E_{th} in the resistor as a function of time t. The vertical scale is set by $E_{th,s} = 2.50$ mJ, and the horizontal scale is set by $t_s = 4.0$ s. What is the electric potential across the battery?

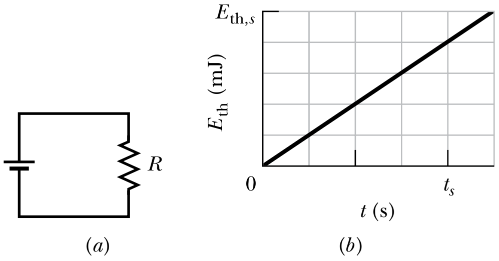


Figure 26-22 Problem 9.

10. Nichrome wire consists of a nickel–chromium–iron alloy, is commonly used in heating elements such as on a stove, and has conductivity 2.0×10^6 $(\Omega\cdot\text{m})^{-1}$. If a Nichrome wire with a cross-sectional area of 2.3 mm^2 carries a current of 5.5 A when a 1.4 V potential difference is applied between its ends, what is the wire's length?

11. A heater contains a Nichrome wire (resistivity 5.0×10^{-7} Ω·m) of length 5.85 m, with an end-to-end potential difference of 112 V, and with a dissipation power of 4000 W. (a) What is the wire's cross-sectional area? (b) If 100 V is used to obtain the same dissipation rate, what should the length be?

12. An unknown resistor is connected between the terminals of a 3.00 V battery. Energy is dissipated in the resistor at the rate of 0.707 W. The same resistor is then connected between the terminals of a 12.0 V battery. At what rate is energy now dissipated?

13. In Fig. 26-23, a battery of potential difference $V = 12$ V is connected to a resistive strip of resistance $R = 4.0$ Ω. When an electron moves through the strip from one end to the other, (a) in which direction in the figure does the electron move, (b) how much work is done on the electron by the electric field in the strip, and (c) how much energy is transferred to the thermal energy of the strip by the electron?

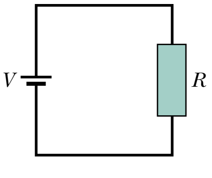


Figure 26-23 Problem 13.

14. Figure 26-24 gives the electric potential $V(x)$ along a copper wire carrying uniform current, from a point of higher potential $V_s = 12.0$ μV at $x = 0$ to a point of zero potential at $x_s = 3.00$ m. The wire has a radius of 2.20 mm. What is the current in the wire?.

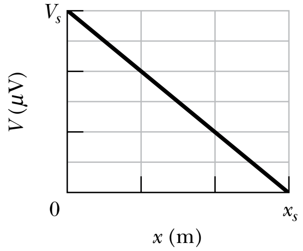


Figure 26-24 Problem 14.

15. A 890 W radiant heater is constructed to operate at 115 V. (a) What is the current in the heater when the unit is operating? (b) What is the resistance of the heating coil? (c) How much thermal energy is produced in 5.00 h?

16. Wire C and wire D are made from different materials and have length $L_C = L_D = 1.0$ m. The resistivity and radius of wire C are 2.0×10^{-6} Ω·m and 1.00 mm, and those of wire D are 1.0×10^{-6} Ω·m and 0.50 mm. The wires are joined as shown in Fig. 26-25, and a current of 2.0 A is set up in them. What is the electric potential difference between (a) points 1 and 2 and (b) points 2 and 3? What is the rate at which energy is dissipated between (c) points 1 and 2 and (d) points 2 and 3?

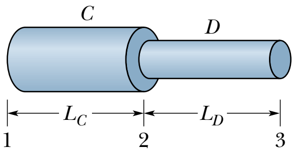


Figure 26-25 Problem 16.

17. A small but measurable current of 1.2×10^{-10} A exists in a copper wire whose diameter is 3.0 mm. The number of charge carriers per unit volume is 8.49×10^{28} m^{-3}. Assuming the current is uniform, calculate the (a) current density and (b) electron drift speed.

18. The current through the battery and resistors 1 and 2 in Fig. 26-26*a* is 1.50 A. Energy is transferred from the current to thermal energy E_{th} in both resistors. Curves 1 and 2 in Fig. 26-26*b* give that thermal energy E_{th} for resistors 1 and 2, respectively, as a function of time t. The vertical scale is set by $E_{th,s} = 40.0$ mJ, and the horizontal scale is set by $t_s = 5.00$ s. What is the power of the battery?

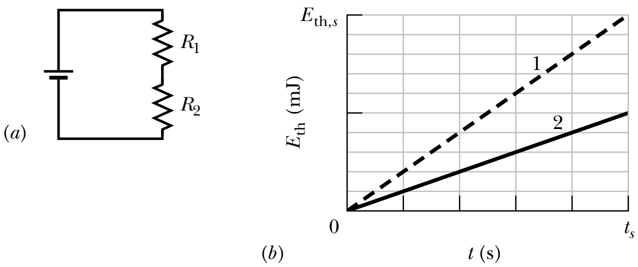


Figure 26-26 Problem 18.

19. A copper wire of cross-sectional area 2.40×10^{-6} m^2 and length 4.00 m has a current of 2.00 A uniformly distributed across that area. (a) What is the magnitude of the electric field along the wire? (b) How much electrical energy is transferred to thermal energy in 30 min?

20. For a current set up in wire for 28.0 d, a total of 1.36×10^{26} electrons pass through any cross section across the wire's width at a steady rate. What is the current?

21. The current-density magnitude in a certain circular wire is $J = (2.75 \times 10^{10}\ \text{A/m}^4)r^2$, where r is the radial distance out to the wire's radius of 3.00 mm. The potential applied to the wire (end to end) is 80.0 V. How much energy is converted to thermal energy in 1.00 h?

22. A charged belt, 50 cm wide, travels at 30 m/s between a source of charge (electrons) and a sphere. The belt carries charge into the sphere at a rate corresponding to 76 μA. (a) Compute the surface charge density on the belt. (b) What is the number density (number per unit area) of the electrons on the belt?

23. A wire initially has length L_0 and resistance 5.00 Ω. The resistance is to be increased to 45.0 Ω by stretching the wire. Assuming that the resistivity and density of the material are unaffected by the stretching, find the ratio of the new length to L_0.

24. A human being can be electrocuted if a current as small as 50 mA passes near the heart. An electrician working with sweaty hands makes good contact with the two conductors he is holding, one in each hand. If his resistance is 2100 Ω, what might the fatal voltage be?

25. A 120 V potential difference is applied to a space heater that dissipates 1500 W during operation. (a) What is its resistance during operation? (b) At what rate do electrons flow through any cross section of the heater element?

26. A certain wire has a resistance R. What is the resistance of a second wire, made of the same material, that is twice as long and has twice the diameter?

27. Figure 26-27*a* shows a rod of resistive material. The resistance per unit length of the rod increases in the positive direction of the x axis. At any position x along the rod, the resistance dR of a narrow (differential) section of width dx is given by $dR = 5.00x\ dx$, where dR is in ohms and x is in meters. Figure 26-27*b* shows such a narrow section. You are to slice off a length of the rod between $x = 0$ and some position $x = L$ and then connect that length to a battery with potential difference $V = 8.0$ V (Fig. 26-27*c*). You want the current in the length to transfer energy to thermal energy at the rate of 180 W. At what position $x = L$ should you cut the rod?

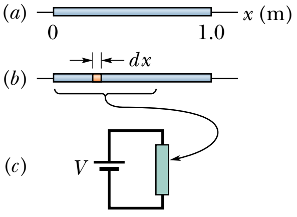


Figure 26-27 Problem 27.

28. A fuse in an electric circuit is a wire that is designed to melt, and thereby open the circuit, if the current exceeds a predetermined value. Suppose that the material to be used in a fuse melts when the current density rises to 440 A/cm². What radius of cylindrical wire should be used to make a fuse that will limit the current to 6.0 A?

29. How long does it take electrons to get from a car battery to the starting motor? Assume the current is 285 A and the electrons travel through a copper wire with cross-sectional area 0.17 cm² and length 0.43 m. The number of charge carriers per unit volume is $8.49 \times 10^{28}\ \text{m}^{-3}$.

30. What is the current in a wire of radius $R = 2.67$ mm if the magnitude of the current density is given by (a) $J_a = J_0 r/R$ and (b) $J_b = J_0(1 - r/R)$, in which r is the radial distance and $J_0 = 5.50 \times 10^4\ \text{A/m}^2$? (c) Which function maximizes the current density near the wire's surface?

31. A certain cylindrical wire carries current. We draw a circle of radius r around its central axis in Fig. 26-28*a* to determine the current i within the circle. Figure 26-28*b* shows current i as a function of r^2. The vertical scale is set by $i_s = 4.0$ mA, and the horizontal scale is set by $r_s^2 = 8.0\ \text{mm}^2$. (a) Is the current density uniform? (b) If so, what is its magnitude? (c) What is the current between $r = 0$ and $r = 2.0$ mm?

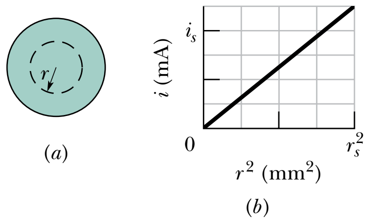


Figure 26-28 Problem 31.

32. Figure 26-29*a* gives the magnitude $E(x)$ of the electric fields that have been set up by a battery along a resistive rod of length 9.00 mm (Fig. 26-29*b*). The vertical scale is set by $E_s = 8.00 \times 10^3$ V/m. The rod consists of three sections of the same material but with different radii. (The schematic diagram of Fig. 26-29*b* does not indicate the different radii.) The radius of section 3 is 1.70 mm. What is the radius of (a) section 1 and (b) section 2?

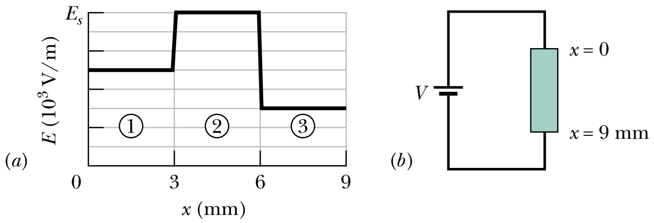


Figure 26-29 Problem 32.

33. Two conductors are made of the same material and have the same length. Conductor A is a solid wire of radius 1.0 mm. Conductor B is a hollow tube of outside radius 2.2 mm and inside radius 1.0 mm. What is the resistance ratio R_A/R_B, measured between their ends?

34. Near Earth, the density of protons in the solar wind (a stream of particles from the Sun) can be 4.63 cm⁻³, and their speed can be 391 km/s. (a) Find the current density of these protons. (b) If Earth's magnetic field did not deflect the protons, what total current would Earth receive?

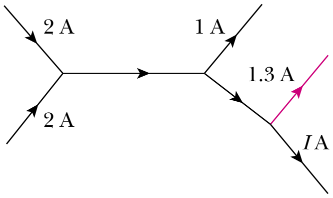
35. An isolated conducting sphere has a 20 cm radius. One wire carries a current of 1.000 002 0 A into it. Another wire carries a current of 1.000 000 0 A out of it. How long would it take for the sphere to increase in potential by 1000 V?

36. The magnitude J of the current density in a certain lab wire with a circular cross section of radius $R = 2.50$ mm is given by $J = (3.00 \times 10^8)r^2$, with J in amperes per square meter and radial distance r in meters. What is the current through the outer section bounded by $r = 0.900R$ and $r = R$?

PRACTICE QUESTIONS

Single Correct Choice Type

1. The following figure shows current in a part of electric circuit, then current I is



(a) 1.7 A
(b) 3.7 A
(c) 1.3 A
(d) 1 A

2. A wire of length l is drawn such that its diameter is reduced to half of its original diameter. If the initial resistance of the wire were 10 Ω, its resistance would be

(a) 40 Ω
(b) 60 Ω
(c) 120 Ω
(d) 160 Ω

3. A wire of resistance 1 Ω is elongated by 10%. The resistance of the elongated wire is

(a) 1.1 Ω
(b) 11.1 Ω
(c) 1.21 Ω
(d) 13.1 Ω

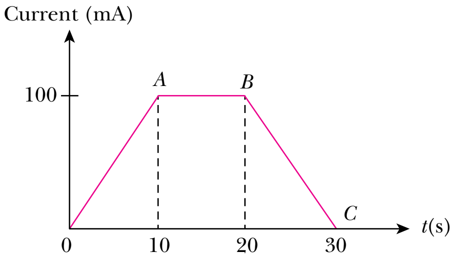
4. A 5°C rise in temperature is observed in a conductor by passing a current. When the current is doubled, the approximate rise in temperature is

(a) 5°C
(b) 20°C
(c) 15°C
(d) 25°C

5. The masses of three copper wires are in the ratio 5:3:1 and their lengths are in the ratio 1:3:5. The ratio of their electrical resistance is

(a) 1:3:5
(b) 5:3:1
(c) 1:15:125
(d) 125:15:1

6. In a copper voltameter, mass deposited in 30 s is m g. If the time–current graph is as shown in the following figure, then ECE of copper is



(a) $m/2$
(b) $2m$
(c) $0.5m$
(d) $0.7m$

7. The resistivity of a wire

(a) Varies with its length
(b) Varies with its mass
(c) Varies with it cross section
(d) Is independent of length, cross section and mass of the wire

8. When a current flows in a conductor, the order of magnitude of drift velocity of electrons through it is:

(a) 10^{-7} cm/s
(b) 10^{-2} cm/s
(c) 10^4 cm/s
(d) 10^{10} cm/s

9. Assume that each atom of copper contributes one free electron. Density of Cu is 9 g/cm^3 and atomic weight is 63 g. If current flowing through a Cu wire of 1 mm diameter is 1.1 A, drift velocity of electrons will be

(a) 0.1 mm/s
(b) 0.2 mm/s
(c) 0.3 mm/s
(d) 0.5 mm/s

10. A cable has single wire of radius 9 mm of resistance 5 Ω. It is replaced by six different well-insulated copper wires each of radius 3 mm. Total resistance of cable now be equal to:

(a) 7·5 Ω
(b) 45 Ω
(c) 90 Ω
(d) 270 Ω

11. If n, e, τ, and m represent electron density, charge, relaxation time, and mass of an electron, respectively, the resistance of a wire of length l and cross-sectional area A is

(a) $\dfrac{ml}{ne^2\tau A}$
(b) $\dfrac{2m\tau A}{ne^2 l}$
(c) $\dfrac{ne^2\tau}{2m}\dfrac{A}{l}$
(d) $\dfrac{ne^2 m}{2\tau}\dfrac{l}{A}$

12. In which one of the following situations does a conventional electric current flow due north?

(a) Protons in a beam are moving due south
(b) A water molecule is moving due north
(c) Electrons in a beam are moving due south
(d) Electrons in a wire connected to a battery are moving from south to north

13. A nonconducting ring of radius R has charge Q distributed over it. If it rotates with an angular velocity ω about its axis, the equivalent electric current will be

(a) Zero
(b) $Q\omega$
(c) $\dfrac{Q\omega}{2\pi}$
(d) $\dfrac{Q\omega}{2\pi R}$

14. A long conductor having charge q, with charge density λ is moving with a velocity $2v$ parallel to its own axis. The convectional current I due to motion of conductor is

(a) $I = \lambda v/2$
(b) $I = 2\lambda v$
(c) $I = \lambda v$
(d) $I = 3\lambda v$

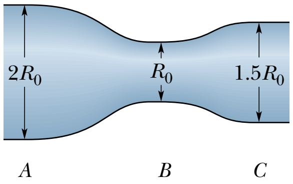
15. In an ionic solution, sodium ions (Na^+) are moving to the right and chlorine ions (Cl^-) are moving to the left. In which direction is the current due to the motion of (1) the sodium ions and (2) the chlorine ions?

(a) Both are to the right
(b) Current due to Na^+ is to the left; current due to Cl^- is to the right
(c) Current due to Na^+ is to the right; current due to Cl^- is to the left
(d) Both are to the left

16. When you flip a switch to turn on a light, the delay before the light turns on is determined by:

(a) The speed of the electric field moving in the wire
(b) The density of electrons in the wire
(c) The drift speed of the electrons in the wire
(d) The number of electron collisions per second in the wire

17. The following figure shows the cross section of a wire. Which of the following statement(s) about current (I) and drift velocity (v) is true?

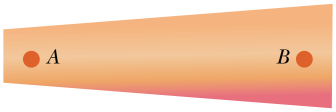


(a) $I_A = I_B = I_C$ and $v_A = v_C > v_B$
(b) $I_A = I_B = I_C$ and $v_A > v_C > v_B$
(c) $I_A = I_B < I_C$ and $v_A = v_C = v_B$
(d) $I_A = I_B = I_C$ and $v_A < v_C < v_B$

18. A hollow copper tube of 5 m length has got external diameter equal to 10 cm and its walls are 5 mm thick. The specific resistance of copper is $1.7 \times 10^{-8}\ \Omega$ m. The resistance of the copper tube, approximately

(a) $5.4 \times 10^{-3}\ \Omega$ **(b)** $5.4 \times 10^{-9}\ \Omega$
(c) $5.4 \times 10^{-5}\ \Omega$ **(d)** $5.4 \times 10^{-7}\ \Omega$

19. A copper wire is fabricated that has a gradually increasing diameter along its length as shown in the following figure. If an electric current is moving through the wire, which quantities vary along the length of the wire?



(a) Current only
(b) Current and current density only
(c) Current density and electric field only
(d) Resistivity and current only

20. The magnitude of the electric field within a conducting wire depends on which of the following quantities?

(a) Potential difference, wire diameter, wire length, and wire conductivity
(b) Potential difference and wire conductivity only
(c) Potential difference, wire length, and wire conductivity only
(d) Wire diameter, wire length, and wire conductivity only

21. Which one of the following statements concerning the electric field inside a conductor is true if electrons are moving from right to left in a conducting wire?

(a) The electric field must be zero in this case
(b) The electric field is directed perpendicular to the direction the electrons are moving
(c) The electric field is directed toward the left
(d) The electric field is directed toward the right

22. Wires A and B are identical except that each is made from a different material. The one end of wire A is connected to one end of wire B. The two remaining ends are connected across a battery and current flows through the two wires. Which one of the following statements concerning this situation is true?

(a) The potential difference across the two wires is the same, but the current through each wire is different
(b) The potential difference across the two wires is the same; and the current through each wire is the same
(c) The potential difference across the two wires is different, but the current through each wire is the same
(d) Both the potential difference across the two wires and the current through each wire is different

23. A conductor of area of cross-section A having charge carries, each having a charge q is subjected to a potential difference V. The number density of charge carries in the conductor is n and the charge carries (along with their random) are moving with a velocity v. If s is the conductivity of the conductor and τ is the average relaxation time, then

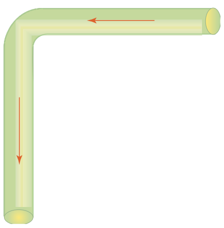
(a) $\tau = \dfrac{m}{nq^2\sigma}$ **(b)** $\tau = \dfrac{m\sigma}{nq^2}$

(c) $\tau = \dfrac{2m\sigma}{nq^2}$ **(d)** $\tau = \dfrac{m\sigma}{2nq^2}$

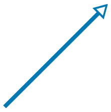
24. How does the resistivity of a metal wire change if either the number of electrons per unit volume increases or the mean free time increases?

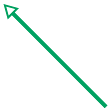
(a) In both cases, the resistivity will increase
(b) In both cases, the resistivity will decrease
(c) Increasing the number of electrons will increase the resistivity, but it will decrease if the mean free time increases
(d) Increasing the number of electrons will decrease the resistivity, but it will increase if the mean free time increases

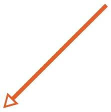
25. The given figure represents a portion of a wire in a circuit. A current is flowing in the wire in the direction shown. Under the convention that it is positive charge that flows, the electric field points in the direction of the current. There will be some charge

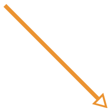


accumulation at the bend to change the direction of the electric field. What is the direction of the electric field due to charges at the bend?

(a)


(b)


(c)


(d)


26. Lengths and cross-sectional areas of four pieces of nichrome wire are (L, A), $(2L, A)$, $(L, 2A)$ and $(2L, 2A)$, respectively. If the same voltage difference V is applied across their lengths, which of the wires will get the hottest in steady state?

(a) Wire (L, A)
(b) Wire $(2L, A)$
(c) Wire $(L, 2A)$
(d) Wire $(2L, 2A)$

27. The insulated wiring in a house can safely carry a maximum current of 18 A. The electrical outlets in the house provide an alternating voltage of 120 V. A space heater when plugged into the outlet operates at an average power of 1500 W. How many space heaters can safely be plugged into a single electrical outlet and turned on for an extended period of time?

(a) Zero
(b) One
(c) Two
(d) Three

28. If the current in an electric bulb decreases by 0.5%, then the power in the bulb decreases approximately by

(a) 0.5% **(b)** 1%
(c) 2% **(d)** 0.2%

29. When a 500 W electric bulb and 500 W heater operate at their rated voltages, the filament of the bulb reaches a much higher temperature than the filament of the heater. The most important reason for this is that

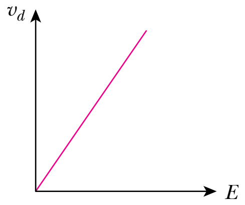
(a) Their resistances are not equal
(b) They are made of different materials
(c) Their dimensions are very different
(d) The emissivity of their surface is different

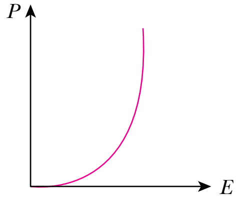
More than One Correct Choice Type

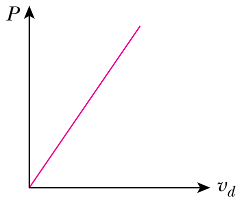
30. The charge flowing in a conductor varies with time as $Q = at - bt^2$. Then, the current

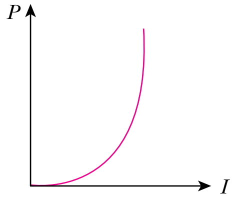
(a) Decreases linearly with time
(b) Reaches a maximum and then decreases
(c) Falls to zero after time $t = a/2b$
(d) Changes at the rate of $-2b$

31. If E denotes the electric field in a uniform conductor, I denotes the corresponding current through it, v_d denotes the drift velocity of electrons, and P denotes the thermal power produced in the conductor, which graph/graphs among the ones shown in the below figure is/are correct?

(a)


(b)


(c)


(d)


32. A voltmeter and an ammeter are joined in series to an ideal cell, giving readings V and A, respectively. If a resistance equal to the resistance of ammeter is now joined in parallel to the ammeter, then

(a) V will not change.
(b) V will increase slightly.
(c) A will become exactly half of its initial value.
(d) A will become slightly more than half of its initial value.

33. A current passes through a wire of non-uniform cross section. Which of the following quantities are independent of the cross section?

(a) Free electron density
(b) Current density
(c) Drift speed
(d) The charge crossing in a given time interval

34. A uniform wire of resistance R is shaped into a regular n-sided polygon (n is even). The equivalent resistance between any two corners can have

(a) The maximum value $R/4$
(b) The maximum value R/n
(c) The minimum value $R\left(\frac{n-1}{n^2}\right)$
(d) The minimum value R/n

35. A copper strip AB and an iron strip AC are joined at A. The junction A is maintained at 0°C and the free ends B and C are maintained at 100°C. There is a potential difference between

(a) The free ends B and C
(b) The two ends of copper strip
(c) The two ends of iron strip
(d) The junction of copper end and iron end

36. When no current is passed through a conductor

(a) The free electrons do not move
(b) The average speed of a free electron over a large period of time is zero
(c) The average thermal velocity of a free electron over a large period of time is zero
(d) The average of thermal velocity of all the free electrons at an instant is zero

37. Which of the following statements is/are correct?
- **(a)** Resistivity of electrolytes decreases on increasing temperature
- **(b)** Resistance of mercury falls on decreasing its temperature
- **(c)** When joined in series, a 40 W bulb glows more than a 60 W bulb
- **(d)** Resistance of a 40 W bulb is less than the resistance of a 60 W bulb

38. Which one of the following statements is false?
- **(a)** The electrons in a wire carrying an electrical current normally move very slowly (< 1 m/s)
- **(b)** A battery supplies the same current to any object to which it is connected
- **(c)** The electric current leaving the positive terminal of a battery is the same as the current returning to the negative terminal
- **(d)** A conductor in which an electric current is flowing has an electric field in its interior

39. Which of the following statements are true for a metallic conductor?
- **(a)** The electrical conductivity depends on the mass of atoms
- **(b)** The electrical conductivity decreases with rise in temperature
- **(c)** The current density depends upon the drift velocity of electrons
- **(d)** The electrical conductivity increases with increase in voltage across it

Linked Comprehension

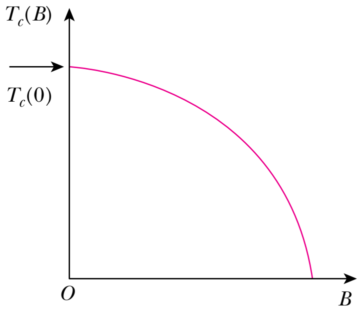
Paragraph for Questions 40–42: The alternate discs of iron and carbon, having same area of cross-section are cemented together to make a cylinder whose temperature coefficient of resistivity is zero. The change in temperature in two alternate discs is same. The ratio of their thickness and ratio of heat produced in them is found out. The resistivity of iron and carbon at 20°C are 1×10^{-7} and 3×10^{-5} Ω m and their temperature coefficient of resistance are 5×10^{-3}°C and -7.5×10^{-4}°C, respectively. Thermal expansion is neglected. Here, $\Delta R_1 + \Delta R_2 = 0$ (where ΔR_1 and ΔR_2 are the increase in resistances of iron and carbon, respectively, with the rise in temperature) because combined temperature coefficient of resistivity is given as zero.

40. Ratio of their thickness is
- **(a)** 54
- **(b)** 45
- **(c)** 35
- **(d)** 21

41. Ratio of heat produced in them is
- **(a)** 0.51
- **(b)** 1
- **(c)** 0.15
- **(d)** 2

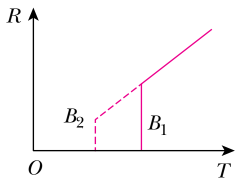
42. A copper wire is stretched to make it 1% longer. The percentage change in its resistance is
- **(a)** 0.2%
- **(b)** 1%
- **(c)** 1.5%
- **(d)** 2.5%

Paragraph for Questions 43 and 44: Electrical resistance of certain materials, known as superconductors, changes abruptly from a non-zero value to zero as their temperature is lowered below a critical temperature $T_c(0)$ An interesting property of superconductors is that their critical temperature becomes smaller than $T_c(0)$ if they are placed in a magnetic field, that is, the critical temperature $T_c(B)$ is a function of the magnetic field strength B. The dependence of $T_c(B)$ on magnetic field is shown in the below figure.

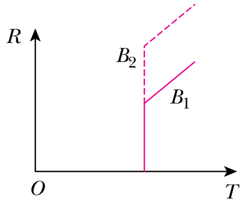


43. In the graphs shown in the below figure, the resistance R of a superconductor is shown as a friction of its temperature T for two different magnetic fields B_1 (solid line) and B_2 (dashed line). If B_2 is larger than B_1, which of the graph/graphs shows/show the correct variation of R with T in these fields?

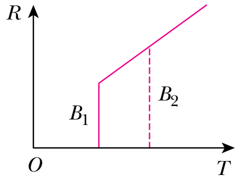
(a)



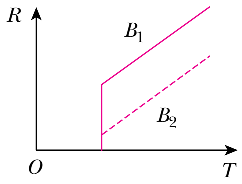
(b)



(c)



(d)



44. A superconductor has $T_c(0) = 100$ K. When a magnetic field of 7.5 T is applied, its T_c decreases to 75 K. For this material, one can definitely say that when
- **(a)** $B = 5$ T, $T_c(B) = 80$ K
- **(b)** $B = 5$ T, 75 K $< T_c(B) <$ 100 K
- **(c)** $B = 10$ T, 75 K $< T_c <$ 100 K
- **(d)** $B = 10$ T, $T_c = 70$ K

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

45.

Column I	Column II
(a) A conductor of larger length is connected across the battery instead of conductor of shorter length, made of same material	**(p)** A constant current flows through the conductor
(b) The temperature of the connected conductor does not change when connected to the battery	**(q)** The current density is not same throughout conductor
(c) The graph between V and I is a straight line passing through the length of the origin	**(r)** The conductor is a conductor but does not be an ohmic resistor
(d) A conductor of variable area of cross-section is connected across a battery of voltage V	**(s)** The resistivity of the conductor does not change

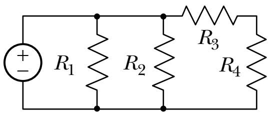
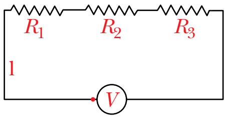
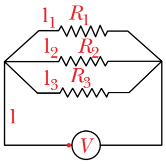
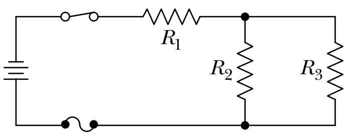
46.

Column I	Column II
(a) emf	**(p)** Motion of electrons in a closed path
(b) current	**(q)** Motion of electrons through a conductor without any collision
(c) Superconductor	**(r)** Motion of electrons in a definite direction
(d) Resistance	**(s)** The collision of electrons while moving through the conductor

47.

Column I	Column II
(a) The fuse wire	**(p)** of nickel–chromium alloy
(b) The heating wire	**(q)** of tin–lead alloy
(c) The best connecting wire	**(r)** of silver metal
(d) The standard resistance coil	**(s)** of copper manganese–nickel alloy

Directions for Questions 48 and 49: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

Column I	Column II	Column III
(I) Voltage remains the same across each resistor in the circuit.	**(i)** 	**(J)** Current in the circuit is different through each resistor.
(II) Voltage across resistors in a series connection will be different.	**(ii)** 	**(K)** Current will be the same which passes through each resistor.
(III) Voltage across resistors R_2 and R_3 are same.	**(iii)** 	**(L)** Current in the circuit is different through resistors R_1 and R_2.
(IV) Voltage across resistors R_1 and R_2 are same.	**(iv)** 	**(M)** Current in the circuit is different through resistors R_2 and R_3.

48. There are mainly two types of combinations of resistors in a circuit, series and parallel. In the given table, Column I shows voltages of different arrangements of resistors, Column II shows different arrangement of resistors and Column III shows currents of different arrangements of resistors

(1) What are the conditions when resistor is in the parallel network?

(a) (I) (ii) (J)
(b) (IV) (ii) (M)
(c) (II) (ii) (J)
(d) (I) (iii) (J)

(2) What are the conditions when resistors are in mixed combination, that is, parallel and series combination respectively?

(a) (IV) (iv) (K)
(b) (III) (iv) (M)
(c) (II) (iii) (L)
(d) (I) (i) (M)

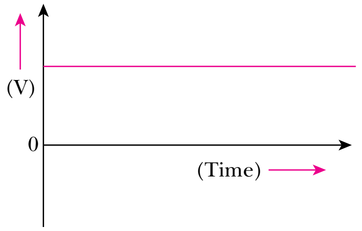
(3) What are the conditions when resistors are in mixed combination, that is series and parallel combination respectively?

(a) (III) (i) (L) **(b)** (I) (i) (J)
(c) (IV) (i) (L) **(d)** (II) (iii) (M)

49. Electrical current is the flow of charged particles. The electric current flows through a wire the same way as how water moves in a river. The electricity is nothing but the flow of electrons.

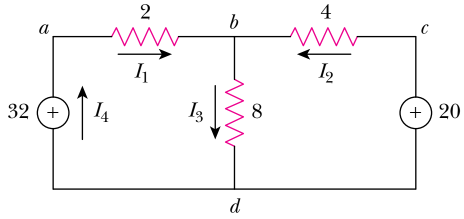
Column I	Column II	Column III
(I) Direct Current	**(i)** based on Kirchoff's current law.	**(J)** $\sum V_{abda} = 0$
(II) Loop Current	**(ii)** electric charge changes the direction periodically	**(K)** $\sum_i I_i = 0 = I_1 + I_2 + I_3$
(III) Alternating Current	**(iii)** unidirectional flow of electric charge	**(L)** produced by the sources such as batteries
(IV) Node Voltage	**(iv)** based on Kirchoff's voltage law.	**(M)** produced by alternator

(1) What does the graph signify?



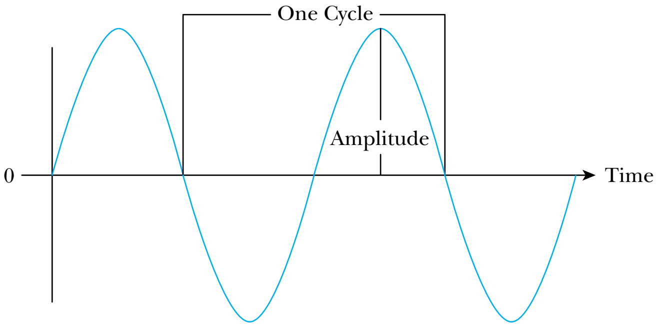
(a) (III) (iv) (K) **(b)** (IV) (i) (M)
(c) (III) (ii) (L) **(d)** (I) (iii) (L)

(2) What does the circuit diagram signify?



(a) (II) (iv) (J) **(b)** (IV) (i) (M)
(c) (I) (ii) (L) **(d)** (III) (iv) (K)

(3) What does the graph signify?

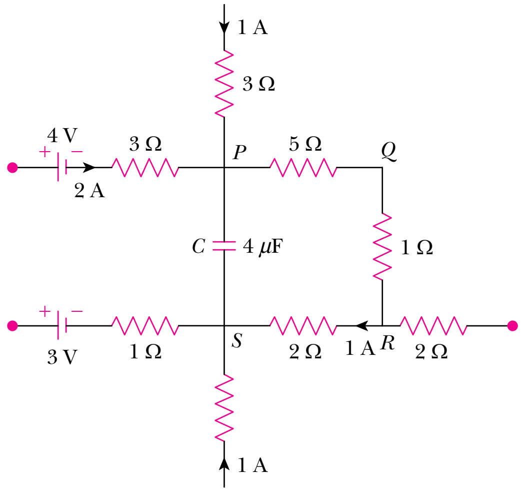


(a) (III) (i) (K) **(b)** (IV) (i) (J)
(c) (III) (iii) (M) **(d)** (III) (ii) (M)

Integer Type

50. The length of a potentiometer wire is 10 cm. A cell of emf ε is balanced at a length 10/3 cm from the positive end of the wire. If the length of the wire is increased by 5 cm, at what distance (in cm) from positive end will the same cell give a balance point?

51. The following figure shows a part of complete circuit. The current in various branches in steady state are shown in the figure. What is the energy stored in capacitor in unit of 10^{-4} J?



52. A spherical drop of capacitance 2 μF is broken into eight drops of equal radius. Then the capacitance of each small drop is (in μF).

ANSWER KEY

Checkpoints

1. 8A, rightward
2. (a) – (c) rightward
3. *a* and *c* tie, then *b*
4. device 2
5. $E_{\text{Cu}} < E_{\text{Ni}}$, positive
6. (a) and (b) tie, then (d), then (c)

Problems

1. (a) 767 A; (b) 2.71×10^{7} A/m^{2}; (c) 10.6×10^{-8} $\Omega \cdot$m
2. (a) 8.0×10^{7} $(\Omega \cdot \text{m})^{-1}$; (b) 1.0×10^{7} $(\Omega \cdot \text{m})^{-1}$
3. $2.08 \times 10^{6}\,\Omega$
4. 5.58×10^{-3} A
5. (a) 3.24×10^{-12} A/m^{2}; (b) 1.70 cm/s
6. 1.4×10^{-6} C
7. 6.22×10^{-9} m/s
8. $8.2 \times 10^{-8}\,\Omega \cdot$m
9. 0.087 Vs
10. 1.2 m
11. (a) 9.33×10^{-7} m^{2}; (b) 4.66 m
12. 11.3 W
13. (a) upward; (b) 12 eV; (c) 12 eV or 1.9×10^{-18} J
14. 0.00360 A ≈ 3.6 mA
15. (a) 7.74 A; (b) 14.9 Ω; (c) 1.60×10^{7} J
16. (a) 1.3 V; (b) 2.546 V ≈ 2.5 V; (c) 2.5 W; (d) 5.1 W
17. (a) 1.7×10^{-5} A/m^{2}; (b) 1.3×10^{-15} m/s
18. 12 mW
19. (a) 1.41×10^{-2} V/m; (b) 2.03×10^{2} J
20. 9.00 A
21. 1.01×10^{6} J
22. (a) 5.1×10^{-6} C/m^{2} (b) 3.2×10^{13}/ m^{2}
23. 3
24. 105 V
25. (a) 9.60 Ω; (b) 7.81×10^{19}/ s
26. $R/2$
27. 0.377 m
28. 6.6×10^{-4} m
29. 5.8 min
30. (a) 0.821 A; (b) 0.411 A; (c) *Ja* has its maximum value near the surface of the wire
31. (a) yes; (b) 199 A/m^{2} ≈ 2.0×10^{13} A/m^{2}
32. (a) 1.32 mm; (b) 104 mm
33. 3.84 ≈ 3.8
34. (a) 2.90×10^{-7} A/m^{2}; (b) 3.69×10^{7} A/m^{2}
35. 1.1×10^{-2} s
36. 6.33×10^{-3} A

Practice Questions

Single Correct Choice Type

1. (a)
2. (d)
3. (d)
4. (b)
5. (c)
6. (a)
7. (d)
8. (b)
9. (a)
10. (a)
11. (a)
12. (c)
13. (c)
14. (b)
15. (a)
16. (c)
17. (d)
18. (c)
19. (c)
20. (c)
21. (d)
22. (c)
23. (b)
24. (b)
25. (d)
26. (c)
27. (b)
28. (b)
29. (c)

More than One Correct Choice Type

30. (a), (c), (d)
31. (a), (b), (d)
32. (b), (d)
33. (a), (d)
34. (a), (c)
35. (a), (b), (c)
36. (c), (d)
37. (a), (b), (c)
38. (a), (b)
39. (a), (b), (c)

Linked Comprehension

40. (b)
41. (c)
42. (a)
43. (a)
44. (b)

Matrix-Match

45. (a) → (s); (b) → (p); (c) → (r); (d) → (p),(q)
46. (a) → (p); (b) → (p),(r); (c) → (q); (d) → (s)
47. (a) → (q); (b) → (p); (c) → (r); (d) → (s)
48. (1) → (d); (2) → (b); (3) → (c)
49. (1) → (d); (2) → (a); (3) → (d)

Integer Type

50. 5
51. 8
52. 1

CHAPTER

27

Circuits

Contents

27.1 What is Physics?
27.2 Work, Energy, and EMF
27.3 Calculating the Current in a Single-Loop Circuit
27.4 Potential Difference Between Two Points
27.5 Multiloop Circuits
27.6 Electricity-Related Instruments
27.7 *RC* Circuits

27.1 | WHAT IS PHYSICS?

You are surrounded by electric circuits. You might take pride in the number of electrical devices you own and might even carry a mental list of the devices you wish you owned. Every one of those devices, as well as the electrical grid that powers your home, depends on modern electrical engineering. We cannot easily estimate the current financial worth of electrical engineering and its products, but we can be certain that the financial worth continues to grow yearly as more and more tasks are handled electrically. Radios are now tuned electronically instead of manually. Messages are now sent by email instead of through the postal system. Research journals are now read on a computer instead of in a library building, and research papers are now copied and filed electronically instead of photocopied and tucked into a filing cabinet. Indeed, you may be reading an electronic version of this book.

The basic science of electrical engineering is physics. In this chapter we cover the physics of electric circuits that are combinations of resistors and batteries (and, in Section 25.4, capacitors). We restrict our discussion to circuits through which charge flows in one direction, which are called either *direct-current circuits* or *DC circuits*. We begin with the question: How can you get charges to flow?

27.2 | WORK, ENERGY, AND EMF

Key Concepts

- An emf device does work on charges to maintain a potential difference between its output terminals. If dW is the work the device does to force positive charge dq from the negative to the positive terminal, then the emf (work per unit charge) of the device is

$$\mathscr{E} = \frac{dW}{dq} \quad (\text{definition of } \mathscr{E}).$$

- An ideal emf device is one that lacks any internal resistance. The potential difference between its terminals is equal to the emf.
- A real emf device has internal resistance. The potential difference between its terminals is equal to the emf only if there is no current through the device.

If you want to make charge carriers flow through a resistor, you must establish a potential difference between the ends of the device. One way to do this is to connect each end of the resistor to one plate of a charged capacitor. The trouble with this scheme is that the flow of charge acts to discharge the capacitor, quickly bringing the plates to the same potential. When that happens, there is no longer an electric field in the resistor, and thus the flow of charge stops.

To produce a steady flow of charge, you need a "charge pump," a device that—by doing work on the charge carriers—maintains a potential difference between a pair of terminals. We call such a device an **emf device**, and the device is said to provide an **emf** $\mathscr{E}$, which means that it does work on charge carriers. An emf device is sometimes called a *seat of emf*. The term *emf* comes from the outdated phrase *electromotive force*, which was adopted before scientists clearly understood the function of an emf device.

In Chapter 26, we discussed the motion of charge carriers through a circuit in terms of the electric field set up in the circuit—the field produces forces that move the charge carriers. In this chapter we take a different approach: We discuss the motion of the charge carriers in terms of the required energy—an emf device supplies the energy for the motion via the work it does.

A common emf device is the *battery*, used to power a wide variety of machines from wristwatches to submarines. The emf device that most influences our daily lives, however, is the *electric generator*, which, by means of electrical connections (wires) from a generating plant, creates a potential difference in our homes and workplaces. The emf devices known as *solar cells*, long familiar as the wing-like panels on spacecraft, also dot the countryside for domestic applications. Less familiar emf devices are the *fuel cells* that powered the space shuttles and the *thermopiles* that provide onboard electrical power for some spacecraft and for remote stations in Antarctica and elsewhere. An emf device does not have to be an instrument—living systems, ranging from electric eels and human beings to plants, have physiological emf devices.

Although the devices we have listed differ widely in their modes of operation, they all perform the same basic function—they do work on charge carriers and thus maintain a potential difference between their terminals.

Figure 27-1 shows an emf device (consider it to be a battery) that is part of a simple circuit containing a single resistance R (the symbol for resistance and a resistor is -⩘⩘-). The emf device keeps one of its terminals (called the positive terminal and often labeled +) at a higher electric potential than the other terminal (called the negative terminal and labeled −). We can represent the emf of the device with an arrow that points from the negative terminal toward the positive terminal as in Fig. 27-1. A small circle on the tail of the emf arrow distinguishes it from the arrows that indicate current direction.

When an emf device is not connected to a circuit, the internal chemistry of the device does not cause any net flow of charge carriers within it. However, when it is connected to a circuit as in Fig. 27-1, its internal chemistry causes a net flow of positive charge carriers from the negative terminal to the positive terminal, in the direction of the emf arrow. This flow is part of the current that is set up around the circuit in that same direction (clockwise in Fig. 27-1).

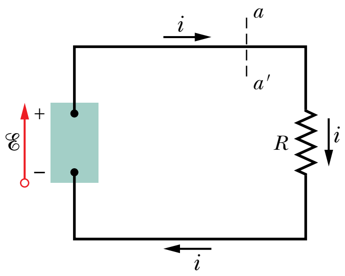


Figure 27-1 A simple electric circuit, in which a device of emf $\mathscr{E}$ does work on the charge carriers and maintains a steady current i in a resistor of resistance R.

Within the emf device, positive charge carriers move from a region of low electric potential and thus low electric potential energy (at the negative terminal) to a region of higher electric potential and higher electric potential energy (at the positive terminal). This motion is just the opposite of what the electric field between the terminals (which is directed from the positive terminal toward the negative terminal) would cause the charge carriers to do.

Thus, there must be some source of energy within the device, enabling it to do work on the charges by forcing them to move as they do. The energy source may be chemical, as in a battery or a fuel cell. It may involve mechanical forces, as in an electric generator. Temperature differences may supply the energy, as in a thermopile; or the Sun may supply it, as in a solar cell.

Let us now analyze the circuit of Fig. 27-1 from the point of view of work and energy transfers. In any time interval dt, a charge dq passes through any cross section of this circuit, such as aa'. This same amount of charge must enter the emf device at its low-potential end and leave at its high-potential end. The device must do an amount of work dW on the charge dq to force it to move in this way. We define the emf of the emf device in terms of this work:

$$\mathscr{E} = \frac{dW}{dq} \quad \text{(definition of } \mathscr{E}\text{).} \tag{27-1}$$

In words, the emf of an emf device is the work per unit charge that the device does in moving charge from its low-potential terminal to its high-potential terminal. The SI unit for emf is the joule per coulomb; in Chapter 24 we defined that unit as the *volt*.

An **ideal emf device** is one that lacks any internal resistance to the internal movement of charge from terminal to terminal. The potential difference between the terminals of an ideal emf device is equal to the emf of the device. For example, an ideal battery with an emf of 12.0 V always has a potential difference of 12.0 V between its terminals.

A **real emf device**, such as any real battery, has internal resistance to the internal movement of charge. When a real emf device is not connected to a circuit, and thus does not have current through it, the potential difference between its terminals is equal to its emf. However, when that device has current through it, the potential difference between its terminals differs from its emf. We shall discuss such real batteries near the end of this module.

When an emf device is connected to a circuit, the device transfers energy to the charge carriers passing through it. This energy can then be transferred from the charge carriers to other devices in the circuit, for example, to light a bulb. Figure 27-2*a* shows a circuit containing two ideal rechargeable (*storage*) batteries A and B, a resistance R, and an electric motor M that can lift an object by using energy it obtains from charge carriers in the circuit. Note that the batteries are connected so that they tend to send charges around the circuit in opposite directions. The actual direction of the current in the circuit is determined by the battery with the larger emf, which happens to be battery B, so the chemical energy within battery B is decreasing as energy is transferred to the charge carriers passing through it. However, the chemical energy within battery A is increasing because the current in it is directed from the positive terminal to the negative terminal. Thus, battery B is charging battery A. Battery B is also providing energy to motor M and energy that is being dissipated by resistance R. Figure 27-2*b* shows all three energy transfers from battery B; each decreases that battery's chemical energy.

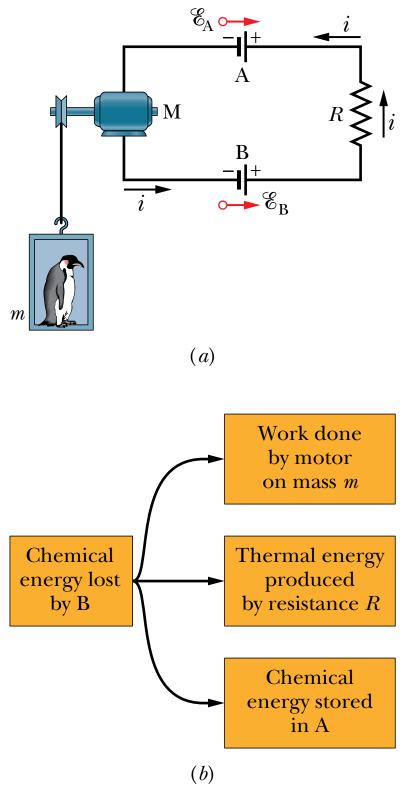

$\mathscr{E}_A$
i
A
M
R
i
B
i
$\mathscr{E}_B$
m
(*a*)
Work done by motor on mass m
Chemical energy lost by B
Thermal energy produced by resistance R
Chemical energy stored in A
(*b*)

Figure 27-2 (*a*) In the circuit, $\mathscr{E}_B > \mathscr{E}_A$; so battery B determines the direction of the current. (*b*) The energy transfers in the circuit.

27.3 | CALCULATING THE CURRENT IN A SINGLE-LOOP CIRCUIT

Key Concepts

- The change in potential in traversing a resistance R in the direction of the current is $-iR$; in the opposite direction it is $+iR$ (resistance rule).
- The change in potential in traversing an ideal emf device in the direction of the emf arrow is $+\mathscr{E}$; in the opposite direction it is $-\mathscr{E}$ (emf rule).
- Conservation of energy leads to the loop rule: *Loop Rule. The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.*
- When resistances are in series, they have the same current. The equivalent resistance that can replace a series combination of resistances is

$$R_{eq} = \sum_{j=1}^{n} R_j \quad (n \text{ resistances in series}).$$

The battery drives current through the resistor, from high potential to low potential.

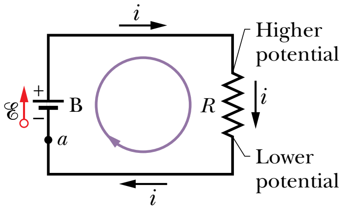


Figure 27-3 A single-loop circuit in which a resistance R is connected across an ideal battery B with emf $\mathscr{E}$. The resulting current i is the same throughout the circuit.

We discuss here two equivalent ways to calculate the current in the simple *single-loop* circuit of Fig. 27-3; one method is based on energy conservation considerations, and the other on the concept of potential. The circuit consists of an ideal battery B with emf $\mathscr{E}$, a resistor of resistance R, and two connecting wires. (Unless otherwise indicated, we assume that wires in circuits have negligible resistance. Their function, then, is merely to provide pathways along which charge carriers can move.)

Energy Method

Equation 26-27 ($P = i^2R$) tells us that in a time interval dt an amount of energy given by $i^2R\,dt$ will appear in the resistor of Fig. 27-3 as thermal energy. As noted in Section 26.5, this energy is said to be *dissipated*. (Because we assume the wires to have negligible resistance, no thermal energy will appear in them.) During the same interval, a charge $dq = i\,dt$ will have moved through battery B, and the work that the battery will have done on this charge, according to Eq. 27-1, is

$$dW = \mathscr{E}\,dq = \mathscr{E}i\,dt.$$

From the principle of conservation of energy, the work done by the (ideal) battery must equal the thermal energy that appears in the resistor:

$$\mathscr{E}i\,dt = i^2Rdt.$$

This gives us

$$\mathscr{E} = iR.$$

The emf $\mathscr{E}$ is the energy per unit charge transferred to the moving charges by the battery. The quantity iR is the energy per unit charge transferred *from* the moving charges to thermal energy within the resistor. Therefore, this equation means that the energy per unit charge transferred to the moving charges is equal to the energy per unit charge transferred from them. Solving for i, we find

$$i = \frac{\mathscr{E}}{R}. \qquad (27\text{-}2)$$

Potential Method

Suppose we start at any point in the circuit of Fig. 27-3 and mentally proceed around the circuit in either direction, adding algebraically the potential differences that we encounter. Then when we return to our starting point, we must also have returned to our starting potential. Before actually doing so, we shall formalize this idea in a statement that holds not only for single-loop circuits such as that of Fig. 27-3 but also for any complete loop in a *multiloop* circuit, as we shall discuss in later section:

LOOP RULE: The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.

This is often referred to as *Kirchhoff's loop rule* (or *Kirchhoff's voltage law*), after German physicist Gustav Robert Kirchhoff. This rule is equivalent to saying that each point on a mountain has only one elevation above sea level. If you start from any point and return to it after walking around the mountain, the algebraic sum of the changes in elevation that you encounter must be zero.

In Fig. 27-3, let us start at point a, whose potential is V_a, and mentally walk clockwise around the circuit until we are back at a, keeping track of potential changes as we move. Our starting point is at the low-potential terminal of the battery. Because the battery is ideal, the potential difference between its terminals is equal to $\mathscr{E}$. When we pass through the battery to the high-potential terminal, the change in potential is $+\mathscr{E}$.

As we walk along the top wire to the top end of the resistor, there is no potential change because the wire has negligible resistance; it is at the same potential as the high-potential terminal of the battery. So too is the top end of the resistor. When we pass through the resistor, however, the potential changes according to Eq. 26-8 (which we can rewrite as $V = iR$). Moreover, the potential must decrease because we are moving from the higher potential side of the resistor. Thus, the change in potential is $-iR$.

We return to point *a* by moving along the bottom wire. Because this wire also has negligible resistance, we again find no potential change. Back at point *a*, the potential is again V_a. Because we traversed a complete loop, our initial potential, as modified for potential changes along the way, must be equal to our final potential; that is,

$$V_a + \mathscr{E} - iR = V_a.$$

The value of V_a cancels from this equation, which becomes

$$\mathscr{E} - iR = 0.$$

Solving this equation for *i* gives us the same result, $i = \mathscr{E}/R$, as the energy method (Eq. 27-2).

If we apply the loop rule to a complete *counterclockwise* walk around the circuit, the rule gives us

$$-\mathscr{E} + iR = 0$$

and we again find that $i = \mathscr{E}/R$. Thus, you may mentally circle a loop in either direction to apply the loop rule.

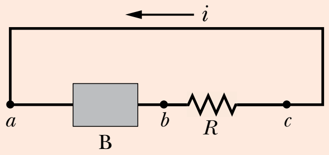
To prepare for circuits more complex than that of Fig. 27-3, let us set down two rules for finding potential differences as we move around a loop:

RESISTANCE RULE: For a move through a resistance in the direction of the current, the change in potential is $-iR$; in the opposite direction it is $+iR$.

EMF RULE: For a move through an ideal emf device in the direction of the emf arrow, the change in potential is $+\mathscr{E}$; in the opposite direction it is $-\mathscr{E}$.

CHECKPOINT 1

The figure shows the current *i* in a single-loop circuit with a battery B and a resistance *R* (and wires of negligible resistance). (a) Should the emf arrow at B be drawn pointing leftward or rightward? At points *a*, *b*, and *c*, rank (b) the magnitude of the current, (c) the electric potential, and (d) the electric potential energy of the charge carriers, greatest first.



Internal Resistance

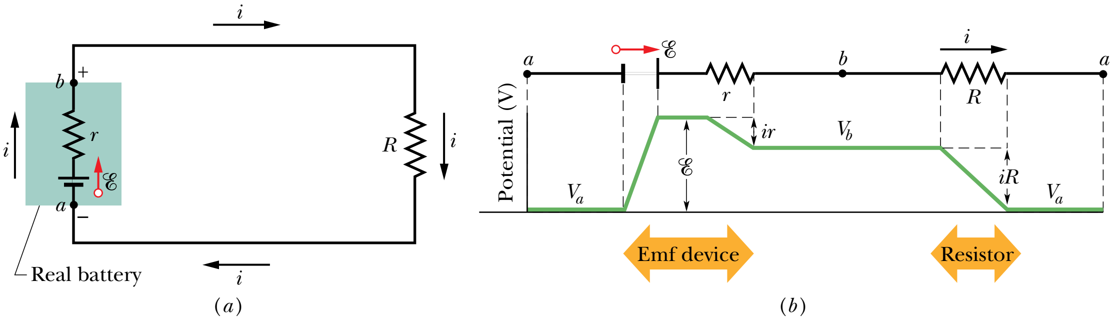
Figure 27-4*a* shows a real battery, with internal resistance *r*, wired to an external resistor of resistance *R*. The internal resistance of the battery is the electrical resistance of the conducting materials of the battery and thus is an unremovable feature of the battery. In Fig. 27-4*a*, however, the battery is drawn as if it could be separated into an ideal battery with emf $\mathscr{E}$ and a resistor of resistance *r*. The order in which the symbols for these separated parts are drawn does not matter.

If we apply the loop rule clockwise beginning at point *a*, the *changes* in potential give us

$$\mathscr{E} - ir - iR = 0. \tag{27-3}$$

Solving for the current, we find

$$i = \frac{\mathscr{E}}{R + r}. \tag{27-4}$$



(*a*) (*b*)

Figure 27-4 (*a*) A single-loop circuit containing a real battery having internal resistance r and emf ℰ. (*b*) The same circuit, now spread out in a line. The potentials encountered in traversing the circuit clockwise from a are also shown. The potential V_a is arbitrarily assigned a value of zero, and other potentials in the circuit are graphed relative to V_a.

Note that this equation reduces to Eq. 27-2 if the battery is ideal—that is, if $r = 0$.

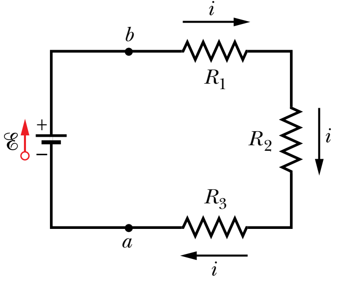
Figure 27-4*b* shows graphically the changes in electric potential around the circuit. (To better link Fig. 27-4*b* with the *closed circuit* in Fig. 27-4*a*, imagine curling the graph into a cylinder with point a at the left overlapping point a at the right.) Note how traversing the circuit is like walking around a (potential) mountain back to your starting point—you return to the starting elevation.

In this book, when a battery is not described as real or if no internal resistance is indicated, you can generally assume that it is ideal—but, of course, in the real world batteries are always real and have internal resistance.

Resistances in Series

Figure 27-5*a* shows three resistances connected **in series** to an ideal battery with emf ℰ. This description has little to do with how the resistances are drawn. Rather, "in series" means that the resistances are wired one after another and that a potential difference V is applied across the two ends of the series. In Fig. 27-5*a*, the resistances are connected one after another between a and b, and a potential difference is maintained across a and b by the battery. The potential differences that then exist across the resistances in the series produce identical currents i in them. In general,

When a potential difference V is applied across resistances connected in series, the resistances have identical currents i. The sum of the potential differences across the resistances is equal to the applied potential difference V.

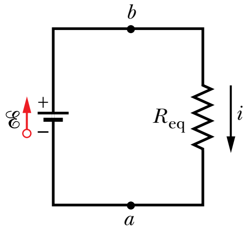


(*a*)

Series resistors and their equivalent have the same current ("ser-i").

Note that charge moving through the series resistances can move along only a single route. If there are additional routes, so that the currents in different resistances are different, the resistances are not connected in series.

Resistances connected in series can be replaced with an equivalent resistance R_{eq} that has the same current i and the same *total* potential difference V as the actual resistances.



(*b*)

You might remember that R_{eq} and all the actual series resistances have the same current i with the nonsense word "ser-i." Figure 27-5*b* shows the equivalent resistance R_{eq} that can replace the three resistances of Fig. 27-5*a*.

Figure 27-5 (*a*) Three resistors are connected in series between points a and b. (*b*) An equivalent circuit, with the three resistors replaced with their equivalent resistance R_{eq}.

To derive an expression for R_{eq} in Fig. 27-5*b*, we apply the loop rule to both circuits. For Fig. 27-5*a*, starting at *a* and going clockwise around the circuit, we find

$$\mathscr{E} - iR_1 - iR_2 - iR_3 = 0,$$

or

$$i = \frac{\mathscr{E}}{R_1 + R_2 + R_3}. \tag{27-5}$$

For Fig. 27-5*b*, with the three resistances replaced with a single equivalent resistance R_{eq}, we find

$$\mathscr{E} - iR_{\text{eq}} = 0,$$

or

$$i = \frac{\mathscr{E}}{R_{\text{eq}}}. \tag{27-6}$$

Comparison of Eqs. 27-5 and 27-6 shows that

$$R_{\text{eq}} = R_1 + R_2 + R_3.$$

The extension to *n* resistances is straightforward and is

$$R_{\text{eq}} = \sum_{j=1}^{n} R_j \quad (n \text{ resistances in series}). \tag{27-7}$$

Note that when resistances are in series, their equivalent resistance is greater than any of the individual resistances.

CHECKPOINT 2

In Fig. 27-5*a*, if $R_1 > R_2 > R_3$, rank the three resistances according to (a) the current through them and (b) the potential difference across them, greatest first.

27.4 | POTENTIAL DIFFERENCE BETWEEN TWO POINTS

Key Concepts

- When a real battery of emf $\mathscr{E}$ and internal resistance r does work on the charge carriers in a current i through the battery, the rate P of energy transfer to the charge carriers is

$$P = iV,$$

where V is the potential across the terminals of the battery.

- The rate P_r at which energy is dissipated as thermal energy in the battery is

$$P_r = i^2 r.$$

- The rate P_{emf} at which the chemical energy in the battery changes is

$$P_{\text{emf}} = i\mathscr{E}.$$

We often want to find the potential difference between two points in a circuit. For example, in Fig. 27-6, what is the potential difference $V_b - V_a$ between points *a* and *b*? To find out, let's start at point *a* (at potential V_a) and move through the battery to point *b* (at potential V_b) while keeping track of the potential changes we encounter. When we pass through the battery's emf, the potential increases by $\mathscr{E}$. When we pass through the battery's internal

The internal resistance reduces the potential difference between the terminals.

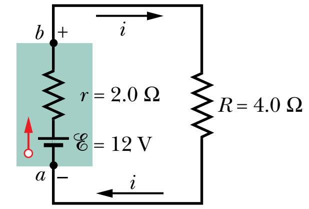


Figure 27-6 Points *a* and *b*, which are at the terminals of a real battery, differ in potential.

resistance r, we move in the direction of the current and thus the potential decreases by ir. We are then at the potential of point b and we have

$$V_a + \mathscr{E} - ir = V_b,$$

or

$$V_b - V_a = \mathscr{E} - ir. \tag{27-8}$$

To evaluate this expression, we need the current i. Note that the circuit is the same as in Fig. 27-4*a*, for which Eq. 27-4 gives the current as

$$i = \frac{\mathscr{E}}{R + r}. \tag{27-9}$$

Substituting this equation into Eq. 27-8 gives us

$$\begin{aligned} V_b - V_a &= \mathscr{E} - \frac{\mathscr{E}}{R + r} r \\ &= \frac{\mathscr{E}}{R + r} R. \end{aligned} \tag{27-10}$$

Now substituting the data given in Fig. 27-6, we have

$$V_b - V_a = \frac{12\text{ V}}{4.0\ \Omega + 2.0\ \Omega} 4.0\ \Omega = 8.0\text{ V}. \tag{27-11}$$

Suppose, instead, we move from a to b counterclockwise, passing through resistor R rather than through the battery. Because we move opposite the current, the potential increases by iR. Thus,

$$V_a + iR = V_b$$

or

$$V_b - V_a = iR. \tag{27-12}$$

Substituting for i from Eq. 27-9, we again find Eq. 27-10. Hence, substitution of the data in Fig. 27-6 yields the same result, $V_b - V_a = 8.0$ V. In general,

To find the potential between any two points in a circuit, start at one point and traverse the circuit to the other point, following any path, and add algebraically the changes in potential you encounter.

Potential Difference Across a Real Battery

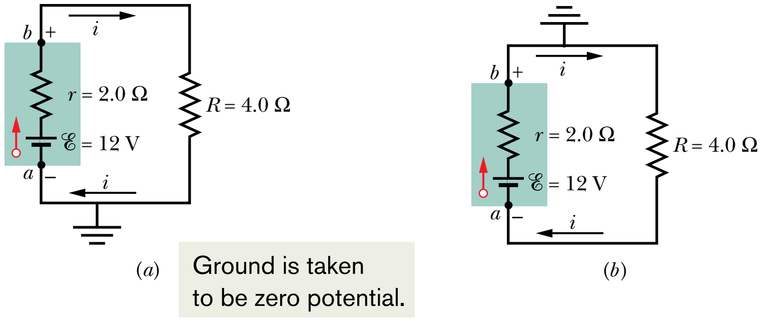
In Fig. 27-6, points a and b are located at the terminals of the battery. Thus, the potential difference $V_b - V_a$ is the terminal-to-terminal potential difference V across the battery. From Eq. 27-8, we see that

$$V = \mathscr{E} - ir. \tag{27-13}$$

If the internal resistance r of the battery in Fig. 27-6 were zero, Eq. 27-13 tells us that V would be equal to the emf $\mathscr{E}$ of the battery—namely, 12 V. However, because $r = 2.0\ \Omega$, Eq. 27-13 tells us that V is less than $\mathscr{E}$. From Eq. 27-11, we know that V is only 8.0 V. Note that the result depends on the value of the current through the battery. If the same battery were in a different circuit and had a different current through it, V would have some other value.

Grounding a Circuit

Figure 27-7*a* shows the same circuit as Fig. 27-6 except that here point a is directly connected to *ground*, as indicated by the common symbol ⏚. *Grounding a circuit* usually means connecting the circuit to a conducting path to Earth's surface (actually to the electrically conducting moist dirt and rock below ground). Here, such a connection means only that the potential is defined to be zero at the grounding point in the circuit. Thus in Fig. 27-7*a*, the potential at a is defined to be $V_a = 0$. Equation 27-11 then tells us that the potential at b is $V_b = 8.0$ V.



Ground is taken to be zero potential.

Figure 27-7 (*a*) Point *a* is directly connected to ground. (*b*) Point *b* is directly connected to ground.

Figure 27-7*b* is the same circuit except that point *b* is now directly connected to ground. Thus, the potential there is defined to be $V_b = 0$. Equation 27-11 now tells us that the potential at *a* is $V_a = -8.0$ V.

Power, Potential, and Emf

When a battery or some other type of emf device does work on the charge carriers to establish a current *i*, the device transfers energy from its source of energy (such as the chemical source in a battery) to the charge carriers. Because a real emf device has an internal resistance *r*, it also transfers energy to internal thermal energy via resistive dissipation (Section 26.7). Let us relate these transfers.

The net rate *P* of energy transfer from the emf device to the charge carriers is given by Eq. 26-26:

$$P = iV, \tag{27-14}$$

where *V* is the potential across the terminals of the emf device. From Eq. 27-13, we can substitute $V = \mathscr{E} - ir$ into Eq. 27-14 to find

$$P = i(\mathscr{E} - ir) = i\mathscr{E} - i^2r. \tag{27-15}$$

From Eq. 26-27, we recognize the term i^2r in Eq. 27-15 as the rate P_r of energy transfer to thermal energy within the emf device:

$$P_r = i^2r \quad \text{(internal dissipation rate).} \tag{27-16}$$

Then the term $i\mathscr{E}$ in Eq. 27-15 must be the rate P_{emf} at which the emf device transfers energy *both* to the charge carriers and to internal thermal energy. Thus,

$$P_{\text{emf}} = i\mathscr{E} \quad \text{(power of emf device).} \tag{27-17}$$

If a battery is being *recharged*, with a "wrong way" current through it, the energy transfer is then *from* the charge carriers *to* the battery—both to the battery's chemical energy and to the energy dissipated in the internal resistance *r*. The rate of change of the chemical energy is given by Eq. 27-17, the rate of dissipation is given by Eq. 27-16, and the rate at which the carriers supply energy is given by Eq. 27-14.

CHECKPOINT 3

A battery has an emf of 12 V and an internal resistance of 2 Ω. Is the terminal-to-terminal potential difference greater than, less than, or equal to 12 V if the current in the battery is (a) from the negative to the positive terminal, (b) from the positive to the negative terminal, and (c) zero?

SAMPLE PROBLEM 27.01

Single-loop circuit with two real batteries

The emfs and resistances in the circuit of Fig. 27-8*a* have the following values:

$$\mathscr{E}_1 = 4.4\text{ V},\ \mathscr{E}_2 = 2.1\text{ V},$$

$$r_1 = 2.3\ \Omega,\ r_2 = 1.8\ \Omega,\ R = 5.5\ \Omega.$$

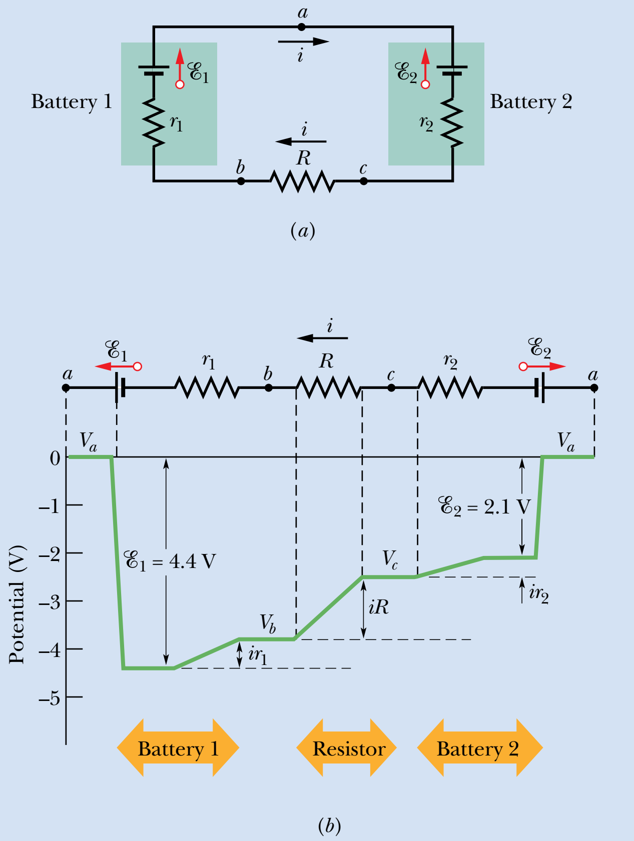
(a) What is the current i in the circuit?

KEY IDEA

We can get an expression involving the current i in this single-loop circuit by applying the loop rule, in which we sum the potential changes around the full loop.

Calculations: Although knowing the direction of i is not necessary, we can easily determine it from the emfs of the two batteries. Because $\mathscr{E}_1$ is greater than $\mathscr{E}_2$, battery 1 controls the direction of i, so the direction is clockwise. Let us then apply the loop rule by going counterclockwise—against the current—and starting at point a. (These decisions about where to start and which way you go are arbitrary but, once made, you must be consistent with decisions about the plus and minus signs.) We find

$$-\mathscr{E}_1 + ir_1 + iR + ir_2 + \mathscr{E}_2 = 0.$$



(*a*)

(*b*)

Figure 27-8 (*a*) A single-loop circuit containing two real batteries and a resistor. The batteries oppose each other; that is, they tend to send current in opposite directions through the resistor. (*b*) A graph of the potentials, counterclockwise from point *a*, with the potential at *a* arbitrarily taken to be zero. (To better link the circuit with the graph, mentally cut the circuit at *a* and then unfold the left side of the circuit toward the left and the right side of the circuit toward the right.)

Check that this equation also results if we apply the loop rule clockwise or start at some point other than a. Also, take the time to compare this equation term by term with Fig. 27-8*b*, which shows the potential changes graphically (with the potential at point a arbitrarily taken to be zero).

Solving the above loop equation for the current i, we obtain

$$i = \frac{\mathscr{E}_1 - \mathscr{E}_2}{R + r_1 + r_2} = \frac{4.4\text{ V} - 2.1\text{ V}}{5.5\ \Omega + 2.3\ \Omega + 1.8\ \Omega}$$

$$= 0.2396\text{ A} \approx 240\text{ mA}. \qquad \text{(Answer)}$$

(b) What is the potential difference between the terminals of battery 1 in Fig. 27-8*a*?

KEY IDEA

We need to sum the potential differences between points a and b.

Calculations: Let us start at point b (effectively the negative terminal of battery 1) and travel clockwise through battery 1 to point a (effectively the positive terminal), keeping track of potential changes. We find that

$$V_b - ir_1 + \mathscr{E}_1 = V_a,$$

which gives us

$$V_a - V_b = -ir_1 + \mathscr{E}_1$$

$$= -(0.2396\text{ A})(2.3\ \Omega) + 4.4\text{ V}$$

$$= +3.84\text{ V} \approx 3.8\text{ V}, \qquad \text{(Answer)}$$

which is less than the emf of the battery. You can verify this result by starting at point b in Fig. 27-8*a* and traversing the circuit counterclockwise to point a. We learn two points here. (1) The potential difference between two points in a circuit is independent of the path we choose to go from one to the other. (2) When the current in the battery is in the "proper" direction, the terminal-to-terminal potential difference is low, that is, lower than the stated emf for the battery that you might find printed on the battery.

PROBLEM-SOLVING TACTIC

Tactic 1: Assuming the Direction of a Current Can be an Arbitrary Decision In solving circuit problems, you do not need to know the direction of a current in advance. Instead, you can just assume its direction, although that may take some physics courage. To show this, assume the current in Fig. 27-8*a* is counterclockwise; that is, reverse the direction of the current arrows shown. Applying the loop rule counterclockwise from point *a* now yields

$$-\xi_1 - ir_1 - iR - ir_2 + \xi_2 = 0,$$

or

$$i = -\frac{\xi_1 - \xi_2}{R + r_1 + r_2}.$$

Substituting the numerical values of Sample Problem 27.01 yields $i = -240$ mA for the current. The minus sign is a signal that the current is opposite the direction we initially assumed.

27.5 | MULTILOOP CIRCUITS

Key Concept

- When resistances are in parallel, they have the same potential difference. The equivalent resistance that can replace a parallel combination of resistances is given by

$$\frac{1}{R_{\text{eq}}} = \sum_{j=1}^{n} \frac{1}{R_j} \quad (n \text{ resistances in parallel}).$$

Conservation of charge leads to the junction rule:

Junction Rule. The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.

Figure 27-9 shows a circuit containing more than one loop. For simplicity, we assume the batteries are ideal. There are two *junctions* in this circuit, at *b* and *d*, and there are three *branches* connecting these junctions. The branches are the left branch (*bad*), the right branch (*bcd*), and the central branch (*bd*). What are the currents in the three branches?

We arbitrarily label the currents, using a different subscript for each branch. Current i_1 has the same value everywhere in branch *bad*, i_2 has the same value everywhere in branch *bcd*, and i_3 is the current through branch *bd*. The directions of the currents are assumed arbitrarily.

Consider junction *d* for a moment: Charge comes into that junction via incoming currents i_1 and i_3, and it leaves via outgoing current i_2. Because there is no variation in the charge at the junction, the total incoming current must equal the total outgoing current:

$$i_1 + i_3 = i_2. \tag{27-18}$$

You can easily check that applying this condition to junction *b* leads to exactly the same equation. Equation 27-18 thus suggests a general principle:

The current into the junction must equal the current out (charge is conserved).

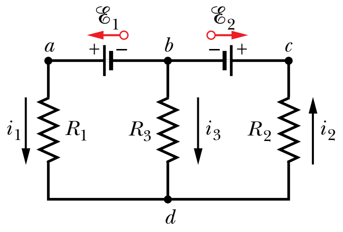


Figure 27-9 A multiloop circuit consisting of three branches: left-hand branch *bad*, righthand branch *bcd*, and central branch *bd*. The circuit also consists of three loops: left-hand loop *badb*, right-hand loop *bcdb*, and big loop *badcb*.

JUNCTION RULE: The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.

This rule is often called *Kirchhoff's junction rule* (or *Kirchhoff's current law*). It is simply a statement of the conservation of charge for a steady flow of charge—there is neither a buildup nor a depletion of charge at a junction.

Thus, our basic tools for solving complex circuits are the *loop rule* (based on the conservation of energy) and the *junction rule* (based on the conservation of charge).

Equation 27-18 is a single equation involving three unknowns. To solve the circuit completely (that is, to find all three currents), we need two more equations involving those same unknowns. We obtain them by applying the loop rule twice. In the circuit of Fig. 27-9, we have three loops from which to choose: the left-hand loop (*badb*), the right-hand loop (*bcdb*), and the big loop (*badcb*). Which two loops we choose does not matter—let's choose the left-hand loop and the righthand loop.

If we traverse the left-hand loop in a counterclockwise direction from point *b*, the loop rule gives us

$$\mathscr{E}_1 - i_1R_1 + i_3R_3 = 0. \tag{27-19}$$

If we traverse the right-hand loop in a counterclockwise direction from point *b*, the loop rule gives us

$$-i_3R_3 - i_2R_2 - \mathscr{E}_2 = 0. \tag{27-20}$$

We now have three equations (Eqs. 27-18, 27-19, and 27-20) in the three unknown currents, and they can be solved by a variety of techniques.

If we had applied the loop rule to the big loop, we would have obtained (moving counterclockwise from *b*) the equation

$$\mathscr{E}_1 - i_1R_1 - i_2R_2 - \mathscr{E}_2 = 0.$$

However, this is merely the sum of Eqs. 27-19 and 27-20.

Resistances in Parallel

Figure 27-10*a* shows three resistances connected *in parallel* to an ideal battery of emf $\mathscr{E}$. The term "in parallel" means that the resistances are directly wired together on one side and directly wired together on the other side, and that a potential difference V is applied across the pair of connected sides. Thus, all three resistances have the same potential difference V across them, producing a current through each. In general,

When a potential difference V is applied across resistances connected in parallel, the resistances all have that same potential difference V.

In Fig. 27-10*a*, the applied potential difference V is maintained by the battery. In Fig. 27-10*b*, the three parallel resistances have been replaced with an equivalent resistance R_{eq}.

Resistances connected in parallel can be replaced with an equivalent resistance R_{eq} that has the same potential difference V and the same *total* current i as the actual resistances.

You might remember that R_{eq} and all the actual parallel resistances have the same potential difference V with the nonsense word "par-V."

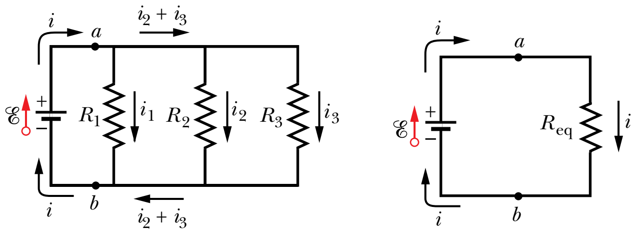
To derive an expression for R_{eq} in Fig. 27-10*b*, we first write the current in each actual resistance in Fig. 27-10*a* as

$$i_1 = \frac{V}{R_1}, \; i_2 = \frac{V}{R_2}, \quad \text{and} \quad i_3 = \frac{V}{R_3},$$

where V is the potential difference between *a* and *b*. If we apply the junction rule at point *a* in Fig. 27-10*a* and then substitute these values, we find

$$i = i_1 + i_2 + i_3 = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right). \tag{27-21}$$

Parallel resistors and their equivalent have the same potential difference ("par-V").



(a) (b)

Figure 27-10 (a) Three resistors connected in parallel across points a and b. (b) An equivalent circuit, with the three resistors replaced with their equivalent resistance R_{eq}.

If we replaced the parallel combination with the equivalent resistance R_{eq} (Fig. 27-10b), we would have

$$i = \frac{V}{R_{eq}}. \tag{27-22}$$

Comparing Eqs. 27-21 and 27-22 leads to

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \tag{27-23}$$

Extending this result to the case of n resistances, we have

$$\frac{1}{R_{eq}} = \sum_{j=1}^{n} \frac{1}{R_j} \quad (n \text{ resistances in parallel}). \tag{27-24}$$

For the case of two resistances, the equivalent resistance is their product divided by their sum; that is,

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}. \tag{27-25}$$

Note that when two or more resistances are connected in parallel, the equivalent resistance is smaller than any of the combining resistances. Table 27-1 summarizes the equivalence relations for resistors and capacitors in series and in parallel.

Table 27-1 Series and Parallel Resistors and Capacitors

Series	Parallel	Series	Parallel
Resistors		Capacitors	
$R_{eq} = \sum_{j=1}^{n} R_j$ Eq. 27-7	$\frac{1}{R_{eq}} = \sum_{j=1}^{n} \frac{1}{R_j}$ Eq. 27-24	$\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j}$ Eq. 25-20	$C_{eq} = \sum_{j=1}^{n} C_j$ Eq. 25-19
Same current through all resistors	Same potential difference across all resistors	Same charge on all capacitors	Same potential difference across all capacitors

CHECKPOINT 4

A battery, with potential V across it, is connected to a combination of two identical resistors and then has current i through it. What are the potential difference across and the current through either resistor if the resistors are (a) in series and (b) in parallel?

SAMPLE PROBLEM 27.02

Resistors in parallel and in series

Figure 27-11*a* shows a multiloop circuit containing one ideal battery and four resistances with the following values:

$$R_1 = 20\ \Omega,\ R_2 = 20\ \Omega,\ \mathscr{E} = 12\ \text{V},$$
$$R_3 = 30\ \Omega,\ R_4 = 8.0\ \Omega.$$

(a) What is the current through the battery?

KEY IDEA

Noting that the current through the battery must also be the current through R_1, we see that we might find the current by applying the loop rule to a loop that includes R_1 because the current would be included in the potential difference across R_1.

Incorrect method: Either the left-hand loop or the big loop should do. Noting that the emf arrow of the battery points upward, so the current the battery supplies is clockwise, we might apply the loop rule to the left-hand loop, clockwise from point *a*. With *i* being the current through the battery, we would get

$$+\mathscr{E} - iR_1 - iR_2 - iR_4 = 0 \quad \text{(incorrect).}$$

However, this equation is incorrect because it assumes that R_1, R_2, and R_4 all have the same current *i*. Resistances R_1 and R_4 do have the same current, because the current passing through R_4 must pass through the battery and then through R_1 with no change in value. However, that current splits at junction point *b*—only part passes through R_2, the rest through R_3.

Dead-end method: To distinguish the several currents in the circuit, we must label them individually as in Fig. 27-11*b*. Then, circling clockwise from *a*, we can write the loop rule for the left-hand loop as

$$+\mathscr{E} - i_1R_1 - i_2R_2 - i_1R_4 = 0.$$

Unfortunately, this equation contains two unknowns, i_1 and I_2; we would need at least one more equation to find them.

Successful method: A much easier option is to simplify the circuit of Fig. 27-11*b* by finding equivalent resistances. Note carefully that R_1 and R_2 are *not* in series and thus cannot be replaced with an equivalent resistance. However, R_2 and R_3 are in parallel, so we can use either Eq. 27-24 or Eq. 27-25 to find their equivalent resistance R_{23}. From the latter,

$$R_{23} = \frac{R_2R_3}{R_2 + R_3} = \frac{(20\ \Omega)(30\ \Omega)}{50\ \Omega} = 12\ \Omega.$$

We can now redraw the circuit as in Fig. 27-11*c*; note that the current through R_{23} must be i_1 because charge that moves through R_1 and R_4 must also move through R_{23}. For this simple one-loop circuit, the loop rule (applied clockwise from point *a* as in Fig. 27-11*d*) yields

$$+\mathscr{E} - i_1R_1 - i_1R_{23} - i_1R_4 = 0.$$

Substituting the given data, we find

$$12\ \text{V} - i_1(20\ \Omega) - i_1(12\ \Omega) - i_1(8.0\ \Omega) = 0,$$

which gives us

$$i_1 = \frac{12\ \text{V}}{40\ \Omega} = 0.30\ \text{A}. \qquad \text{(Answer)}$$

(b) What is the current i_2 through R_2?

KEY IDEAS

(1) We must now work backward from the equivalent circuit of Fig. 27-11*d*, where R_{23} has replaced R_2 and R_3. (2) Because R_2 and R_3 are in parallel, they both have the same potential difference across them as R_{23}.

Working backward: We know that the current through R_{23} is $i_1 = 0.30$ A. Thus, we can use Eq. 26-8 ($R = V/i$) and Fig. 27-11*e* to find the potential difference V_{23} across R_{23}. Setting $R_{23} = 12\ \Omega$ from (a), we write Eq. 26-8 as

$$V_{23} = i_1R_{23} = (0.30\ \text{A})(12\ \Omega) = 3.6\ \text{V}.$$

The potential difference across R_2 is thus also 3.6 V (Fig. 27-11*f*), so the current i_2 in R_2 must be, by Eq. 26-8 and Fig. 27-11*g*,

$$i_2 = \frac{V_2}{R_2} = \frac{3.6\ \text{V}}{20\ \Omega} = 0.18\ \text{A}. \qquad \text{(Answer)}$$

(c) What is the current i_3 through R_3?

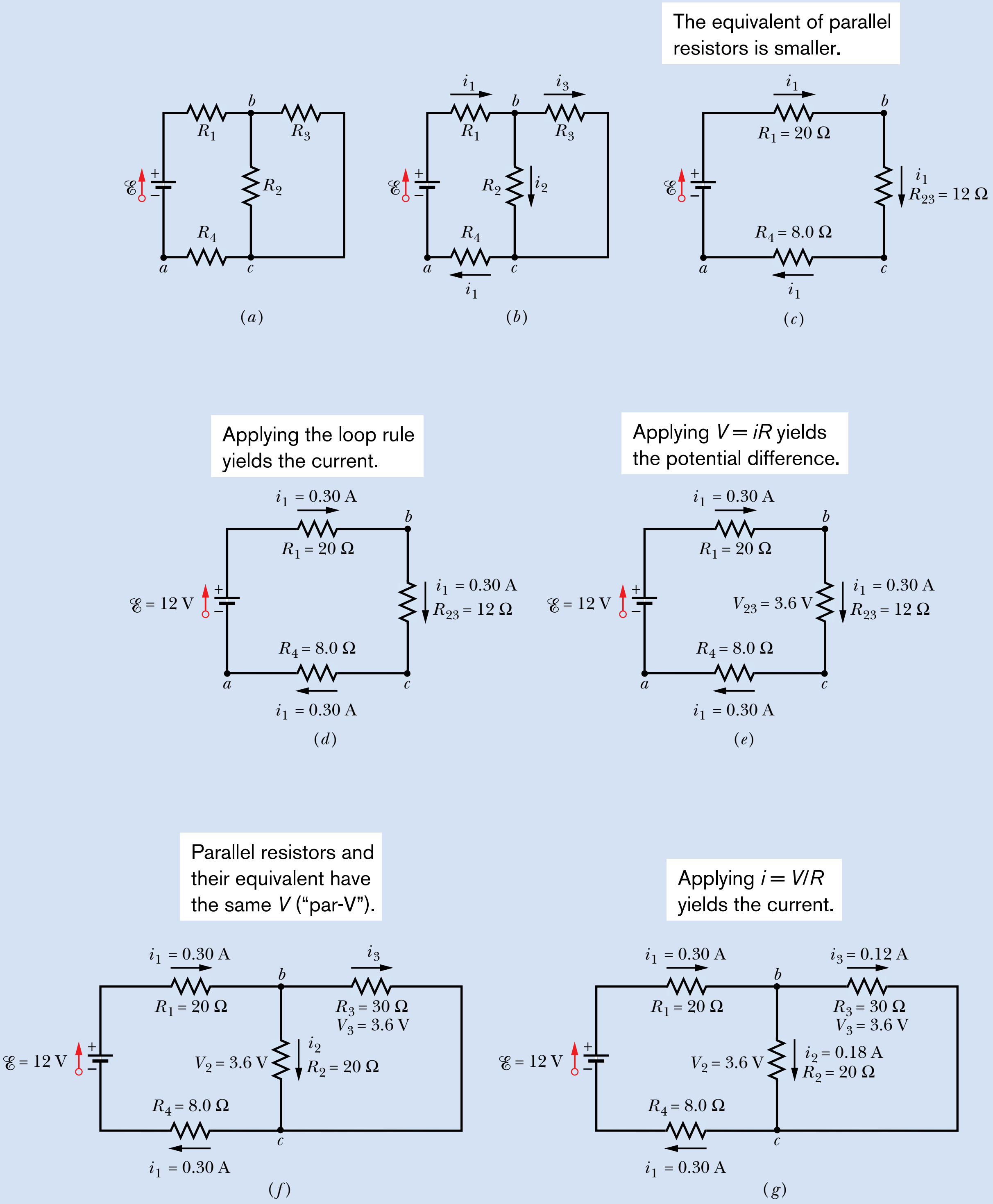
KEY IDEAS

We can answer by using either of two techniques: (1) Apply Eq. 26-8 as we just did. (2) Use the junction rule, which tells us that at point *b* in Fig. 27-11*b*, the incoming current i_1 and the outgoing currents i_2 and i_3 are related by

$$i_1 = i_2 + i_3.$$

Calculation: Rearranging this junction-rule result yields the result displayed in Fig. 27-11*g*:

$$\begin{aligned} i_3 &= i_1 - i_2 = 0.30\ \text{A} - 0.18\ \text{A} \\ &= 0.12\ \text{A}. \qquad \text{(Answer)} \end{aligned}$$



The equivalent of parallel resistors is smaller.

Applying the loop rule yields the current.

Applying $V = iR$ yields the potential difference.

Parallel resistors and their equivalent have the same V ("par-V").

Applying $i = V/R$ yields the current.

Figure 27-11 (*a*) A circuit with an ideal battery. (*b*) Label the currents. (*c*) Replacing the parallel resistors with their equivalent. (*d*)–(*g*) Working backward to find the currents through the parallel resistors.

SAMPLE PROBLEM 27.03

Multiloop circuit and simultaneous loop equations

Figure 27-12 shows a circuit whose elements have the following values: $\mathscr{E}_1 = 3.0$ V, $\mathscr{E}_2 = 6.0$ V, $R_1 = 2.0\ \Omega$, $R_2 = 2.0\ \Omega$. The three batteries are ideal batteries. Find the magnitude and direction of the current in each of the three branches.

KEY IDEAS

It is not worthwhile to try to simplify this circuit, because no two resistors are in parallel, and the resistors that are in series (those in the right branch or those in the left branch) present no problem. So, our plan is to apply the junction and loop rules.

Junction rule: Using arbitrarily chosen directions for the currents as shown in Fig. 27-12, we apply the junction rule at point *a* by writing

$$i_3 = i_1 + i_2. \qquad (27\text{-}26)$$

An application of the junction rule at junction *b* gives only the same equation, so we next apply the loop rule to any two of the three loops of the circuit.

Left-hand loop: We first arbitrarily choose the left-hand loop, arbitrarily start at point *b*, and arbitrarily traverse the loop in the clockwise direction, obtaining

$$-i_1R_1 + \mathscr{E}_1 - i_1R_1 - (i_1 + i_2)R_2 - \mathscr{E}_2 = 0,$$

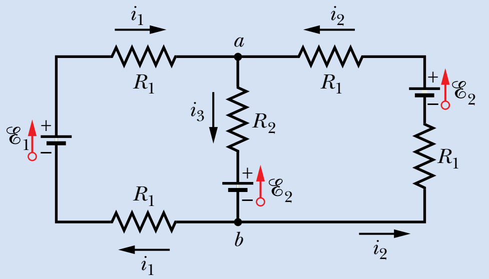


Figure 27-12 A multiloop circuit with three ideal batteries and five resistances.

where we have used $(i_1 + i_2)$ instead of i_3 in the middle branch. Substituting the given data and simplifying yield

$$i_1(8.0\ \Omega) + i_2(4.0\ \Omega) = -3.0\text{ V}. \qquad (27\text{-}27)$$

Right-hand loop: For our second application of the loop rule, we arbitrarily choose to traverse the right-hand loop counterclockwise from point *b*, finding

$$-i_2R_1 + \mathscr{E}_2 - i_2R_1 - (i_1 + i_2)R_2 - \mathscr{E}_2 = 0.$$

Substituting the given data and simplifying yield

$$i_1(4.0\ \Omega) + i_2(8.0\ \Omega) = 0. \qquad (27\text{-}28)$$

Combining equations: We now have a system of two equations (Eqs. 27-27 and 27-28) in two unknowns (i_1 and i_2) to solve either "by hand" (which is easy enough here) or with a "math package." (given in Appendix E.) We find

$$i_1 = -0.50\text{ A}. \qquad (27\text{-}29)$$

(The minus sign signals that our arbitrary choice of direction for i_1 in Fig. 27-12 is wrong, but we must wait to correct it.) Substituting $i_1 = -0.50$ A into Eq. 27-28 and solving for i_2 then give us

$$i_2 = 0.25\text{ A}. \qquad \text{(Answer)}$$

With Eq. 27-26 we then find that

$$i_3 = i_1 + i_2 = -0.50\text{ A} + 0.25\text{ A}$$
$$= -0.25\text{ A}.$$

The positive answer we obtained for i_2 signals that our choice of direction for that current is correct. However, the negative answers for i_1 and i_3 indicate that our choices for those currents are wrong. Thus, as a *last step* here, we correct the answers by reversing the arrows for i_1 and i_3 in Fig. 27-12 and then writing

$$i_1 = 0.50\text{ A} \quad \text{and} \quad i_3 = 0.25\text{ A}. \qquad \text{(Answer)}$$

Caution: Always make any such correction as the last step and not before calculating *all* the currents.

Another useful method here is nodal analysis. A node is another name for a junction or a point in a circuit. For example, in Sample Problem 27.03, points *a* and *b* are the nodes. The first step in nodal analysis is to choose a reference node. Recall that in Chapter 24 we stated that any point in space can be taken to be at zero potential and all the other potentials measured with respect to it. Here we choose a reference node and assign it zero potential. So, the node voltage is the potential difference between the node and the reference node. If we do not know the potential difference, we assign an arbitrary potential to the node. The current through all the resistances connected to the node can then be calculated and we can then use Kirchhoff's current law to evaluate the unknown potential.

Let us illustrate this method by solving Sample Problem 27.03 by this method (Fig. 27-12). Here we note that the resistance R_1 at the lower left corner is like an internal resistance of the battery ξ_1. There will be no change in the currents if we swap the battery and R_1 to simplify the circuit a little. Similarly, the circuit will not change if we do the same with the right-hand side of the circuit. This new figure is shown in Fig. 27-13.

In this new circuit, we will assign zero potential to point b and potential V to point a. We can see that there are two nodes in the circuit, and applying Kirchhoff's first law to the node a will yield a relation that can be solved to get the potential at point a. From that we can find the current through each branch.

The negative terminal of battery ξ_1 is connected to b. So, at the lower end of resistance R_2 the potential is 6 V. Similarly, we have marked out the potentials at different points of the circuit as shown in Fig. 27-14.

We do not know which point is at a higher potential, point a or any of the other three points connecting it to batteries. So, we arbitrarily assume that a is at a higher potential so that current will be flowing away from node a through each of the three resistances.

Kirchhoff's junction rule yields $i_1 + i_2 + i_3 = 0$; therefore

$$\frac{V-3}{2R_1} + \frac{V-6}{R_2} + \frac{V-6}{2R_1} = 0.$$

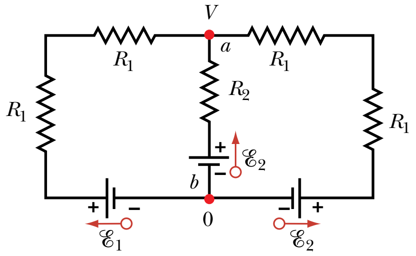


Figure 28-13 Circuit 27-12 rearranged.

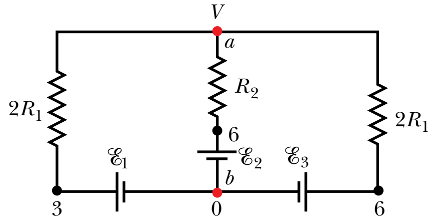


Figure 27-14 Potential at the nodes.

Each of the terms represents the current in the left branch, middle branch and right branch, respectively. Solving this, we get

$$V = 5 \text{ volts.}$$

So, the current in each of the branch is

$$i_1 = \frac{V-3}{2R_1} = 0.5 \text{ A},$$

$$i_2 = \frac{V-6}{R_2} = -0.25 \text{ A},$$

$$i_3 = \frac{V-6}{2R_1} = -0.25 \text{ A},$$

which is the same as we obtained in Sample Problem 27.03 using a lengthier method.

Note that even if we had assumed point a to be at potential V_1 and point b to be at a potential V_2, the answer would be the same because in a circuit current flows through the resistance only on account of the potential difference. The negative current implies that we chose a wrong direction for the current in that branch. The current will flow in that branch toward node b, not away from it.

Wheatstone Bridge

We know that Wheatstone bridge is a simple circuit in the form of a quadrilateral, used for measuring an unknown resistance by connecting it to three known resistances and applying a voltage between a pair of opposite corners. As shown in Fig. 27-15, the equivalent circuit diagram of a Wheatstone bridge consists of three known resistances P, Q and R and the unknown resistance X, connected to form the four sides of a quadrilateral. A battery is connected across the junctions of P and R and the junctions of Q and X, while a galvanometer connected across the junctions P–Q and R–X completes the circuit.

In general, P and Q, which are said to constitute the ratio arms, may be varied within the simple limits $P = Q$, $P = 10Q$ or $P = 100Q$, respectively. The relation is reciprocal, that is, Q may be made equal to $10P$ or $100P$; the basic value when $P = Q$ is usually 10 Ω. The resistance R is variable over a wide range, usually from 1 Ω to several thousand ohms. Any sensitive galvanometer is suitable.

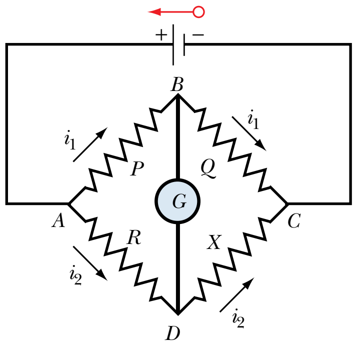


Figure 27-15 Circuit diagram for multi-loop circuit consisting of a network of resistors connected across a battery. The network of resistors includes a variable resistor R, the resistance of which can be adjusted. There is also a galvanometer connected in the middle of the network, which shows deflection if the value of the variable resistor is such that nonzero current flows through the galvanometer. This type of circuit, called a Wheatstone bridge, is used to determine the value of the unknown resistor X by adjusting the variable resistor to such a value that no current flows through the galvanometer.

The action of the instrument depends on the adjustment of the value of R until there is no deflection in the galvanometer. This null condition is realized when no current flows in the galvanometer, that is, when the potential at B equals the potential of D. From Fig. 27-15 we have

$$V_{AB} = i_1 P,\ V_{AD} = i_2 R.$$

Now the potential of A is common to both P and R, and if no current flows in the galvanometer, the potentials of B and D are equal. Hence,

$$V_{AB} = V_{AD};$$

and therefore,

$$i_1 P = i_2 R. \tag{27-30}$$

Similarly, for the resistance Q and X,

$$V_{BC} = V_{DC},\ i_1 Q = i_2 X. \tag{27-31}$$

Hence, finally from Eqs. 27-30 and 27-31, we have

$$\frac{Q}{P} = \frac{X}{R}. \tag{27-32}$$

From which X is easily determined if P, Q, and R are known.

Grouping of Batteries

Series Connection

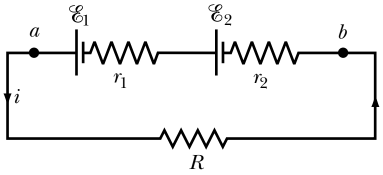


Figure 27-16 Connecting two batteries in series.

Suppose two batteries having emfs ξ_1 and ξ_2 and internal resistances r_1 and r_2 are connected in series as shown in Fig. 27-16. The points a and b act as the terminals of the combination. Suppose an external resistance R is connected across the combination. From Kirchhoff's loop law,

$$Ri + r_2 i - \xi_2 + r_1 i - \xi_1 = 0.$$

or

$$i = \frac{\xi_1 + \xi_2}{R + (r_1 + r_2)} = \frac{\xi}{R + r_0},$$

where i is the current through the resistance R.

We see that the combination acts as a battery of emf $\xi_0 = \xi_1 + \xi_2$ having an internal resistance $r_0 = r_1 + r_2$. If the polarity of one of the batteries is reversed, the equivalent emf will be $|\xi_1 - \xi_2|$.

Parallel Connection

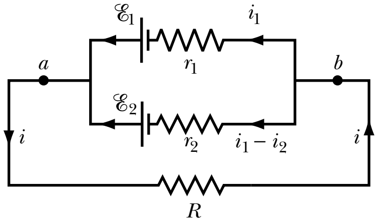


Figure 27-17 Connecting two batteries in parallel.

Now suppose the batteries are connected in parallel as shown in Fig. 27-17. The currents are also shown in the figure. Applying Kirchhoff's loop law in the loop containing ξ_1, r_1, and R,

$$iR + r_1 i_1 - \xi_1 = 0. \tag{27-33}$$

Similarly, applying Kirchhoff's law in the loop containing ξ_2, r_2, and R, we get

$$iR + r_2(i - i_1) - \xi_2 = 0. \tag{27-34}$$

By multiplying Eq. 27-33 by r_2, Eq. 27-34 by r_1, and then adding, we get

$$iR(r_1 + r_2) + r_1 r_2 i - \xi_1 r_2 - \xi_2 r_1 = 0,$$

or

$$i = \frac{\xi_1 r_2 + \xi_2 r_1}{R(r_1 + r_2) + r_1 r_2} = \frac{(\xi_1 r_2 + \xi_2 r_1)/(r_1 + r_2)}{R + (r_1 r_2)/(r_1 + r_2)} = \frac{\xi_0}{R + r_0}. \tag{27-35}$$

We see that combination acts as a battery of emf

$$\xi_0 = \frac{\xi_1 r_2 + \xi_2 r_1}{r_1 + r_2}$$

and internal resistance

$$r_0 = \frac{r_1 r_2}{r_1 + r_2}.$$

SAMPLE PROBLEM 27.04

Equivalent resistance in a symmetric loop

Twelve wires, each having resistance r, are joined to form a cube as shown in Fig. 27-18. Find the equivalent resistance between the ends of a face diagonal such as a and c.

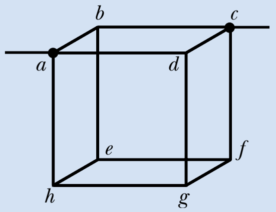


Figure 27-18 Twelve equal resistors are arranged in the shape of a cube. The battery is connected between points a and c.

KEY IDEAS

This circuit is quite complicated and use of nodal analysis or loop law will yield a lot of equations which cannot be solved so easily. But the circuit is symmetric and symmetry considerations will help us in simplifying the circuit.

Suppose a potential difference V is applied between the points a and c so that a current i enters at a and the same current leaves at c. The current distribution is shown in Fig. 27-19*a*.

By symmetry, the paths *ad* and *ab* are equivalent and hence will carry the same current i_1. The path *ah* will carry the remaining current $i - 2i_1$ (using Kirchhoff's junction law). Similarly, at junction c, currents coming from *dc* and *bc* will be i_1 each and from *fc* will be $i - 2i_1$. Kirchhoff's junction law at b and d shows that current through *be* and *dg* will be zero and hence may be ignored for further analysis. Omitting these two wires, the circuit is redrawn in Fig. 27-19*b*.

Calculations: The wire *hef* and *hgf* are joined in parallel and have equivalent resistance $(2r)(2r)/(2r + 2r) = r$ between h and f. This equivalent is joined in series with *ah* and *fc* giving equivalent resistance $r + r + r = 3r$. This $3r$ is joined in parallel with *adc* $(2r)$ and *abc* $(2r)$ between a and c.

The equivalent resistance R between a and c is, therefore, given by

$$\frac{1}{R} = \frac{1}{3r} + \frac{1}{2r} + \frac{1}{2r},$$

or

$$R = \frac{3}{4}r.$$

In many of the situations, we can considerably simplify the circuit by guessing the branches in which the current is the same.

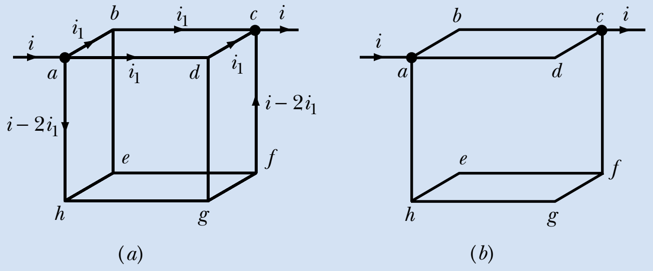


Figure 27-19 (*a*) Current distribution in the cube and (*b*) equivalent circuit.

SAMPLE PROBLEM 27.05

Symmetry of a different kind

Find the equivalent resistance between points *a* and *b* of the circuit shown in Fig. 27-20.

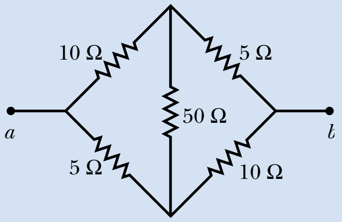


Figure 27-20 A network of five resistances.

KEY IDEA

Here we can imagine the circuit to be connected to a battery of 10 V (Chosen arbitrarily). Since resistances follow ohm's law, the current through the circuit is proportional to the applied voltage, the constant of proportionality will be $1/R_{\text{eq}}$.

Caution: Do not mistake this to be an example of Wheatstone bridge. Note here that

$$\frac{P}{Q} \neq \frac{R}{X}.$$

Instead we use nodal analysis to find the current through the source. Let's assume the right end of battery to be at zero potential. Other potentials are as indicated in Fig. 27-21.

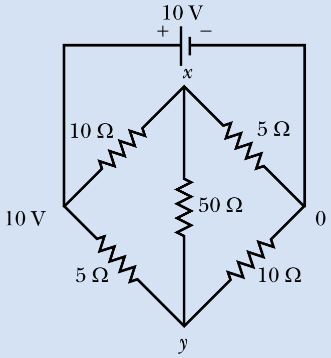


Figure 27-21 Circuit for nodal analysis.

Calculations: For applying Kirchhoff's junction rule at the node *x*, we assume that *x* is greater than 10 V as well as *y*.

$$\frac{x-10}{10} = i_{10\Omega}, \quad \frac{x-y}{5} = i_{5\Omega}, \quad \frac{x-0}{5} = i_{5\Omega}.$$

By summing up and equating to zero, we get

$$5x - 2y = 10. \qquad (27\text{-}36)$$

Let us now apply the Kirchhoff's junction rule at the node *y*. We get

$$\frac{y-10}{5} = i_{5\Omega}, \quad \frac{y-x}{5} = i_{5\Omega}, \quad \frac{y-10}{10} = i_{10\Omega}$$

By summing up and equating to zero, we get

$$5y - 2x = 20. \qquad (27\text{-}37)$$

By multiplying Eq. 27-36 by 2 and Eq. 27-37 by 5 and then adding, we get

$$21y = 120 \quad \Rightarrow \quad y = \frac{40}{7}\ \text{V}.$$

Substitute $x = (30/7)$ V. Thus, current through the battery is equal to

$$i_{10\Omega} + i_{5\Omega} = \frac{10-(30/7)}{10} + \frac{10-(40/7)}{5} = \frac{10}{7}\ \text{A}.$$

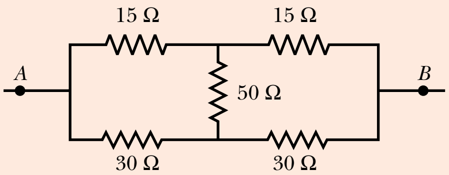
Thus,

$$R_{\text{eq}} = \frac{V}{i} = 7\ \Omega. \qquad \text{(Answer)}$$

Note: An interesting point that can be noted is that the potential difference across both the 10 Ω resistance is equal (10 – 30/7) and (40/7 – 0). This is no coincidence. Suppose we were to reverse the terminals of the battery. The current, instead of entering from a into left 10 Ω resistor and 5 Ω resistor would enter from right 10 Ω resistor and 5 Ω resistor. The magnitude of potential difference across each of the resistors would remain the same. Also, the circuit has a certain symmetry: the resistors at the input are the same as resistors at the output. We can safely conclude that the current through these resistors would be the same. This kind of symmetry is known as "input–output symmetry." When such symmetry occurs, we need to write the potential at one node, say *x* and the potential at the other node would be 10 – *x*. This indeed eases the problem at hand. So, we can solve such a circuit using only one equation.

CHECKPOINT 5

In the adjoining circuit, find the branches in which the current is the same. Hence, redraw the circuit so that we can find the equivalent resistance of the circuit by simple series parallel analysis.



27.6 | ELECTRICITY-RELATED INSTRUMENTS

Key Concepts

- Galvanometer is a device used for detecting currents. There are three measurement instruments used with circuits: An ammeter measures current. A voltmeter measures voltage (potential differences). A multimeter can be used to measure current, voltage, or resistance.
- Current-limiting rheostats are used to vary and control a current.
- Wheatstone bridges in different forms are used to find unknown resistance.
- Potentiometer is a device that allows measurement of potential without taking any current from the system.

Galvanometer shown in Fig. 27-22 is a device used for detecting currents. Normally, commercially available galvanometers are very sensitive instruments and can detect very small currents and also detect currents flowing in either direction. When connected suitably to a high resistance or a low resistance, it can act as a voltmeter or an ammeter.

Ammeter and Voltmeter

Ammeter is a device to measure an electric current and **voltmeter** is a device to measure a potential difference. In both the instruments there is a coil, suspended between the poles of a magnet as shown in Fig. 27-22. When a current is passed through the coil, it deflects. The angle of deflection is proportional to the current going through the coil. A needle is fixed to the coil. When the coil deflects, the needle moves on a graduated scale.

Ammeter

In an ammeter, a resistor having a small resistance is connected in parallel with the coil. This resistor is called the *shunt*. The current to be measured is passed through the ammeter by connecting it in series with the segment which carries the current. Plus and minus signs are marked near the terminals of the ammeter. The current should enter the ammeter through the terminal marked "plus." When no current passes through the ammeter, the needle stays at zero which is marked at the left end of the scale. The equivalent circuit is shown in Fig. 27-23.

Suppose the coil has a resistance R_c and the small resistance connected in parallel (shunt) has a value r. When a current i is sent through the ammeter, the current gets divided into two parts. A part i_1 goes through the coil and the rest, $i - i_1$, through the shunt. As the potential difference across R_c is the same as that across r,

$$i_1 R_c = (i - i_1) r,$$

or

$$i_1 = \frac{r}{R_c + r} i. \tag{27-38}$$

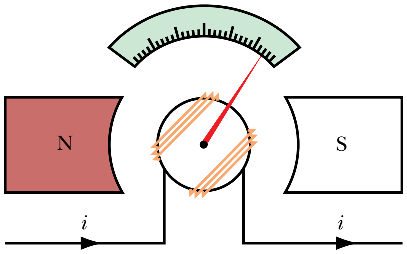


Figure 27-22 A typical galvanometer.

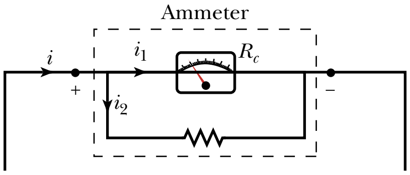


Figure 27-23 The equivalent circuit of an ammeter. The dotted lines indicate the ammeter.

The deflection is proportional to i_1 and hence to i. The scale is graduated to read the value of i directly.

The equivalent resistance of an ammeter is given by

$$R_{\text{eq}} = \frac{R_c r}{R_c + r}.$$

When the ammeter is connected in a segment of a circuit, the resistance of the segment increases by this amount R_{eq}. This reduces the main current which we wish to measure. To minimize this error, the equivalent resistant R_{eq} should be small. This is one reason why the shunt having a small resistance r is connected in parallel to the coil. This reduces the value of R_{eq}.

Voltmeter

In a voltmeter, a resistor having a high resistance R is connected in series with the coil. The end points (terminals) are connected to the points A and B between which the potential difference is to be measured. Plus and minus signs are marked on the terminals. The terminal marked "plus" should be connected to the point at higher potential. When no potential difference is applied between the terminals, the needle stays at zero which is marked at the left extreme of the scale. Its equivalent circuit is shown in Fig. 27-24.

When the potential difference is applied, current passes through the coil and the high resistance. If R_c be the resistance of the coil and V be the potential difference applied to the voltmeter, the current in the coil is

$$i = \frac{V}{R_c + R}.$$

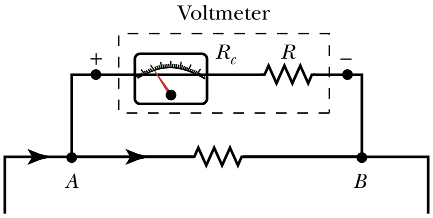


Figure 27-24 The equivalent circuit of a voltmeter. The dotted lines indicate the voltmeter.

The deflection is proportional to the current i and hence to V. The scale is graduated to read to potential difference directly.

When the voltmeter is used in a circuit, its resistance $R_{\text{eq}} = R_c + R$ is connected in parallel to the element of the circuit across which we want to measure the potential difference. This changes the overall current in the circuit and hence, the potential difference to be measured is also changed. To minimize the error due to this, the equivalent resistance R_{eq} of the voltmeter should be large. (When a large resistance is connected in parallel to a small resistance, the equivalent resistance is only slightly less than the smaller one.) That is why, a large resistance R is added in series with the coil of a voltmeter.

Often a single meter is packaged so that, by means of a switch, it can be made to serve as either an ammeter or a voltmeter—and usually also as an *ohmmeter*, designed to measure the resistance of any element connected between its terminals. Such a versatile unit is called a *multimeter*.

CHECKPOINT 6

A galvanometer has a coil of resistance 100 Ω showing a full-scale deflection at 50 μA. What resistance should be added to use it as (a) a voltmeter of range 50 V and (b) an ammeter of range 10 mA?

Current-Limiting Rheostats

In many cases it is necessary to vary and control a current. The nature of the variation required determines the type of resistance to choose for the purpose. Resistances designed for this purpose are called rheostats.

In general, we can divide rheostats into two classes: slide-wire types as shown in Fig. 27-25 and carbon compression types. The first kind is used when the range of variation is to be large and, for practical purposes, it is satisfactory if the resistance change occurs in very small jumps. Slide-wire rheostats are used nowadays almost exclusively. When large amounts of current have to be controlled and finer continuous adjustment is necessary, a carbon rheostat is used. The resistance change can be made practically infinitesimal by changing the pressure on the blocks slightly.

Most of the variable rheostats have connections as shown in Fig. 27-26. When the instrument is connected into a circuit at *A* and *C*, we have a variable rheostat. In this connection, the rheostat is acting as a current controller.

Another use of rheostat is in acting as a potential divider. To demonstrate the use of rheostat as a potential divider, we are showing here an experiment which is used to calibrate a voltmeter to read from 0 to 1.5 V. Figure 27-27 shows the connections. Above the dotted line we have our home-made voltmeter, consisting of galvanometer and a variable high resistance R'. We shall calibrate this voltmeter by comparison with a standard voltmeter V. The 150 Ω rheostat serves as a potential divider. By varying the position of the slider, the potential difference applied to the two voltmeters can be varied from 0 to 2 V. Adjust the Rheostat so that the standard voltmeter reads 1.5 V. Adjust R' so that G reads full scale. Now reduce the current in the main circuit and take readings on both instruments for about 10 points on the scale. In taking these readings set the standard voltmeter as exactly as possible on a division line and read the galvanometer (estimating to tenths of a division). Calibrate for various points on the scale, tabulating your results.

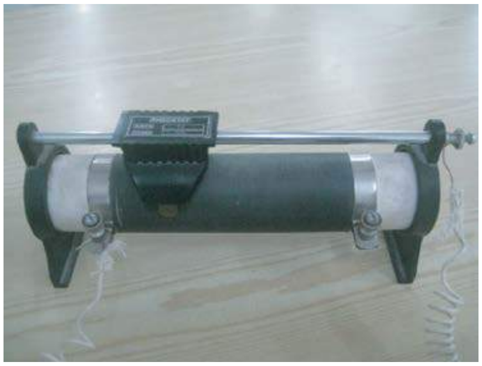


Figure 27-25 A rheostat.

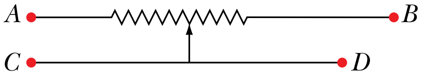


Figure 27-26 Rheostat.

Verification of Ohm's law

Perhaps the simplest way of measuring the resistance of a wire is to connect it in series with an ammeter and a cell and place a voltmeter across the resistance as shown in Fig. 27-28. Then application of Ohm's law gives $R = \xi/i$, where ξ is the reading on the voltmeter and i is the reading on the ammeter. The result can only be approximately correct, because when the voltmeter is placed in parallel with the resistance alone as in Fig. 27-28, the current i registered by the ammeter is only partially going through the resistance, and partly going through the voltmeter itself. Strictly Ohm's law would require $R = \xi/iR$, where iR is the current in the resistance. If, however, iR is determined by placing the voltmeter across the ammeter and resistance together as in Fig. 27-29, the voltage ξ is now no longer that across the resistance alone but across ammeter and resistance together. However, the lower the resistance of the ammeter and the higher the resistance of the voltmeter, the more accurate is the result, and if an ideal voltmeter is used, an accurate value for the resistance may be obtained, as the ideal voltmeter takes no current.

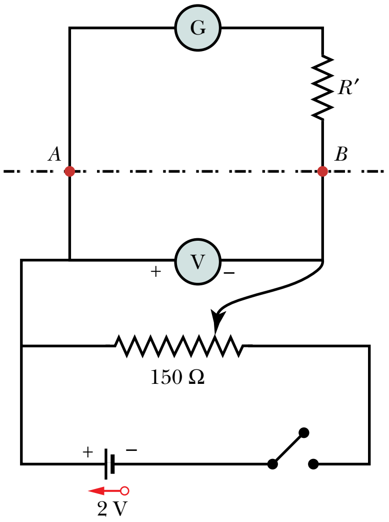


Figure 27-27 Calibration of a voltmeter.

Use of Wheatstone Bridge to Find the Unknown Resistance

Very accurate values of resistance may be obtained with a special arrangement of resistances termed as the Wheatstone bridge that we discussed earlier in this chapter.

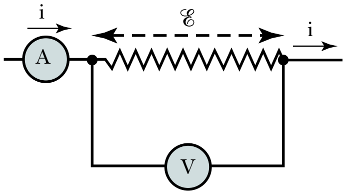


Figure 27-28 Approximate measurement of resistance placing a voltmeter across the resistance.

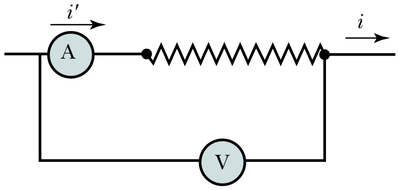


Figure 27-29 Approximate measurement of resistance placing the voltmeter across the ammeter and resistance together.

In using the bridge, the resistance R is changed until ideally the galvanometer shows no deflection. In practice, for values of R slightly greater than a certain amount, the galvanometer will generally show a small deflection in one direction, whereas for values of R slightly less than the same amount, the galvanometer will be deflected in the reverse direction. As we said in Section 27.5 (Eq. 27-32),

$$\frac{Q}{P} = \frac{X}{R}.$$

Post Office Box and Meter Bridge

The Wheatstone bridge is available in many different forms, two of which are **the post office box** and **the meter bridge**. The former, which is shown diagrammatically in Fig. 27-30, is fitted with two tapping keys which are situated in the battery and galvanometer circuits, respectively. Incidentally, the bridge is symmetrical in that the battery and galvanometer may be interchanged without affecting the balance point. Owing to the existence of electromagnetic induction (see Chapter 30), the battery circuit should be closed before the galvanometer circuit, so that steady current conditions are already established in the circuit.

The resistances in the box are usually non-inductively wound to reduce induction effects to a minimum. In any case, it is the final steady deflection of the galvanometer which is observed and not any temporary kick which may occur when the keys are depressed (see Chapter 30). Since resistance varies with temperature, only small currents are permissible, and boxes are usually rated at a definite maximum current which can be carried with safety. This should on no account be exceeded or the resistances may be permanently damaged and acquire values differing from their normal amounts. Figure 27-30 is same as that represented by the circuit diagram in Fig. 27-15.

If the ratio arms are equal, so that $P = Q$, from Eq. 27-32, $R = X$ for a balance to be obtained, so X will usually lie somewhere between two integral values of R differing by 1 Ω.

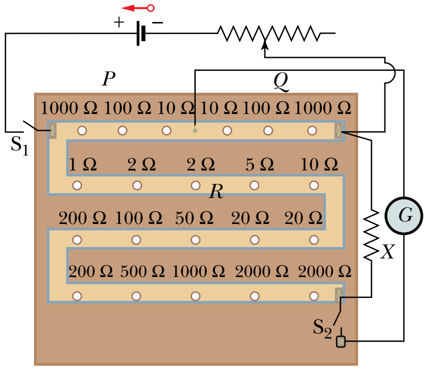

+
−
P
Q
1000 Ω 100 Ω 10 Ω 10 Ω 100 Ω 1000 Ω
S_1
1 Ω 2 Ω 2 Ω 5 Ω 10 Ω
R
200 Ω 100 Ω 50 Ω 20 Ω 20 Ω
G
X
200 Ω 500 Ω 1000 Ω 2000 Ω 2000 Ω
S_2

Figure 27-30 Post office box.

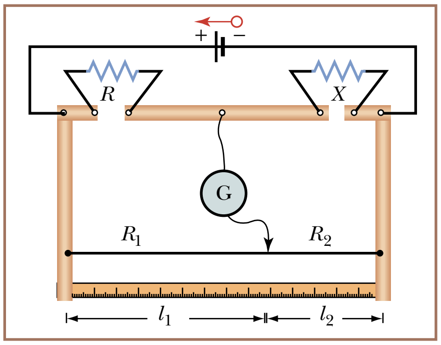


Figure 27-31 Meter bridge. Here the switch in the galvanometer branch, the switch in series with the battery and the rheostat are omitted for simplicity.

When an approximate value for X has been obtained as above, the ratio arms are changed so that $P = 10Q$. The value of R, which now gives a balance, from Eq. 28-32 is equal to $10X$. Hence, the accuracy of determination of X is carried to one-tenth of an ohm. By further increasing the ratio to $P = 100Q$, the value of $R = 100X$, and X is determined to one-hundredth of an ohm. Of course, the value of X must be such that the resistances available for R in the given instrument are sufficient to cover the range $R = 100X$, or alternatively $R = X/100$ if the ratio arms are reversed. Theoretically, by having greater ratio arms the accuracy could be extended indefinitely. In practice, the introduction of very large resistances reduces the current in the circuit to such an extent that the galvanometer sensitivity becomes insufficient to respond to small changes in R. So, a natural limit is set to the accuracy obtainable. In general, R can be varied by steps of 1 Ω or larger so that the value of X can be found only within certain accuracy.

The **meter bridge** or **slide-wire bridge** was devised to allow a continuous change in the value of the resistances instead of only integral values such as the post office box affords. As shown in Fig. 27-31, it consists of a rectangular board to which a meter rule is attached, stretched beside which is a wire of about 1 m long. The wire is made of some resistance material, such as Eureka alloy, so that the whole wire has a resistance of about 1 Ω. The ends of the wire are soldered to terminals which are in turn connected to stout copper strips of negligible resistance. These strips are provided with two gaps in which a known resistance R and the unknown resistance X are inserted, respectively. A third copper strip joins R and X, and a third terminal is attached to the center of this strip. This terminal is joined by a wire to a galvanometer, the other lead of which goes to a metal slider or jockey which slides along the bare wire of the meter bridge. The battery is joined to the extremities of the bridge wire, and the whole constitutes the usual Wheatstone bridge. A balance is obtained by sliding the jockey until no deflection is shown in the galvanometer. If the two resistances into which the bridge wire is then divided are R_1 and R_2, we have by Eq. 27-32

$$\frac{R_1}{R_2} = \frac{R}{X}.$$

The wire is assumed to be of uniform cross section; hence, the resistance of any portion of it is proportional to the length. Therefore $R_1 \propto l_1$ and $R_2 \propto l_2$. So it follows from the above that

$$X = R\frac{l_2}{l_1}.$$

If λ is the resistance of the wire of length 1 m, we can write by Eq. 27-32

$$\frac{X}{R} = \frac{\lambda(1-l)}{\lambda l}.$$

The value of distance "l" at which the current in the galvanometer is zero is known as null point. The error in the value of "X" will be minimum when the null point is obtained at the center of meter bridge. To show this, we assume that the resistance of the left and right copper strips is each equal to 'α'. If the null point is obtained at the center, then

$$\frac{X+\alpha}{R+\alpha} = \frac{1-0.5}{0.5} = 1.$$

Then the effect of "α" gets canceled. But if null point is not at the center, there will be some error because X will then not be equal to R. Another reason for this is related to the sensitivity of the instrument. Let us apply Eq. 27-32 to analyze for the indeterminate error in the unknown resistance X:

$$X = R\frac{(1-l)}{l}$$

$$\frac{dX}{X} = \frac{dR}{R} - \frac{dl}{1-l} - \frac{dl}{l} = \frac{dR}{R} - \frac{dl}{(1-l)l}.$$

The error in R and dR is zero because R is known accurately. The error in X and dX is minimum when denominator is maximum.

$$\frac{dX}{X} = -\frac{dl}{(1-l)l}.$$

We know that this is minimum when $l = 0.5$ m.

Potentiometer

In Section 27.3, we established that a real battery always has some internal resistance. So, if we want to measure the emf of a real battery, we cannot use a voltmeter. The ordinary moving-coil voltmeter is fairly satisfactory if its resistance is high, but clearly it will cause great disturbance if it is introduced across a circuit in which resistances of several thousand ohms are present. The potentiometer is a device which allows the measurement of potential without taking any current from the system. So, it can be used to measure the emf of a real battery also.

In construction, the potentiometer is identical with the meter bridge, which is often used as a potentiometer, except that the resistances R and X are omitted, these spaces being left unoccupied. The instrument therefore consists, in its simplest form, of a wire stretched beside a meter scale and connected to a steady supply battery (ξ_p) as shown in Fig. 27-32.

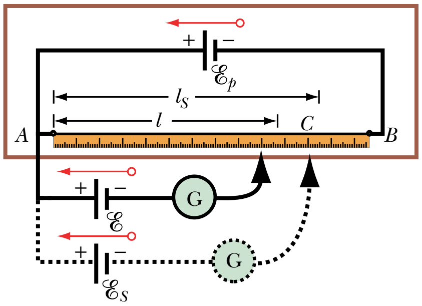


Figure 27-32 Potentiometer.

The potentiometer will only compare voltages (potential differences, emf), that is, a standard voltage is first necessary to calibrate the potentiometer wire. Such a standard voltage is provided by the emf of a standard cell. We denote this by ξ. The positive pole of the supply battery (ξ_p) is connected to that end of the potentiometer wire to which the positive pole of the standard cell is attached. The negative pole of the standard cell is connected through a sensitive galvanometer to a jockey sliding on the potentiometer wire. Now at certain positions of the jockey, it will be found that the galvanometer shows no deflection. This means that the drop in potential between the common point A and the point of contact C of the jockey, due to the current supplied from the supply battery (ξ_p) to the bridge wire, is exactly equal to the potential ξ_s between the poles of the standard cell. Remember that, by definition, the emf of a battery is the potential difference between its poles when no current is flowing through it.

To determine any other potential, therefore, the source to be measured simply replaces the standard cell and the new balance point l is obtained. Then from Eq. 27-39, its emf ξ can be measured. Since the wire is uniform, the drop ξ in potential across any length l is known by simple ratio,

$$\frac{\xi}{\xi_S} = \frac{l}{l_S}, \tag{27-39}$$

where l_S is the distance AC.

The voltage to be measured must not exceed that of the supply battery (ξ_p); otherwise no balance point will be found anywhere along the wire. Commercial potentiometers are available for measuring potentials over a very wide range.

27.7 | *RC* CIRCUITS

Key Concepts

- When an emf $\mathscr{E}$ is applied to a resistance R and capacitance C in series, the charge on the capacitor increases according to

$$q = C\mathscr{E}(1 - e^{-t/RC}) \quad \text{(charging a capacitor)},$$

in which $C\mathscr{E} = q_0$ is the equilibrium (final) charge and $RC = \tau$ is the capacitive time constant of the circuit.

- During the charging, the current is

$$i = \frac{dq}{dt} = \left(\frac{\mathscr{E}}{R}\right)e^{-t/RC} \quad \text{(charging a capacitor)}.$$

- When a capacitor discharges through a resistance R, the charge on the capacitor decays according to

$$q = q_0 e^{-t/RC} \quad \text{(discharging a capacitor)}.$$

- During the discharging, the current is

$$i = \frac{dq}{dt} = -\left(\frac{q_0}{RC}\right)e^{-t/RC} \quad \text{(discharging a capacitor)}.$$

In preceding sections we dealt only with circuits in which the currents did not vary with time. Here we begin a discussion of time-varying currents.

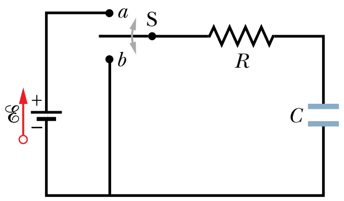


Figure 27-33 When switch S is closed on *a*, the capacitor is *charged* through the resistor. When the switch is afterward closed on *b*, the capacitor *discharges* through the resistor.

Charging a Capacitor

The capacitor of capacitance C in Fig. 27-33 is initially uncharged. To charge it, we close switch S on point a. This completes an *RC series circuit* consisting of the capacitor, an ideal battery of emf $\mathscr{E}$, and a resistance R.

We already know that as soon as the circuit is complete, charge begins to flow (current exists) between a capacitor plate and a battery terminal on each side of the capacitor. This current increases the charge q on the plates and the potential difference V_C ($= q/C$) across the capacitor. When that potential difference equals the potential difference across the battery (which here is equal to the emf $\mathscr{E}$), the current is zero. From Eq. 25-1 ($q = CV$), the *equilibrium* (final) *charge* on the then fully charged capacitor is equal to $C\mathscr{E}$.

Here we want to examine the charging process. In particular we want to know how the charge $q(t)$ on the capacitor plates, the potential difference $V_C(t)$ across the capacitor, and the current $i(t)$ in the circuit vary with time during the charging process. We begin by applying the loop rule to the circuit, traversing it clockwise from the negative terminal of the battery. We find

$$\mathscr{E} - iR - \frac{q}{C} = 0. \tag{27-40}$$

The last term on the left side represents the potential difference across the capacitor. The term is negative because the capacitor's top plate, which is connected to the battery's positive terminal, is at a higher potential than the lower plate. Thus, there is a drop in potential as we move down through the capacitor.

We cannot immediately solve Eq. 27-40 because it contains two variables, i and q. However, those variables are not independent but are related by

$$i = \frac{dq}{dt}. \tag{27-41}$$

Substituting this for i in Eq. 27-40 and rearranging, we find

$$R\frac{dq}{dt} + \frac{q}{C} = \mathscr{E} \quad \text{(charging equation).} \tag{27-42}$$

This differential equation describes the time variation of the charge q on the capacitor in Fig. 27-33. To solve it, we need to find the function $q(t)$ that satisfies this equation and also satisfies the condition that the capacitor be initially uncharged; that is, $q = 0$ at $t = 0$.

We shall soon show that the solution to Eq. 27-42 is

$$q = C\mathscr{E}(1 - e^{-t/RC}) \quad \text{(charging a capacitor).} \tag{27-43}$$

(Here e is the exponential base, 2.718 …, and not the elementary charge.) Note that Eq. 27-43 does indeed satisfy our required initial condition, because at $t = 0$ the term $e^{-t/RC}$ is unity; so the equation gives $q = 0$. Note also that as t goes to infinity (that is, a long time later), the term $e^{-t/RC}$ goes to zero; so the equation gives the proper value for the full (equilibrium) charge on the capacitor—namely, $q = C\mathscr{E}$. A plot of $q(t)$ for the charging process is given in Fig. 27-34*a*.

The derivative of $q(t)$ is the current $i(t)$ charging the capacitor:

$$i = \frac{dq}{dt} = \left(\frac{\mathscr{E}}{R}\right)e^{-t/RC} \quad \text{(charging a capacitor).} \tag{27-44}$$

A plot of $i(t)$ for the charging process is given in Fig. 27-34*b*. Note that the current has the initial value $\mathscr{E}/R$ and that it decreases to zero as the capacitor becomes fully charged.

> A capacitor that is being charged initially acts like ordinary connecting wire relative to the charging current. A long time later, it acts like a broken wire.

By combining Eq. 25-1 ($q = CV$) and Eq. 27-43, we find that the potential difference $V_C(t)$ across the capacitor during the charging process is

$$V_C = \frac{q}{C} = \mathscr{E}(1 - e^{-t/RC}) \quad \text{(charging a capacitor).} \tag{27-45}$$

This tells us that $V_C = 0$ at $t = 0$ and that $V_C = \mathscr{E}$ when the capacitor becomes fully charged as $t \to \infty$.

The capacitor's charge grows as the resistor's current dies out.

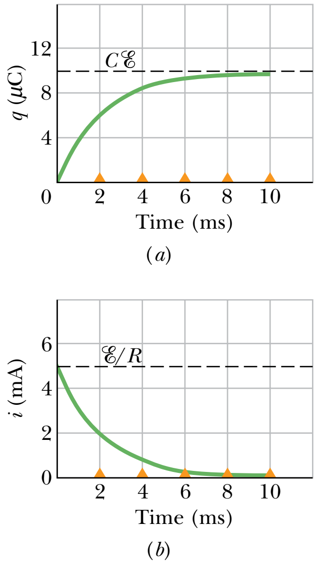


Figure 27-34 (*a*) A plot of Eq. 27-43, which shows the buildup of charge on the capacitor of Fig. 27-33. (*b*) A plot of Eq. 27-44, which shows the decline of the charging current in the circuit of Fig. 27-33. The curves are plotted for $R = 2000\ \Omega$, $C = 1\ \mu$F, and $\mathscr{E} = 10$ V; the small triangles represent successive intervals of one time constant τ.

The Time Constant

The product RC that appears in Eqs. 27-43, 27-44, and 27-45 has the dimensions of time (both because the argument of an exponential must be dimensionless and because, in fact, $1.0\ \Omega \times 1.0\ \mathrm{F} = 1.0\ \mathrm{s}$). The product RC is called the **capacitive time constant** of the circuit and is represented with the symbol τ.

$$\tau = RC \quad \text{(time constant).} \tag{27-46}$$

From Eq. 27-43, we can now see that at time $t = \tau\,(= RC)$, the charge on the initially uncharged capacitor of Fig. 27-33 has increased from zero to

$$q = C\mathscr{E}(1 - e^{-1}) = 0.63C\mathscr{E}. \tag{27-47}$$

In words, during the first time constant τ the charge has increased from zero to 63% of its final value $C\mathscr{E}$. In Fig. 27-34, the small triangles along the time axes mark successive intervals of one time constant during the charging of the capacitor. The charging times for RC circuits are often stated in terms of τ. For example, a circuit with $\tau = 1\ \mu\mathrm{s}$ charges quickly while one with $\tau = 100$ s charges much more slowly.

Discharging a Capacitor

Assume now that the capacitor of Fig. 27-33 is fully charged to a potential V_0 equal to the emf $\mathscr{E}$ of the battery. At a new time $t = 0$, switch S is thrown from a to b so that the capacitor can *discharge* through resistance R. How do the charge $q(t)$ on the capacitor and the current $i(t)$ through the discharge loop of capacitor and resistance now vary with time?

The differential equation describing $q(t)$ is like Eq. 27-42 except that now, with no battery in the discharge loop, $\mathscr{E} = 0$. Thus,

$$R\frac{dq}{dt} + \frac{q}{C} = 0 \quad \text{(discharging equation).} \tag{27-48}$$

The solution to this differential equation is

$$q = q_0 e^{-t/RC} \quad \text{(discharging a capacitor),} \tag{27-49}$$

where $q_0\ (= CV_0)$ is the initial charge on the capacitor. You can verify by substitution that Eq. 27-49 is indeed a solution of Eq. 27-48.

Equation 27-49 tells us that q decreases exponentially with time, at a rate that is set by the capacitive time constant $\tau = RC$. At time $t = \tau$, the capacitor's charge has been reduced to q_0e^{-1}, or about 37% of the initial value. Note that a greater τ means a greater discharge time.

Differentiating Eq. 27-49 gives us the current $i(t)$:

$$i = \frac{dq}{dt} = -\left(\frac{q_0}{RC}\right)e^{-t/RC} \quad \text{(discharging a capacitor).} \tag{27-50}$$

This tells us that the current also decreases exponentially with time, at a rate set by τ. The initial current i_0 is equal to q_0/RC. Note that you can find i_0 by simply applying the loop rule to the circuit at $t = 0$; just then the capacitor's initial potential V_0 is connected across the resistance R, so the current must be $i_0 = V_0/R = (q_0/C)/R = q_0/RC$. The minus sign in Eq. 27-50 can be ignored; it merely means that the capacitor's charge q is decreasing.

Derivation of Eq. 27-43

To solve Eq. 27-42, we first rewrite it as

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{\mathscr{E}}{R}. \tag{27-51}$$

The general solution to this differential equation is of the form

$$q = q_\mathrm{p} + Ke^{-at}, \tag{27-52}$$

where q_p is a *particular solution* of the differential equation, K is a constant to be evaluated from the initial conditions, and $a = 1/RC$ is the coefficient of q in Eq. 27-51. To find q_p, we set $dq/dt = 0$ in Eq. 27-51 (corresponding to the final condition of no further charging), let $q = q_\mathrm{p}$, and solve, obtaining

$$q_\mathrm{p} = C\mathscr{E}. \tag{27-53}$$

To evaluate K, we first substitute this into Eq. 27-52 to get

$$q = C\mathscr{E} + Ke^{-at}.$$

Then substituting the initial conditions $q = 0$ and $t = 0$ yields

$$0 = C\mathscr{E} + K,$$

or $K = -C\mathscr{E}$. Finally, with the values of q_p, a, and K inserted, Eq. 27-52 becomes

$$q = C\mathscr{E} - C\mathscr{E}e^{-t/RC},$$

which, with a slight modification, is Eq. 27-43.

☑ CHECKPOINT 7

The table gives four sets of values for the circuit elements in Fig. 27-33. Rank the sets according to (a) the initial current (as the switch is closed on a) and (b) the time required for the current to decrease to half its initial value, greatest first.

	1	**2**	**3**	**4**
$\mathscr{E}$(V)	12	12	10	10
R(Ω)	2	3	10	5
C(μF)	3	2	0.5	2

SAMPLE PROBLEM 27.06

Discharging an *RC* circuit to avoid a fire in a race car pit stop

As a car rolls along pavement, electrons move from the pavement first onto the tires and then onto the car body. The car stores this excess charge and the associated electric potential energy as if the car body were one plate of a capacitor and the pavement were the other plate (Fig. 27-35*a*). When the car stops, it discharges its excess charge and energy through the tires, just as a capacitor can discharge through a resistor. If a conducting object comes within a few centimeters of the car before the car is discharged, the remaining energy can be suddenly transferred to a spark between the car and the object. Suppose the conducting object is a fuel dispenser. The spark will not ignite the fuel and cause a fire if the spark energy is less than the critical value $U_\mathrm{fire} = 50$ mJ.

When the car of Fig. 27-35*a* stops at time $t = 0$, the car–ground potential difference is $V_0 = 30$ kV. The car–ground capacitance is $C = 500$ pF, and the resistance of *each* tire is $R_\mathrm{tire} = 100$ GΩ. How much time does the car take to discharge through the tires to drop below the critical value U_fire?

KEY IDEAS

(1) At any time t, a capacitor's stored electric potential energy U is related to its stored charge q according to Eq. 25-21 ($U = q^2/2C$). (2) While a capacitor is discharging, the charge decreases with time according to Eq. 27-49 ($q = q_0e^{-t/RC}$).

Calculations: We can treat the tires as resistors that are connected to one another at their tops via the car body and at their bottoms via the pavement. Figure 27-35*b* shows how the four resistors are connected in parallel across the car's capacitance, and Fig. 27-35*c* shows

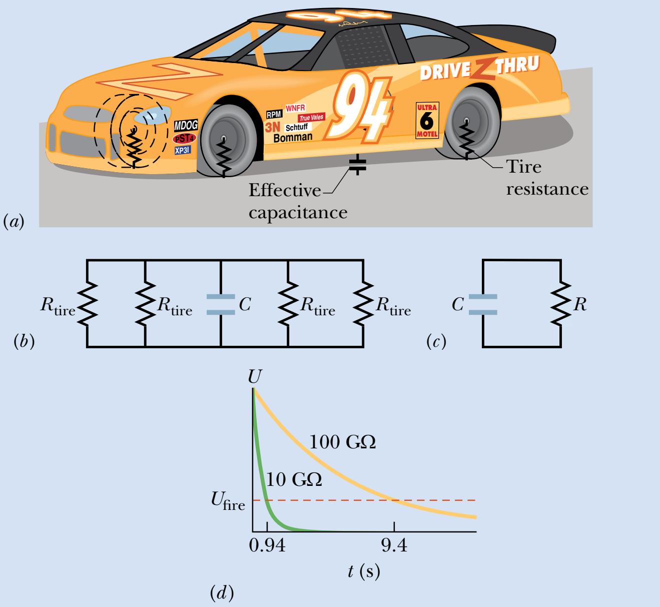


Figure 27-35 (*a*) A charged car and the pavement acts like a capacitor that can discharge through the tires. (*b*) The effective circuit of the car–pavement capacitor, with four tire resistances R_{tire} connected in parallel. (*c*) The equivalent resistance R of the tires. (*d*) The electric potential energy U in the car–pavement capacitor decreases during discharge.

their equivalent resistance R. From Eq. 27-24, R is given by

$$\frac{1}{R}=\frac{1}{R_{\text{tire}}}+\frac{1}{R_{\text{tire}}}+\frac{1}{R_{\text{tire}}}+\frac{1}{R_{\text{tire}}},$$

or

$$R=\frac{R_{\text{tire}}}{4}=\frac{100\times10^{9}\ \Omega}{4}=25\times10^{9}\ \Omega. \qquad (27\text{-}54)$$

When the car stops, it discharges its excess charge and energy through R. We now use our two Key Ideas to analyze the discharge. Substituting Eq. 27-49 into Eq. 25-21 gives

$$U=\frac{q^2}{2C}=\frac{(q_0e^{-t/RC})^2}{2C}=\frac{q_0^2}{2C}e^{-2t/RC}. \qquad (27\text{-}55)$$

From Eq. 25-1 ($q = CV$), we can relate the initial charge q_0 on the car to the given initial potential difference V_0: $q_0 = CV_0$. Substituting this equation into Eq. 27-55 brings us to

$$U=\frac{(CV_0)^2}{2C}e^{-2t/RC}=\frac{CV_0^2}{2}e^{-2t/RC},$$

or

$$e^{-2t/RC}=\frac{2U}{CV_0^2}. \qquad (27\text{-}56)$$

Taking the natural logarithms of both sides, we obtain

$$-\frac{2t}{RC}=\ln\left(\frac{2U}{CV_0^2}\right),$$

or

$$t=-\frac{RC}{2}\ln\left(\frac{2U}{CV_0^2}\right). \qquad (27\text{-}57)$$

Substituting the given data, we find that the time the car takes to discharge to the energy level $U_{\text{fire}} = 50$ mJ is

$$\begin{aligned} t &= -\frac{(25\times10^{9}\ \Omega)(500\times10^{-12}\ \text{F})}{2} \\ &\quad\times\ln\left(\frac{2(50\times10^{-3}\ \text{J})}{(500\times10^{-12}\ \text{F})(30\times10^{3}\ \text{V})^2}\right) \\ &= 9.4\ \text{s}. \end{aligned} \qquad \text{(Answer)}$$

Fire or no fire: This car requires at least 9.4 s before fuel can be brought safely near it. A pit crew cannot wait that long. So the tires include some type of conducting material (such as carbon black) to lower the tire resistance and thus increase the discharge rate. Figure 27-35*d* shows the stored energy U versus time t for tire resistances of R = 100 GΩ (our value) and R = 10 GΩ. Note how much more rapidly a car discharges to level U_{fire} with the lower R value.

SAMPLE PROBLEM 27.07

Current and charge in complicated *RC* circuit

(a) Find the current i supplied by the battery in the network as shown in Fig. 27-36 in steady state. (b) Find the steady state charge on the capacitor. (c) Find the initial current through the battery.

KEY IDEA

Here we do not have a simple RC circuit as seen in Section 27.7. So, we cannot apply Eqs. 27-43 and 27-44 here. We have studied in Section 27.7 that an uncharged

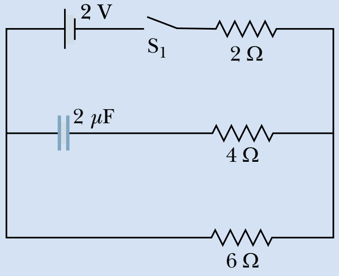


Figure 27-36 A complicated *RC* circuit.

capacitor will initially behave like short circuit and after a long time behave like a broken wire.

Calculations: (a) Once the capacitor is charged, no current will go through it and the current through the middle branch of the circuit is zero in steady state. The 4 Ω resistor will have no current in it and may be omitted for current analysis. The 2 Ω and 6 Ω resistors are therefore connected in series and hence

$$i = \frac{2\ \text{V}}{2\ \Omega + 6\ \Omega} = 0.25\ \text{A}. \qquad \text{(Answer)}$$

(b) The potential drop across the 6 Ω resistor is 6 Ω × 0.25 A = 1.5 V. As there is no current in the 4 Ω resistor, there is no potential drop across it. The potential difference across the capacitor is, therefore, 1.5 V. The charge on this capacitor is

$$Q = CV = 2\ \mu\text{F} \times 1.5\ \text{V} = 3\ \mu\text{C}. \qquad \text{(Answer)}$$

(c) At the initial moment, an uncharged capacitor can be assumed to be short circuited. So, the equivalent resistance of the circuit will be 4.4 Ω. The current through the battery will be

$$\frac{2}{4.4}\ \text{A} = 0.45\ \text{A}. \qquad \text{(Answer)}$$

REVIEW AND SUMMARY

Emf An **emf device** does work on charges to maintain a potential difference between its output terminals. If dW is the work the device does to force positive charge dq from the negative to the positive terminal, then the **emf** (work per unit charge) of the device is

$$\mathscr{E} = \frac{dW}{dq} \quad (\text{definition of } \mathscr{E}). \qquad (27\text{-}1)$$

The volt is the SI unit of emf as well as of potential difference. An **ideal emf device** is one that lacks any internal resistance. The potential difference between its terminals is equal to the emf. A **real emf device** has internal resistance. The potential difference between its terminals is equal to the emf only if there is no current through the device.

Analyzing Circuits The change in potential in traversing a resistance R in the direction of the current is $-iR$; in the opposite direction it is $+iR$ (resistance rule). The change in potential in traversing an ideal emf device in the direction of the emf arrow is $+\mathscr{E}$; in the opposite direction it is $-\mathscr{E}$ (emf rule). Conservation of energy leads to the loop rule:

***Loop Rule**. The algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero.*

Conservation of charge gives us the junction rule:

***Junction Rule**. The sum of the currents entering any junction must be equal to the sum of the currents leaving that junction.*

Single-Loop Circuits The current in a single-loop circuit containing a single resistance R and an emf device with emf $\mathscr{E}$ and internal resistance r is

$$i = \frac{\mathscr{E}}{R + r}, \qquad (27\text{-}4)$$

which reduces to $i = \mathscr{E}/R$ for an ideal emf device with $r = 0$.

Power When a real battery of emf $\mathscr{E}$ and internal resistance r does work on the charge carriers in a current i through the battery, the rate P of energy transfer to the charge carriers is

$$P = iV, \qquad (27\text{-}14)$$

where V is the potential across the terminals of the battery. The rate P_r at which energy is dissipated as thermal energy in the battery is

$$P_r = i^2 r. \qquad (27\text{-}16)$$

The rate P_{emf} at which the chemical energy in the battery changes is

$$P_{\text{emf}} = i\mathscr{E}. \qquad (27\text{-}17)$$

Series Resistances When resistances are in **series**, they have the same current. The equivalent resistance that can replace a series combination of resistances is

$$R_{\text{eq}} = \sum_{j=1}^{n} R_j \quad (n \text{ resistances in series}). \qquad (27\text{-}7)$$

Parallel Resistances When resistances are in **parallel**, they have the same potential difference. The equivalent resistance that can replace a parallel combination of resistances is given by

$$\frac{1}{R_{\text{eq}}}=\sum_{j=1}^{n}\frac{1}{R_j} \quad (n \text{ resistances in parallel}). \tag{27-24}$$

Electricity-Related Instruments In a Wheatstone bridge, the unknown resistance (X) can be determined as

$$\frac{Q}{P}=\frac{X}{R}. \tag{27-32}$$

In a meter bridge or slide-wire bridge, if λ is the resistance of the wire of length 1 m, we can write by Eq. 27-32

$$\frac{X}{R}=\frac{\lambda(1-l)}{\lambda l}.$$

In a potentiometer, to determine any potential, the source to be measured simply replaces the standard cell and the new balance point l is obtained. Then from Eq. 27-39, its emf ξ can be measured. Since the wire is uniform, the drop ξ in potential across any length l is known by simple ratio,

$$\frac{\xi}{\xi_S}=\frac{l}{l_s}, \tag{27-39}$$

***RC* Circuits** When an emf $\mathscr{E}$ is applied to a resistance R and capacitance C in series, as in Fig. 27-33 with the switch at a, the charge on the capacitor increases according to

$$q=C\mathscr{E}(1-e^{-t/RC}) \quad \text{(charging a capacitor)}, \tag{27-43}$$

in which $C\mathscr{E}=q_0$ is the equilibrium (final) charge and $RC=\tau$ is the **capacitive time constant** of the circuit. During the charging, the current is

$$i=\frac{dq}{dt}=\left(\frac{\mathscr{E}}{R}\right)e^{-t/RC} \quad \text{(charging a capacitor)}. \tag{27-44}$$

When a capacitor discharges through a resistance R, the charge on the capacitor decays according to

$$q=q_0e^{-t/RC} \quad \text{(discharging a capacitor)}. \tag{27-49}$$

During the discharging, the current is

$$i=\frac{dq}{dt}=-\left(\frac{q_0}{RC}\right)e^{-t/RC} \quad \text{(discharging a capacitor)}. \tag{27-50}$$

PROBLEMS

1. An isolated charged capacitor may gradually discharge as charge leaks from one plate to the other through the intermediate material, as if it were discharging through an external resistor. (a) What is the resistance of such an external resistor if a 2.00 μF capacitor's potential difference decreases to 60.0% of its initial value of 50.0 V in 2.40 d? (b) What is the corresponding loss of potential energy in that time interval? (c) At the end of that interval, at what rate is the capacitor losing potential energy?

2. In Fig. 27-37, the ideal batteries have emfs $\mathscr{E}_1 = 12.0$ V and $\mathscr{E}_2 = 0.500\,\mathscr{E}_1$ and the resistances are each 4.00 Ω. What is the current in (a) resistance 2 and (b) resistance 3?

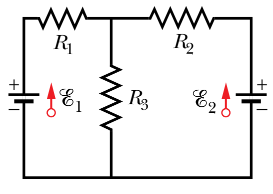


Figure 27-37 Problems 2 and 3.

3. In Fig. 27-37, $\mathscr{E}_1 = 1.00$ V, $\mathscr{E}_2 = 3.00$ V, $R_1 = 4.00$ Ω, $R_2 = 2.00$ Ω, $R_3 = 5.00$ Ω, and both batteries are ideal. What is the rate at which energy is dissipated in (a) R_1, (b) R_2, and (c) R_3? What is the power of (d) battery 1 and (e) battery 2? Is energy being absorbed or provided in (f) battery 1 and (g) battery 2?

4. In Fig. 27-38, two batteries with an emf $\mathscr{E} = 12.0$ V and an internal resistance $r = 0.500$ Ω are connected in parallel across a resistance R. (a) For what value of R is the dissipation rate in the resistor a maximum? (b)What is that maximum? and (c) what is the total dissipation rate in the batteries? In terms of r, what are (d) the effective (internal) resistance of the two-battery system and (e) the resistance R that is required to maximize the dissipation rate? This is a general result.

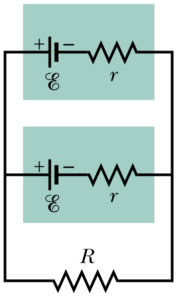


Figure 27-38 Problem 4.

5. In Fig. 27-39, a voltmeter of resistance $R_V = 300$ Ω and an ammeter of resistance $R_A = 3.00$ Ω are being used to measure a resistance R in a circuit that also contains a resistance $R_0 = 100$ Ω and an ideal battery with an emf of $\mathscr{E} = 18.0$ V. Resistance R is given by $R = V/i$, where V is the potential across R and i is the ammeter reading. The voltmeter reading is V', which is V plus the potential difference across the ammeter. Thus, the ratio of the two meter readings is not R but only an *apparent* resistance $R' = V'/i$. If $R = 85.0$ Ω, what are (a) the ammeter reading,

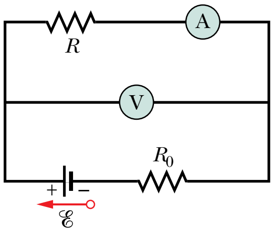


Figure 27-39 Problem 5.

(b) the voltmeter reading, and (c) R'? (d) If R_A is decreased, does the difference between R' and R increase, decrease, or remain the same?

6. In Fig. 27-40, $R_1 = 100\ \Omega$, $R_2 = R_3 = 50.0\ \Omega$, $R_4 = 75.0\ \Omega$, and the ideal battery has emf $\mathcal{E} = 12.0$ V. (a) What is the equivalent resistance? What is i in (b) resistance 1, (c) resistance 2, (d) resistance 3, and (e) resistance 4?

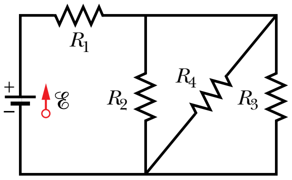


Figure 27-40 Problems 6 and 7.

7. In Fig. 27-40, the resistors have the values $R_1 = 7.00\ \Omega$, $R_2 = 12.0\ \Omega$, and $R_3 = 4.00\ \Omega$, and the ideal battery's emf is $\mathcal{E} = 22.0$ V. For what value of R_4 will the rate at which the battery transfers energy to the resistors equal (a) 60.0 W, (b) the maximum possible rate P_{max}, and (c) the minimum possible rate P_{min}? What are (d) P_{max} and (e) P_{min}?

8. Figure 27-41 shows a battery connected across a uniform resistor R_0. A sliding contact can move across the resistor from $x = 0$ at the left to $x = 10$ cm at the right. Moving the contact changes how much resistance is to the left of the contact and how much is to the right. Find the rate at which energy is dissipated in resistor R as a function of x. Plot the function for $\mathcal{E} = 50$ V, $R = 2000\ \Omega$, and $R_0 = 100\ \Omega$.

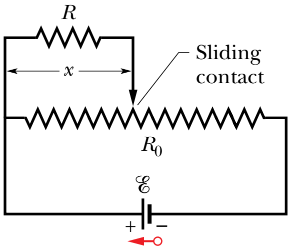


Figure 27-41 Problem 8.

9. In Fig. 27-42, the resistances are $R_1 = 3.00\ \Omega$, $R_2 = 7.00\ \Omega$, and the battery is ideal. What value of R_3 maximizes the dissipation rate in resistance 3?

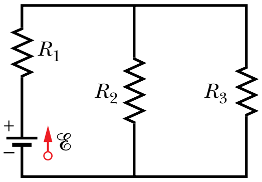


Figure 27-42 Problem 9.

10. A capacitor with an initial potential difference of 80.0 V is discharged through a resistor when a switch between them is closed at $t = 0$. At $t = 10.0$ s, the potential difference across the capacitor is 1.00 V. (a) What is the time constant of the circuit? (b) What is the potential difference across the capacitor at $t = 17.0$ s?

11. In Fig. 27-43, the current in resistance 6 is $i_6 = 2.80$ A and the resistances are $R_1 = R_2 = R_3 = 2.00\ \Omega$, $R_4 = 16.0\ \Omega$, $R_5 = 8.00\ \Omega$, and $R_6 = 4.00\ \Omega$. What is the emf of the ideal battery?

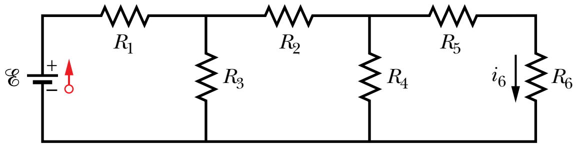


Figure 27-43 Problem 11.

12. In Fig. 27-44, $\mathcal{E}_1 = 8.00$ V, $\mathcal{E}_2 = 12.0$ V, $R_1 = 100\ \Omega$, $R_2 = 200\ \Omega$, and $R_3 = 300\ \Omega$. One point of the circuit is grounded ($V = 0$). What are the (a) size and (b) direction (up or down) of the current through resistance 1, the (c) size and (d) direction (left or right) of the current through resistance 2, and the (e) size and (f) direction of the current through resistance 3? (g) What is the electric potential at point A?

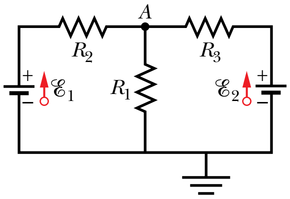


Figure 27-44 Problem 12.

13. In Fig. 27-45, assume that $\xi = 3.0$ V, $r = 100\ \Omega$, $R_1 = 250\ \Omega$, and $R_2 = 300\ \Omega$. If the voltmeter resistance RV is 5.0 kΩ, what percent error does it introduce into the measurement of the potential difference across R_1? Ignore the presence of the ammeter.

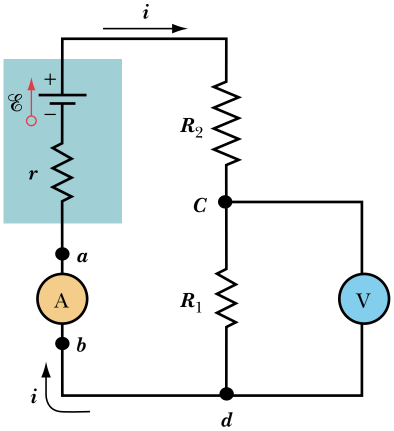


Figure 27-45 Problem 13.

14. Switch S in Fig. 27-46 is closed at time $t = 0$, to begin charging an initially uncharged capacitor of capacitance $C = 49.0\ \mu$F through a resistor of resistance $R = 32.0\ \Omega$. At what time is the potential across the capacitor equal to that across the resistor?

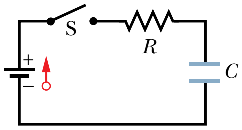


Figure 27-46 Problem 14.

15. In the circuit shown in Fig. 27-47, there is a cube of resistances which are all having the same resistance of R. Find the equivalent resistance of the network between diagonally opposite points as shown in the figure.

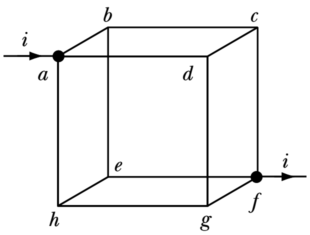


Figure 27-47 Problem 15.

16. Find the equivalent resistance of the network as shown in Fig. 27-48 between points *a* and *b*.

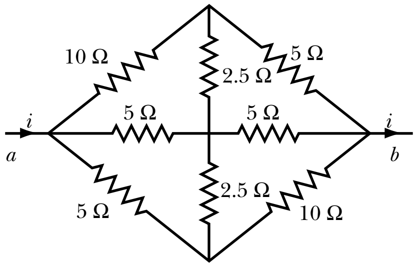


Figure 27-48 Problem 16.

17. An infinite ladder is constructed with 1 Ω and 2 Ω resistors as shown in Fig. 27-49. Find the effective resistance between the points *A* and *B*.

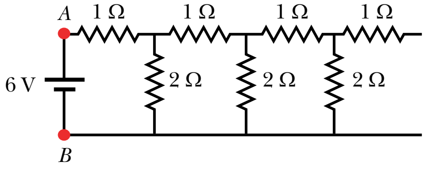


Figure 27-49 Problem 17.

18. In Fig. 27-50, $\mathscr{E} = 24.0$ V, $R_1 = 2000\ \Omega$, $R_2 = 3000\ \Omega$, and $R_3 = 4000\ \Omega$. What are the potential differences (a) $V_A - V_B$, (b) $V_B - V_C$, (c) $V_C - V_D$, and (d) $V_A - V_C$?

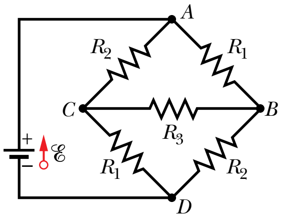


Figure 27-50 Problem 18.

19. In Fig. 27-51, battery 1 has emf $\mathscr{E}_1 = 12.0$ V and internal resistance $r_1 = 0.025\ \Omega$ and battery 2 has emf $\mathscr{E}_2 = 12.0$ V and internal resistance $r_2 = 0.012\ \Omega$. The batteries are connected in series with an external resistance *R*. (a) What *R* value makes the terminal-to-terminal potential difference of one of the batteries zero? (b) Which battery is that?

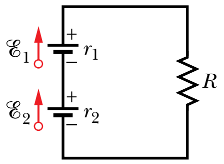


Figure 27-51 Problem 19.

20. A capacitor with initial charge q_0 is discharged through a resistor. What multiple of the time constant τ gives the time the capacitor takes to lose (a) the first 25% of its charge and (b) 50% of its charge?

21. In Fig. 27-52, circuit section *AB* absorbs energy at a rate of 50 W when current $i = 2.0$ A through it is in the indicated direction. Resistance $R = 2.0\ \Omega$. (a) What is the potential difference between *A* and *B*? Emf device *X* lacks internal resistance. (b) What is its emf? (c) Is point *B* connected to the positive terminal of *X* or to the negative terminal?

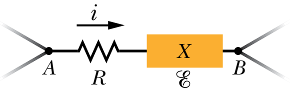


Figure 27-52 Problem 21.

22. Sixteen copper wires of length *l* and diameter *d* are connected in parallel to form a single composite conductor of resistance *R*. What must the diameter *D* of a single copper wire of length *l* be if it is to have the same resistance?

23. In Fig. 27-53, a voltmeter of resistance $R_V = 300\ \Omega$ and an ammeter of resistance $R_A = 3.00\ \Omega$ are being used to measure a resistance *R* in a circuit that also contains a resistance $R_0 = 100\ \Omega$ and an ideal battery of emf $\mathscr{E} = 28.5$ V. Resistance *R* is given by $R = V/i$, where *V* is the voltmeter reading and *i* is the current in resistance *R*. However, the ammeter reading is not *i* but rather *i*′, which is *i* plus the current through the voltmeter. Thus, the ratio of the two meter readings is not *R* but only an *apparent* resistance $R' = V/i'$. If $R = 85.0\ \Omega$, what are (a) the ammeter reading, (b) the voltmeter reading, and (c) *R*′? (d) If R_V is increased, does the difference between *R*′ and *R* increase, decrease, or remain the same?

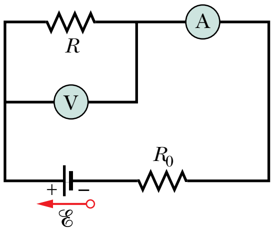


Figure 27-53 Problem 23.

24. Figure 27-54 shows an arrangement to measure the emf ξ and internal resistance *r* of a battery. The voltmeter has resistance. The voltmeter reads 1.52 V when the switch *S* is open. When the switch is closed the voltmeter reading drops to 1.45 V and the ammeter reads 1.0 A. Find the (a) emf and (b) the internal resistance of the battery.

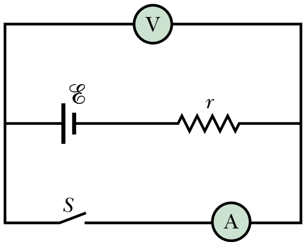


Figure 27-54 Problem 24.

25. A voltmeter consists of a 25 Ω coil connected in series with a 575 Ω resistor. The coil takes 10 mA for full-scale deflection. What maximum potential difference can be measured on this voltmeter?

26. An ammeter is to be constructed which can read currents up to 2.0 A. If the coil has a resistance of 25 Ω and takes 1 mA for full-scale deflection, what should be the resistance of the shunt used?

27. A voltmeter coil has resistance 50.0 Ω and a resistor of 1.15 kΩ is connected in series. It can read potential differences up to 12 V. If this same coil is used to construct an ammeter which can measure currents up to 2.0 A, what should be the resistance of the shunt used?

28. A 3.00 MΩ resistor and a 1.00 μF capacitor are connected in series with an ideal battery of emf $\mathscr{E} = 4.00$ V. At 6.00 s after the connection is made, what is the rate at which (a) the charge of the capacitor is increasing, (b) energy is being stored in the capacitor, (c) thermal energy is appearing in the resistor, and (d) energy is being delivered by the battery?

29. In an RC series circuit, emf $\mathscr{E} = 12.0$ V, resistance $R = 1.40$ MΩ, and capacitance $C = 2.70$ μF. (a) Calculate the time constant. (b) Find the maximum charge that will appear on the capacitor during charging. (c) How long does it take for the charge to build up to 16.0 μC?

30. In Fig. 27-55, the ideal batteries have emfs $\mathscr{E}_1 = 200$ V and $\mathscr{E}_2 = 50$ V and the resistances are $R_1 = 3.0\ \Omega$ and $R_2 = 2.0\ \Omega$. If the potential at P is 100 V, what is it at Q?

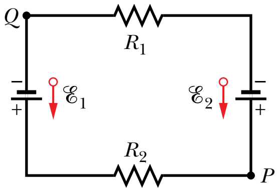


Figure 27-55 Problem 30.

31. Figure 27-56 shows the circuit of a flashing lamp, like those attached to barrels at highway construction sites. The fluorescent lamp L (of negligible capacitance) is connected in parallel across the capacitor C of an RC circuit. There is a current through the lamp only when the potential difference across it reaches the breakdown voltage V_L; then the capacitor discharges completely through the lamp and the lamp flashes briefly. For a lamp with breakdown voltage $V_L = 75.0$ V, wired to a 95.0 V ideal battery and a 0.150 μF capacitor, what resistance R is needed for two flashes per second?

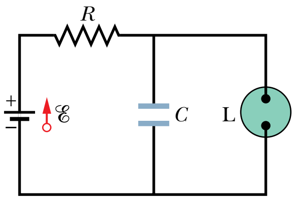


Figure 27-56 Problem 31.

32. Six 18.0 Ω resistors are connected in parallel across a 12.0 V ideal battery. What is the current through the battery?

33. In Fig. 27-9, what is the potential difference $V_d - V_c$ between points d and c if $\mathscr{E}_1 = 4.0$ V, $\mathscr{E}_2 = 1.0$ V, $R_1 = R_2 = 10\ \Omega$, and $R_3 = 8.0\ \Omega$, and the battery is ideal?

34. In the circuit of Fig. 27-57, $\mathscr{E} = 1.2$ kV, $C = 6.5$ μF, $R_1 = R_2 = R_3 = 0.73$ MΩ. With C completely uncharged, switch S is suddenly closed (at $t = 0$). At $t = 0$, what are (a) current i_1 in resistor 1, (b) current i_2 in resistor 2, and (c) current i_3 in resistor 3? At $t = \infty$ (that is, after many time constants), what are (d) i_1, (e) i_2, and (f) i_3? What is the potential difference V_2 across resistor 2 at (g) $t = 0$ and (h) $t = \infty$? (i) Sketch V_2 versus t between these two extreme times.

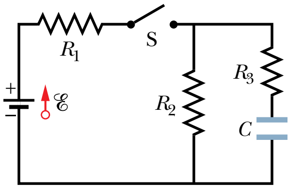


Figure 27-57 Problem 34.

35. When resistors 1 and 2 are connected in series, the equivalent resistance is 20.0 Ω. When they are connected in parallel, the equivalent resistance is 3.75 Ω. What are (a) the smaller resistance and (b) the larger resistance of these two resistors?

36. A car battery with a 12 V emf and an internal resistance of 0.030 Ω is being charged with a current of 40 A. What are (a) the potential difference V across the terminals, (b) the rate P_r of energy dissipation inside the battery, and (c) the rate P_{emf} of energy conversion to chemical form? When the battery is used to supply 40 A to the starter motor, what are (d) V and (e) P_r?

37. In Fig. 27-58, $R_1 = R_2 = 4.00\ \Omega$ and $R_3 = 1.50\ \Omega$. Find the equivalent resistance between points D and E. (*Hint:* Imagine that a battery is connected across those points.)

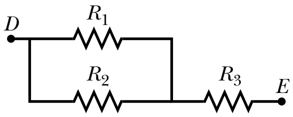


Figure 27-58 Problem 37.

38. (a) In Fig. 27-59, what current does the ammeter read if $\mathscr{E} = 5.0$ V (ideal battery), $R_1 = 2.0\ \Omega$, $R_2 = 9.0\ \Omega$, and $R_3 = 6.0\ \Omega$? (b) The ammeter and battery are now interchanged. Show that the ammeter reading is unchanged.

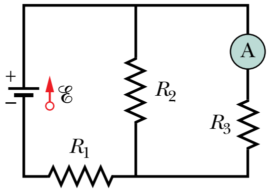


Figure 27-59 Problem 38.

39. A total resistance of 5.00 Ω is to be produced by connecting an unknown resistance to a 15.0 Ω resistance. (a) What must be the value of the unknown resistance, and (b) should it be connected in series or in parallel? (c) What is the total resistance if the unknown resistance is connected the other way?

40. A 5.7 A current is set up in a circuit for 15.0 min by a rechargeable battery with a 6.0 V emf. By how much is the chemical energy of the battery reduced?

41. Consider the potentiometer circuit arranged as shown in Fig. 27-60. The potentiometer wire is 600 cm long. (a) At what distance from the point A should the jockey touch the wire to get zero deflection in the galvanometer? (b) If the jockey touches the wire at a distance of 560 cm from A, what will be the current in the galvanometer?

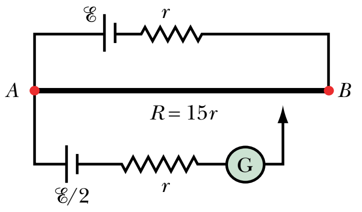


Figure 27-60 Problem 41.

42. A 6 V battery of negligible internal resistance is connected across a uniform wire AB of length 100 cm. The positive terminal of another battery of emf 4 V and internal resistance 1 Ω is joined to the point A as shown in

Fig. 27-61. Take the potential at B to be zero. (a) What are the potentials at the points A and C? (b) At which point D of the wire AB, the potential is equal to the potential at C? (c) If the points C and D are connected by a wire, what will be the current through it? (d) If the 4 V battery is replaced by 7.5 V battery, what would be the answers of part (a) and (b)?

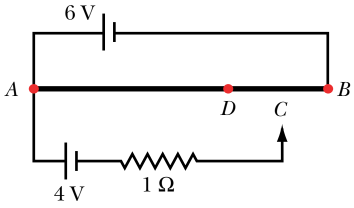


Figure 27-61 Problem 42.

43. The wire AB of an arrangement like that of meter bridge as shown in Fig. 27-62 is 40 cm long. Where should the free end of the galvanometer be connected on AB so that the galvanometer may show zero deflection?

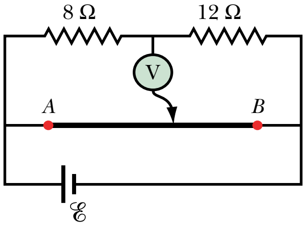


Figure 27-62 Problem 43.

44. In Fig. 27-63, $R_1 = 100\ \Omega$, $R_2 = 50\ \Omega$, and the ideal batteries have emfs $\mathscr{E}_1 = 6.0$ V, $\mathscr{E}_2 = 10$ V, and $\mathscr{E}_3 = 4.0$ V. Find (a) the current in resistor 1, (b) the current in resistor 2, and (c) the potential difference between points a and b.

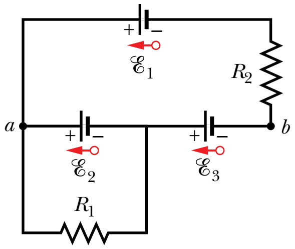


Figure 27-63 Problem 44.

45. Figure 27-64 shows a conductor of length l having a circular cross section. The radius of cross section varies linearly from a to b. The resistivity of the material is ρ. Assuming that $b - a \ll l$, find the resistance of the conductor.

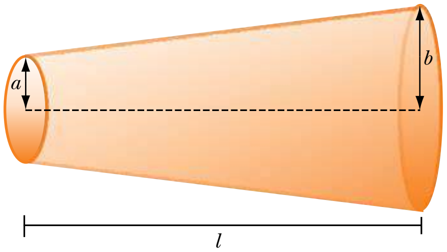
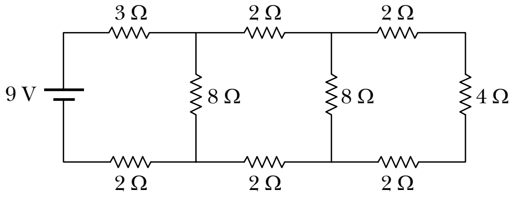


Figure 27-64 Problem 45.

PRACTICE QUESTIONS

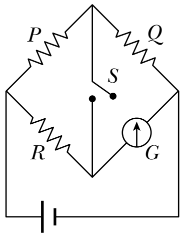
Single Correct Choice Type

1. In the circuit shown in the figure, the current through



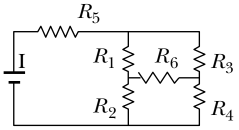
(a) The 3 Ω resistor is 0.50 A
(b) The 3 Ω resistor is 0.25 A
(c) The 4 Ω resistor is 0.50 A
(d) The 4 Ω resistor is 0.25 A

2. In the circuit shown $P \neq R$, the reading of galvanometer is same with switch S open or closed. Then



(a) $I_R = I_G$ **(b)** $I_P = I_G$
(c) $I_Q = I_G$ **(d)** $I_Q = I_R$

3. In the given circuit, it is observed that the current I is independent of the value of the resistance R_6. The resistance values must satisfy



(a) $R_1R_2R_5 = R_3R_4R_6$

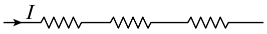
(b) $\dfrac{1}{R_5} + \dfrac{1}{R_6} = \dfrac{1}{R_1 + R_2} + \dfrac{1}{R_3 + R_4}$

(c) $R_1R_4 = R_2R_3$

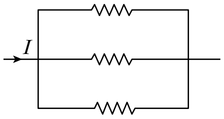
(d) $R_1R_3 = R_2R_4 = R_5R_6$

4. The three resistances of equal value are arranged in the different combinations shown in the figure. Arrange them in the increasing order of dissipation of power.

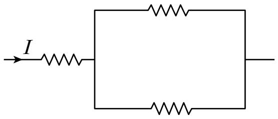
(I)



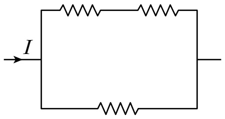
(II)



(III)



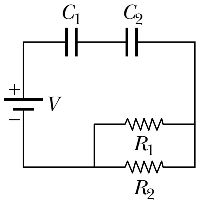
(IV)



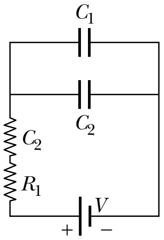
(a) II < IV < I < III **(b)** II < IV < III < I
(c) IV < II < I < III **(d)** IV < I < II < III

5. In the following circuits, it is given that $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $C_1 = 2\ \mu$F and $C_2 = 4\ \mu$F.

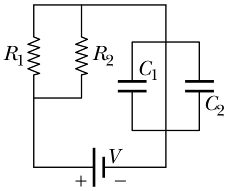
(I)



(II)



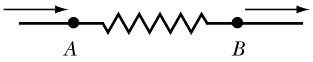
(III)



The time constants (in μs) for the circuits I, II, III are, respectively,

(a) 18, 8/9, 4 **(b)** 18, 4, 8/9
(c) 4, 8/9, 18 **(d)** 8/9, 18, 4

6. Somewhere in a circuit is a resistor. A constant current is flowing in the direction as indicated in the following figure. In going from A to B, we measure $\int_A^B \vec{E}\cdot d\vec{l}$. What do we find?



(a) A positive value
(b) A negative value
(c) Zero
(d) We do not have enough information to answer

7. If we go from A to B through the wire to the resistor, or choose a random routing through free space (thus not through the resistor itself), starting at A and ending at B. Which of the following statement is correct?

(a) There is no difference in both cases
(b) There is a major difference due to path chosen
(c) We cannot choose a random routine path
(d) None of these

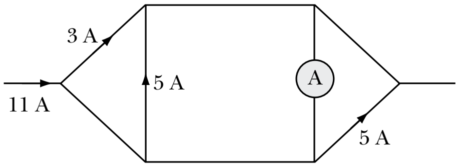
8. Ten coulombs of charge start from the negative terminal of a battery, flow through the battery and then leave the positive terminal through a wire, flow through a resistor and then return to the starting point on this closed circuit. In this complete process, the 10 C

(a) Have no net change in potential energy
(b) Receive heat energy from the resistor
(c) Have a net loss of potential energy
(d) Do positive work on the battery

9. A real battery has an emf ξ and an internal resistance r. A variable resistor R is connected across the terminals of the battery. A current i is drawn from the battery and the potential difference across the terminals of the battery is V. If R is slowly decreased to zero, which of the following best describes i and V?

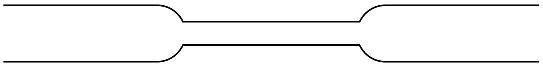
(a) i decreases to zero; V approaches ξ
(b) i approaches an infinite value; V decreases to zero
(c) i approaches ξr; V approaches ξ
(d) i approaches ξ/r; V approaches zero

10. The given figure shows a network of wires carrying various currents. What is the current through A?



(a) 1 A **(b)** 2 A
(c) 6 A **(d)** 8 A

11. A current-carrying wire got pulled, stretching it and making it thinner in its middle. The cross-section of that wire is shown in the following figure. Which of the following statements about the current and the temperature of the wire is true?



(a) The current and temperature at the thin section of the wire is higher than at the thick section
(b) The current and temperature at the thick section of the wire is higher than at the thin section
(c) The current and temperature at the wire is the same everywhere
(d) The current in the wire is the same everywhere, but the temperature of the wire is higher in the thin section

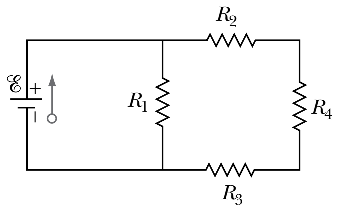
12. Four unequal resistors are connected in parallel. Which one of the following statements is correct?

(a) The total resistance is less than the smallest resistor
(b) The total resistance is equal to the average of the individual resistances
(c) The total resistance is more than the largest resistor
(d) The total resistance is in between the smallest and largest resistor but not the average of the individual resistances

13. Three 100 W light bulbs are connected in series to a 220 V power source. If two of the light bulbs are replaced by 60 W bulbs, the brightness of the remaining 100 W bulb is

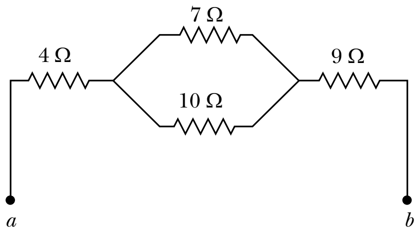
(a) Brighter than it was before
(b) Dimmer than it was before
(c) The same brightness as it was before
(d) Will not illuminate

14. An ideal battery is connected to a circuit of four resistors as shown in the following figure. If the resistance R_4 is increased, then



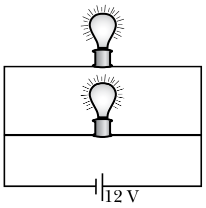
(a) The current through R_1 decreases
(b) The potential difference across R_1 increases
(c) The potential difference across R_4 increases
(d) The potential difference across R_2 decreases

15. If voltage V is applied across terminals a and b of the circuit shown in the following figure, which of the statements below is true? (Notation: i_4 is current through 4.00-Ω resistor, …)

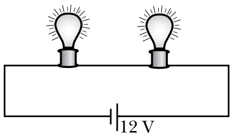


(a) $i_7 < i_{10} < i_9$
(b) $i_4 = i_7 = i_9$
(c) $i_4 > i_7 + i_{10} > i_9$
(d) $i_{10} < i_7$

16. If the four light bulbs in the following figure are identical, which circuit gives out more light?



Circuit I

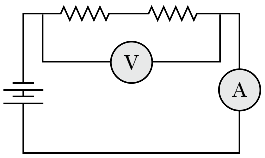


Circuit II

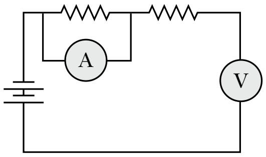
(a) I
(b) The two emit the same amount of light
(c) II
(d) None of these

17. Which circuit diagram shows voltmeter V and ammeter A correctly positioned to measure the total potential difference of the circuit and the current through each resistor?

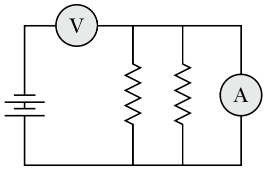
(a)



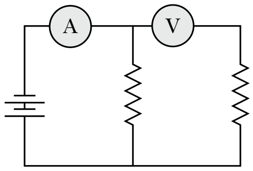
(b)



(c)



(d)



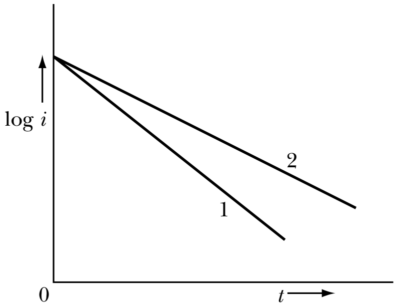
18. When a capacitor is discharged through a resistor, which of the following is false?

(a) The current in the capacitor is equal to the current in the resistor
(b) The capacitor voltage decreases by half after a time equal to RC
(c) The voltage across the capacitor is equal to the voltage across the resistor at any instant
(d) The current in the circuit takes an infinite time to reach zero

19. Two identical resistors of magnitude R and two identical capacitors of magnitude C are used to form an RC circuit. In which case the time constant of that RC circuit is the highest?

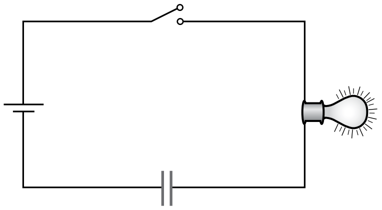
(a) Two resistors and two capacitors connected in series with a power supply
(b) Two resistors and two capacitors connected in parallel with a power supply
(c) Two resistors connected in parallel, two capacitors connected in parallel, and those two combinations are connected in series with a power supply
(d) Two resistors connected in series, two capacitors connected in parallel, and those two combinations are connected in series with a power supply

20. A capacitor of capacity C is charged to a steady potential difference V and connected in series with an open key and a pure resistor R. At time $t = 0$, the key is closed if i = current at time t, a plot of log i against t is as shown in (1) in the following figure. Later one of the parameter, that is, V. R or C is changed keeping the other two constant, and the graph (2) is recorded. Then



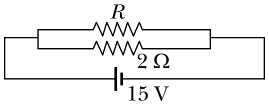
(a) C is reduced
(b) C is increased
(c) R is reduced
(d) R is increased

21. When the switch is closed in the circuit as shown in the following figure, the light bulb becomes bright in an imperceptibly short-time interval, and then gradually goes out. We can interpret this to mean that



(a) Electrons on the negative terminal of the battery reach a capacitor plate almost instantaneously
(b) Electrons from a capacitor plate travel to the positive terminal of the battery almost instantaneously
(c) The capacitor must already have had charge on its plates
(d) The battery establishes an electric field in the circuit almost instantaneously

22. If in the circuit, power dissipation is 150 W, then R is



(a) 2 Ω **(b)** 6 Ω
(c) 5 Ω **(d)** 4 Ω

23. A wire when connected to 220 V mains supply has power dissipation P_1. Now the wire is cut into two equal pieces which are connected in parallel to the same supply. Power dissipation in this case is P_2. Then $P_2 : P_1$ is
(a) 1 **(b)** 4
(c) 2 **(d)** 3

24. The mass of product liberated on anode in an electrochemical cell depends on
(a) $(It)^{1/2}$ **(b)** It
(c) I/t **(d)** I^2t
(where t is the time period, for which the current is passed)

25. The negative Zn pole of a Daniel cell, sending a constant current through a circuit, decreases in mass by 0.13 g in 30 minutes. If the electrochemical equivalent of Zn and Cu are 32.5 and 31.5, respectively, the increase in the mass of the positive Cu pole in this time is
(a) 0.180 g **(b)** 0.141 g
(c) 0.126 g **(d)** 0.242 g

26. An ammeter reads upto 1 ampere. Its internal resistance is 0.81 Ω. To increase the range to 10 A the value of the required shunt is
(a) 0.03 Ω **(b)** 0.3 Ω
(c) 0.9 Ω **(d)** 0.09 Ω

27. The length of a given cylindrical wire is increased by 100%. Due to the consequent decrease in diameter the change in the resistance of the wire will be
(a) 200% **(b)** 100%
(c) 50% **(d)** 300%

28. The thermo emf of a thermo-couple is 25 μV/°C at room temperature. A galvanometer of 40 ohm resistance, capable of detecting current as low as 10^{-5} A, is connected with the thermocouple. The smallest temperature difference that can be detected by this system is
(a) 16°C **(b)** 12°C
(c) 8°C **(d)** 20°C

29. The length of a wire of a potentiometer is 100 cm, and the emf of its stand and cell is E volt. It is employed to measure the emf of a battery whose internal resistance is 0.5 Ω. If the balance point is obtained at $l = 30$ cm from the positive end, the emf of the battery is

(a) $\dfrac{30\,E}{100.5}$

(b) $\dfrac{30\,E}{100-0.5}$

(c) $\dfrac{30(E-0.5i)}{100}$, where i is the current in the potentiometer wire

(d) $\dfrac{30\,E}{100}$

30. The electrochemical equivalent of a metal is 3.3×10^{-7} kg per coulomb. The mass of the metal liberated at the cathode when a 3 A current is passed for 2 s will be
(a) 19.8×10^{-7} kg **(b)** 9.9×10^{-7} kg
(c) 6.6×10^{-7} kg **(d)** 1.1×10^{-7} kg

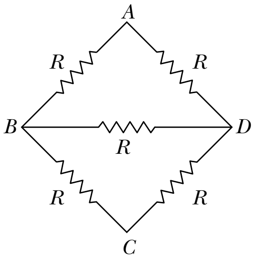
31. The thermistors are usually made of
(a) Metals with low temperature coefficient of resistivity
(b) Metals with high temperature coefficient of resistivity
(c) Metal oxides with high temperature coefficient of resistivity
(d) Semiconducting materials having low temperature coefficient of resistivity

32. A heater of 220 V heats a volume of water in 5 min time. A heater of 110 V heats the same volume of water. Time of heating is
(a) 1000 s **(b)** 1200 s
(c) 1100 s **(d)** 1250 s

33. Two bulbs of wattage 25 and 100, respectively, each rated at 220 V are connected in series with the supply of 440 V. The bulb that will fuse is
(a) 100 W **(b)** 25 W
(c) Both of them **(d)** None of them

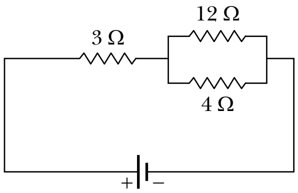
More than One Correct Choice Type

34. Five equal resistances each of value R are connected in a form shown in the following figure. The equivalent resistance of the network



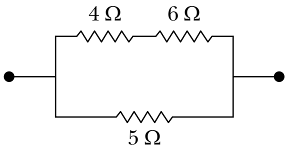
(a) Between points B and D is $R/2$
(b) Between points A and C is R
(c) Between points B and D is R
(d) Between points A and C is $R/2$

35. For the circuit shown in the following figure,



(a) The heat dissipated in 4 Ω resistance is more than that dissipated in 12 Ω resistance
(b) Maximum heat is dissipated in 3 Ω resistance
(c) The heat dissipated in 12 Ω resistance is more than that dissipated in 4 Ω resistance
(d) The power dissipated in 3 Ω resistance is 18 W

36. In the circuit shown in the following figure, the heat product in the 5 Ω resistance due to the current flowing it is 10 cal/s. The heat generated in 4 Ω resistor and 6 Ω resistor is

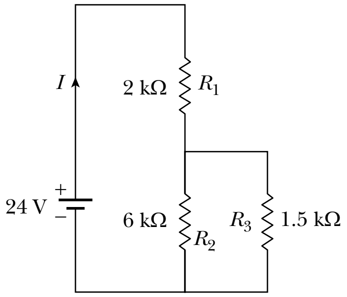


(a) 1 cal/s
(b) 2 cal/s
(c) 3 cal/s
(d) 4 cal/s

37. In a typical Wheatstone's network, the resistances in cyclic order are $P = 10\ \Omega$, $Q = 5\ \Omega$, $S = 4\ \Omega$, and $R = 4\ \Omega$. For the bridge to balance,

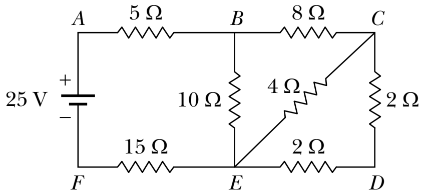
(a) 10 Ω should be connected in series with P
(b) 10 Ω should be connected in parallel with P
(c) 5 Ω should be connected in series with Q
(d) 5 Ω should be connected in parallel with Q

38. For the circuit shown in the following figure,



(a) The current I through the battery is 7.5 mA
(b) The potential difference across R_3 is 18 V
(c) Ratio of powers dissipated in R_1 and R_2 is 3
(d) If R_1 and R_2 are interchanged, magnitude of the power dissipated in R_3 will decrease by a factor of 9

39. For the circuit shown in the following figure, the

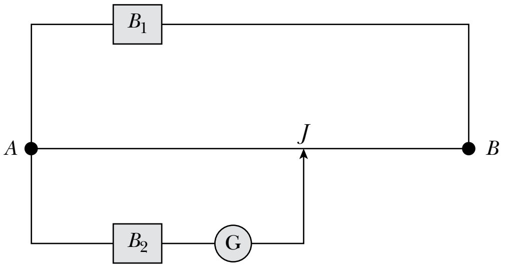


(a) Potential difference between points B and E is 5 V
(b) Potential difference between points A and B is 10 V
(c) Current in the 4 Ω resistor is 0.25 A
(d) Current in the 4 Ω resistor is 0.5 A

40. Which of the following statements is/are correct?

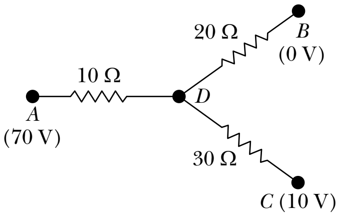
(a) If n identical cells are connected in series and then the battery thus formed is short circuited by a conduction wire, current through the wire will be independent of n
(b) If n identical cells are connected in parallel and then the battery thus formed is short circuited by conducting wire, the current through the wire will be proportional to n
(c) If n identical cells are connected in parallel are then the battery thus formed is short circuited by a wire having a constant resistance, the current through the wire will increase as n increases
(d) None of these

41. The below figure shows a potentiometer arrangement, where B_1 is the driving cell, B_2 is the cell whose emf is to be determined. AB is the potentiometer wire and G is the galvanometer. J is a sliding contact which can touch any point on AB. Which of the following are essential conditions for obtaining balance?



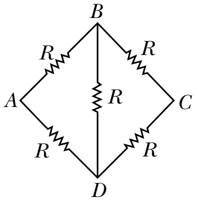
(a) The emf of B_1 must be greater than the emf of B_2
(b) Either the positive terminals of both B_1 and B_2 or the negative terminals of both B_1 and B_2 must be joined at A
(c) The positive terminals of B_1 and B_2 must be joined to A
(d) The resistance of G must be less than the resistance of AB

42. In the network shown in the below figure, points A, B, and C are at potentials of 70 V, zero, and 10 V, respectively. Then



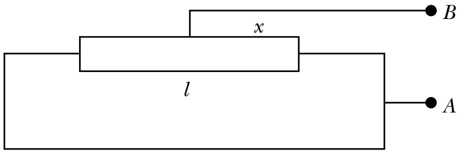
(a) The potential of point D is 40 V
(b) The currents in the sections AD, DB, and DC are in the ratio 3:2:1
(c) The currents in the sections AD, DB, and DC are in the ratio of 1:2:3
(d) The network draws a total power of 200 W

43. Five equal resistance each of value R are connected in a form shown in the following figure. The equivalent resistance of the network



(a) Between the points B and D is R
(b) Between the points B and D is $R/2$
(c) Between the points A and C is R
(d) Between the points A and C is $R/2$

44. Two ends of an 0.1 MΩ potentiometer are connected in the way as shown in the following figure.



(a) $R_{AB} = 0.1$ MΩ for all values of x
(b) $R_{AB} > 0.1$ MΩ for all values of x
(c) $R_{AB} \leq 0.1$ MΩ for all values of x
(d) $R_{AB} \leq 0.05$ MΩ for all values of x

45. Capacitor C_1 of capacitance 1 microfarad and capacitor C_2 of capacitance 2 μF are separately charged fully by a common battery. The two capacitors are then separately allowed to discharge through equal resistors at time $t = 0$.
(a) The current in each of the two discharging circuits is zero at $t = 0$
(b) The current in the two discharging circuits at $t = 0$ are equal but not zero
(c) The current in the two discharging circuits at $t = 0$ are unequal
(d) Capacitor C_1 loses 50% of its initial charge sooner than C_2 loses 50% of its initial charge

Linked Comprehension

Paragraph for Questions 46–49: An electrolytic cell containing a solution of $CuSO_4$ has an internal resistance of 1 Ω. It is connected in series with 3 V battery of negligible resistance and a 4 Ω resistance. The mass of copper which will be deposited on the copper electrode in 30 min is calculated. If 4 Ω resistance is connected in parallel across the electrolytic cell and the same battery is used, the amount of copper deposited in 30 min is calculated ECE of Cu = 0.00033 g/C. Using Faraday's first law of electrolysis, the mass deposited on the copper electrode can be calculated.

46. For series connection of 4 Ω resistor, the current flowing through electrolyte is
(a) 6 A **(b)** 0.6 A
(c) 0.2 A **(d)** 1 A

47. Mass of copper deposited in 30 min when resistance of 4 Ω is in series is
(a) 0.214 g **(b)** 0.156 g
(c) 0.356 g **(d)** 0.653 g

48. For the resistance of 4 Ω connected in parallel to cell, the current in the circuit is
(a) 3.75 A **(b)** 7.35 A
(c) 5.73 A **(d)** 5 A

49. Mass of copper deposited in 30 min when resistance of 4 Ω is in parallel is
(a) 4.214 g **(b)** 3.337 g
(c) 2.227 g **(d)** 1.117 g

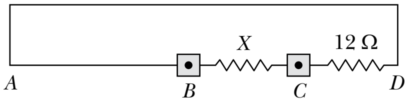
Paragraph for Questions 50–52: The alternate discs of iron and carbon, having same area of cross-section are cemented together to make a cylinder whose temperature coefficient of resistivity is zero. The change in temperature in two alternate discs is same. The ratio of their thickness and ratio of heat produced in them is found out. The resistivity of iron and carbon at 20°C are 1×10^{-7} and 3×10^{-5} Ωm and their temperature coefficient of resistance are 5×10^{-3}°C and -7.5×10^{-4}°C, respectively. Thermal expansion is neglected. Here, $\Delta R_1 + \Delta R_2 = 0$ (where ΔR_1 and ΔR_2 are the increase in resistances of iron and carbon, respectively, with the rise in temperature) because combined temperature coefficient of resistivity is given as zero.

50. Ratio of their thickness is
(a) 54 **(b)** 45
(c) 35 **(d)** 21

51. Ratio of heat produced in them is
(a) 0.51 **(b)** 1
(c) 0.15 **(d)** 2

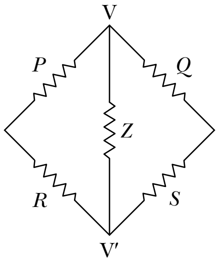
52. A copper wire is stretched to make it 1% longer. The percentage change in its resistance is
(a) 0.2% **(b)** 1%
(c) 1.5% **(d)** 2.5%

Paragraph for Questions 53–55: A thin uniform wire AB of length 1 m, an unknown resistance X and a resistance of 12 Ω are connected by thick conducting strips, as shown in the following figure.



A battery and a galvanometer (with a sliding jockey connected to it) are also available. Connections are to be made to measure the unknown resistance X using the principle of Wheatstone's bridge. For a balanced Wheatstone's bridge as shown in the following figure, we have

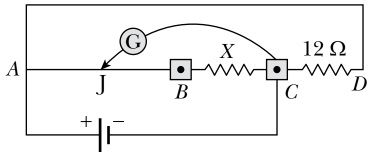
$$\frac{P}{Q} = \frac{R}{S}$$



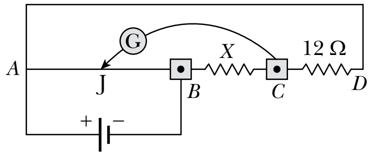
53. The galvanometer has
- **(a)** Positive terminal marked on it
- **(b)** Negative terminal marked on it
- **(c)** Both *A* and *B*
- **(d)** None of these

54. Among those shown in the following figures, the correct figure showing the battery and the galvanometer (with jockey) connected at appropriate points is

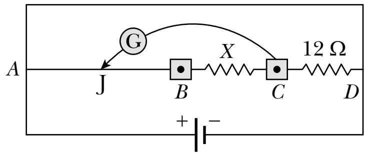
(a)



(b)



(c)



(d) None of these

55. After appropriate connections are made, it is found that no deflection takes place in the galvanometer when the sliding jockey touches the wire at a distance of 60 cm from A. The value of the resistance *X*:
- **(a)** 6 Ω
- **(b)** 8 Ω
- **(c)** 12 Ω
- **(d)** 9 Ω

Paragraph for Questions 56–57: A resistor, capacitor, switch, and ideal battery are in series. Originally the capacitor is uncharged. The switch is then closed, allowing current of flow.

56. While the current is flowing, the potential difference across the resistor is
- **(a)** Increasing
- **(b)** Decreasing
- **(c)** Fixed
- **(d)** First increase then decreases

57. While the current is flowing, the potential difference across the capacitor is
- **(a)** Increasing
- **(b)** Decreasing
- **(c)** Fixed
- **(d)** First increase then decreases

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

58.

Column I	Column II
(a) The capacity of supplying the current in the circuit	**(p)** Voltmeter
(b) The electric energy consumed in the circuit	**(q)** Potentiometer
(c) emf of the cell in a closed circuit	**(r)** watt-hour
(d) Maximum potential difference between two points of a circuit	**(s)** ampere-hour

Directions for Questions 59 and 60: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

59. Electrical current is the flow of charged particles. The electric current flows through a wire the same way as how water moves in a river. The electricity is nothing but the flow of electrons. There are two main combinations of resistors in circuit, that is series and parallel. In the given table, Column I shows different combinations of resistors, Column II shows voltages of different arrangements of resistors and Column III shows currents and voltage division needed or not in different arrangements of resistors.

Column I	Column II	Column III
(I) Series connected resistors	**(i)** voltage is different while current is same	**(J)** voltage division needed
(II) Parallel connected resistors	**(ii)** voltage and current are different	**(K)** voltage division not needed
(III) When resistors are equal	**(iii)** voltage and current are same	**(L)** current division not needed
(IV) Both series and parallel resistors.	**(iv)** voltage is same while current is different	**(M)** current division needed

(1) When does the voltage division problem arise?
- **(a)** (I) (i) (J)
- **(b)** (IV) (i) (M)
- **(c)** (III) (ii) (L)
- **(d)** (I) (iii) (L)

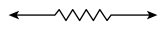
(2) When there is no voltage division problem arise?
- **(a)** (II) (iv) (K)
- **(b)** (IV) (i) (M)
- **(c)** (I) (ii) (L)
- **(d)** (III) (iv) (K)

(3) When does the current division problem arises?
- **(a)** (III) (i) (K)
- **(b)** (IV) (i) (J)
- **(c)** (III) (iii) (M)
- **(d)** (II) (iv) (M)

60. Electrical symbols and electronic circuit symbols are used for drawing schematic diagram. The symbols represent electrical and electronic components. In the given table, Column I shows different components of electric circuit, Column II shows special property of the different components of electric circuit and Column III shows units of different components of electric circuit or effect of the different components of electric circuit.

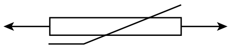
Column I	Column II	Column III
(I) Battery Cell	**(i)** store electric charge	**(J)** measured in units of Farad
(II) Capacitor	**(ii)** Generates constant voltage	**(K)** measured in units of ohms
(III) Resistor	**(iii)** change resistance when temperature changes	**(L)** one or more electrochemical cells with external connections
(IV) Thermistor	**(iv)** reduces the current flow	**(M)** reduced by heating

(1) What does the symbol depict?



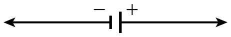
(a) (I) (ii) (J) **(b)** (I) (ii) (L)
(c) (III) (iv) (K) **(d)** (I) (iv) (K)

(2) What does the symbol depict?



(a) (III) (iv) (K) **(b)** (IV) (iii) (M)
(c) (III) (iii) (L) **(d)** (I) (i) (M)

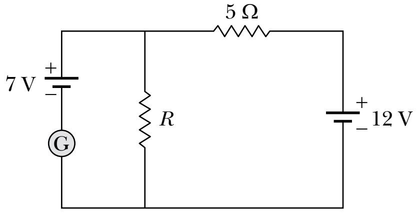
(3) What does the symbol depict?



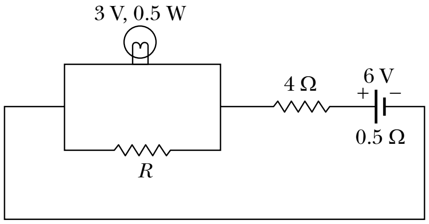
(a) (III) (i) (L) **(b)** (IV) (i) (J)
(c) (III) (iii) (K) **(d)** (I) (ii) (L)

Integer Type

61. In the circuit shown in the following figure, for what value of R (Ω) will the current in galvanometer G be zero?



62. In the circuit shown in the following figure, what would be the value of resistance R (Ω) so that the electric bulb consumes the rated power?



63. When two identical batteries of internal resistance 1 Ω each are connected in series across a resistor R, the rate of heat produced in R is J_1. When the same batteries are connected in parallel across R, the rate is J_2. If $J_1 = 2.25\ J_2$, then what is the value of R in ohms?

64. A battery of emf ε and internal resistance r is connected to an external resistance R. The maximum power in the external circuit is 9 W. The current flowing in the circuit under the conditions is 3 A. What is the value of ε in volt?

65. Thirty-two cells each of emf 3 V are connected in series and kept in a box. Externally, the combination shows an emf of 84 V. How many numbers of cells are connected reversely?

ANSWER KEY

Checkpoints

1. (a) rightward; (b) all tie; (c) *b*, then *a* and *c* tie; (d) *b*, then *a* and *c* tie

2. (a) all tie; (b) R_1, R_2, R_3

3. (a) less; (b) greater; (c) equal

4. (a) $V/2, i$; (b) $V, i/2$

5. 20 Ω

6. (a) $R = 10^6\,\Omega - 100\,\Omega \approx 10^6\,\Omega$; (b) $r = 0.5\,\Omega$

7. (a) 1, 2, 4, 3; (b) 4, tie of 1 and 2; then 3

Problems

1. (a) $2.03 \times 10^{11}\,\Omega$; (b) 1.60×10^{-3} J; (c) 4.43×10^{-9} J/s

2. −0.167 A; (b) 1.41 A

3. (a) 0.177 W; (b) 0.670 W; (c) 0.679 W; (d) −0.211 W; (e) +1.74 W; (f) The negative sign indicates that ξ_1 is actually absorbing energy from the circuit; (g) The positive sign indicates that ξ_2 is actually providing energy to the circuit.

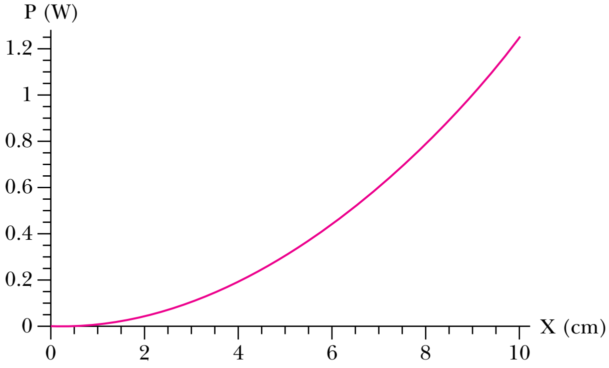
4. (a) 0.250 Ω; (b)144 W; (c) 144 W; (d) $r/2$; (e) $r/2$

5. (a) 0.828 A; (b) 7.29 V; (c) 88.0 Ω; (d) remain the same

6. (a) 119 Ω; (b)1.01×10^{-1} A; (c) 3.79×10^{-2} A; (d) 3.79×10^{-2} A; (e) 2.53×10^{-2} A

7. (a) 1.66 Ω; (b) 0; (c) ∞; (d) 69.1 W; (e) 48.4 W

8.



9. 2.10 Ω

10. (a) 2.28 s; (b) 46.5 mV

11. 96.6 V

12. (a) 0.04365 A (downward); (b) downward; (c) 0.0182 A; (d) rightward; (e) 0.0254 A; (f) leftward; (g) +4.37 V

13. –3%

14. 1.09×10^{-3} s

15. $\frac{5}{6}R$

16. 4.1 Ω

17. 2 Ω

18. (a) 10.5 V; (b) 3.00 V; (c) 10.5 V; (d) 13.5 V

19. (a) 0.013 Ω; (b) battery 1

20. (a) 0.29; (b) 0.69

21. (a) 25 V; (b) 21 V; (c) The negative terminal is at *B*

22. 4*d*

23. (a) 0.168 A; (b) 11.2 V; (c) 66.2 Ω; (d) $R' \rightarrow R$ as $R_V \rightarrow \infty$

24. (a) 1.52 V; (b) 0.07 Ω

25. 6 V

26. 1.25×10^{-2} V

27. 0251 Ω

28. (a) 1.80×10^{-7} C/s; (b) 62.4×10^{-7} W; (c) 9.77×10^{-8} W; (d) 7.22×10^{-7} W

29. (a) 3.78 s; (b) 32.4 μC; (c) 2.57 s

30. –10 V

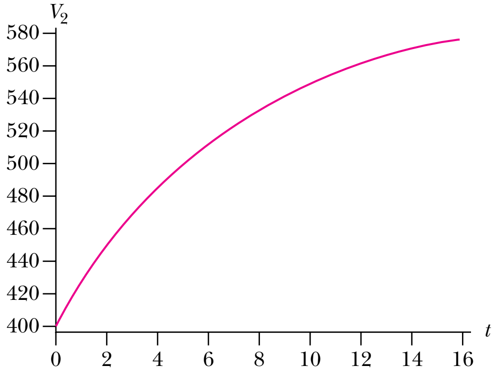
31. 2.14×10^{6} Ω

32. 4.00 A

33. +0.54 V

34. (a) 1.1×10^{-3} A; (b) 5.5×10^{-4} A; (c) 5×10^{-4} A; (d) 8.2 .5 $\times 10^{-4}$ A; (e) 8.2 .5 $\times 10^{-4}$ A; (f) 0; (g) 4.2×10^{2} V; (h) 6.0×10^{2} V;

(i)



35. (a) 5.0 Ω; (b) 15 Ω

36. (a) 13.2 V ≈ 13 V; (b) 48 W; (c) 4.8×10^{2} W; (d) 10.8 V ≈ 11 V; (e) 48 W

37. 3.5 Ω

38. (a) 0.54 A

39. (a) 7.50 Ω; (b) parallel; (c) 22.5 Ω

40. 31 kJ

41. (a) 320 cm; (b) $3\xi/22r$

42. (a) 6 V, 2 V; (b) *AD* = 66.7 cm; (c) zero; (d) 6 V, –1.5 V, no such point *D* exists

43. 16 cm from *A*

44. (a) 0.10 A; (b) –0.16 A; (c) 14 V

45. $\rho l / \pi ab$

Practice Questions

Single Correct Choice Type

1. (d)
2. (a)
3. (c)
4. (b)
5. (d)
6. (a)
7. (a)
8. (a)
9. (d)
10. (b)
11. (d)
12. (a)
13. (a)
14. (a)
15. (d)
16. (a)
17. (a)
18. (d)
19. (d)
20. (b)
21. (d)
22. (b)
23. (b)
24. (b)
25. (c)

26. (d)
27. (d)
28. (a)
29. (d)
30. (a)
31. (c)
32. (b)
33. (b)

More than One Correct Choice Type

34. (a), (b)
35. (a), (b)
36. (b), (c)
37. (b), (c)
38. (a), (d)
39. (a), (d)
40. (a), (b), (c)
41. (a), (b)
42. (a), (b), (d)
43. (b), (c)
44. (c), (d)
45. (b), (d)

Linked Comprehension

46. (b)
47. (c)
48. (a)
49. (c)
50. (b)
51. (c)
52. (a)
53. (c)
54. (b)
55. (b)
56. (b)
57. (a)

Matrix-Match

58. (a) → (s); (b) → (r); (c) → (p); (d) → (p), (q)
59. (1) → (a); (2) → (a); (3) → (d)
60. (1) → (c); (2) → (b); (3) → (d)

Integer Type

61. 7
62. 6
63. 4
64. 6
65. 2

CHAPTER

28

Magnetic Force

Contents

28.1 What is Physics?
28.2 What Produces a Magnetic Field?
28.3 The Definition of $\vec{B}$
28.4 A Circulating Charged Particle
28.5 Cyclotrons
28.6 Crossed Fields: Discovery of the Electron
28.7 The Hall Effect
28.8 Magnetic Force on a Current-Carrying Wire
28.9 Torque on a Current Loop
28.10 The Magnetic Dipole Moment

28.1 | WHAT IS PHYSICS?

As we have discussed, one major goal of physics is the study of how an *electric field* can produce an *electric force* on a charged object. A closely related goal is the study of how a *magnetic field* can produce a *magnetic force* on a (moving) charged particle or on a magnetic object such as a magnet. You may already have a hint of what a magnetic field is if you have ever attached a note to a refrigerator door with a small magnet or accidentally erased a credit card by moving it near a magnet. The magnet acts on the door or credit card via its magnetic field.

The applications of magnetic fields and magnetic forces are countless and changing rapidly every year. Here are just a few examples. For decades, the entertainment industry depended on the magnetic recording of music and images on audiotape and videotape. Although digital technology has largely replaced magnetic recording, the industry still depends on the magnets that control CD and DVD players and computer hard drives; magnets also drive the speaker cones in headphones, TVs, computers, and telephones. A modern car comes equipped with dozens of magnets because they are required in the motors for engine ignition, automatic window control, sunroof control, and windshield wiper control. Most security alarm systems, doorbells, and automatic door latches employ magnets. In short, you are surrounded by magnets.

The science of magnetic fields is physics; the application of magnetic fields is engineering. Both the science and the application begin with the question "What produces a magnetic field?"

28.2 | WHAT PRODUCES A MAGNETIC FIELD?

Key Concept

- When a charged particle moves through a magnetic field $\vec{B}$, a magnetic force acts on the particle as given by

$$\vec{F}_B = q(\vec{v} \times \vec{B}),$$

where q is the particle's charge (sign included) and $\vec{v}$ is the particle's velocity.

Digital Vision/Getty Images, Inc.

Figure 28-1 Using an electromagnet to collect and transport scrap metal at a steel mill.

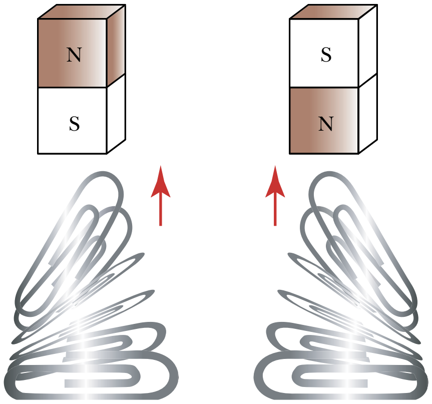


Figure 28-2 Bar magnets and paper clips.

Figure 28-3 A commercially available compass.

Because an electric field $\vec{E}$ is produced by an electric charge, we might reasonably expect that a magnetic field $\vec{B}$ is produced by a magnetic charge. Although individual magnetic charges (called *magnetic monopoles*) are predicted by certain theories, their existence has not been confirmed. How then are magnetic fields produced? There are two ways.

One way is to use moving electrically charged particles, such as a current in a wire, to make an **electromagnet**. The current produces a magnetic field that can be used, for example, to control a computer hard drive or to sort scrap metal (Fig. 28-1). In Chapter 29, we discuss the magnetic field due to a current.

The other way to produce a magnetic field is by means of elementary particles such as electrons because these particles have an *intrinsic* magnetic field around them. That is, the magnetic field is a basic characteristic of each particle just as mass and electric charge (or lack of charge) are basic characteristics. As we discuss in Chapter 32, the magnetic fields of the electrons in certain materials add together to give a net magnetic field around the material. Such addition is the reason why a **permanent magnet**, the type used to hang refrigerator notes, has a permanent magnetic field. In other materials, the magnetic fields of the electrons cancel out, giving no net magnetic field surrounding the material. Such cancellation is the reason you do not have a permanent field around your body, which is good because otherwise you might be slammed up against a refrigerator door every time you passed one.

If you touch the surface of a magnet with an object containing iron, such as a paper clip, you will discover that certain parts of the magnets, called magnetic poles, interact more strongly with the paper clip than other parts of the magnet. Disk-shaped refrigerator magnets usually have poles on either end; paper clips stick to the flat ends, but don't stick to the other surface.

Bar magnets (Fig. 28-2) also have poles at the ends; moreover, the sides of the bar don't interact very strongly with a paper clip. In fact, some magnets (like the magnetic strips) may have more than one pole. It just has as many north poles as south poles. Like charges, magnets can attract or repel without touching. This tells us that magnetic interactions are of long range and that there must be two types of magnetic poles.

A magnet interacts with an iron object via a magnetic field it produces in space. When a bar magnet with one pole on each end is suspended to rotate freely, it aligns itself so that its poles lie on a line roughly in south–north direction of Earth. Actually, the Earth's magnetic field exerts a torque on this magnet to align it in the direction of stable equilibrium (and minimum energy). This north–south alignment of a freely rotating magnet is the basic operational principle of a compass needle, which is simply a freely rotating magnet (Fig. 28-3).

This alignment provides a way of distinguishing between the two types of poles. The end of the magnet which aligns toward the north pole is called the north pole (denoted by N), and the opposite end is called south pole (denoted by S). Commercially available magnets are painted red on the portion which acts as a north pole. To understand what causes this alignment, we must first examine how one magnet interacts with other. If you hold the poles of two magnets near one another, you find that like poles repel each other and opposite poles attract each other (see Fig. 28-4).

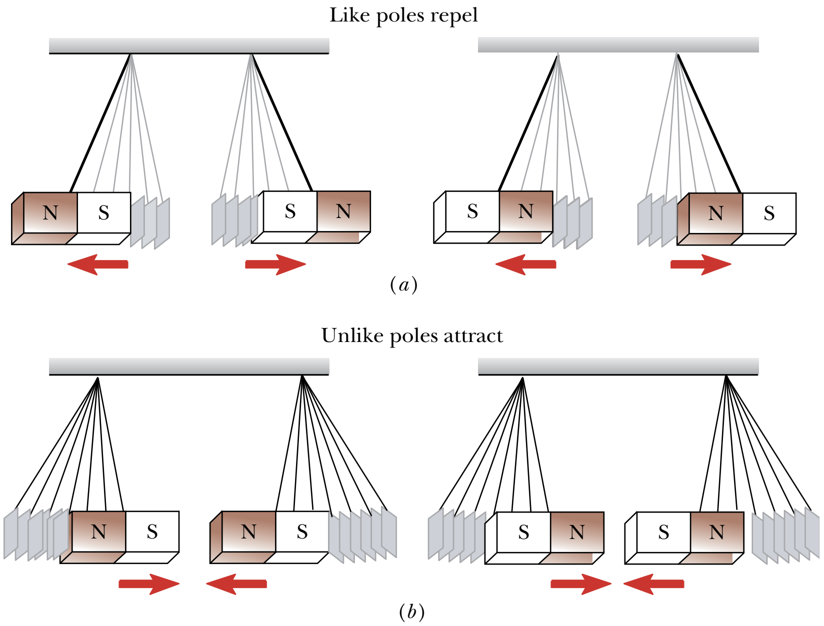


Figure 28-4 (*a*) Like poles repel, (*b*) unlike poles attract.

Materials that are attracted by both types of magnetic poles are iron, nickel, cobalt and certain alloys such as steel (but not stainless steel). These materials are called magnetic materials. Normally, two objects made from these materials do not interact with one another. For example, one paper clip containing iron does not attract another one. However, just like an object carrying charge induces a charge polarization on a neutral object, the magnetic field induces a magnetic polarization in a paper clip or any other object made from magnetic materials. The poles of the magnet then interact with the induced poles of the magnetized paper clip (see Fig. 28-2).

However, unlike induced polarizations, which vanish as soon as the charged object is removed, some magnetic materials retain their induced magnetic polarization. There is a fundamental difference between magnetic and electrostatic interactions: it is not possible to isolate the N pole or the S pole of a magnet the way one can separate positive and negative charges from each other. For example, if we cut a bar magnet in two, we find that two new poles have appeared and that the resulting pieces each have two opposite poles at the ends (see Fig. 28-5). The cutting has created an additional pair of opposite poles. Remarkably, if we carefully place the two pieces together again, the two new poles seem to vanish.

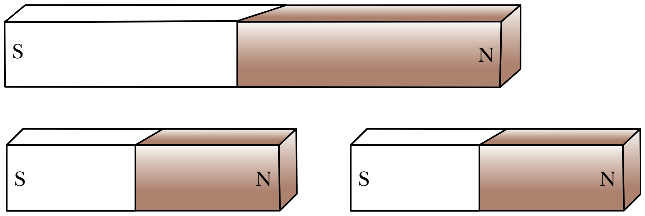


Figure 28-5 If you break a magnet, each fragment becomes a separate magnet, with its own north and south poles.

CHECKPOINT 1

If we have two similar looking bars, one of which is a magnet and the other a bar of iron, how will we distinguish between the two by using these two bars? We cannot suspend them in the Earth's magnetic field.

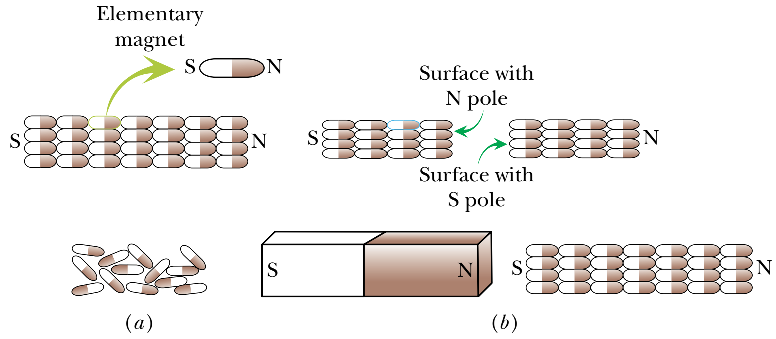


Figure 28-6 (*a*) Unmagnetized and (*b*) magnetized piece of magnetic material.

Experiments show that the iron, nickel, or cobalt atoms which make up magnets are indeed "elementary magnets" each with an N pole and an S pole. A magnet thus simply consists of a large number of elementary magnets whose alignment is such that their combined effect is reinforced at the poles. Because we cannot cut the atoms that are the elementary magnets themselves, a pair of new poles appears every time we cut through the magnet. In an unmagnetized piece of magnetic material, the elementary magnets are randomly aligned–some push or pull in one direction while others push or pull in other directions – and so their effects cancel, and the piece as a whole does not act like a magnet (Fig. 28-6*a*). When another magnet is brought nearby, however, the elementary magnets align themselves to the nearby magnet and the material becomes magnetized (Fig. 28-6*b*).

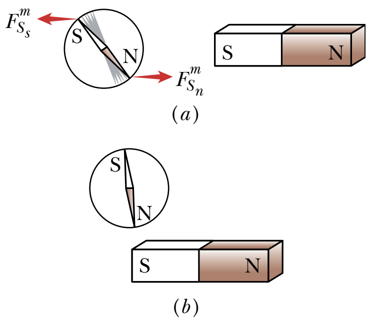


Figure 28-7 (*a*) A magnet exerts a torque on a compass. We can safely assume that the magnetic field exerts a force on the north pole in the direction of the field and that on the south pole in the opposite direction. (*b*) In the equilibrium position, the compass will point in the direction of the magnetic field.

You can also use this model to visualize the demagnetization of a magnet. If a magnetized piece of material is heated or roughly handled, the elementary magnets are jarred around and they lose their alignment. Because there is no such thing as a "magnetic charge," it is not possible, as in electrostatics, to map out a magnetic field using a "test magnetic charge." We can, however, use a compass to map out the direction of the magnetic field. When a compass is placed near a bar magnet, as illustrated in Fig. 28-7, the S pole of the bar magnet exerts an attractive force on the N pole of the compass needle and a repulsive force on the S pole. These two forces cause a torque that tends to align the needle with its N pole pointing toward the magnet's S pole. Even in a general magnetic field, the N pole of a magnet experiences a force in the direction of the magnetic field and the S pole experience a force in the direction opposite to the magnetic field.

Magnetic Field Lines

We can represent magnetic fields with field lines, as we did for electric fields. Similar rules apply: (1) the direction of the tangent to a magnetic field line at any point gives the direction of magnetic field at that point, and (2) the spacing of the lines represents the magnitude of magnetic field–the magnetic field is stronger where the lines are closer together, and conversely.

Figure 28-8 shows how the magnetic field near a *bar magnet* (a permanent magnet in the shape of a bar) can be represented by magnetic field lines. The lines all pass through the magnet, and they all form closed loops (even those that are not shown closed in the figure). The external magnetic effects of a bar magnet are strongest near its ends, where the field lines are most closely spaced. Thus, the magnetic fields of the bar magnets in Figs. 28-8 and 28-9 collect the iron filings mainly near the two ends of the magnet. You might have done this experiment in your junior classes. The pattern which emerges is something similar to the electric field lines in space.

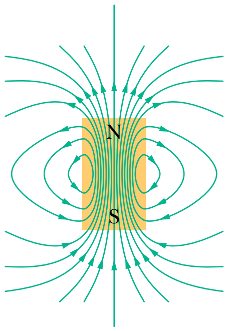


Figure 28-8 The magnetic field lines for a bar magnet.

Two points can be noted here. At the points near the poles, the density of these magnetic field lines is high, and near the center of the magnet, the density of these lines is low. Another point is that if we keep a magnetic compass at some point, it will align itself in the same direction as the magnetic field lines.

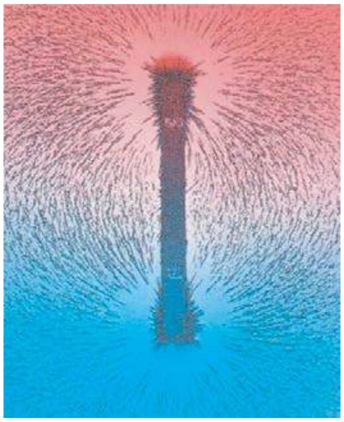


Figure 28-9 Using iron fillings to map the magnetic field.

The Magnetism of Earth

Earth is a huge magnet; for points near Earth's surface, its magnetic field can be approximated as the field of a huge bar magnet–a magnetic dipole–that straddles the center of the planet. Figure 28-10 is an idealized symmetric depiction of the magnetic field, without the distortion caused by passing charged particles from the Sun.

For the idealized field of Fig. 28-10, the magnet's axis makes an angle of 11.5° with the rotation axis (RR) of Earth. The *Magnet's axis* (MM in Fig. 28-10) lies along and intersects Earth's surface at the *geomagnetic north pole* in northwest Greenland and the *geomagnetic south pole* in Antarctica. The lines of the magnetic field generally emerge in the Southern Hemisphere and reenter Earth in the Northern Hemisphere. Thus, the magnetic pole that is in Earth's Northern Hemisphere and known as a "north magnetic pole" *is really the south pole of Earth's magnet.*

The direction of the magnetic field at any location on Earth's surface is commonly specified in terms of two angles. The **angle of declination** (ϕ) is the angle (left or right) between geographic north and the horizontal component of the field. The **angle of dip (or field inclination)** (θ) is the angle (up or down) between a horizontal plane and the field's direction. If we include the **horizontal component of the Earth's magnetic field** as well, these three are called the elements of the Earth's magnetic field. So, if the magnitude of the magnetic field of Earth is B at a certain point, then its horizontal component is given by

$$B_h = B\cos\theta.$$

The vertical component of the magnetic field is given by

$$B_v = B\sin\theta.$$

So, if you are navigating in sea using only a compass, you are sure to land at a wrong place. You need to know the angle of declination at your location to find the true north direction (Fig. 28-11).

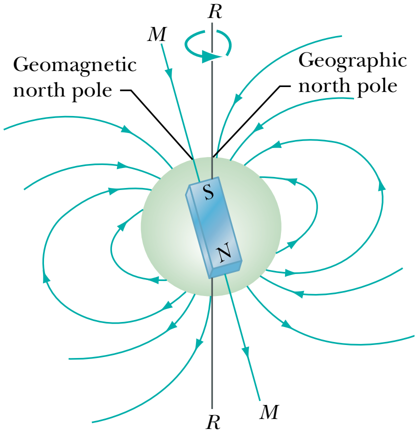


Figure 28-10 Magnetic field lines due to Earth. The dipole axis MM makes an angle of 11.5° with Earth's rotational axis RR. The south pole of the magnet is in Earth's Northern Hemisphere.

At any point on Earth's surface, the measured magnetic field may differ appreciably, in both magnitude and direction, from the idealized field of Fig. 28-11. In fact, the point where the field is actually perpendicular to Earth's

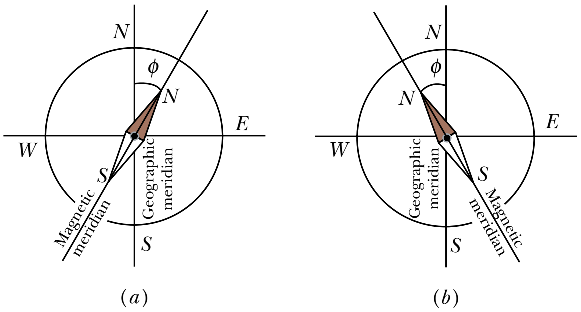


Figure 28-11 The arrangement of a compass: (a) in the region with eastern magnetic declination; (b) in the region with western magnetic declination. The eastern declination is often denoted by the plus sign and the western declination by the minus sign.

surface and inward is not located at the geomagnetic North Pole in Greenland as we would expect; instead, this so-called *dip north pole* is located in the Queen Elizabeth Islands in northern Canada, far from Greenland. In addition, the field observed at any location on the surface of Earth varies with time, by measurable amounts over a period of a few years and by substantial amounts over, say, 100 years. In fact, the mechanism that produces Earth's magnetic field is only vaguely understood.

Our first job in this chapter is to define the magnetic field $\vec{B}$. We do so by using the experimental fact that when a charged particle moves through a magnetic field, a magnetic force $\vec{F}_B$ acts on the particle.

28.3 | THE DEFINITION OF $\vec{B}$

Key Concepts

- The right-hand rule for cross products gives the direction of $\vec{v} \times \vec{B}$. The sign of q then determines whether $\vec{F}_B$ is in the same direction as $\vec{v} \times \vec{B}$ or in the opposite direction.
- The magnitude of $\vec{F}_B$ is given by
$$F_B = |q|vB\sin\phi,$$
where ϕ is the angle between $\vec{v}$ and $\vec{B}$.

We determined the electric field $\vec{E}$ at a point by putting a test particle of charge q at rest at that point and measuring the electric force $\vec{F}_E$ acting on the particle. We then defined $\vec{E}$ as

$$\vec{E} = \frac{\vec{F}_E}{q}. \tag{28-1}$$

If a magnetic monopole were available, we could define $\vec{B}$ in a similar way. Because such particles have not been found, we must define $\vec{B}$ in another way, in terms of the magnetic force $\vec{F}_B$ exerted on a moving electrically charged test particle.

Moving Charged Particle. In principle, we do this by firing a charged particle through the point at which $\vec{B}$ is to be defined, using various directions and speeds for the particle and determining the force $\vec{F}_B$ that acts on the particle at that point. After many such trials we would find that when the particle's velocity $\vec{v}$ is along a particular axis through the point, force $\vec{F}_B$ is zero. For all other directions of $\vec{v}$, the magnitude of $\vec{F}_B$ is always proportional to $v \sin \phi$, where ϕ is the angle between the zero-force axis and the direction of $\vec{v}$. Furthermore, the direction of $\vec{F}_B$ is always perpendicular to the direction of $\vec{v}$. (These results suggest that a cross product is involved.)

The Field. We can then define a **magnetic field** $\vec{B}$ to be a vector quantity that is directed along the zero-force axis. We can next measure the magnitude of $\vec{F}_B$ when $\vec{v}$ is directed perpendicular to that axis and then define the magnitude of $\vec{B}$ in terms of that force magnitude:

$$B = \frac{F_B}{|q|v},$$

where q is the charge of the particle.

We can summarize all these results with the following vector equation:

$$\vec{F}_B = q\vec{v} \times \vec{B}; \tag{28-2}$$

that is, the force $\vec{F}_B$ on the particle is equal to the charge q times the cross product of its velocity $\vec{v}$ and the field $\vec{B}$ (all measured in the same reference frame). Using Eq. 3-24 for the cross product, we can write the magnitude of $\vec{F}_B$ as

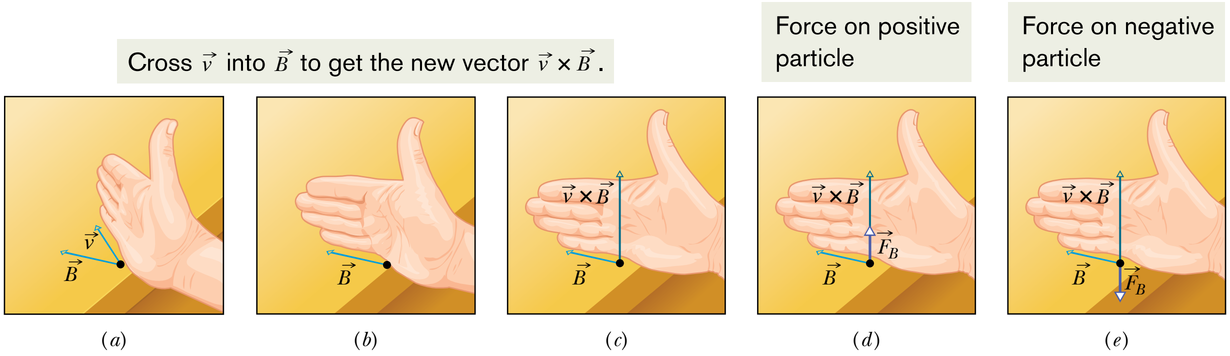
$$F_B = |q|vB\sin\phi, \tag{28-3}$$

where ϕ is the angle between the directions of velocity $\vec{v}$ and magnetic field $\vec{B}$.

Finding the Magnetic Force on a Particle

Equation 28-3 tells us that the magnitude of the force $\vec{F}_B$ acting on a particle in a magnetic field is proportional to the charge q and speed v of the particle. Thus, the force is equal to zero if the charge is zero or if the particle is stationary. Equation 28-3 also tells us that the magnitude of the force is zero if $\vec{v}$ and $\vec{B}$ are either parallel ($\phi = 0°$) or antiparallel ($\phi = 180°$), and the force is at its maximum when $\vec{v}$ and $\vec{B}$ are perpendicular to each other.

Directions. Equation 28-2 tells us all this plus the direction of $\vec{F}_B$. We know that the cross product $\vec{v} \times \vec{B}$ in Eq. 28-2 is a vector that is perpendicular to the two vectors $\vec{v}$ and $\vec{B}$. The right-hand rule (Figs. 28-12*a* through *c*) tells us that the thumb of the right hand points in the direction of $\vec{v} \times \vec{B}$ when the fingers sweep $\vec{v}$ into $\vec{B}$. If q is positive, then (by Eq. 28-2) the force $\vec{F}_B$ has the same sign as $\vec{v} \times \vec{B}$ and thus must be in the same direction; that is, for positive q, $\vec{F}_B$ is directed along the thumb (Fig. 28-12*d*). If q is negative, then the force $\vec{F}_B$ and cross product $\vec{v} \times \vec{B}$ have opposite signs and thus must be in opposite directions. For negative q, $\vec{F}_B$ is directed opposite the thumb (Fig. 28-12*e*). *Heads up:* Neglect of this effect of negative q is a very common error on exams.


Cross $\vec{v}$ into $\vec{B}$ to get the new vector $\vec{v} \times \vec{B}$.

Force on positive particle

Force on negative particle

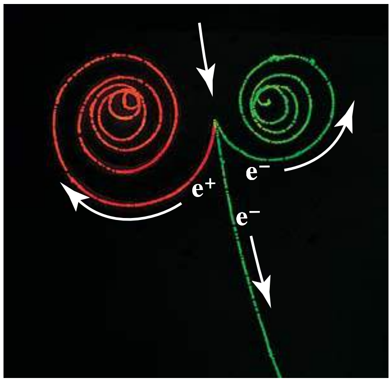
Figure 28-12 (*a*)–(*c*) The right-hand rule (in which $\vec{v}$ is swept into $\vec{B}$ through the smaller angle ϕ between them) gives the direction of $\vec{v} \times \vec{B}$ as the direction of the thumb. (*d*) If q is positive, then the direction of $\vec{F}_B = q\vec{v} \times \vec{B}$ is in the direction of $\vec{v} \times \vec{B}$. (*e*) If q is negative, then the direction of $\vec{F}_B$ is opposite that of $\vec{v} \times \vec{B}$.

Regardless of the sign of the charge, however,

> The force $\vec{F}_B$ acting on a charged particle moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ is *always* perpendicular to $\vec{v}$ and $\vec{B}$.

Thus, $\vec{F}_B$ *never* has a component parallel to $\vec{v}$. This means that $\vec{F}_B$ cannot change the particle's speed v (and thus it cannot change the particle's kinetic energy). The force can change only the direction of $\vec{v}$ (and thus the direction of travel); only in this sense can $\vec{F}_B$ accelerate the particle.

To develop a feeling for Eq. 28-2, consider Fig. 28-13, which shows some tracks left by charged particles moving rapidly through a *bubble chamber*. The chamber, which is filled with liquid hydrogen, is immersed in a strong uniform magnetic field that is directed out of the plane of the figure. An incoming gamma ray particle—which leaves no track because it is uncharged—transforms into an electron (spiral track marked e^-) and a positron (track marked e^+) while it knocks an electron out of a hydrogen atom (long track marked e^-). Check with Eq. 28-2 and Fig. 28-12 that the three tracks made by these two negative particles and one positive particle curve in the proper directions.



Lawrence Berkeley Laboratory/Photo Researchers, Inc.

Figure 28-13 The tracks of two electrons (e^-) and a positron (e^+) in a bubble chamber that is immersed in a uniform magnetic field that is directed out of the plane of the page.

Unit. The SI unit for $\vec{B}$ that follows from Eqs. 28-2 and 28-3 is the newton per coulomb-meter per second. For convenience, this is called the **tesla** (T):

$$1 \text{ tesla} = 1 \text{ T} = 1 \frac{\text{newton}}{(\text{coulomb})(\text{meter/second})}.$$

Recalling that a coulomb per second is an ampere, we have

$$1\ \mathrm{T} = 1\frac{\text{newton}}{(\text{coulomb/second})(\text{meter})} = 1\frac{\mathrm{N}}{\mathrm{A \cdot m}}. \tag{28-4}$$

An earlier (non-SI) unit for $\vec{B}$, still in common use, is the *gauss* (G), and

$$1\ \text{tesla} = 10^4\ \text{gauss}. \tag{28-5}$$

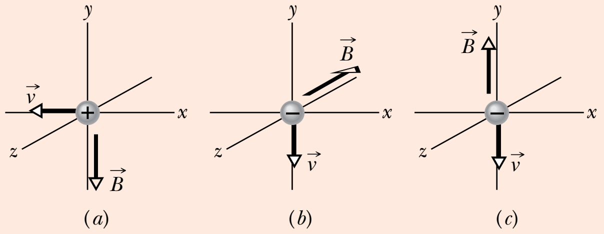
Table 28-1 lists the magnetic fields that occur in a few situations. Note that Earth's magnetic field near the planet's surface is about 10^{-4} T (= 100 μT or 1 G).

Table 28-1 Some Approximate Magnetic Fields

At surface of neutron star	10^8 T
Near big electromagnet	1.5 T
Near small bar magnet	10^{-2} T
At Earth's surface	10^{-4} T
In interstellar space	10^{-10} T
Smallest value in magnetically shielded room	10^{-14} T

CHECKPOINT 2

The figure shows three situations in which a charged particle with velocity $\vec{v}$ travels through a uniform magnetic field $\vec{B}$. In each situation, what is the direction of the magnetic force $\vec{F}_B$ on the particle?



SAMPLE PROBLEM 28.01

Magnetic force on a moving charged particle

A uniform magnetic field $\vec{B}$, with magnitude 1.2 mT, is directed vertically upward throughout the volume of a laboratory chamber. A proton with kinetic energy 5.3 MeV enters the chamber, moving horizontally from south to north. What magnetic deflecting force acts on the proton as it enters the chamber? The proton mass is 1.67×10^{-27} kg. (Neglect Earth's magnetic field.)

KEY IDEAS

Because the proton is charged and moving through a magnetic field, a magnetic force $\vec{F}_B$ can act on it. Because the initial direction of the proton's velocity is not along a magnetic field line, $\vec{F}_B$ is not simply zero.

Magnitude: To find the magnitude of $\vec{F}_B$, we can use Eq. 28-3 ($F_B = |q|vB\sin\phi$) provided we first find the proton's speed v. We can find v from the given kinetic energy because $K = 1/2mv^2$. Solving for v, we obtain

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{(2)(5.3\ \mathrm{MeV})(1.60\times10^{-13}\ \mathrm{J/MeV})}{1.67\times10^{-27}\ \mathrm{kg}}}$$
$$= 3.2\times10^{7}\ \mathrm{m/s}.$$

Equation 28-3 then yields

$$F_B = |q|vB\sin\phi$$
$$= (1.60\times10^{-19}\ \mathrm{C})(3.2\times10^{7}\ \mathrm{m/s})\times(1.2\times10^{-3}\ \mathrm{T})(\sin 90°)$$
$$= 6.1\times10^{-15}\ \mathrm{N}. \qquad \text{(Answer)}$$

This may seem like a small force, but it acts on a particle of small mass, producing a large acceleration; namely,

$$a = \frac{F_B}{m} = \frac{6.1\times10^{-15}\ \mathrm{N}}{1.67\times10^{-27}\ \mathrm{kg}} = 3.7\times10^{12}\ \mathrm{m/s^2}.$$

Direction: To find the direction of $\vec{F}_B$, we use the fact that $\vec{F}_B$ has the direction of the cross product $q\vec{v} \times \vec{B}$. Because the charge q is positive, $\vec{F}_B$ must have the same direction as $\vec{v} \times \vec{B}$, which can be determined with the right-hand rule for cross products (as in Fig. 28-12*d*). We know that $\vec{v}$ is directed horizontally from south to north and $\vec{B}$ is directed vertically up. The right-hand rule shows us that the deflecting force $\vec{F}_B$ must be directed horizontally from west to east, as Fig. 28-14 shows. (The array of dots in the figure represents a magnetic field directed out of the plane of the figure. An array of Xs would have represented a magnetic field directed into that plane.)

If the charge of the particle were negative, the magnetic deflecting force would be directed in the opposite direction—that is, horizontally from east to west. This is predicted automatically by Eq. 28-2 if we substitute a negative value for q.

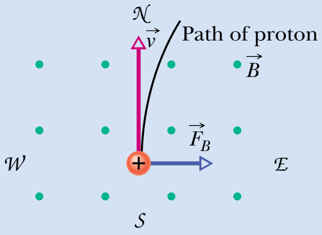


Figure 28-14 An overhead view of a proton moving from south to north with velocity $\vec{v}$ in a chamber. A magnetic field is directed vertically upward in the chamber, as represented by the array of dots (which resemble the tips of arrows). The proton is deflected toward the east.

28.4 | A CIRCULATING CHARGED PARTICLE

Key Concepts

- ◆ A charged particle with mass m and charge magnitude $|q|$ moving with velocity $\vec{v}$ perpendicular to a uniform magnetic field $\vec{B}$ will travel in a circle.
- ◆ Applying Newton's second law to the circular motion yields

$$|q|vB = \frac{mv^2}{r},$$

from which we find the radius r of the circle to be

$$r = \frac{mv}{|q|B}.$$

- ◆ The frequency of revolution f, the angular frequency ω, and the period of the motion T are given by

$$f = \frac{\omega}{2\pi} = \frac{1}{T} = \frac{|q|B}{2\pi m}.$$

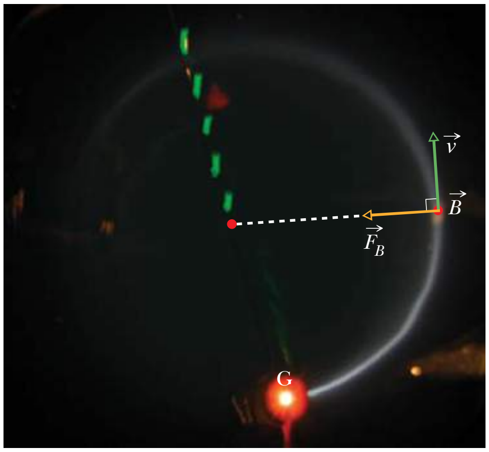
- ◆ If the velocity of the particle has a component parallel to the magnetic field, the particle moves in a helical path about field vector $\vec{B}$.

If a particle moves in a circle at constant speed, we can be sure that the net force acting on the particle is constant in magnitude and points toward the center of the circle, always perpendicular to the particle's velocity. Think of a stone tied to a string and whirled in a circle on a smooth horizontal surface, or of a satellite moving in a circular orbit around Earth. In the first case, the tension in the string provides the necessary force and centripetal acceleration. In the second case, Earth's gravitational attraction provides the force and acceleration.

Figure 28-15 shows another example: A beam of electrons is projected into a chamber by an *electron gun* G. The electrons enter in the plane of the page with speed v and then move in a region of uniform magnetic field $\vec{B}$ directed out of that plane. As a result, a magnetic force $\vec{F}_B = q\vec{v} \times \vec{B}$ continuously deflects the electrons, and because $\vec{v}$ and $\vec{B}$ are always perpendicular to each other, this deflection causes the electrons to follow a circular path. The path is visible in the photo because atoms of gas in the chamber emit light when some of the circulating electrons collide with them.

We would like to determine the parameters that characterize the circular motion of these electrons, or of any particle of charge magnitude $|q|$ and mass m moving perpendicular to a uniform magnetic field $\vec{B}$ at speed v. From Eq. 28-3, the force acting on the particle has a magnitude of $|q|vB$. From Newton's second law ($\vec{F} = m\vec{a}$) applied to uniform circular motion,

$$F = m\frac{v^2}{r}, \tag{28-6}$$



Courtesy Jearl Walker

Figure 28-15 Electrons circulating in a chamber containing gas at low pressure (their path is the glowing circle). A uniform magnetic field $\vec{B}$, pointing directly out of the plane of the page, fills the chamber. Note the radially directed magnetic force $\vec{F}_B$; for circular motion to occur, $\vec{F}_B$ *must* point toward the center of the circle. Use the right-hand rule for cross products to confirm that $\vec{F}_B = q\vec{v} \times \vec{B}$ gives $\vec{F}_B$ the proper direction. (Don't forget the sign of q.)

we have

$$|q|vB = \frac{mv^2}{r}. \tag{28-7}$$

Solving for r, we find the radius of the circular path as

$$r = \frac{mv}{|q|B} \quad \text{(radius)}. \tag{28-8}$$

The period T (the time for one full revolution) is equal to the circumference divided by the speed:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{|q|B} = \frac{2\pi m}{|q|B} \quad \text{(period)}. \tag{28-9}$$

The frequency f (the number of revolutions per unit time) is

$$f = \frac{1}{T} = \frac{|q|B}{2\pi m} \quad \text{(frequency)}. \tag{28-10}$$

The angular frequency ω of the motion is then

$$\omega = 2\pi f = \frac{|q|B}{m} \quad \text{(angular frequency)}. \tag{28-11}$$

The quantities T, f, and ω do not depend on the speed of the particle (provided the speed is much less than the speed of light). Fast particles move in large circles and slow ones in small circles, but all particles with the same charge-to-mass ratio $|q|/m$ take the same time T (the period) to complete one round trip. Using Eq. 28-2, you can show that if you are looking in the direction of $\vec{B}$, the direction of rotation for a positive particle is always counterclockwise, and the direction for a negative particle is always clockwise.

Helical Paths

If the velocity of a charged particle has a component parallel to the (uniform) magnetic field, the particle will move in a helical path about the direction of the field vector. Figure 28-16*a*, for example, shows the velocity vector $\vec{v}$ of such a particle resolved into two components, one parallel to $\vec{B}$ and one perpendicular to it:

$$v_{\parallel} = v\cos\phi \quad \text{and} \quad v_{\perp} = v\sin\phi. \tag{28-12}$$

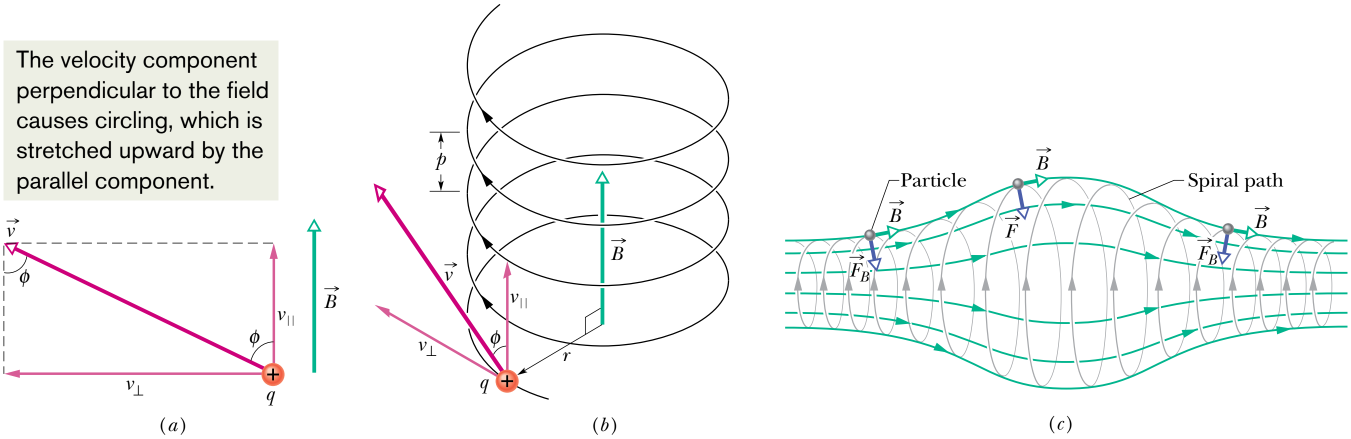

The velocity component perpendicular to the field causes circling, which is stretched upward by the parallel component.

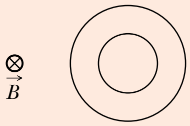
Figure 28-16 (*a*) A charged particle moves in a uniform magnetic field $\vec{B}$, the particle's velocity $\vec{v}$ making an angle ϕ with the field direction. (*b*) The particle follows a helical path of radius r and pitch p. (*c*) A charged particle spiraling in a nonuniform magnetic field. (The particle can become trapped in this *magnetic bottle*, spiraling back and forth between the strong field regions at either end.) Note that the magnetic force vectors at the left and right sides have a component pointing toward the center of the figure.

The parallel component determines the *pitch* p of the helix—that is, the distance between adjacent turns (Fig. 28-16*b*). The perpendicular component determines the radius of the helix and is the quantity to be substituted for v in Eq. 28-8.

Figure 28-16*c* shows a charged particle spiraling in a nonuniform magnetic field. The more closely spaced field lines at the left and right sides indicate that the magnetic field is stronger there. When the field at an end is strong enough, the particle "reflects" from that end. If the particle reflects from both the ends, it is said to be trapped in a *magnetic bottle*. Electrons and protons are trapped in this way by the terrestrial magnetic fields and form the *Van Allen radiation belts*, which loop above Earth's atomsphere between Earth's north south geomagnetic poles.

CHECKPOINT 3

The figure here shows the circular paths of two particles that travel at the same speed in a uniform magnetic field $\vec{B}$, which is directed into the page. One particle is a proton; the other is an electron (which is less massive). (a) Which particle follows the smaller circle, and (b) does that particle travel clockwise or counterclockwise?



SAMPLE PROBLEM 28.02

Helical motion of a charged particle in a magnetic field

An electron with a kinetic energy of 22.5 eV moves into a region of uniform magnetic field $\vec{B}$ of magnitude 4.55×10^{-4} T. The angle between the directions of $\vec{B}$ and the electron's velocity $\vec{v}$ is 65.5°. What is the pitch of the helical path taken by the electron?

KEY IDEAS

(1) The pitch p is the distance the electron travels parallel to the magnetic field $\vec{B}$ during one period T of circulation. (2) The period T is given by Eq. 28-9 for any nonzero angle between $\vec{v}$ and $\vec{B}$.

Calculations: Using Eqs. 28-12 and 28-9, we find

$$p = v_{\parallel} T = (v \cos \phi) \frac{2\pi m}{|q| B}. \tag{28-13}$$

Calculating the electron's speed v from its kinetic energy, we find that $v = 2.81 \times 10^6$ m/s, and so Eq. 28-13 gives us

$$\begin{aligned} p &= (2.81 \times 10^6 \text{ m/s})(\cos 65.5°) \\ &\quad \times \frac{2\pi(9.11 \times 10^{-31} \text{ kg})}{(1.60 \times 10^{-19} \text{ C})(4.55 \times 10^{-4} \text{ T})} \\ &= 9.16 \text{ cm} \end{aligned} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 28.03

Uniform circular motion of a charged particle in a magnetic field

Figure 28-17 shows the essentials of a *mass spectrometer*, which can be used to measure the mass of an ion; an ion of mass m (to be measured) and charge q is produced in source S. The initially stationary ion is accelerated by the electric field due to a potential difference V. The ion leaves S and enters a separator chamber in which a uniform magnetic field $\vec{B}$ is perpendicular to the path of the ion. A wide detector lines the bottom wall of the chamber, and the $\vec{B}$ causes the ion to move in a semicircle and thus strike the detector. Suppose that $B = 80.000$ mT, $V = 1000.0$ V, and ions of charge $q = +1.6022 \times 10^{-19}$ C strike the detector at a point that lies at $x = 1.6254$ m. What is the mass m of the individual ions, in atomic mass units ($\text{u} = 1.6605 \times 10^{-27}$ kg)?

KEY IDEAS

(1) Because the (uniform) magnetic field causes the (charged) ion to follow a circular path, we can relate the ion's mass m to the path's radius r with Eq. 28-8 ($r = mv/|q|B$). From Fig. 28-17 we see that $r = x/2$ (the radius is half the diameter). From the problem statement,

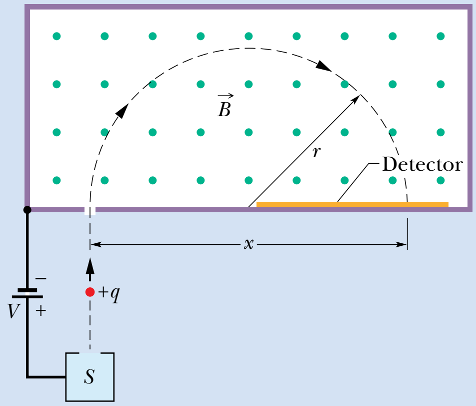


Figure 28-17 A positive ion is accelerated from its source S by a potential difference V, enters a chamber of uniform magnetic field $\vec{B}$, travels through a semicircle of radius r, and strikes a detector at a distance x.

we know the magnitude B of the magnetic field. However, we lack the ion's speed v in the magnetic field after the ion has been accelerated due to the potential difference V. (2) To relate v and V, we use the fact that mechanical energy ($E_{\text{mec}} = K + U$) is conserved during the acceleration.

Finding speed: When the ion emerges from the source, its kinetic energy is approximately zero. At the end of the acceleration, its kinetic energy is $1/2mv^2$. Also, during the acceleration, the positive ion moves through a change in potential of $-V$. Thus, because the ion has positive charge q, its potential energy changes by $-qV$. If we now write the conservation of mechanical energy as

$$\Delta K + \Delta U = 0,$$

we get

$$\frac{1}{2}mv^2 - qV = 0$$

or

$$v = \sqrt{\frac{2qV}{m}}. \tag{28-14}$$

Finding mass: Substituting this value for v into Eq. 28-8 gives us

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{q}}.$$

Thus,

$$x = 2r = \frac{2}{B}\sqrt{\frac{2mV}{q}}.$$

Solving this for m and substituting the given data yield

$$m = \frac{B^2qx^2}{8V}$$
$$= \frac{(0.080000\text{ T})^2(1.6022\times10^{-19}\text{ C})(1.6254\text{ m})^2}{8(1000.0\text{ V})}$$
$$= 3.3863\times10^{-25}\text{ kg} = 203.93\text{ u.} \qquad \text{(Answer)}$$

SAMPLE PROBLEM 28.04

Path of a charged particle in a uniform magnetic field

A particle of mass $m = 1.6 \times 10^{-27}$ kg and charge $q = 1.6 \times 10^{-19}$ C enters a region of uniform magnetic field of strength 1 T along the direction shown in Fig. 28-18. The speed of the particle is 10^7 m/s. The magnetic field is directed along the inward normal to the plane of the paper. The particle leaves the region of the field at point F. Find the distance EF and the time spent by the charge in the magnetic field.

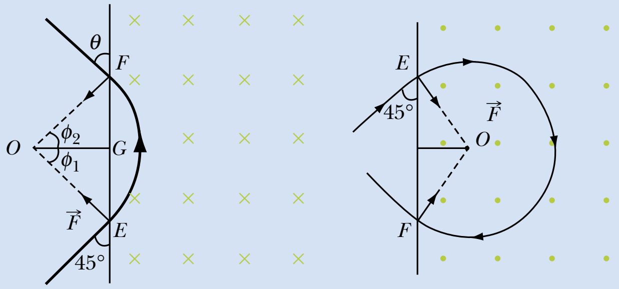
KEY IDEAS

First of all let us see how the path will look like. At least for the part when the particle is inside the magnetic field, the path will be part of a circle. To locate the center, let us find the direction of $q(\vec{v} \times \vec{B})$.

As can be seen in the figure, it is directed perpendicular to velocity and is toward O. Since this force is centripetal in nature, the force is directed toward the center. The coordinates of center can be easily found out. The charge particle will move in anti-clockwise direction and exit the magnetic field at the boundary.

When a charged particle moves perpendicular to a magnetic field the path is a circle with magnetic force qvB directed radially toward the center of the path. So, if we draw normal at E and F they will meet at O which is the center of the circle. By geometry of Fig. 28-18a:

$$\theta = \phi_2 = \phi_1 = 45°.$$



(*a*) (*b*)

Figure 28-18 (*a*) A positively charged particle is projected in a magnetic field. (*b*) If the direction of the magnetic field is reversed, the particle follows a different path.

Calculation: As $\phi_1 = \phi_2 = \theta$, therefore

$$EF = 2EG = 2r\sin\theta.$$

Since

$$r = \frac{mv}{qB} = \frac{1.6\times10^{-27}\times10^{7}}{1.6\times10^{-19}\times1} = 0.1\ \text{m},$$

therefore

$$EF = (2\times0.1\times\sin45^\circ)\,\text{m} = 10\sqrt{2}\ \text{cm}. \quad \text{(Answer)}$$

Now, as in a magnetic field the speed of a particle remains constant, so if t is the time taken by the particle in the field, then

$$S = v\times t = r\theta$$

$$t = \frac{r\theta}{v}.$$

Hence,

$$t = \frac{0.1\times\pi/2}{10^{7}} = 5\pi\times10^{-9} = 1.57\times10^{-8}\ \text{s}. \quad \text{(Answer)}$$

Note: If the direction of the field is out of the page the force will act as shown in Fig. 28-18*b* and so now *O* will be the center and the angle subtended by the path at *O* will be $[2\pi - (\pi/2)] = (3/2)\pi$. The time taken will be thrice.

28.5 | CYCLOTRONS

Key Concept

- In a cyclotron, charged particles are accelerated by electric forces as they circle in a magnetic field.

Beams of high-energy particles, such as high-energy electrons and protons, have been enormously useful in probing atoms and nuclei to reveal the fundamental structure of matter. Such beams were instrumental in the discovery that atomic nuclei consist of protons and neutrons and in the discovery that protons and neutrons consist of quarks and gluons. Because electrons and protons are charged, they can be accelerated to the required high energy if they move through large potential differences. The required acceleration distance is reasonable for electrons (low mass) but unreasonable for protons (greater mass).

A clever solution to this problem is first to let protons and other massive particles move through a modest potential difference (so that they gain a modest amount of energy) and then use a magnetic field to cause them to circle back and move through a modest potential difference again. If this procedure is repeated thousands of times, the particles end up with a very large energy.

Here we discuss two *accelerators* that employ a magnetic field to repeatedly bring particles back to an accelerating region, where they gain more and more energy until they finally emerge as a high-energy beam.

The Cyclotron

Figure 28-19 is a top view of the region of a *cyclotron* in which the particles (protons, say) circulate. The two hollow **D**-shaped objects (each open on its straight edge) are made of sheet copper. These *dees*, as they are called, are part of an electrical oscillator that alternates the electric potential difference across the gap between the dees. The electrical signs of the dees are alternated so that the electric field in the gap alternates in direction, first toward one dee and then toward the other dee, back and forth. The dees are immersed in a large magnetic field directed out of the plane of the page. The magnitude B of this field is set via a control on the electromagnet producing the field.

Suppose that a proton, injected by source S at the center of the cyclotron in Fig. 28-19, initially moves toward a negatively charged dee. It will accelerate toward this dee and enter it. Once inside, it is shielded from electric fields by the copper walls of the dee; that is, the electric field does not enter the dee. The magnetic field, however, is not screened by the (nonmagnetic) copper dee, so the proton moves in a circular path whose radius, which depends on its speed, is given by Eq. 28-8 ($r = mv/|q|B$).

Let us assume that at the instant the proton emerges into the center gap from the first dee, the potential difference between the dees is reversed. Thus, the proton *again* faces a negatively charged dee and is *again* accelerated. This process continues, the circulating proton always being in step with the oscillations of the dee potential, until the proton has spiraled out to the edge of the dee system. There a deflector plate sends it out through a portal.

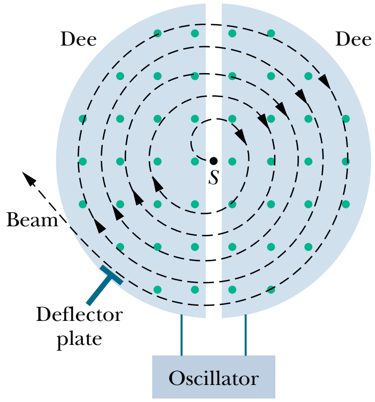

The protons spiral outward in a cyclotron, picking up energy in the gap.

Figure 28-19 The elements of a cyclotron, showing the particle source S and the dees. A uniform magnetic field is directed up from the plane of the page. Circulating protons spiral outward within the hollow dees, gaining energy every time they cross the gap between the dees.

Frequency. The key to the operation of the cyclotron is that the frequency f at which the proton circulates in the magnetic field (and that does *not* depend on its speed) must be equal to the fixed frequency f_{osc} of the electrical oscillator, or

$$f = f_{\text{osc}} \quad \text{(resonance condition).} \tag{28-15}$$

This *resonance condition* says that, if the energy of the circulating proton is to increase, energy must be fed to it at a frequency f_{osc} that is equal to the natural frequency f at which the proton circulates in the magnetic field.

Combining Eqs. 28-10 ($f = |q|B/2\pi m$) and 28-15 allows us to write the resonance condition as

$$|q|B = 2\pi m f_{\text{osc}}. \tag{28-16}$$

The oscillator (we assume) is designed to work at a single fixed frequency f_{osc}. We then "tune" the cyclotron by varying B until Eq. 28-16 is satisfied, and then many protons circulate through the magnetic field, to emerge as a beam.

Other types of charged particles can be accelarated by a cyclotron to emerge as high energy beam. For example *deuterons* (bundles of one proton and one neutron) can be accelerated because they have the same charge as a proton. However, electrically neutral particles, such as neutrons, cannot be accelerated by a cyclotron.

SAMPLE PROBLEM 28.05

Accelerating a charged particle in a cyclotron

Suppose a cyclotron is operated at an oscillator frequency of 12 MHz and has a dee radius $R = 53$ cm.

(a) What is the magnitude of the magnetic field needed for deuterons to be accelerated in the cyclotron? The deuteron mass is $m = 3.34 \times 10^{-27}$ kg (twice the proton mass).

KEY IDEA

For a given oscillator frequency f_{osc}, the magnetic field magnitude B required to accelerate any particle in a cyclotron depends on the ratio $m/|q|$ of mass to charge for the particle, according to Eq. 28-16 ($|q|B = 2\pi m f_{\text{osc}}$).

Calculations: For deuterons and the oscillator frequency $f_{\text{osc}} = 12$ MHz, we find

$$B = \frac{2\pi m f_{\text{osc}}}{|q|} = \frac{(2\pi)(3.34 \times 10^{-27}\ \text{kg})(12 \times 10^{6}\ \text{s}^{-1})}{1.60 \times 10^{-19}\ \text{C}}$$

$$= 1.57\ \text{T} \approx 1.6\ \text{T}. \qquad \text{(Answer)}$$

Note that, to accelerate protons, B would have to be reduced by a factor of 2, provided the oscillator frequency remained fixed at 12 MHz.

(b) What is the resulting kinetic energy of the deuterons?

KEY IDEAS

(1) The kinetic energy ($1/2mv^2$) of a deuteron exiting the cyclotron is equal to the kinetic energy it had just before exiting, when it was traveling in a circular path with a radius approximately equal to the radius R of the cyclotron dees. (2) We can find the speed v of the deuteron in that circular path with Eq. 28-8 ($r = mv/|q|B$).

Calculations: Solving that equation for v, substituting R for r, and then substituting known data, we find

$$v = \frac{R|q|B}{m} = \frac{(0.53\text{ m})(1.60\times10^{-19}\text{ C})(1.57\text{ T})}{3.34\times10^{-27}\text{ kg}}$$
$$= 3.99\times10^{7}\text{ m/s}.$$

This speed corresponds to a kinetic energy of

$$K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}(3.34\times10^{-27}\text{ kg})(3.99\times10^{7}\text{ m/s})^2$$
$$= 2.7\times10^{-12}\text{ J}, \qquad \text{(Answer)}$$

or about 17 MeV.

28.6 | CROSSED FIELDS: DISCOVERY OF THE ELECTRON

Key Concepts

- If a charged particle moves through a region containing both an electric field and a magnetic field, it can be affected by both an electric force and a magnetic force.
- If the fields are perpendicular to each other, they are said to be crossed fields.
- If the forces are in opposite directions, a particular speed will result in no deflection of the particle.

Both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ can produce a force on a charged particle. The electric force is given by Eq. 22-18 and the magnetic force is given by Eq. 28-2. Taken together, they constitute all of the forces that can be exerted on a particle by virtue of its electric charge. Their sum F is called Lorentz force and is given by

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}). \tag{28-17}$$

We will now consider the Lorentz force in some important cases.

When the two fields are perpendicular to each other, they are said to be *crossed fields*. Here we shall examine what happens to charged particles—namely, electrons—as they move through crossed fields. We use as our example the experiment that led to the discovery of the electron in 1897 by J. J. Thomson at Cambridge University.

Two Forces. Figure 28-20 shows a modern, simplified version of Thomson's experimental apparatus—a *cathode ray tube* (which is like the picture tube in an old-type television set). Charged particles (which we now know

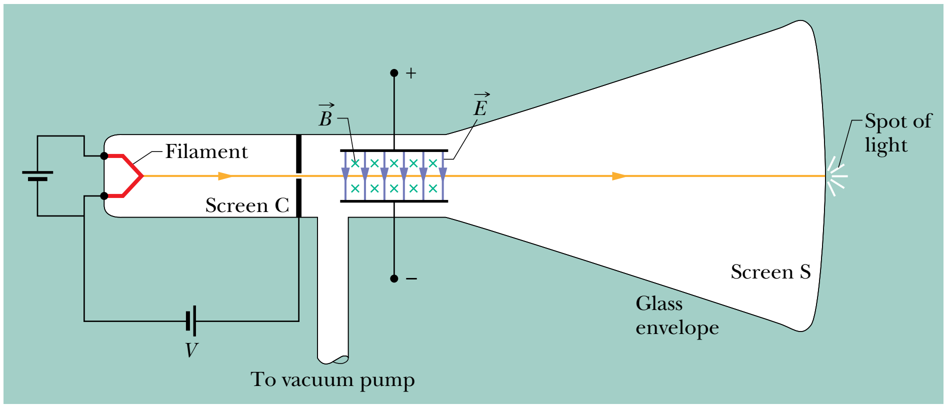


Figure 28-20 A modern version of J. J. Thomson's apparatus for measuring the ratio of mass to charge for the electron. An electric field $\vec{E}$ is established by connecting a battery across the deflecting-plate terminals. The magnetic field $\vec{B}$ is set up by means of a current in a system of coils (not shown). The magnetic field shown is into the plane of the figure, as represented by the array of **X**s (which resemble the feathered ends of arrows).

as electrons) are emitted by a hot filament at the rear of the evacuated tube and are accelerated by an applied potential difference V. After they pass through a slit in screen C, they form a narrow beam. They then pass through a region of crossed $\vec{E}$ and $\vec{B}$ fields, headed toward a fluorescent screen S, where they produce a spot of light (on a television screen the spot is part of the picture). The forces on the charged particles in the crossed-fields region can deflect them from the center of the screen. By controlling the magnitudes and directions of the fields, Thomson could thus control where the spot of light appeared on the screen. Recall that the force on a negatively charged particle due to an electric field is directed opposite the field. Thus, for the arrangement of Fig. 28-20, electrons are forced up the page by electric field $\vec{E}$ and down the page by magnetic field $\vec{B}$; that is, the forces are *in opposition*. Thomson's procedure was equivalent to the following series of steps.

1. Set $E = 0$ and $B = 0$ and note the position of the spot on screen S due to the undeflected beam.
2. Turn on $\vec{E}$ and measure the resulting beam deflection.
3. Maintaining $\vec{E}$, now turn on $\vec{B}$ and adjust its value until the beam returns to the undeflected position. (With the forces in opposition, they can be made to cancel.)

We discussed the deflection of a charged particle moving through an electric field $\vec{E}$ between two plates (step 2 here) in Sample Problem 22.08. We found that the deflection of the particle at the far end of the plates is

$$y = \frac{|q|EL^2}{2mv^2}, \tag{28-18}$$

where v is the particle's speed, m its mass, and q its charge, and L is the length of the plates. We can apply this same equation to the beam of electrons in Fig. 28-20; if need be, we can calculate the deflection by measuring the deflection of the beam on screen S and then working back to calculate the deflection y at the end of the plates. (Because the direction of the deflection is set by the sign of the particle's charge, Thomson was able to show that the particles that were lighting up his screen were negatively charged.)

Canceling Forces. When the two fields in Fig. 28-20 are adjusted so that the two deflecting forces cancel (step 3), we have from Eqs. 28-1 and 28-3

$$|q|E = |q|vB\sin(90°) = |q|vB$$

or

$$v = \frac{E}{B} \quad \text{(opposite forces canceling).} \tag{28-19}$$

Thus, the crossed fields allow us to measure the speed of the charged particles passing through them. Substituting Eq. 28-19 for v in Eq. 28-18 and rearranging yield

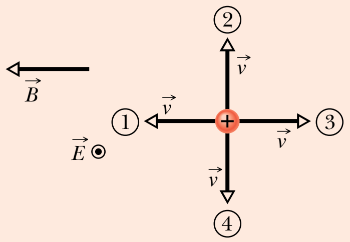
$$\frac{m}{|q|} = \frac{B^2L^2}{2yE}, \tag{28-20}$$

in which all quantities on the right can be measured. Thus, the crossed fields allow us to measure the ratio $m/|q|$ of the particles moving through Thomson's apparatus. (*Caution:* Equation 28-19 applies only when the electric and magnetic forces are in opposite directions.)

Thomson claimed that these particles are found in all matter. He also claimed that they are lighter than the lightest known atom (hydrogen) by a factor of more than 1000. (The exact ratio proved later to be 1836.15.) His $m/|q|$ measurement, coupled with the boldness of his two claims, is considered to be the "discovery of the electron."

CHECKPOINT 4

The figure shows four directions for the velocity vector $\vec{v}$ of a positively charged particle moving through a uniform electric field $\vec{E}$ (directed out of the page and represented with an encircled dot) and a uniform magnetic field $\vec{B}$. (a) Rank directions 1, 2, and 3 according to the magnitude of the net force on the particle, greatest first. (b) Of all four directions, which might result in a net force of zero?



CHECKPOINT 5

Of the three vectors in the equation $\vec{F} = q(\vec{v} \times \vec{B})$, which pairs are always at right angles? Which may have any angle between them?

SAMPLE PROBLEM 28.06

Velocity of a charged particle in a region with uniform, constant electric and magnetic fields

A particle of mass m and charge q is moving in a region where uniform, constant electric and magnetic fields $\vec{E}$ and $\vec{B}$ are present. $\vec{E}$ and $\vec{B}$ are parallel to each other. At time $t = 0$ the velocity $\vec{v}_0$ of the particle is perpendicular to $\vec{E}$. Find the velocity $\vec{v}$ of the particle at time t. You must express your answer in terms of t, q, m, $\vec{v}_0$, $\vec{E}$, and $\vec{B}$, and their magnitudes v_0, E, and B.

KEY IDEAS

(1) In this case, we cannot say that the path is circular or a helix with uniform pitch. The motion is rather complicated; here we have to use the Lorentz force (Eq. 28-17).

(2) The forces due to parallel electric and magnetic fields on a charged particle moving perpendicular to the fields will be at right angles to each other (electric force being along the direction of $\vec{E}$ while magnetic force being perpendicular to the plane containing $\vec{v}$ and $\vec{B}$). Therefore, the magnetic force will not affect the motion of charged particle in the direction of electric field and vice versa. So, the problem is equivalent to superposition of two independent motions as shown in Fig. 28-21 and Fig. 28-22.

Calculations: For motion of particle under electric field alone,

$$a_y = \frac{qE}{m}, \quad \text{i.e.,} \quad \frac{dv_y}{dt} = \frac{qE}{m} \left(\text{as } a = \frac{F}{m} \quad \text{and} \quad F = qE \right)$$

or

$$\int_0^{v_y} dv_y = \int_0^t \frac{qE}{m} dt, \quad \text{i.e.,} \quad v_y = \frac{qE}{m} t. \qquad (28\text{-}21)$$

While at the same time the charged particle under the action of magnetic field will describe a circle in the xz plane with

$$r = \frac{mv_0}{qB}, \quad \text{i.e.,} \quad \omega = \frac{v_0 qB}{m}.$$

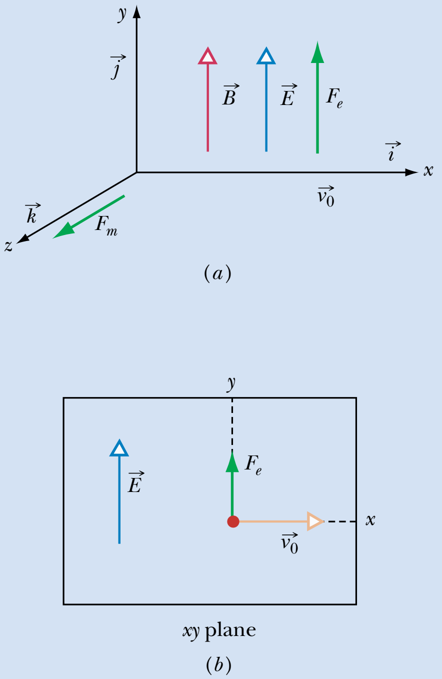


Figure 28-21 (*a*) Force on a charged particle in uniform electric and magnetic field. (*b*) Force on it due to electric field alone. Note that the charge is positive, so the force is in the direction of the electric field. This force is independent of the velocity.

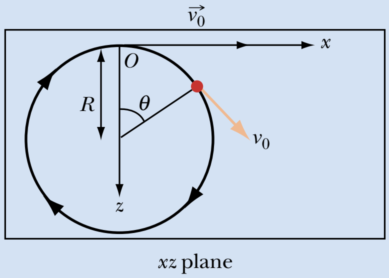


Figure 28-22 The force due to the magnetic field is always perpendicular to the velocity. So, under its influence, the particle will move in a circular path.

So, the angular position of the particle at time t in the xz plane (or in a plane parallel to xz plane) will be given by

$$\theta = \omega t = \frac{qB}{m}t.$$

Hence, in accordance with Fig. 28-22, we have

$$v_x = v_0\cos\theta = v_0\cos\omega t = v_0\cos\left(\frac{qB}{m}\right)t, \qquad (28\text{-}22)$$

and

$$v_z = v_0\sin\theta = v_0\sin\omega t = v_0\sin\left(\frac{qB}{m}\right)t. \qquad (28\text{-}23)$$

Note that magnetic field does not do any work on the charged particle, so it can change the direction of the velocity v_0, but not its magnitude. Also, the angular velocity of its circular motion is independent of its velocity.

So, in the light of Eqs. 28-21 to 28-23 we have

$$\begin{aligned}\vec v &= v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}\\ &= v_0\cos\left(\frac{qB}{m}t\right)\hat{\imath} + \left(\frac{qE}{m}t\right)\hat{\jmath} + v_0\sin\left(\frac{qB}{m}t\right)\hat{k}.\end{aligned}$$

Since in this case

$$\hat{\imath} = \frac{\vec v_0}{v_0},\ \hat{\jmath} = \frac{\vec E}{E} = \frac{\vec B}{B}, \quad \text{and} \quad \hat{k} = \frac{\vec v_0\times\vec B}{v_0 B};$$

Therefore, we have

$$\vec v = v_0\cos\left[\frac{qB}{m}t\right]\left[\frac{\vec v_0}{v_0}\right] + \frac{qE}{m}t\left[\frac{\vec E}{E}\right] + v_0\sin\left(\frac{qB}{m}t\right)\left[\frac{\vec v_0\times\vec B}{v_0 B}\right].$$

The answer seems to be quite complicated, on analysis of the motion, we see that in the y-direction, the motion is uniformly accelerated. The motion in xz plane is circular. So, the resultant path is a helix with non-uniform pitch. The path looks like that shown in Fig. 28-23. As before, the particle repeatedly touches the y axis (if it was projected from the origin) after regular time intervals of T when it has completed one circle. But now the motion along the y axis is accelerated, so the distance between successive points on the y axis where it touches is not the same.

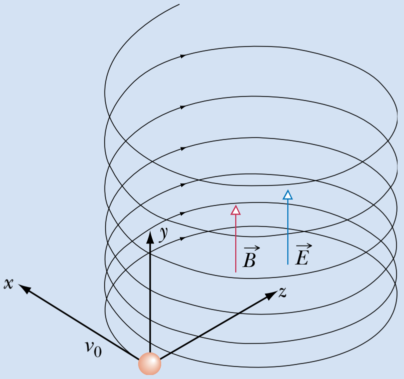


Figure 28-23 In a uniform magnetic field which is parallel to a uniform electric field, the charged particle follows a helical path with variable pitch.

SAMPLE PROBLEM 28.07

Charged particle in crossed electric and magnetic field

If a uniform electric field is at right angles to the magnetic field and a charge is released from the origin as shown in Fig. 28-24. Suppose, for instance, that B points in the x direction and E in the z direction. What path will the charge follow?

KEY IDEAS

(1) Let's think it through qualitatively. Initially, the particle is at rest, so the magnetic force on it is zero and the electric field accelerates the charge in the z direction. As it speeds up, a magnetic force develops which, according to Eq. 28-17, rotates its velocity vector. The faster it goes, the stronger the magnetic field force

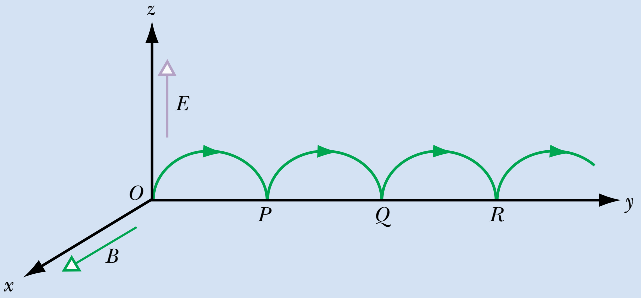


Figure 28-24 A charged particle starts from rest under the influence of a uniform electric field and a uniform magnetic field. Note that its average displacement is not in the direction of the electric field but perpendicular to it.

becomes; eventually, it curves the particle back around the y axis. At this point the charge is moving against the electrical force. So it begins to slow down. The magnetic force then decreases, and the electrical force takes over, bringing the charge to rest at point P (see Fig. 28-24). There the entire process commences anew, carrying the particle over to point Q, and so on.

(2) This is a case where we need to apply Eq. 28-17 to decide on the path followed by the charge.

Calculate: There being no force in the x direction, the velocity of the particle at any time t can be described by the vector

$$\vec{v} = v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}.$$

By applying the equation of Lorentz force, we get

$$\begin{aligned}\vec{F} &= q[E\hat{\mathrm{k}} + (v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}) \times B\hat{\mathrm{i}}]\\ &= q(E - v_yB)\hat{\mathrm{k}} + qv_zB\hat{\mathrm{j}}\end{aligned}$$

Treating the y and z components separately, we get

$$\begin{aligned}ma_y &= qv_zB\\ ma_z &= q(E - v_yB).\end{aligned}$$

Let us take

$$\omega = \frac{qB}{m}.$$

Then the equation becomes

$$\begin{aligned}a_y &= \omega v_z\\ a_z &= \omega\left(\frac{E}{B} - v_y\right).\end{aligned}$$

By differentiating the second and using the first to eliminate a_y, we get

$$\frac{d^2v_z}{dt^2} = -\omega^2 v_z.$$

This is the familiar equation of simple harmonic motion. Its solution is given by

$$v_z = v_0 \sin(\omega t + \phi).$$

At $t = 0$, $v_z = 0$. This means that $\phi = 0$. Substituting this expression of v_z in the second equation, we get

$$v_y = \frac{E}{B} - v_0\cos(\omega t).$$

At $t = 0$, v_y is also zero. This means that

$$v_0 = \frac{E}{B}.$$

Therefore, we get

$$\begin{aligned}v_y &= \frac{E}{B}[1 - \cos(\omega t)],\\ v_z &= \frac{E}{B}\sin(\omega t).\end{aligned}$$

By integrating these two expressions, we get

$$\begin{aligned}y &= \frac{E}{\omega B}[\omega t - \sin(\omega t)],\\ z &= \frac{E}{\omega B}[1 - \cos(\omega t)].\end{aligned}$$

In this expression, if we let

$$R = \frac{E}{\omega B}$$

and eliminate sine and cosine, we find that

$$(y - R\omega t)^2 + (z - R)^2 = R^2.$$

This is the equation of a circle of radius R whose center is at $(0, R\omega t, R)$. This means that the center travels in y direction at a constant speed of E/B.

The particle moves as though it were a spot on the rim of a wheel, rolling down the y axis at a speed v. This path is called a cycloid.

28.7 | THE HALL EFFECT

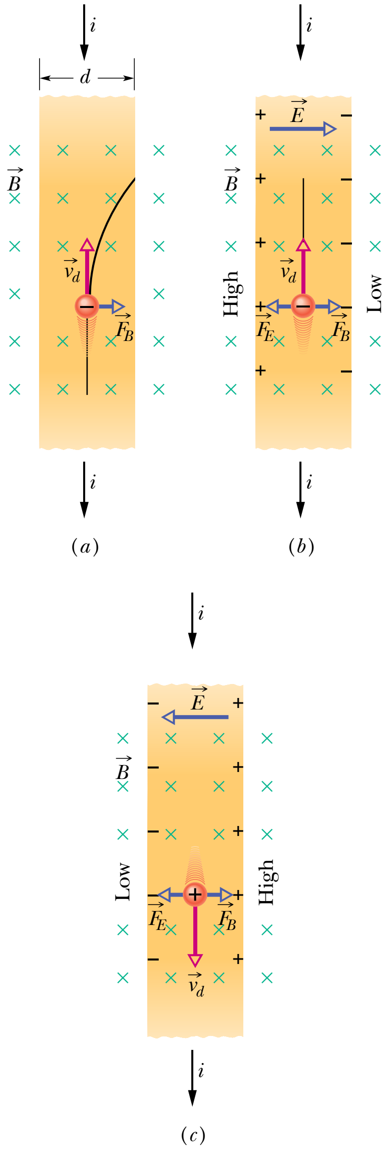
Key Concepts

- When a uniform magnetic field $\vec{B}$ is applied to a conducting strip carrying current i, with the field perpendicular to the direction of the current, a Hall-effect potential difference V is set up across the strip.
- The electric force $\vec{F}_E$ on the charge carriers is then balanced by the magnetic force $\vec{F}_B$ on them.
- The number density n of the charge carriers can then be determined from

$$n = \frac{Bi}{Vle},$$

where l is the thickness of the strip (parallel to $\vec{B}$).

As we just discussed, a beam of electrons in a vacuum can be deflected by a magnetic field. Can the drifting conduction electrons in a copper wire also be deflected by a magnetic field? In 1879, Edwin H. Hall, then a 24-year-old graduate student at the Johns Hopkins University, showed that they can. This **Hall effect** allows us to find out whether the charge carriers in a conductor are positively or negatively charged. Beyond that, we can measure the number of such carriers per unit volume of the conductor.



(*a*) (*b*) (*c*)

Figure 28-25 A strip of copper carrying a current i is immersed in a magnetic field $\vec{B}$. (*a*) The situation immediately after the magnetic field is turned on. The curved path that will then be taken by an electron is shown. (*b*) The situation at equilibrium, which quickly follows. Note that negative charges pile up on the right side of the strip, leaving uncompensated positive charges on the left. Thus, the left side is at a higher potential than the right side. (*c*) For the same current direction, if the charge carriers were positively charged, *they* would pile up on the right side, and the right side would be at the higher potential.

Figure 28-25*a* shows a copper strip of width d, carrying a current i whose conventional direction is from the top of the figure to the bottom. The charge carriers are electrons and, as we know, they drift (with drift speed v_d) in the opposite direction, from bottom to top. At the instant shown in Fig. 28-25*a*, an external magnetic field $\vec{B}$, pointing into the plane of the figure, has just been turned on. From Eq. 28-2 we see that a magnetic deflecting force $\vec{F}_B$ will act on each drifting electron, pushing it toward the right edge of the strip.

As time goes on, electrons move to the right, mostly piling up on the right edge of the strip, leaving uncompensated positive charges in fixed positions at the left edge. The separation of positive charges on the left edge and negative charges on the right edge produces an electric field $\vec{E}$ within the strip, pointing from left to right in Fig. 28-25*b*. This field exerts an electric force $\vec{F}_E$ on each electron, tending to push it to the left. Thus, this electric force on the electrons, which opposes the magnetic force on them, begins to build up.

Equilibrium. An equilibrium quickly develops in which the electric force on each electron has increased enough to match the magnetic force. When this happens, as Fig. 28-25*b* shows, the force due to $\vec{B}$ and the force due to $\vec{E}$ are in balance. The drifting electrons then move along the strip toward the top of the page at velocity $\vec{v}_d$ with no further collection of electrons on the right edge of the strip and thus no further increase in the electric field $\vec{E}$.

A *Hall potential difference V* is associated with the electric field across strip width d. From Eq. 24-21, the magnitude of that potential difference is

$$V = Ed. \tag{28-24}$$

By connecting a voltmeter across the width, we can measure the potential difference between the two edges of the strip. Moreover, the voltmeter can tell us which edge is at higher potential. For the situation of Fig. 28-25*b*, we would find that the left edge is at higher potential, which is consistent with our assumption that the charge carriers are negatively charged.

For a moment, let us make the opposite assumption, that the charge carriers in current i are positively charged (Fig. 28-25*c*). Convince yourself that as these charge carriers move from top to bottom in the strip, they are pushed to the right edge by $\vec{F}_B$ and thus that the *right* edge is at higher potential. Because that last statement is contradicted by our voltmeter reading, the charge carriers must be negatively charged.

Number Density. Now for the quantitative part. When the electric and magnetic forces are in balance (Fig. 28-25*b*), Eqs. 28-1 and 28-3 give us

$$eE = ev_dB. \tag{28-25}$$

From Eq. 26-7, the drift speed v_d is

$$v_d = \frac{J}{ne} = \frac{i}{neA}, \tag{28-26}$$

in which $J(= i/A)$ is the current density in the strip, A is the cross-sectional area of the strip, and n is the *number density* of charge carriers (number per unit volume).

In Eq. 28-25, substituting for E with Eq. 28-24 and substituting for v_d with Eq. 28-26, we obtain

$$n = \frac{Bi}{Vle}, \tag{28-27}$$

in which $l\,(= A/d)$ is the thickness of the strip. With this equation we can find n from measurable quantities.

Drift Speed. It is also possible to use the Hall effect to measure directly the drift speed v_d of the charge carriers, which you may recall is of the order of centimeters per hour. In this clever experiment, the metal strip is moved mechanically through the magnetic field in a direction opposite that of the drift velocity of the charge carriers. The speed of the moving strip is then adjusted until the Hall potential difference vanishes. At this condition, with no Hall effect, the velocity of the charge carriers *with respect to the laboratory frame* must be zero, so the velocity of the strip must be equal in magnitude but opposite the direction of the velocity of the negative charge carriers.

28.8 | MAGNETIC FORCE ON A CURRENT-CARRYING WIRE

Key Concepts

- A straight wire carrying a current i in a uniform magnetic field experiences a sideways force

$$\vec{F}_B = i\vec{L} \times \vec{B}.$$

- The force acting on a current element $i\,d\vec{L}$ in a magnetic field is

$$d\vec{F}_B = i\,d\vec{L} \times \vec{B}.$$

- The direction of the length vector $\vec{L}$ or $d\vec{L}$ is that of the current i.

We have seen in connection with the Hall effect that a magnetic field exerts a sideways force on electrons moving in a wire. This force must then be transmitted to the wire itself, because the conduction electrons cannot escape sideways out of the wire.

In Fig. 28-26*a*, a vertical wire, carrying no current and fixed in place at both ends, extends through the gap between the vertical pole faces of a magnet. The magnetic field between the faces is directed outward from the page. In Fig. 28-26*b*, a current is sent upward through the wire; the wire deflects to the right. In Fig. 28-26*c*, we reverse the direction of the current and the wire deflects to the left.

Figure 28-27 shows what happens inside the wire of Fig. 28-26*b*. We see one of the conduction electrons, drifting downward with an assumed drift speed v_d. Equation 28-3, in which we must put $\phi = 90°$, tells us that a force $\vec{F}_B$ of magnitude ev_dB must act on each such electron. From Eq. 28-2 we see that this force must be directed to the right. We expect then that the wire as a whole will experience a force to the right, in agreement with Fig. 28-26*b*.

If, in Fig. 28-27, we were to reverse *either* the direction of the magnetic field *or* the direction of the current, the force on the wire would reverse, being directed now to the left. Note too that it does not matter whether we consider negative charges drifting downward in the wire (the actual case) or positive charges drifting upward. The direction of the deflecting force on the wire is the same. We are safe then in dealing with a current of positive charge, as we usually do in dealing with circuits.

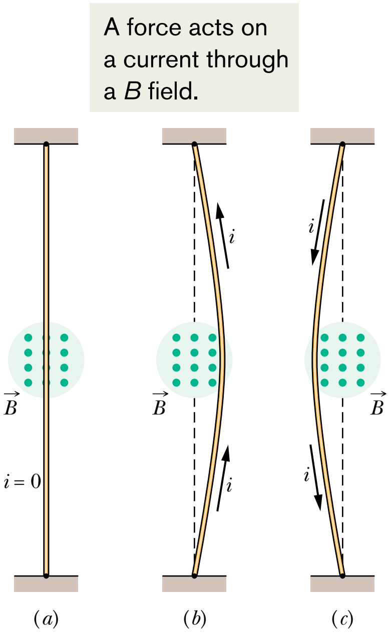

A force acts on a current through a B field.

Figure 28-26 A flexible wire passes between the pole faces of a magnet (only the farther pole face is shown). (*a*) Without current in the wire, the wire is straight. (*b*) With upward current, the wire is deflected rightward. (*c*) With downward current, the deflection is leftward. The connections for getting the current into the wire at one end and out of it at the other end are not shown.

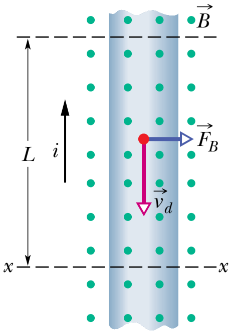


Figure 28-27 A close-up view of a section of the wire of Fig. 28-26*b*. The current direction is upward, which means that electrons drift downward. A magnetic field that emerges from the plane of the page causes the electrons and the wire to be deflected to the right.

Find the Force. Consider a length L of the wire in Fig. 28-27. All the conduction electrons in this section of wire will drift past plane xx in Fig. 28-27 in a time $t = L/v_d$. Thus, in that time a charge given by

$$q = it = i\frac{L}{v_d}$$

will pass through that plane. Substituting this into Eq. 28-3 yields

$$F_B = qv_dB\sin\phi = \frac{iL}{v_d}v_dB\sin 90°$$

or

$$F_B = iLB. \tag{28-28}$$

Note that this equation gives the magnetic force that acts on a length L of straight wire carrying a current i and immersed in a uniform magnetic field $\vec{B}$ that is *perpendicular* to the wire.

If the magnetic field is *not* perpendicular to the wire, as in Fig. 28-28, the magnetic force is given by a generalization of Eq. 28-28:

$$\vec{F}_B = i\vec{L}\times\vec{B} \quad \text{(force on a current).} \tag{28-29}$$

The force is perpendicular to both the field and the length.

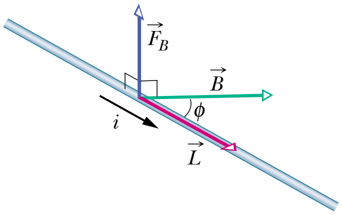


Figure 28-28 A wire carrying current i makes an angle ϕ with magnetic field $\vec{B}$. The wire has length L in the field and length vector $\vec{L}$ (in the direction of the current). A magnetic force $\vec{F}_B = i\vec{L}\times\vec{B}$ acts on the wire.

Here $\vec{L}$ is a *length vector* that has magnitude L and is directed along the wire segment in the direction of the (conventional) current. The force magnitude F_B is

$$F_B = iLB\sin\phi, \tag{28-30}$$

where ϕ is the angle between the directions of $\vec{L}$ and $\vec{B}$. The direction of $\vec{F}_B$ is that of the cross product $\vec{L}\times\vec{B}$ because we take current i to be a positive quantity. Equation 28-29 tells us that $\vec{F}_B$ is always perpendicular to the plane defined by vectors $\vec{L}$ and $\vec{B}$, as indicated in Fig. 28-28.

Equation 28-29 is equivalent to Eq. 28-2 in that either can be taken as the defining equation for $\vec{B}$. In practice, we define $\vec{B}$ from Eq. 28-29 because it is much easier to measure the magnetic force acting on a wire than that on a single moving charge.

Crooked Wire. If a wire is not straight or the field is not uniform, we can imagine the wire broken up into small straight segments and apply Eq. 28-29 to each segment. The force on the wire as a whole is then the vector sum of all the forces on the segments that make it up. In the differential limit, we can write

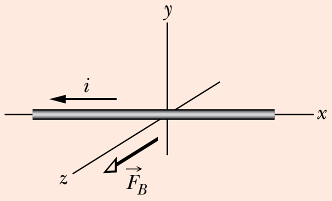
$$d\vec{F}_B = i\,d\vec{L}\times\vec{B}, \tag{28-31}$$

and we can find the resultant force on any given arrangement of currents by integrating Eq. 28-31 over that arrangement.

In using Eq. 28-31, bear in mind that there is no such thing as an isolated current-carrying wire segment of length dL. There must always be a way to introduce the current into the segment at one end and take it out at the other end.

CHECKPOINT 6

The figure shows a current i through a wire in a uniform magnetic field $\vec{B}$, as well as the magnetic force $\vec{F}_B$ acting on the wire. The field is oriented so that the force is maximum. In what direction is the field?



SAMPLE PROBLEM 28.08

Magnetic force on a wire carrying current

A straight, horizontal length of copper wire has a current $i = 28$ A through it. What are the magnitude and direction of the minimum magnetic field $\vec{B}$ needed to suspend the wire—that is, to balance the gravitational force on it? The linear density (mass per unit length) of the wire is 46.6 g/m.

KEY IDEAS

(1) Because the wire carries a current, a magnetic force $\vec{F}_B$ can act on the wire if we place it in a magnetic field $\vec{B}$. To balance the downward gravitational force $\vec{F}_g$ on the wire, we want $\vec{F}_B$ to be directed upward (Fig. 28-29). (2) The direction of $\vec{F}_B$ is related to the directions of $\vec{B}$ and the wire's length vector $\vec{L}$ by Eq. 28-29 ($\vec{F}_B = i\vec{L} \times \vec{B}$).

Calculations: Because $\vec{L}$ is directed horizontally (and the current is taken to be positive), Eq. 28-29 and the right-hand rule for cross products tell us that $\vec{B}$ must be horizontal and rightward (in Fig. 28-29) to give the required upward $\vec{F}_B$.

The magnitude of $\vec{F}_B$ is $F_B = iLB \sin\phi$ (Eq. 28-30). Because we want $\vec{F}_B$ to balance $\vec{F}_g$, we want

$$iLB \sin\phi = mg, \tag{28-32}$$

where mg is the magnitude of $\vec{F}_g$ and m is the mass of the wire. We also want the minimal field magnitude B for $\vec{F}_B$ to balance $\vec{F}_g$. Thus, we need to maximize $\sin\phi$ in Eq. 28-32. To do so, we set $\phi = 90°$, thereby arranging for $\vec{B}$ to be perpendicular to the wire. We then have $\sin\phi = 1$, so Eq. 28-32 yields

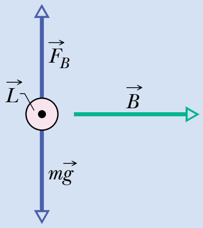


Figure 28-29 A wire (shown in cross section) carrying current out of the page.

$$B = \frac{mg}{iL\sin\phi} = \frac{(m/L)g}{i}. \tag{28-33}$$

We write the result this way because we know m/L, the linear density of the wire. Substituting known data then gives us

$$B = \frac{(46.6 \times 10^{-3}\ \text{kg/m})(9.8\ \text{m/s}^2)}{28\ \text{A}}$$
$$= 1.6 \times 10^{-2}\ \text{T}. \qquad \text{(Answer)}$$

This is about 160 times the strength of Earth's magnetic field.

SAMPLE PROBLEM 28.09

Force on a current carrying ring placed in non-uniform magnetic field

A bar magnet with its north pole up is placed along the axis of symmetry of a horizontal conducting ring carrying current i, as shown in the Fig. 28-30. At the location of the ring, the magnetic field makes an angle θ with the vertical. What is the force on the ring?

KEY IDEA

Here the wire is in a non-uniform magnetic field. The magnitude of the magnetic field at all the points of the ring is the same, but the direction is different. So, we cannot say that the net force on the wire is zero. We have two options: we can either divide the entire ring into small elements and integrate to find the force on the ring, or

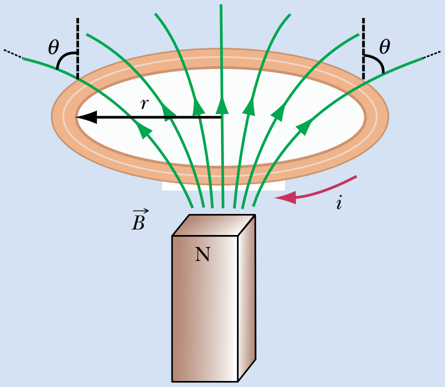


Figure 28-30 A bar magnet kept below a conducting ring.

divide the magnetic field into two components and then integrate the force due to each component of the field. We choose the second option.

Calculations: The magnetic field at each point of the ring is having a vertical component of $B\cos\theta$. This field is uniform and hence the force due to this component of the magnetic field will be zero. The horizontal component of the magnetic field at each element of the ring will be radially outward. This will produce a force on the element in vertically upward direction as can be seen from Fig. 28-31.

The magnitude of this force is given by

$$dF = idLB\sin\theta.$$

The net force acting on the wire is given by

$$F = 2\pi irB\sin\theta.$$

We can see that the vertical component of the magnetic field does not contribute to the net force on the wire; however, each element of the wire will be experiencing a force inward. This will tend to contract the wire. The current-carrying loop experiences a force in a nonuniform magnetic field.

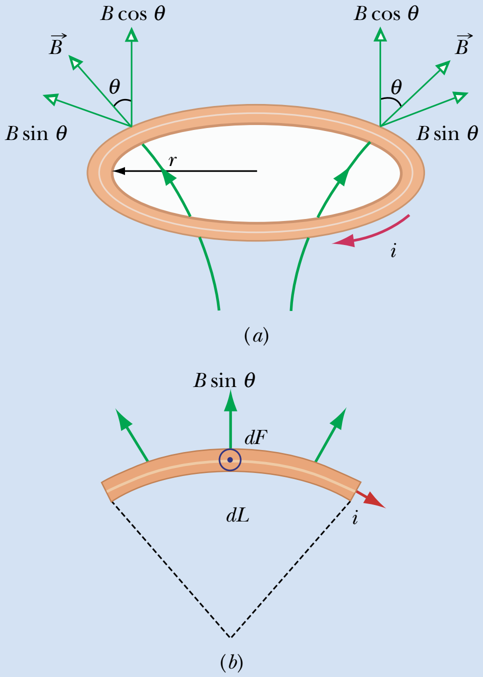


Figure 28-31 Magnetic force acting (*a*) on a wire; (*b*) on the element.

28.9 | TORQUE ON A CURRENT LOOP

Key Concepts

- Various magnetic forces act on the sections of a current-carrying coil lying in a uniform external magnetic field, but the net force is zero.
- The net torque acting on the coil has a magnitude given by

$$\tau = NiAB\sin\theta,$$

where N is the number of turns in the coil, A is the area of each turn, i is the current, B is the field magnitude, and θ is the angle between the magnetic field $\vec{B}$ and the normal vector to the coil $\vec{n}$.

Much of the world's work is done by electric motors. The forces behind this work are the magnetic forces that we studied in the preceding section—that is, the forces that a magnetic field exerts on a wire that carries a current.

Figure 28-32 shows a simple motor, consisting of a single current-carrying loop immersed in a magnetic field $\vec{B}$. The two magnetic forces $\vec{F}$ and $-\vec{F}$ produce a torque on the loop, tending to rotate it about its central axis. Although many essential details have been omitted, the figure does suggest how the action of a magnetic field on a current loop produces rotary motion. Let us analyze that action.

Figure 28-33*a* shows a rectangular loop of sides a and b, carrying current i through uniform magnetic field $\vec{B}$. We place the loop in the field so that its long sides, labeled 1 and 3, are perpendicular to the field direction (which is into the page), but its short sides, labeled 2 and 4, are not. Wires to lead the current into and out of the loop are needed but, for simplicity, are not shown.

To define the orientation of the loop in the magnetic field, we use a normal Vector $\vec{n}$ that is perpendicular to the plane of the loop. Figure 28-33*b* shows a right-hand rule for finding the direction of $\vec{n}$. Point or curl the fingers of your right hand in the direction of the current at any point on the loop. Your extended thumb then points in the direction of the normal vector $\vec{n}$.

In Fig. 28-33*c*, the normal vector of the loop is shown at an arbitrary angle θ to the direction of the magnetic field $\vec{B}$. We wish to find the net force and net torque acting on the loop in this orientation.

Net Torque. The net force on the loop is the vector sum of the forces acting on its four sides. For side 2 the vector $\vec{L}$ in Eq. 28-29 points in the direction of the current and has magnitude b. The angle between $\vec{L}$ and $\vec{B}$ for side 2 (see Fig. 28-33*c*) is $90° - \theta$. Thus, the magnitude of the force acting on this side is

$$F_2 = ibB \sin(90° - \theta) = ibB \cos\theta. \tag{28-34}$$

You can show that the force $\vec{F}_4$ acting on side 4 has the same magnitude as $\vec{F}_2$ but the opposite direction. Thus, $\vec{F}_2$ and $\vec{F}_4$ cancel out exactly. Their net force is zero and, because their common line of action is through the center of the loop, their net torque is also zero.

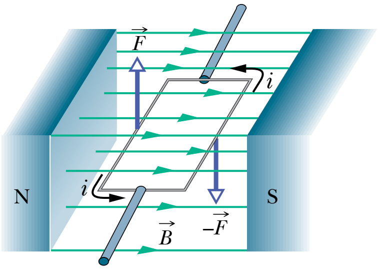


Figure 28-32 The elements of an electric motor. A rectangular loop of wire, carrying a current and free to rotate about a fixed axis, is placed in a magnetic field. Magnetic forces on the wire produce a torque that rotates it. A commutator (not shown) reverses the direction of the current every half-revolution so that the torque always acts in the same direction.

The situation is different for sides 1 and 3. For them, $\vec{L}$ is perpendicular to $\vec{B}$, so the forces $\vec{F}_1$ and $\vec{F}_3$ have the common magnitude iaB. Because these two forces have opposite directions, they do not tend to move the loop up or down. However, as Fig. 28-33*c* shows, these two forces do *not* share the same line of action; so they *do* produce a net torque. The torque tends to rotate the loop so as to align its normal vector $\vec{n}$ with the direction of the magnetic field $\vec{B}$. That torque has moment arm $(b/2)\sin\theta$ about the central axis of the loop. The magnitude τ' of the torque due to forces $\vec{F}_1$ and $\vec{F}_3$ is then (see Fig. 28-33*c*)

$$\tau' = \left(iaB\frac{b}{2}\sin\theta\right) + \left(iaB\frac{b}{2}\sin\theta\right) = iabB\sin\theta. \tag{28-35}$$

Coil. Suppose we replace the single loop of current with a *coil* of N loops, or *turns*. Further, suppose that the turns are wound tightly enough that they can be approximated as all having the same dimensions and lying in a plane. Then the turns form a *flat coil*, and a torque τ' with the magnitude given in Eq. 28-35 acts on each of them. The total torque on the coil then has magnitude

$$\tau = N\tau' = NiabB\sin\theta = (NiA)B\sin\theta, \tag{28-36}$$

in which A $(= ab)$ is the area enclosed by the coil. The quantities in parentheses (NiA) are grouped together because they are all properties of the coil: its number of turns, its area, and the current it carries. Equation 28-36 holds for all

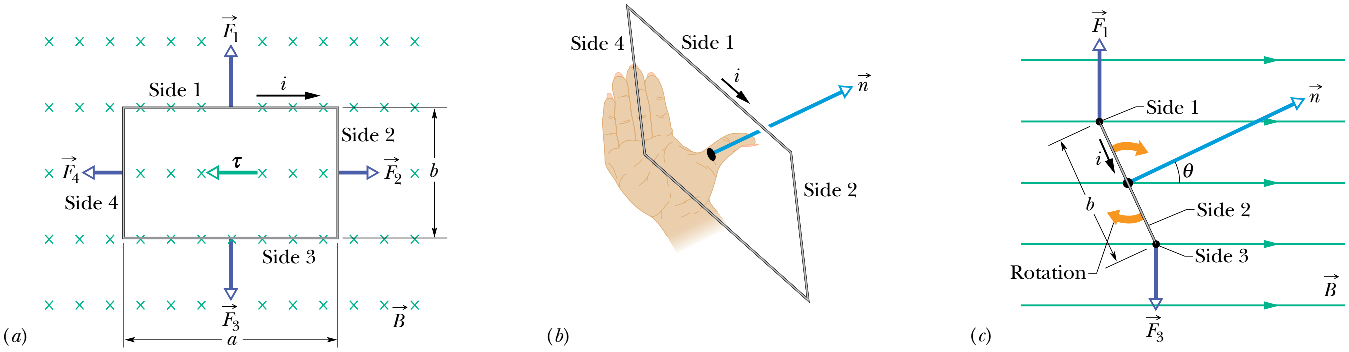


Figure 28-33 A rectangular loop, of length a and width b and carrying a current i, is located in a uniform magnetic field. A torque τ acts to align the normal vector $\vec{n}$ with the direction of the field. (*a*) The loop as seen by looking in the direction of the magnetic field. (*b*) A perspective of the loop showing how the right-hand rule gives the direction of $\vec{n}$, which is perpendicular to the plane of the loop. (*c*) A side view of the loop, from side 2. The loop rotates as indicated.

flat coils, no matter what their shape, provided the magnetic field is uniform. For example, for the common circular coil, with radius r, we have

$$\tau = (Ni\pi r^2)B \sin \theta. \tag{28-37}$$

Normal Vector. Instead of focusing on the motion of the coil, it is simpler to keep track of the vector $\vec{n}$, which is normal to the plane of the coil. Equation 28-36 tells us that a current-carrying flat coil placed in a magnetic field will tend to rotate so that $\vec{n}$ has the same direction as the field. In a motor, the current in the coil is reversed as $\vec{n}$ begins to line up with the field direction, so that a torque continues to rotate the coil. This automatic reversal of the current is done via a commutator that electrically connects the rotating coil with the stationary contacts on the wires that supply the current from some source.

28.10 | THE MAGNETIC DIPOLE MOMENT

Key Concepts

- A coil (of area A and N turns, carrying current i) in a uniform magnetic field $\vec{B}$ will experience a torque $\vec{\tau}$ given by

  $$\vec{\tau} = \vec{\mu} \times \vec{B}.$$

  Here $\vec{\mu}$ is the magnetic dipole moment of the coil, with magnitude $\mu = NiA$ and direction given by the right-hand rule.
- The orientation energy of a magnetic dipole in a magnetic field is

  $$U(\theta) = -\vec{\mu} \cdot \vec{B}.$$
- If an external agent rotates a magnetic dipole from an initial orientation θ_i to some other orientation θ_f and the dipole is stationary both initially and finally, the work W_a done on the dipole by the agent is

  $$W_a = \Delta U = U_f - U_i.$$

As we have just discussed, a torque acts to rotate a current-carrying coil placed in a magnetic field. In that sense, the coil behaves like a bar magnet placed in the magnetic field. Thus, like a bar magnet, a current-carrying coil is said to be a *magnetic dipole*. Moreover, to account for the torque on the coil due to the magnetic field, we assign a **magnetic dipole moment** $\vec{\mu}$ to the coil. The direction of $\vec{\mu}$ is that of the normal vector $\vec{n}$ to the plane of the coil and thus is given by the same right-hand rule shown in Fig. 28-33. That is, grasp the coil with the fingers of your right hand in the direction of current i; the outstretched thumb of that hand gives the direction of $\vec{\mu}$. The magnitude of $\vec{\mu}$ is given by

$$\mu = NiA \quad \text{(magnetic moment)}, \tag{28-38}$$

in which N is the number of turns in the coil, i is the current through the coil, and A is the area enclosed by each turn of the coil. From this equation, with i in amperes and A in square meters, we see that the unit of $\vec{\mu}$ is the ampere – square meter ($\mathrm{A \cdot m^2}$).

Torque. Using $\vec{\mu}$, we can rewrite Eq. 28-36 for the torque on the coil due to a magnetic field as

$$\tau = \mu B \sin \theta, \tag{28-39}$$

in which θ is the angle between the vectors $\vec{\mu}$ and $\vec{B}$.

We can generalize this to the vector relation

$$\vec{\tau} = \vec{\mu} \times \vec{B}, \tag{28-40}$$

which reminds us very much of the corresponding equation for the torque exerted by an *electric* field on an *electric* dipole—namely, Eq. 22-37

$$\vec{\tau} = \vec{p} \times \vec{E}.$$

In each case the torque due to the field—either magnetic or electric—is equal to the vector product of the corresponding dipole moment and the field vector.

Energy. A magnetic dipole in an external magnetic field has an energy that depends on the dipole's orientation in the field. For electric dipoles we have shown (Eq. 22-41) that

$$U(\theta) = -\vec{p} \cdot \vec{E}.$$

In strict analogy, we can write for the magnetic case

$$U(\theta) = -\vec{\mu} \cdot \vec{B}. \tag{28-41}$$

In each case the energy due to the field is equal to the negative of the scalar product of the corresponding dipole moment and the field vector.

A magnetic dipole has its lowest energy ($= -\mu B \cos 0 = -\mu B$) when its dipole moment $\vec{\mu}$ is lined up with the magnetic field (Fig. 28-34). It has its highest energy ($= -\mu B \cos 180° = +\mu B$) when $\vec{\mu}$ is directed opposite the field. From Eq. 28-41, with U in joules and $\vec{B}$ in teslas, we see that the unit of $\vec{\mu}$ can be the joule per tesla (J/T) instead of the ampere–square meter as suggested by Eq. 28-38.

Work. If an applied torque (due to "an external agent") rotates a magnetic dipole from an initial orientation θ_i to another orientation θ_f, then work W_a is done on the dipole by the applied torque. *If the dipole is stationary* before and after the change in its orientation, then work W_a is

$$W_a = U_f - U_i, \tag{28-42}$$

where U_f and U_i are calculated with Eq. 28-41.

So far, we have identified only a current-carrying coil and a permanent magnet as a magnetic dipole. However, a rotating sphere of charge is also a magnetic dipole, as is Earth itself (approximately). Finally, most subatomic particles, including the electron, the proton, and the neutron, have magnetic dipole moments. As you will see in Chapter 32, all these quantities can be viewed as current loops. For comparison, some approximate magnetic dipole moments are shown in Table 28-2.

Language. Some instructors refer to U in Eq. 28-41 as a potential energy and relate it to work done by the magnetic field when the orientation of the dipole changes. Here we shall avoid the debate and say that U is an energy associated with the dipole orientation.

The magnetic moment vector attempts to align with the magnetic field.

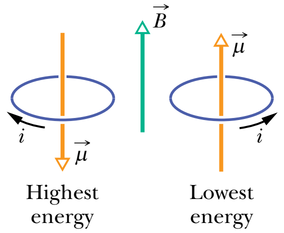


Figure 28-34 The orientations of highest and lowest energy of a magnetic dipole (here a coil carrying current) in an external magnetic field $\vec{B}$. The direction of the current i gives the direction of the magnetic dipole moment $\vec{\mu}$ via the right-hand rule shown for $\vec{n}$ in Fig. 28-33*b*.

Table 28-2 Some Magnetic Dipole Moments

Small bar magnet	5 J/T
Earth	8.0×10^{22} J/T
Proton	1.4×10^{-26} J/T
Electron	9.3×10^{-24} J/T

Loop Model for Electron Orbits

We do not need a conducting wire loop to make a magnetic dipole. Rather, a circulating electron can also form a magnetic dipole. We imagine an electron moving at constant speed v in a circular path of radius r, counterclockwise as shown in Fig. 28-35.

The motion of the negative charge of the electron is equivalent to a conventional current i (of positive charge) that is clockwise, as also shown in Fig. 28-35. The magnitude of the orbital magnetic dipole moment of such a *current loop* is obtained from Eq. 28-38 with $N = 1$:

$$\mu_{\text{orb}} = iA, \tag{28-43}$$

where A is the area enclosed by the loop. The direction of this magnetic dipole moment is, from the right-hand rule of Fig. 28-28, downward in Fig. 28-35.

To evaluate Eq. 28-43, we need the current i. Current is, generally, the rate at which charge passes some point in a circuit. Here, the charge of magnitude e takes a time $T = 2\pi r/v$ to circle from any point back through that point, so

$$i = \frac{e}{2\pi r/v}.$$

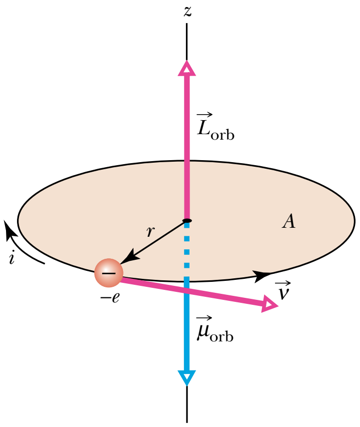


Figure 28.35 An electron moving at constant speed v in a circular path of radius r that encloses an area A. The electron has an orbital angular momentum and an associated orbital magnetic dipole moment. A clockwise current i (of positive charge) is equivalent to the counterclockwise circulation of the negatively charged electron.

Substituting this and the area $A = \pi r^2$ of the loop into Eq. 28-43 gives us

$$\mu_{orb} = \frac{e}{2\pi r/v}\pi r^2 = \frac{evr}{2}. \tag{28-44}$$

To find the electron's orbital angular momentum $\vec{L}_{orb}$, we use the definition of angular momentum, $\vec{L} = m(\vec{r} \times \vec{v})$. Because $\vec{r}$ and $\vec{v}$ are perpendicular, $\vec{L}$ has the magnitude

$$L_{orb} = mrv. \tag{28-45}$$

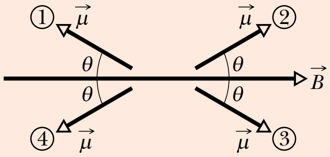
The vector is directed upward in Fig. 28-35. Combining Eqs. 28-44 and 28-45, generalizing to a vector formulation, and indicating the opposite directions of the vectors with a minus sign yield

$$\vec{\mu}_{orb} = -\frac{e}{2m}\vec{L}_{orb}.$$

This relation can be readily seen to be valid for any rotating system of charges. The idea is that the calculation of angular momentum of a system is easy if we know its moment of inertia about the axis of rotation. By this formula, we can easily calculate its magnetic moment. Note that the magnetic dipole moment is independent of the choice of origin. As we saw in Sample Problem 28.09, a magnetic dipole may feel a net force in a non-uniform magnetic field. In any atom, there are electrons revolving around the nucleus. This gives a magnetic dipole moment to the atom, and so the atom may get attracted or repelled in a magnetic field. The magnetic force experienced by so many atoms put together gives magnetic substances their magnetic properties.

CHECKPOINT 7

The figure shows four orientations, at angle θ, of a magnetic dipole moment $\vec{\mu}$ in a magnetic field. Rank the orientations according to (a) the magnitude of the torque on the dipole and (b) the orientation energy of the dipole, greatest first.



SAMPLE PROBLEM 28.10

Rotating a magnetic dipole in a magnetic field

Figure 28-36 shows a circular coil with 250 turns, an area A of 2.52×10^{-4} m^2, and a current of 100 μA. The coil is at rest in a uniform magnetic field of magnitude $B = 0.85$ T, with its magnetic dipole moment $\vec{\mu}$ initially aligned with $\vec{B}$.

(a) In Fig. 28-36, what is the direction of the current in the coil?

Right-hand rule: Imagine cupping the coil with your right hand so that your right thumb is outstretched in the direction of $\vec{\mu}$. The direction in which your fingers curl around the coil is the direction of the current in the coil. Thus, in the wires on the near side of the coil—those we see in Fig. 28-36—the current is from top to bottom.

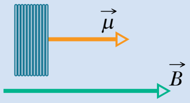


Figure 28-36 A side view of a circular coil carrying a current and oriented so that its magnetic dipole moment is aligned with magnetic field $\vec{B}$.

(b) How much work would the torque applied by an external agent have to do on the coil to rotate it 90° from its initial orientation, so that $\vec{\mu}$ is perpendicular to $\vec{B}$ and the coil is again at rest?

KEY IDEA

The work W_a done by the applied torque would be equal to the change in the coil's orientation energy due to its change in orientation.

Calculations: From Eq. 28-42 ($W_a = U_f - U_i$), we find

$$\begin{aligned} W_a &= U(90°) - U(0°) \\ &= -\mu B \cos 90° - (-\mu B \cos 0°) = 0 + \mu B \\ &= \mu B. \end{aligned}$$

Substituting for μ from Eq. 28-38 ($\mu = NiA$), we find that

$$\begin{aligned} W_a &= (NiA)B \\ &= (250)(100 \times 10^{-6}\,\text{A})(2.52 \times 10^{-4}\,\text{m}^2)(0.85\,\text{T}) \\ &= 5.355 \times 10^{-6}\,\text{J} \approx 5.4\,\mu\text{J}. \qquad \text{(Answer)} \end{aligned}$$

Similarly, we can show that to change the orientation by another 90°, so that the dipole moment is opposite the field, another 5.4 μJ is required.

SAMPLE PROBLEM 28.11

A wire frame in uniform magnetic field

A uniform, constant magnetic field B is directed at an angle of 45° to the x-axis in the xy plane. $PQRS$ is a rigid square wire frame carrying a steady current i (clockwise), with its center at the origin O. At time $t = 0$, the frame is at rest in the position shown in Fig. 28-37, with its sides parallel to the x and y axes. Each side of the frame is of mass M and length L.

(a) What is the torque acting on the frame due to the magnetic field?

KEY IDEA

Since this frame is a closed current-carrying loop, it will experience a torque in a constant magnetic field. The torque due to the magnetic field can be found by Eq. 28-37.

Calculations: The magnetic moment of the loop is given by $\mu = iL^2$ because the loop has only one turn. This is a vector whose direction is negative z direction by right hand rule.

The torque acting on the loop is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B}.$$

Since $\vec{\mu}$ and $\vec{B}$ are perpendicular to each other, its magnitude is given by

$$\tau = \mu \times B \sin 90° = iL^2B.$$

This torque is directed in xy plane at an angle of 45º to the x axis as seen in Fig. 28-38. Use right-hand rule to convince yourself!

(b) Find the initial angular acceleration of the frame. Assume that the frame is in a gravity-free space free from any other influence.

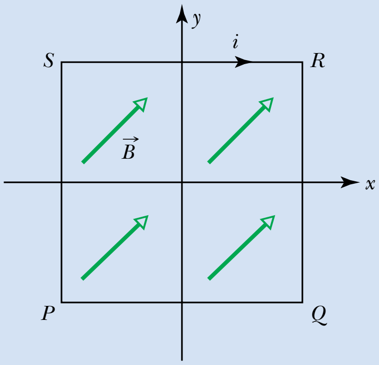


Figure 28-37 A wire frame kept in a uniform magnetic field.

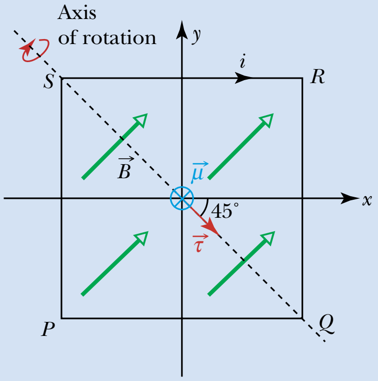


Figure 28-38 The magnetic moment, torque, and the direction of axis of rotation of the wire frame.

KEY IDEAS

(1) To find the angular acceleration of the frame, we need to know the moment of inertia of the frame for which we further need to know the axis about which the

frame is rotating. Now, this can be a little tricky because as per the information given, the frame is not hinged at any point. We know that the force on a magnetic dipole in a uniform magnetic field is zero. So, the torque that we have calculated is actually a couple. So, whatever be the axis of rotation, the torque will turn out to be the same.

(2) Since there is no net force, the center of mass of the body should be at rest. So, the frame should rotate about its center of mass.

(3) The torque vector is in the direction of the axis of rotation. So, this means that the frame is rotating about an axis in the *xy* plane about its diagonal. We can use the perpendicular axis theorem to find the angular acceleration.

Calculations: Using perpendicular axis theorem we can say that

$$I_1 + I_2 = I_Z,$$

where I_1 and I_2 are the moments of inertia of the frame about the diagonals. Also, by symmetry, $I_1 = I_2$.

By parallel axis theorem, the moment of inertia of each of the rods about the *z* axis is given by

$$\frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}.$$

So, the moment of inertia of the entire frame about the *z* axis is given by

$$I_z = \frac{4ML^2}{3}.$$

But we can say that

$$I_1 = \frac{I_z}{2} = \frac{2ML^2}{3}.$$

By using the expression $\tau = I_1\alpha$, we get

$$\alpha = \frac{\tau}{I_1} = \frac{iL^2B}{2ML^2/3} = \frac{3iB}{2M}.$$

Note: We should note that this is the instantaneous angular acceleration. As the frame rotates, the angle between the magnetic moment vector and the magnetic field changes. So, the angular acceleration will change with time.

Moving Coil Galvanometer

Analog voltmeters and ammeters work by measuring the torque exerted by a magnetic field on a current-carrying coil. The reading is displayed by means of the deflection of a pointer over a scale. Figure 28-39 shows the essentials of a *galvanometer*, a device on which both analog ammeters and analog voltmeters are based. For any orientation of the coil, the net magnetic field through the coil is perpendicular to the normal vector of the coil (and thus parallel to the plane of the coil). A spring "Sp" provides a counter torque that balances the magnetic torque, so that a given steady current *i* in the coil results in a steady angular deflection ϕ. The greater the current is, the greater the deflection is, and thus the greater the torque required of the spring is.

We write this balance as

$$NiAB = c\phi, \tag{28-46}$$

where *c* is the torsional constant of the spring Sp. Equation 28-46 then reduces to

$$i = \left(\frac{c}{NAB}\right)\phi. \tag{28-47}$$

Thus, we can see that the deflection of the needle is directly proportional to the current passing through the coil.

The current sensitivity is the current necessary (usually given in amperes) to give a deflection of one scale. From Eq. 28-47 this current sensitivity is

$$\frac{\phi}{i} = \frac{NAB}{c}. \tag{28-48}$$

A convenient way to increase the sensitivity is to increase the number of turns *N*.

The moving coil galvanometer can also be used as a voltmeter to measure the voltage across a given section of the circuit. The scale of the voltmeter is

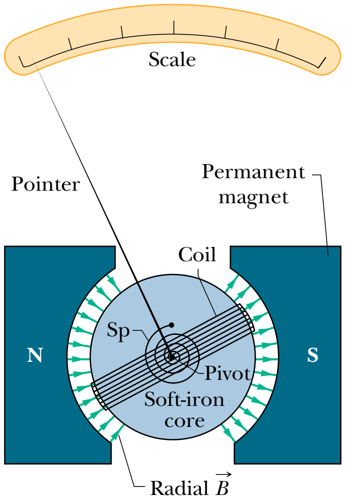


Figure 28-39 The elements of a galvanometer. Depending on the external circuit, this device can be wired up as either a voltmeter or an ammeter With a constant current through the device, the resulting magnetic torque (Eq. 28-36) is balanced by the spring torque.

calibrated and then graduated to read off the voltage value with ease. We define the voltage sensitivity as the deflection per unit voltage. From Eq. 28-41:

$$\frac{\phi}{V}=\left(\frac{NAB}{k}\right)\frac{i}{V}=\left(\frac{NAB}{k}\right)\frac{1}{R}. \tag{28-49}$$

An interesting point to note is that increasing the current sensitivity may not necessarily increase the voltage sensitivity. Let us take Eq. 28-48 which provides a measure of current sensitivity. If $N \to 2N$, that is, we double the number of turns; then

$$\frac{\phi}{i}\to 2\frac{\phi}{i}.$$

Thus, the current sensitivity doubles. However, the resistance of the galvanometer is also likely to double, since it is proportional to the length of the wire. In Eq. 29-49, $N \to 2N$ and $R \to 2R$; thus the voltage sensitivity remains unchanged.

REVIEW AND SUMMARY

Magnetic Field $\vec{B}$ A **magnetic field** $\vec{B}$ is defined in terms of the force $\vec{F}_B$ acting on a test particle with charge q moving through the field with velocity $\vec{v}$:

$$\vec{F}_B = q\vec{v}\times\vec{B}. \tag{28-2}$$

The SI unit for $\vec{B}$ is the **tesla** (T): $1\ \text{T} = 1\ \text{N/(A}\cdot\text{m)} = 10^4$ gauss.

A Charged Particle Circulating in a Magnetic Field A charged particle with mass m and charge magnitude $|q|$ moving with velocity $\vec{v}$ perpendicular to a uniform magnetic field $\vec{B}$ will travel in a circle. Applying Newton's second law to the circular motion yields

$$|q|vB=\frac{mv^2}{r}, \tag{28-7}$$

from which we find the radius r of the circle to be

$$r=\frac{mv}{|q|B}. \tag{28-8}$$

The frequency of revolution f, the angular frequency ω, and the period of the motion T are given by

$$f=\frac{\omega}{2\pi}=\frac{1}{T}=\frac{|q|B}{2\pi m}. \qquad (28\text{-}11, 28\text{-}10, 28\text{-}9)$$

Cyclotron Charged particles are accelerated by electric forces as they circle in a magnetic field. If the frequency f at which the proton circulates in the magnetic field (and that does *not* depend on its speed) is equal to the fixed frequency f_{osc} of the electrical oscillator then it is called the resonance condition, given by

$$f=f_{\text{osc}} \tag{28-15}$$

Crossed Fields If the fields are perpendicular to each other, they are said to be *crossed fields*. The crossed fields allow us to measure the speed of the charged particles passing through them, given by

$$v=\frac{E}{B} \tag{28-19}$$

If the forces are in opposite directions, a particular speed will result in no deflection of the particle.

The Hall Effect When a conducting strip carrying a current i is placed in a uniform magnetic field $\vec{B}$, some charge carriers (with charge e) build up on one side of the conductor, creating a potential difference V across the strip. The polarities of the sides indicate the sign of the charge carriers.

The magnitude of that potential difference is

$$V=Ed \tag{28-24}$$

The number density n of the charge carriers can then be determined from

$$n=\frac{Bi}{Vle} \tag{28-27}$$

where l is the thickness of the strip (parallel to $\vec{B}$).

Magnetic Force on a Current-Carrying Wire A straight wire carrying a current i in a uniform magnetic field experiences a sideways force

$$\vec{F}_B=i\vec{L}\times\vec{B}. \tag{28-29}$$

The force acting on a current element $i\,d\vec{L}$ in a magnetic field is

$$d\vec{F}_B=id\vec{L}\times\vec{B}. \tag{28-31}$$

The direction of the length vector $\vec{L}$ or $d\vec{L}$ is that of the current i.

Torque on a Current-Carrying Coil A coil (of area A and N turns, carrying current i) in a uniform magnetic field $\vec{B}$ will experience a torque $\vec{\tau}$ given by

$$\vec{\tau}=\vec{\mu}\times\vec{B}. \tag{28-40}$$

Here $\vec{\mu}$ is the **magnetic dipole moment** of the coil, with magnitude $\mu = NiA$ and direction given by the right-hand rule.

Orientation Energy of a Magnetic Dipole The orientation energy of a magnetic dipole in a magnetic field is

$$U(\theta)=-\vec{\mu}\cdot\vec{B}. \tag{28-41}$$

If an external agent rotates a magnetic dipole from an initial orientation θ_i to some other orientation θ_f and the dipole is stationary both initially and finally, the work W_a done on the dipole by the agent is

$$W_a=\Delta U=U_f-U_i. \tag{28-42}$$

PROBLEMS

1. A conducting rectangular solid of dimensions $d_x = 5.00$ m, $d_y = 3.00$ m, and $d_z = 2.00$ m moves with a constant velocity $\vec{v} = (20.0\text{ m/s})\hat{\text{i}}$ through a uniform magnetic field $\vec{B} = (40.0\text{ mT})\hat{\text{j}}$ (Fig. 28-40). What are the resulting (a) electric field within the solid, in unit-vector notation, and (b) potential difference across the solid? (c) Which face becomes negatively charged?

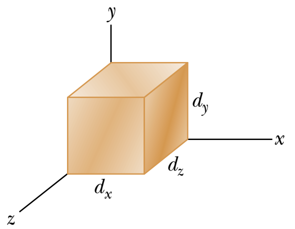


Figure 28-40 Problem 1.

2. A particular type of fundamental particle decays by transforming into an electron e^- and a positron e^+. Suppose the decaying particle is at rest in a uniform magnetic field $\vec{B}$ of magnitude 9.57 mT and the e^- and e^+ move away from the decay point in paths lying in a plane perpendicular to $\vec{B}$. How long after the decay do the e^- and e^+ collide?

3. An electron follows a helical path in a uniform magnetic field given by $\vec{B} = (20\hat{\text{i}} - 50\hat{\text{j}} - 30\hat{\text{k}})$ mT. At time $t = 0$, the electron's velocity is given by $\vec{v} = (40\hat{\text{i}} - 30\hat{\text{j}} + 50\hat{\text{k}})$ m/s. (a) What is the angle ϕ between $\vec{v}$ and $\vec{B}$? The electron's velocity changes with time. Do (b) its speed and (c) the angle ϕ change with time? (d) What is the radius of the helical path?

4. A wire 66.0 cm long carries a 0.750 A current in the positive direction of an x axis through a magnetic field $\vec{B} = (3.00\text{ mT})\hat{\text{j}} + (14.0\text{ mT})\hat{\text{k}}$. In unit-vector notation, what is the magnetic force on the wire?

5. A wire 2.30 m long carries a current of 13.0 A and makes an angle of 35.0° with a uniform magnetic field of magnitude $B = 1.50$ T. Calculate the magnetic force on the wire.

6. Figure 28-41 shows a wood cylinder of mass $m = 0.150$ kg and length $L = 0.100$ m, with $N = 13.0$ turns of wire wrapped around it longitudinally, so that the plane of the wire coil contains the long central axis of the cylinder. The cylinder is released on a plane inclined at an angle θ to the horizontal, with the plane of the coil parallel to the incline plane. If there is a vertical uniform magnetic field of magnitude 0.92 T, what is the least current i through the coil that keeps the cylinder from rolling down the plane?

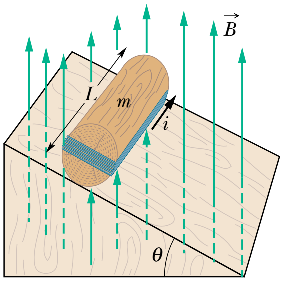


Figure 28-41 Problem 6.

7. In Fig. 28-42, a charged particle moves into a region of uniform magnetic field $\vec{B}$, goes through half a circle, and then exits that region. The particle is either a proton or an electron (you must decide which). It spends 160 ns in the region. (a) What is the magnitude of $\vec{B}$? (b) If the particle is sent back through the magnetic field (along the same initial path) but with 2.00 times its previous kinetic energy, how much time does it spend in the field during this trip?

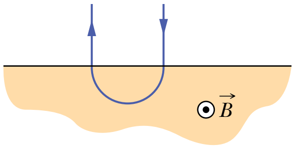


Figure 28-42 Problem 7.

8. Two concentric, circular wire loops, of radii $r_1 = 20.0$ cm and $r_2 = 40.0$ cm, are located in an xy plane; each carries a clockwise current of 11.0 A (Fig. 28-43). (a) Find the magnitude of the net magnetic dipole moment of the system. (b) Repeat for reversed current in the inner loop.

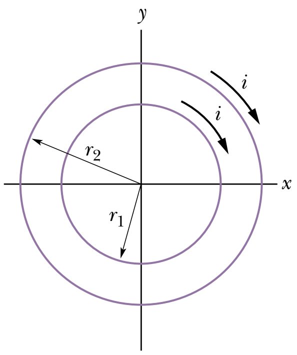


Figure 28-43 Problem 8.

9. A source injects an electron of speed $v = 1.2 \times 10^7$ m/s into a uniform magnetic field of magnitude $B = 1.0 \times 10^{-3}$ T. The velocity of the electron makes an angle $\theta = 10°$ with the direction of the magnetic field. Find the distance d from the point of injection at which the electron next crosses the field line that passes through the injection point.

10. A 0.85 kg copper rod rests on two horizontal rails 1.0 m apart and carries a current of 65 A from one rail to the other. The coefficient of static friction between rod and rails is 0.50. What are the (a) magnitude and (b) angle (relative to the vertical) of the smallest magnetic field that puts the rod on the verge of sliding?

11. Figure 28-44 shows a wire ring of radius $a = 1.8$ cm that is perpendicular to the general direction of a radially symmetric, diverging magnetic field. The magnetic field at the ring is everywhere of the same magnitude $B = 3.4$ mT, and its direction at the ring everywhere makes an angle $\theta = 15°$ with a normal to the plane of the ring. The twisted lead wires have no effect on the problem. Find the magnitude of the force the field exerts on the ring if the ring carries a current $i = 4.6$ mA.

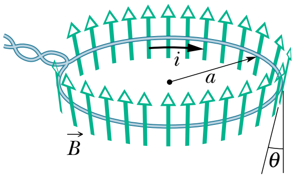


Figure 28-44 Problem 11.

12. (a) Find the frequency of revolution of an electron with an energy of 189 eV in a uniform magnetic field of magnitude 70.0 μT. (b) Calculate the radius of the path of this electron if its velocity is perpendicular to the magnetic field.

13. Figure 28-45 gives the orientation energy U of a magnetic dipole in an external magnetic field $\vec{B}$, as a function of angle ϕ between the directions of $\vec{B}$ and the dipole moment. The vertical axis scale is set by $U_s = 2.0 \times 10^{-4}$ J. The dipole can be rotated about an axle with negligible friction in order to change ϕ. Counterclockwise rotation from $\phi = 0$ yields positive values of ϕ, and clockwise rotations yield negative values. The dipole is to be released at angle $\phi = 0$ with a rotational kinetic energy of 9.0×10^{-4} J, so that it rotates counterclockwise. To what maximum value of ϕ will it rotate?

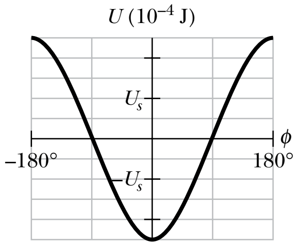


Figure 28-45 Problem 13.

14. A current loop, carrying a current of 7.5 A, is in the shape of a right triangle with sides 30, 40, and 50 cm. The loop is in a uniform magnetic field of magnitude 120 mT whose direction is parallel to the current in the 50 cm side of the loop. Find the magnitude of (a) the magnetic dipole moment of the loop and (b) the torque on the loop.

15. An electron is accelerated from rest by a potential difference of 380 V. It then enters a uniform magnetic field of magnitude 200 mT with its velocity perpendicular to the field. Calculate (a) the speed of the electron and (b) the radius of its path in the magnetic field.

16. Figure 28-46 shows a rectangular 28-turn coil of wire, of dimensions 10 cm by 5.0 cm. It carries a current of 0.80 A and is hinged along one long side. It is mounted in the xy plane, at angle $\theta = 25°$ to the direction of a uniform magnetic field of magnitude 0.50 T. In unit vector notation, what is the torque acting on the coil about the hinge line?

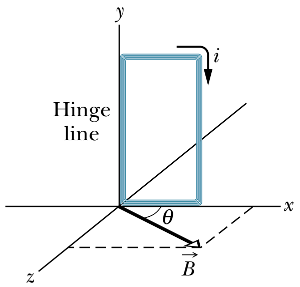


Figure 28-46 Problem 16.

17. A circular wire loop of radius 15.0 cm carries a current of 3.20 A. It is placed so that the normal to its plane makes an angle of 41.0° with a uniform magnetic field of magnitude 12.0 T. (a) Calculate the magnitude of the magnetic dipole moment of the loop. (b) What is the magnitude of the torque acting on the loop?

18. A 6.75 g wire of length $L = 15.0$ cm is suspended by a pair of flexible leads in a uniform magnetic field of magnitude 0.440 T (Fig. 28-47). What are the (a) magnitude and (b) direction (left or right) of the current required to remove the tension in the supporting leads?

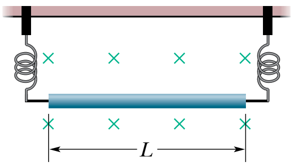


Figure 28-47 Problem 18.

19. A metal strip 6.50 cm long, 0.850 cm wide, and 0.760 mm thick moves with constant velocity $\vec{v}$ through a uniform magnetic field $B = 1.20$ mT directed perpendicular to the strip, as shown in Fig. 28-48. A potential difference of 3.30 μV is measured between points x and y across the strip. Calculate the speed v.

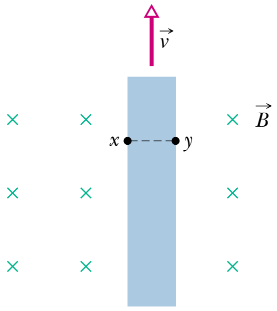


Figure 28-48 Problem 19.

20. An electron moves in a circle of radius $r = 5.29 \times 10^{-11}$ m with speed 4.12×10^6 m/s. Treat the circular path as a current loop with a constant current equal to the ratio of the electron's charge magnitude to the period of the motion. If the circle lies in a uniform magnetic field of magnitude $B = 7.10$ mT, what is the maximum possible magnitude of the torque produced on the loop by the field?

21. A proton circulates in a cyclotron, beginning approximately at rest at the center. Whenever it passes through the gap between dees, the electric potential difference between the dees is 350 V. (a) By how much does its kinetic energy increase with each passage through the gap? (b) What is its kinetic energy as it completes 100 passes through the gap? Let r_{100} be the radius of the proton's circular path as it completes those 100 passes and enters a dee, and let r_{101} be its next radius, as it enters a dee the next time. (c) By what percentage does the radius increase when it changes from r_{100} to r_{101}? That is, what is

$$\text{percentage increase} = \frac{r_{101} - r_{100}}{r_{100}} 100\%?$$

22. Figure 28-49 shows a current loop $ABCDEFA$ carrying a current $i = 3.00$ A. The sides of the loop are parallel to the coordinate axes shown, with $AB = 20.0$ cm, $BC = 30.0$ cm, and $FA = 10.0$ cm. In unit-vector notation, what is the magnetic dipole moment of this loop? (*Hint:* Imagine equal and opposite currents i in the line segment AD; then treat the two rectangular loops $ABCDA$ and $ADEFA$.)

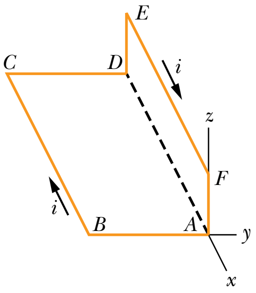


Figure 28-49 Problem 22.

23. A circular coil of 500 turns has a radius of 1.90 cm. (a) Calculate the current that results in a magnetic dipole moment of magnitude 1.90 A·m². (b) Find the maximum magnitude of the torque that the coil, carrying this current, can experience in a uniform 35.0 mT magnetic field.

24. In Fig. 28-50, an electron accelerated from rest through potential difference $V_1 = 2.50$ kV enters the gap between two parallel plates having separation $d = 16.0$ mm and potential difference $V_2 = 100$ V. The lower plate is at the lower potential. Neglect fringing and assume that the electron's velocity vector is perpendicular to the electric field vector between the plates. (a) In unit-vector notation, what uniform magnetic field allows the electron to travel in a straight line in the gap? (b) If the potential difference is increased slightly, in what direction does the electron veer from straight-line motion?

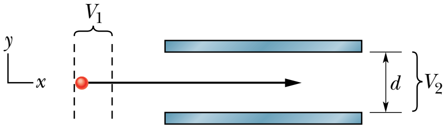


Figure 28-50 Problem 24.

25. In Fig. 28-51, a metal wire of mass $m = 24.1$ mg can slide with negligible friction on two horizontal parallel rails separated by distance $d = 2.56$ cm. The track lies in a vertical uniform magnetic field of magnitude 73.5 mT. At time $t = 0$, device G is connected to the rails, producing a constant current $i = 9.13$ mA in the wire and rails (even as the wire moves). At $t = 61.1$ ms, what are the wire's (a) speed and (b) direction of motion (left or right)?

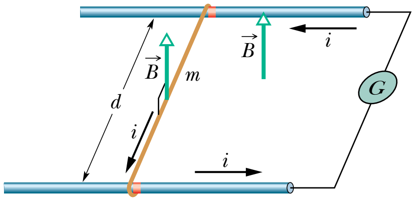


Figure 28-51 Problem 25.

26. A horizontal power line carries a current of 7000 A from south to north. Earth's magnetic field (60.0 μT) is directed toward the north and inclined downward at 70.0° to the horizontal. Find the (a) magnitude and (b) direction of the magnetic force on 100 m of the line due to Earth's field.

27. A circular loop of wire having a radius of 8.0 cm carries a current of 0.20 A. A vector of unit length and parallel to the dipole moment $\vec{\mu}$ of the loop is given by $0.60\hat{\mathrm{i}} - 0.80\hat{\mathrm{j}}$. (This unit vector gives the orientation of the magnetic dipole moment vector.) If the loop is located in a uniform magnetic field given by $\vec{B} = (0.50\ \mathrm{T})\hat{\mathrm{i}} + (0.20\ \mathrm{T})\hat{\mathrm{k}}$, find (a) the torque on the loop (in unit-vector notation) and (b) the orientation energy of the loop.

28. An electron is accelerated from rest through potential difference V and then enters a region of uniform magnetic field, where it undergoes uniform circular motion. Figure 28-52 gives the radius r of that motion versus $V^{1/2}$. The vertical axis scale is set by $r_s = 9.0$ mm, and the horizontal axis scale is set by $V_s^{1/2} = 40.0\ \mathrm{V}^{1/2}$. What is the magnitude of the magnetic field?

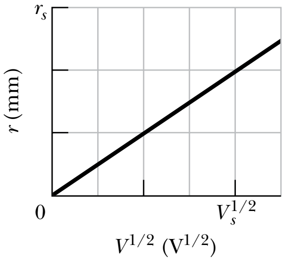


Figure 28-52 Problem 28.

29. An electron moves through a uniform magnetic field given by $\vec{B} = B_x\hat{\mathrm{i}} + (-3.0B_x)\hat{\mathrm{j}}$. At a particular instant, the electron has velocity $\vec{v} = (2.0\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}})$ m/s and the magnetic force acting on it is $(6.4 \times 10^{-19}\ \mathrm{N})\hat{\mathrm{k}}$. Find B_x.

30. A cyclotron with dee radius 47.0 cm is operated at an oscillator frequency of 12.0 MHz to accelerate protons. (a) What magnitude B of magnetic field is required to achieve resonance? (b) At that field magnitude, what is the kinetic energy of a proton emerging from the cyclotron? Suppose, instead, that $B = 1.57$ T. (c) What oscillator frequency is required to achieve resonance now? (d) At that frequency, what is the kinetic energy of an emerging proton?

31. (a) What uniform magnetic field, applied perpendicular to a beam of electrons moving at 1.30×10^6 m/s, is required to make the electrons travel in a circular arc of radius 0.500 m? (b) What is the period of the motion?

32. The magnetic dipole moment of Earth has magnitude 8.00×10^{22} J/T. Assume that this is produced by charges flowing in Earth's molten outer core. If the radius of their circular path is 3700 km, calculate the current they produce.

33. An electron follows a helical path in a uniform magnetic field of magnitude 1.30 T. The pitch of the path is 6.00 μm, and the magnitude of the magnetic force on the electron is 2.00×10^{-14} N. What is the electron's speed?

34. A particle of mass 12 g and charge 80 μC moves through a uniform magnetic field, in a region where the free-fall acceleration is $-9.8\hat{\mathrm{j}}\ \mathrm{m/s^2}$. The velocity of the particle is a constant $20\hat{\mathrm{i}}$ km/s, which is perpendicular to the magnetic field. What, then, is the magnetic field?

35. An electron has an initial velocity of $(12.0\hat{\mathrm{j}} - 15.0\hat{\mathrm{k}})$ km/s and a constant acceleration of $(2.00 \times 10^{12}\ \mathrm{m/s^2})\hat{\mathrm{i}}$ in a region in which uniform electric and magnetic fields are present. If $\vec{B} = (300\ \mu\mathrm{T})\hat{\mathrm{i}}$, find the electric field $\vec{E}$.

36. A particle undergoes uniform circular motion of radius 28.7 μm in a uniform magnetic field. The magnetic force on the particle has a magnitude of 1.60×10^{-17} N. What is the kinetic energy of the particle?

37. A certain particle is sent into a uniform magnetic field, with the particle's velocity vector perpendicular to the direction of the field. Figure 28-53 gives the period T of the particle's motion versus the *inverse* of the field magnitude B. The vertical axis scale is set by $T_s = 80.0$ ns, and the horizontal axis scale is set by $B_s^{-1} = 10.0\ \mathrm{T}^{-1}$. What is the ratio m/q of the particle's mass to the magnitude of its charge?

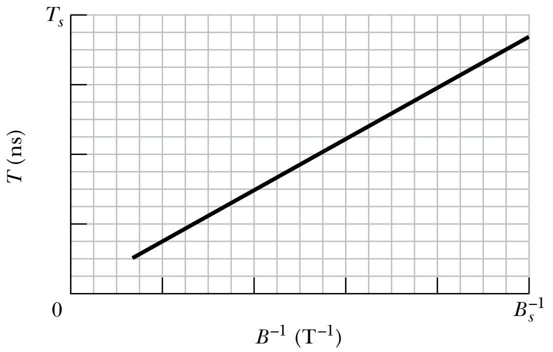


Figure 28-53 Problem 37.

38. A magnetic dipole with a dipole moment of magnitude 0.020 J/T is released from rest in a uniform magnetic field of magnitude 46 mT. The rotation of the dipole due to the magnetic force on it is unimpeded. When the dipole rotates through the orientation where its dipole moment is aligned with the magnetic field, its kinetic energy is 0.80 mJ. (a) What is the initial angle between the dipole moment and the magnetic field? (b) What is the angle when the dipole is next (momentarily) at rest?

39. A strip of copper 75.0 μm thick and 4.5 mm wide is placed in a uniform magnetic field $\vec{B}$ of magnitude 0.65 T, with $\vec{B}$ perpendicular to the strip. A current i = 57 A is then sent through the strip such that a Hall potential difference V appears across the width of the strip. Calculate V. (The number of charge carriers per unit volume for copper is 8.47×10^{28} electrons/m^3.)

40. An alpha particle travels at a velocity $\vec{v}$ of magnitude 620 m/s through a uniform magnetic field $\vec{B}$ of magnitude 0.045 T. (An alpha particle has a charge of $+3.2 \times 10^{-19}$ C and a mass of 6.6×10^{-27} kg.) The angle between $\vec{v}$ and $\vec{B}$ is 52°. What is the magnitude of (a) the force $\vec{F}_B$ acting on the particle due to the field and (b) the acceleration of the particle due to $\vec{F}_B$? (c) Does the speed of the particle increase, decrease, or remain the same?

41. A proton travels through uniform magnetic and electric fields. The magnetic field is $\vec{B} = -3.25\hat{\text{i}}$ mT. At one instant the velocity of the proton is $\vec{v} = 2000\hat{\text{j}}$ m/s. At that instant and in unit-vector notation, what is the net force acting on the proton if the electric field is (a) $4.00\hat{\text{k}}$ V/m, (b) $-4.00\hat{\text{k}}$ V/m, and (c) $4.00\hat{\text{i}}$ V/m?

42. A proton traveling at 42.0° with respect to the direction of a magnetic field of strength 2.60 mT experiences a magnetic force of 1.17×10^{-17} N. Calculate (a) the proton's speed, (b) its kinetic energy in electron-volts, and (c) its momentum.

43. In Fig. 28-54, a particle moves along a circle in a region of uniform magnetic field of magnitude B = 5.00 mT. The particle is either a proton or an electron (you must decide which). It experiences a magnetic force of magnitude 3.20×10^{-15} N. What are (a) the particle's speed, (b) the radius of the circle, and (c) the period of the motion?

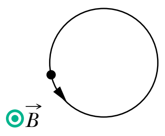


Figure 28-54 Problem 43.

44. An electron of kinetic energy 600 eV circles in a plane perpendicular to a uniform magnetic field. The orbit radius is 12.5 cm. Find (a) the electron's speed, (b) the magnetic field magnitude, (c) the circling frequency, and (d) the period of the motion. (e) Through what potential difference would the electron have to be accelerated from rest to reach this kinetic energy?

45. In a certain cyclotron a proton moves in a circle of radius 0.500 m. The magnitude of the magnetic field is 1.00 T. (a) What is the oscillator frequency? (b) What is the kinetic energy of the proton, in electron-volts?

46. A proton moves through a uniform magnetic field given by $\vec{B} = (10\hat{\text{i}} - 20\hat{\text{j}} + 25\hat{\text{k}})$ mT. At time t_1, the proton has a velocity given by $\vec{v} = v_x\hat{\text{i}} + v_y\hat{\text{j}} + (2.0 \text{ km/s})\hat{\text{k}}$ and the magnetic force on the proton is $\vec{F}_B = (4.0 \times 10^{-17} \text{ N})\hat{\text{i}} + (2.0 \times 10^{-17} \text{ N})\hat{\text{j}}$. At that instant, what are (a) v_x and (b) v_y?

47. A single-turn current loop, carrying a current of 8.00 A, is in the shape of a right triangle with sides 50.0, 120, and 130 cm. The loop is in a uniform magnetic field of magnitude 75.0 mT whose direction is parallel to the current in the 130 cm side of the loop. What is the magnitude of the magnetic force on (a) the 130 cm side, (b) the 50.0 cm side, and (c) the 120 cm side? (d) What is the magnitude of the net force on the loop?

48. Figure 28-55*a*, two concentric coils, lying in the same plane, carry currents in opposite directions. The current in the larger coil 1 is fixed. Current i_2 in coil 2 can be varied. Figure 28-55*b* gives the net magnetic moment of the two-coil system as a function of i_2. The vertical axis scale is set by $\mu_{\text{net},s} = 2.0 \times 10^{-5}$ A·m^2, and the horizontal axis scale is set by i_{2s} = 20.0 mA. If the current in coil 2 is then reversed, what is the magnitude of the net magnetic moment of the two-coil system when i_2 = 7.0 mA?

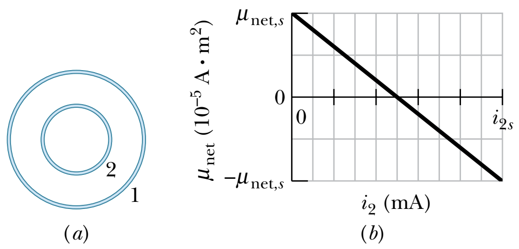


Figure 28-55 Problem 48.

49. An electric field of 1.50 kV/m and a perpendicular magnetic field of 0.350 T act on a moving electron to produce no net force. What is the electron's speed?

50. An ion source is producing ^{6}Li ions, which have charge $+e$ and mass 9.99×10^{-27} kg. The ions are accelerated by a potential difference of 25 kV and pass horizontally into a region in which there is a uniform vertical magnetic field of magnitude B = 1.2 T. Calculate the strength of the electric field, to be set up over the same region, that will allow the ^{6}Li ions to pass through without any deflection.

51. An electron that has an instantaneous velocity of

$$\vec{v} = (-5.0 \times 10^6 \text{ m/s})\hat{\text{i}} + (3.0 \times 10^6 \text{ m/s})\hat{\text{j}}$$

is moving through the uniform magnetic field $\vec{B} = (0.030\ \text{T})\hat{\text{i}} - (0.15\ \text{T})\hat{\text{j}}$. (a) Find the force on the electron due to the magnetic field. (b) Repeat your calculation for a proton having the same velocity.

52. In Fig. 28-56, an electron with an initial kinetic energy of 5.0 keV enters region 1 at time $t = 0$. That region contains a uniform magnetic field directed into the page, with magnitude 0.010 T. The electron goes through a half-circle and then exits region 1, headed toward region 2 across a gap of 25.0 cm. There is an electric potential difference $\Delta V = 2000$ V across the gap, with a polarity such that the electron's speed increases uniformly as it traverses the gap. Region 2 contains a uniform magnetic field directed out of the page, with magnitude 0.020 T. The electron goes through a half-circle and then leaves region 2. At what time t does it leave?

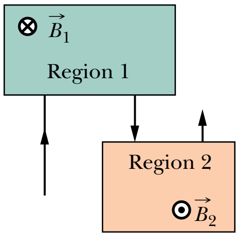


Figure 28-56 Problem 52.

53. An alpha particle can be produced in certain radioactive decays of nuclei and consists of two protons and two neutrons. The particle has a charge of $q = +2e$ and a mass of 4.00 u, where u is the atomic mass unit, with $1\ \text{u} = 1.661 \times 10^{-27}$ kg. Suppose an alpha particle travels in a circular path of radius 4.50 cm in a uniform magnetic field with $B = 1.20$ T. Calculate (a) its speed, (b) its period of revolution, (c) its kinetic energy, and (d) the potential difference through which it would have to be accelerated to achieve this energy. (e) If the field magnitude is doubled, what is the ratio of the new value of kinetic energy to the initial value?

PRACTICE QUESTIONS

Single Correct Choice Type

1. Magnetic intensity for an axial point due to a short bar magnetic of magnetic moment M is given by

(a) $\dfrac{\mu_0}{4\pi} \times \dfrac{M}{d^3}$ **(b)** $\dfrac{\mu_0}{4\pi} \times \dfrac{M}{d^2}$

(c) $\dfrac{\mu_0}{2\pi} \times \dfrac{M}{d^3}$ **(d)** $\dfrac{\mu_0}{2\pi} \times \dfrac{M}{d^2}$

2. The material used for a permanent magnet has

(a) High retentivity, high coercivity
(b) Low retentivity, low coercivity
(c) Low retentivity, high coercivity
(d) High retentivity, low coercivity

3. $\vec{B}$, $\vec{H}$, and $\vec{I}$ are related according to the equation:

(a) $\vec{B} = \mu_0(\vec{H} \times \vec{I})$ **(b)** $|\vec{B}| = \dfrac{|\vec{I}|}{|\vec{H}|}$

(c) $|\vec{B}| = \mu_0(\vec{H} - \vec{I})$ **(d)** $|\vec{B}| = \mu_0(\vec{H} + \vec{I})$

4. A bar magnet suspended by a silk thread is parallel to the magnetic meridian of the Earth with no twist in the thread. When the upper end of the thread is rotated by 130°, the magnet makes an angle of 60° with the magnetic meridian in equilibrium. To turn the magnet at 90° to the magnetic meridian (in equilibrium), what additional angle of rotation should be given to the upper end of the thread?

(a) 82° **(b)** 71°
(c) 90° **(d)** 65°

5. If a bar magnet of pole strength m and magnetic moment M is cut equally in five parts parallel to its axis and four equal parts perpendicular to its axis, then the pole strength and magnetic moment of each piece are, respectively,

(a) $\dfrac{m}{20}, \dfrac{M}{20}$ **(b)** $\dfrac{m}{4}, \dfrac{M}{20}$

(c) $\dfrac{m}{5}, \dfrac{M}{20}$ **(d)** $\dfrac{m}{5}, \dfrac{M}{4}$

6. A compass needle of magnetic moment 60 Am^2 is pointing to the geographical north at a certain place. It experiences a torque of 1.2×10^{-3} Nm. The Earth's magnetic field at that place is 80 μT. Find the declination at that place (angle of dip = 60°).

(a) 30° **(b)** 60°
(c) 45° **(d)** 90°

7. A magnetizing field of 1600 A/m produces a magnetic flux of 2.4×10^{-5} Wb in a bar of iron of cross-section $0.2 \times 10^{-4}\ \text{m}^2$. Calculate the susceptibility of the bar.

(a) 596.8 **(b)** 288.9
(c) 2 **(d)** None of these

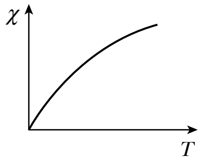
8. Two short magnets placed along the same axis with their like poles facing each other repel each other with a force of 25 N. The distance between the magnets is reduced by a factor of 4. Now, the force of repulsion will be

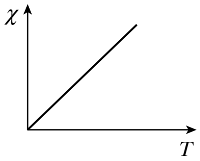
(a) 4 × 25 N
(b) 4 × 4 × 25 N
(c) 4 × 4 × 4 × 25 N
(b) 4 × 4 × 4 × 4 × 25 N

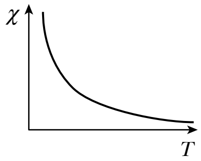
9. Time period of a freely suspended magnet does not depend upon

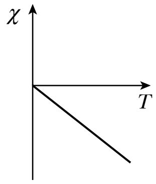
(a) Length of the magnet
(b) Pole strength of the magnet
(c) Horizontal component of Earth's magnetic field
(d) Length of the suspension thread

10. Which among the four χ versus T graphs shown in the following figure is the correct one for a paramagnetic substance will look like?

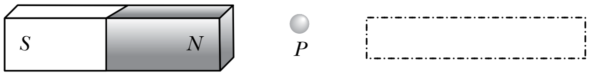
(a)


(b)


(c)


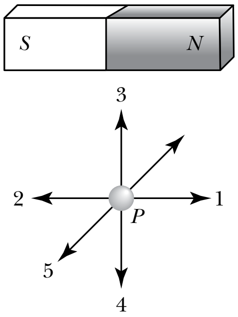
(d)


11. Consider the two rectangular areas shown in the following figure with a point P located at the midpoint between the two areas. The rectangular area on the left contains a bar magnet with the south pole near point P. The rectangle on the right is initially empty. How will the magnetic field at P change, if at all, when a second bar magnet is placed on the right rectangle with its south pole near point P?



(a) The direction of the magnetic field will not change, but its magnitude will decrease
(b) The direction of the magnetic field will not change, but its magnitude will increase
(c) The magnetic field at P will be zero
(d) The direction of the magnetic field will change and its magnitude will increase

12. What is the direction of the magnetic field at the point P, directly below a point at the center of the magnet? The numbered arrows represent various directions. Direction "1" is to the right, "2" to the left, "3" is upward, "4" is downward, and "5" is toward you (in the given figure).

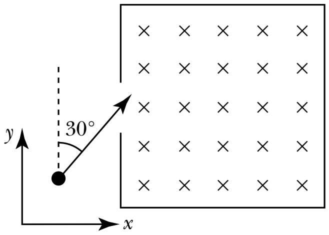


(a) 1 **(b)** 2
(c) 3 **(d)** 4

13. An observer turns on an electric field across a sealed glass tube and the electrons from one of the electrodes move toward the other, horizontally toward right. A magnetic field is applied pointing toward the observer. The force acting upon the electrons points

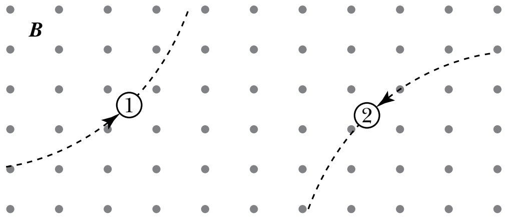
(a) To the right **(b)** To the left
(c) Upward **(d)** Downward

14. An electron enters a region that contains a magnetic field directed into the page as shown in the following figure. The velocity vector of the electron makes an angle of 30° with the $+y$ axis. What is the direction of the magnetic force on the electron when it enters the field?



(a) Up, out of the page
(b) At an angle of 30° below the positive x axis
(c) At an angle of 30° above the positive x axis
(d) At an angle of 60° below the positive x axis

15. Two particles move through a uniform magnetic field that is directed out of the plane of the page. The given figure shows the paths taken by the two particles as they move through the field. The particles are not subject to any other forces or fields. Which one of the following statements concerning these particles is true?



(a) The particles may both be neutral
(b) Particle 1 is positively charged; 2 is negative
(c) Particle 1 as well as 2 is positively charged
(d) Particle 1 as well as 2 is negatively charged

16. A proton traveling at velocity v enters a region with magnetic field B pointing perpendicular to the original direction of the proton. Which of the following statements is true?

(a) Velocity v of the proton remains constant
(b) Speed v of the proton remains constant
(c) Both direction and speed of the proton change
(d) Direction of the proton remains constant although its speed increases

17. If an electron moves downward towards Earth surface near the Earth's equator, which direction is this particle deflected?

(a) Northward **(b)** Southward
(c) Eastward **(d)** Westward

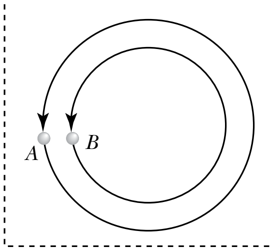
18. Cosmic rays (atomic nuclei stripped bare of their electrons) would continuously bombard Earth's surface if most of them were not deflected by Earth's magnetic field. Given that Earth is, to an excellent approximation, a magnetic dipole, the intensity of cosmic rays bombarding its surface is greatest at the

(a) Poles **(b)** Mid-latitudes
(c) Equator **(d)** Mid-Longitude

19. A positively charged particle enters a region with magnetic field parallel to its direction of motion. The subsequent trajectory of that charged particle track will

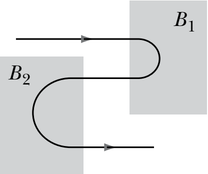
(a) curve to the right with radius, $R = mv/qB$.
(b) curve to the left with radius, $R = mv/qB$.
(c) curve upward with radius, $R = mv/qB$.
(d) not be affected at all.

20. The following figure shows the respective paths taken by two particles, *A* and *B*, in a region containing a constant, uniform magnetic field directed into the plane of motion (into the page). There is no electric field. Both particles have charge $+e$. What can we correctly conclude about particles *A* and *B*?



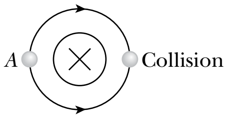
(a) The mass of *A* > the mass of *B*
(b) The speed of *A* > the speed of *B*
(c) The magnitude of momentum of *A* > the magnitude of momentum of *B*
(d) The kinetic energy of *A* > the kinetic energy of *B*

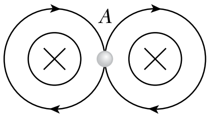
21. The following figure shows the path of an electron that passes through two regions containing uniform magnetic fields of magnitudes B_1 and B_2. Its path in each region is a half circle. The directions of the two fields are

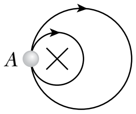


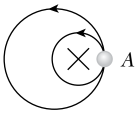
(a) Both B_1 and B_2 are directed into the page
(b) Both B_1 and B_2 are directed out of the page
(c) B_1 is directed into the page and B_2 is directed out of the page
(d) B_1 is directed out of the page and B_2 is directed into the page

22. A neutral particle at rest at *A* decays into two charged particles of mass *m* each. Magnetic field is inside the paper. Which can be the correct path for two particles if we ignore electrostatic force between them?

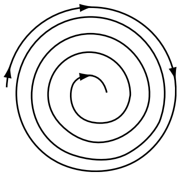
(a)


(b)


(c)


(d)


23. A uniform magnetic field is directed into the page. A charged particle, moving in the plane of the page, follows a clockwise spiral of decreasing radius as shown in the following figure. A reasonable explanation is



(a) The charge is positive and slowing down
(b) The charge is negative and slowing down
(c) The charge is positive and speeding up
(d) The charge is negative and speeding up

24. If an electric dipole is rotating about its center with a uniform angular velocity in the anticlockwise direction in a uniform magnetic field which is in the direction of the angular velocity, then

(a) Net magnetic force as well as torque on the dipole is zero
(b) Net magnetic force as well as torque on the dipole is nonzero
(c) Net magnetic force on the dipole is zero but the net torque on the dipole is nonzero
(d) Net magnetic force on the dipole is not zero but the net torque on the dipole is zero

25. A proton is moving in a uniform electric field, *E*, of unknown magnitude and direction. There is a uniform magnetic field of 0.01 T pointing in the *y* direction. The proton moves in the *x* direction at a constant velocity of 10 km/s. Which of the following is correct?

(a) $E = 100$ V/m, in the *z* direction
(b) $E = 100$ V/m, in the negative *z* direction
(c) $E = 100$ V/m, in the *x* direction
(d) $E = 100$ V/m, in the negative *y* direction

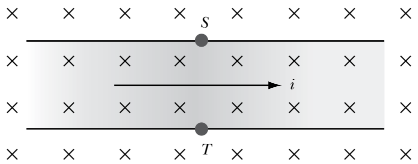
26. Which of the following configurations of electric and magnetic fields in a chamber can allow a positively charged particle to traverse it with no deflection?

(a) Chamber with electric field perpendicular and magnetic field parallel to the velocity of the charged particle
(b) Chamber with electric field parallel, and magnetic field perpendicular to the velocity of the charged particle
(c) Chamber with electric field parallel, and magnetic field anti-parallel to the velocity of the charged particle
(d) Chamber with electric field as well as magnetic field perpendicular to the velocity of the charged particle

27. A proton moves in the $+z$ direction after being accelerated from rest through a potential difference *V*. The proton then passes through a region with a uniform electric field *E* in the $+x$ direction and a uniform magnetic field *B* in the $+y$ direction, but the proton's trajectory is not affected. If the experiment were repeated using a potential difference of 2 V, the proton would then be

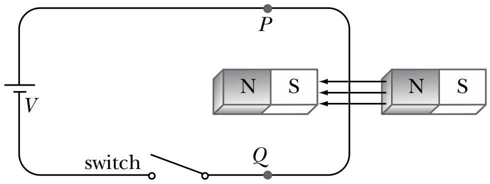
(a) Deflected in the $+x$ direction
(b) Deflected in the $-x$ direction
(c) Deflected in the $+y$ direction
(d) Deflected in the $-y$ direction

28. The current is from left to right in the conductor shown in the following figure. The magnetic field is into the page and point *S* is at a higher potential than point *T*. The charge carriers are



(a) Positive **(b)** Negative
(c) Neutral **(d)** Absent

29. A portion of a loop of wire passes between the poles of a magnet as shown in the following figure. We are viewing the circuit from above. When the switch is closed and a current passes through the circuit, what is the movement, if any, of the wire between the poles of the magnet?



(a) The wire moves toward the north pole of the magnet
(b) The wire moves toward the south pole of the magnet
(c) The wire moves upward (toward us)
(d) The wire moves downward (away from us)

30. Three long, straight, identical wires are inserted one at a time into a magnetic field directed due east. Wire A carries a current of 2 A in the direction of 45° south of east. Wire B carries a current of 8 A, due north. Wire C carries a current of 10 A, due west. Rank the wires in terms of the magnitude of the magnetic force on each wire, with the largest force listed first and the smallest force listed last.

(a) A > B > C **(b)** B > A > C
(c) C > B > A **(d)** A > C > B

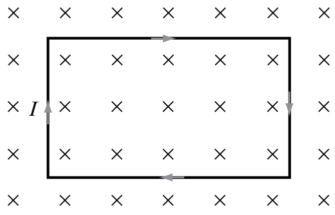
31. Which of the following setups can make a small conductor coil that is placed horizontally float on top of an object?

(a) The object is a magnet that provides a magnetic field pointing straight downward, whereas the coil carries an electric current flowing clockwise as viewed from top down
(b) The object is a magnet that provides a magnetic field pointing straight downward, while the coil carries an electric current flowing counterclockwise as viewed from top down
(c) The object carries positive electric charge while the coil carries negative charge
(d) The object as well as coil carries negative electric charge

32. An electric current runs counterclockwise in a rectangular loop around the outside edge of this page, which lies flat on your table. A uniform magnetic field is then turned on, directed parallel to the page from top to bottom. The magnetic force on the page will cause

(a) The left edge to lift up
(b) The right edge to lift up
(c) The top edge to lift up
(d) The bottom edge to lift up

33. The figure shows a frame carrying a current in a uniform magnetic field.

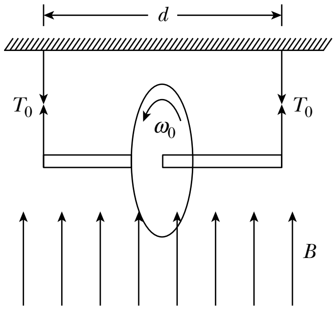


(a) The net force on the loop is zero, but the magnetic field compresses the frame
(b) The net torque on the loop is zero, but the magnetic field tries to extend the frame
(c) The net torque on the loop is zero, but the magnetic field compresses the frame
(d) The energy of interaction of the loop with the magnetic field is zero

34. Two particles, each of mass m and charge q, are attached to the two ends of a light rigid rod of length $2R$. The rod is rotated at constant angular speed about a perpendicular axis passing through its centre. The ratio of the magnitudes of the magnetic moment of the system and its angular momentum about the centre of the rod is

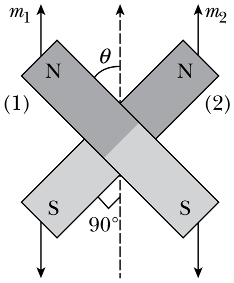
(a) $q/2m$ **(b)** q/m
(c) $2q/m$ **(d)** $q/\pi m$

35. A wheel of radius R having charge Q, uniformly distributed on the rim of the wheel is free to rotate about a light horizontal rod. The rod is suspended by light inextensible strings and a magnetic field B is applied as shown in the figure. The initial tensions in the strings are T_a if the breaking tension of the strings are $3T_0/2$, find the maximum angular velocity ω_0 with which the wheel can be rotated.



(a) $\dfrac{T_0 d}{qR^2 B}$ **(b)** $\dfrac{T_0(d+R)}{qR^2 B}$
(c) $\dfrac{T_0 R}{qd^2 B}$ **(d)** $\dfrac{T_0 d}{q(R-d)^2 B}$

36. Two magnets of equal mass are joined at right angles to each other as shown in the following figure. Magnet 1 has a magnetic moment three times that of magnet 2. This arrangement is pivoted so that it is free to rotate in the horizontal plane. In equilibrium, what angle will the magnet 1 subtend with the magnetic meridian?



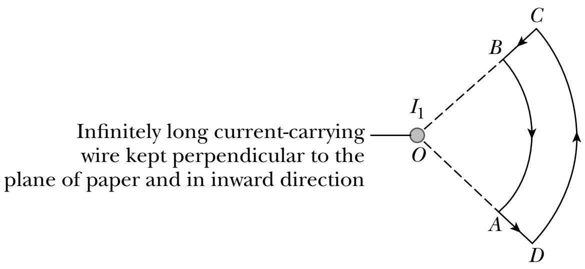
(a) $\tan^{-1}\left(\dfrac{1}{2}\right)$ **(b)** $\tan^{-1}\left(\dfrac{1}{3}\right)$
(c) $\tan^{-1}(1)$ **(d)** 0°

More than One Correct Choice Type

37. Choose the correct answer:
- **(a)** Diamagnetism occurs in all materials
- **(b)** Diamagnetism is the result of partial alignment of permanent magnetic moment
- **(c)** The magnetic field due to induced magnetic moment is opposite to the applied field
- **(d)** The magnetizing field intensity is always zero in free space

38. Choose the correct answer:
- **(a)** All electrons possess magnetic moment
- **(b)** All protons possess magnetic moment
- **(c)** All nuclei possess magnetic moment
- **(d)** All atoms possess magnetic moment

39. Which of the following statement is correct as per the following figure?


C
B
I_1
Infinitely long current-carrying wire kept perpendicular to the plane of paper and in inward direction
O
I_2
A
D

- **(a)** Net force on the loop is zero
- **(b)** Net torque on the loop is zero
- **(c)** Loop will rotate clockwise about axis OO', when seen from O
- **(d)** Loop will rotate anticlockwise about axis OO', when seen from O

40. A beam of electrons moving with a momentum p enters a uniform magnetic field B perpendicular to its motion. Which of the following statement(s) is(are) true?
- **(a)** Energy gained is $p^2/2m$
- **(b)** Centripetal force on the electron is $Be(p/m)$
- **(c)** Radius of the electron's path is p/Be
- **(d)** Work done on the electrons by the magnetic field is zero

41. The radius of curvature of the path of a charged particle moving in a static uniform magnetic field is
- **(a)** Directly proportional to the magnitude of the charge on the particle
- **(b)** Directly proportional to the magnitude of the liner momentum of the particle
- **(c)** Directly proportional to the kinetic energy of the particle
- **(d)** Inversely proportional to the magnitude of the magnetic field

42. Two singly ionized isotopes of lead are accelerated through the same potential difference and enter perpendicular to a uniform magnetic field. Then
- **(a)** Their kinetic energies before entering the magnetic field are equal and remain equal in the magnetic field.
- **(b)** They trace circular orbits of the same radii in the magnetic field.
- **(c)** The radius of the circular orbit of the heavier isotope in the magnetic field is greater than that of the lighter isotope
- **(d)** The radii of the circular orbits in the magnetic field are in the direct ratio of the magnitude of their momenta

43. An electron is moving along the positive x axis. You want to apply a uniform magnetic field for a short time so that the electron may reverse its direction and move parallel to the negative x axis. This can be done by applying magnetic field along
- **(a)** y axis
- **(b)** z axis
- **(c)** Any direction in yz plane
- **(d)** z axis only

44. A charged particle of charge q and mass m is shot into a uniform magnetic field of induction B at an angle θ with the field. The frequency of revolution of the particle
- **(a)** Depends on the angle θ
- **(b)** Is independent of the angle θ
- **(c)** Is proportional to the specific charge q/m of the particle
- **(d)** Is inversely proportional to the value of B

45. A particle of mass m and charge q moving with velocity $\vec{v}$ enters a region of uniform magnetic field $\vec{B}$. Then
- **(a)** Its path in the region of the field is always circular
- **(b)** Its path in the region of the field is circular if $\vec{v}\times\vec{B}=0$
- **(c)** Its path in the region of the field is a straight line if $\vec{v}\times\vec{B}=0$
- **(d)** Distance traveled by the particle in time T does not depend on the angle between $\vec{v}$ and $\vec{B}$

46. A proton moves with constant velocity in a certain region of gravity-free space. Which one of the following conclusions is valid?
- **(a)** There must be no electric or magnetic fields in this region
- **(b)** There could be an electric field without any magnetic field
- **(c)** There could be a magnetic field without any electric field
- **(d)** There can be an electric field as well as magnetic field

47. If the acceleration of a charged particle, moving the velocity $\vec{v}$ through a uniform electric field $\vec{E}$ and a uniform magnetic field $\vec{B}$, is zero, then
- **(a)** $\vec{E}$ must be perpendicular to $\vec{B}$
- **(b)** $\vec{v}$ must be perpendicular to both $\vec{E}$ and $\vec{B}$
- **(c)** $\vec{E}$ must be perpendicular to both $\vec{v}$ and $\vec{B}$
- **(d)** $\vec{E}$ and $\vec{B}$ must be in same direction

48. The current sensitivity of a moving coil galvanometer can be increased by
- **(a)** Increasing the magnetic field of the permanent magnet.
- **(b)** Increasing the area of the deflecting coil
- **(c)** Increasing the number of turns in the coil
- **(d)** Increasing the torsional constant of the coil

Linked Comprehension

Paragraph for Questions 49 and 50: The lines of the magnetic field B generally emerge in the southern hemisphere and reenter the Earth in the northern hemisphere. Thus, the magnetic pole that is in the Earth's northern hemisphere and known as a "north magnetic pole" is really the south pole of the Earth's magnetic dipole.

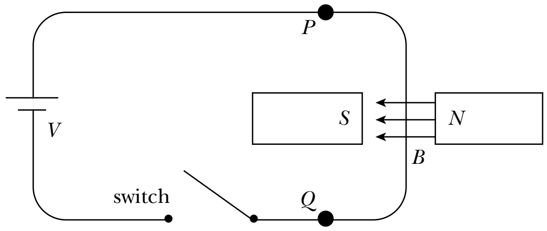
49. The statement in the paragraph means that

- **(a)** The north pole of a compass is attracted to Earth's geographic north pole
- **(b)** The north pole of a compass is attracted to the Earth's geographic south pole
- **(c)** The north pole of a compass is repelled to the Earth's geographic north pole
- **(d)** The north pole of a compass is repelled to the Earth's geographic south pole

50. Choose the correct statement:

- **(a)** The direction of the magnetic field does not vary from location to location on Earth.
- **(b)** The direction of the magnetic field varies from location to location on Earth.
- **(c)** The direction of the magnetic field is same in all locations of Earth.
- **(d)** The direction of the magnetic field is zero on all locations of Earth.

Paragraph for Questions 51 and 52: A long straight vertical segment of wire traverses a magnetic field of magnitude 2.0 T in the direction shown in the diagram. The length of the wire that lies in the magnetic field is 0.060 m. When the switch is closed, a current of 4.0 A flows through the wire from point P to point Q.



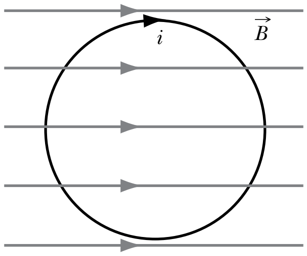
51. Which one of the following statements concerning the effect of the magnetic force on the wire is true?

- **(a)** The wire will be pushed to the left
- **(b)** The wire will be pushed to the right
- **(c)** The wire will have no net force acting on it
- **(d)** The wire will be pushed downward, into the plane of the paper

52. What is the magnitude of the magnetic force acting on the wire?

- **(a)** 0.12 N
- **(b)** 0.48 N
- **(c)** 0.24 N
- **(d)** 67 N

Paragraph for Questions 53 and 55: The wire loop in the following figure carries a clockwise current. There is a uniform magnetic field directed to the right.



53. What is the direction of the net force on the current loop?

- **(a)** Into the page
- **(b)** Out of the page
- **(c)** Up the page
- **(d)** The net force is zero

54. What is the direction of net torque on the current loop?

- **(a)** Into the page
- **(b)** Out of the page
- **(c)** Up the page
- **(d)** Down the page

55. If we assume that the field is not uniform; it is stronger at the top of the page than at the bottom. What is the direction of the net force on the current loop?

- **(a)** Into the page
- **(b)** Out of the page
- **(c)** Up the page
- **(d)** The net force is zero

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

56.

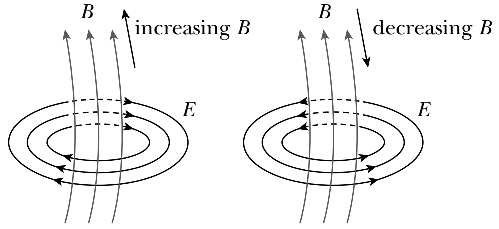
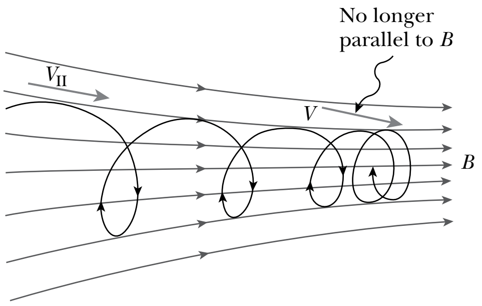
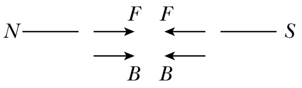
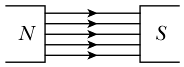
Column I	Column II
(a) The end of a magnet from which the field lines emerge	**(p)** Attract one another
(b) Opposite magnetic poles	**(q)** South pole
(c) The end of a magnet into which the field lines enter	**(r)** Repel one another
(d) Like magnetic poles	**(s)** North Pole

57.

Column I	Column II
(a) Magnetic susceptibility	**(p)** $\Phi_B = \oint \vec{B}\cdot d\vec{A} = 0$
(b) Curie's law	**(q)** $B = \frac{\mu_0}{4\pi}\frac{2m}{r^3}$
(c) magnetic field on axial line of a bar magnet	**(r)** $M = C\frac{B_{\text{ext}}}{T}$
(d) Gauss's law for magnetic fields	**(s)** $\chi = \frac{M}{H}$

Directions for Questions 58 and 59: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

58. A magnetic field can be represented by a set of magnetic lines of force. If the lines are spaced widely apart, it is a weak field. If the lines are situated close to each other, it represents a strong field. In the given table, Column I shows the different types of magnetic field strength, Column II shows the lines of magnetic field and Column III shows the figure of magnetic field lines.

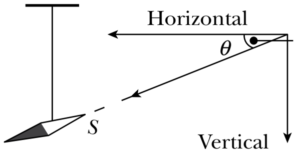
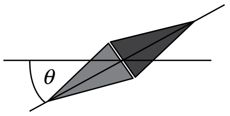
Column I	Column II	Column III
(I) Different field strength at different points	**(i)** represented by a set of parallel lines of force.	**(J)** 
(II) Same strength (in magnitude and direction) at all the points	**(ii)** represented by a set of convergent or divergent lines of force.	**(K)** 
(III) Magnetic field which keeps on changing	**(iii)** with respect to time.	**(L)** 
(IV) Magnetic field which does not change	**(iv)** with respect to temperature.	**(M)** 

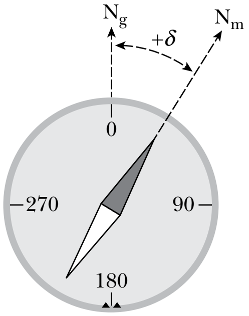
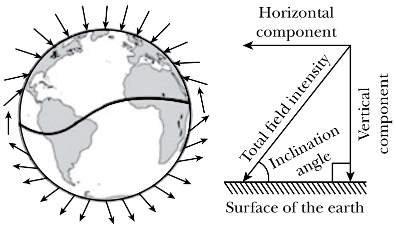
(1) What are the characteristics of uniform field?
- **(a)** (I) (ii) (J)
- **(b)** (IV) (ii) (M)
- **(c)** (II) (i) (M)
- **(d)** (I) (iii) (J)

(2) What are the characteristics of non-varying magnetic field?
- **(a)** (I) (iv) (K)
- **(b)** (IV) (iii) (L)
- **(c)** (II) (iii) (L)
- **(d)** (I) (i) (M)

(3) What are the characteristics of varying magnetic field?
- **(a)** (III) (i) (L)
- **(b)** (I) (i) (J)
- **(c)** (III) (iii) (J)
- **(d)** (II) (iii) (M)

59. Earth's magnetic field, is also known as the geomagnetic field, is the magnetic field that extends from the Earth's interior out into space, where it meets the solar wind, a stream of charged particles emanating from the Sun. In the given table, Column I, II and III have parts of the statements.

Column I	Column II	Column III
(I) The angle made by	**(i)** points in the north-south direction	**(J)** 
(II) The angle made by	**(ii)** radiations cause the atmosphere of the earth to get ionized	**(K)** 

Column I	Column II	Column III
(III) A magnetic needle suspended freely in the air	**(iii)** magnetic meridian with the geographic meridian	**(L)** 
(IV) when the solar activity of the sun is more active	**(iv)** north pole of the needle makes with the horizontal axis	**(M)**  Horizontal component Total field intensity Inclination angle Vertical component Surface of the earth

(1) What do you understand from magnetic declination?
(a) (I) (iii) (L) **(b)** (IV) (ii) (M)
(c) (II) (i) (M) **(d)** (I) (iii) (J)

(2) What are the characteristics of irregular and aperiodic variation?
(a) (IV) (ii) (M) **(b)** (IV) (iii) (L)
(c) (II) (iii) (L) **(d)** (I) (i) (M)

(3) What are the characteristics of angle of dip?
(a) (III) (i) (L)
(b) (I) (i) (J)
(c) (III) (iii) (J)
(d) (II) (iv) (J)

Integer Type

60. A magnetized wire of length of 8 cm has a magnetic moment of 80×10^{-3} J/T. It is then bent into a semicircular arc. Find pole strength. What is the new magnetic moment in terms of π?

61. A bar magnet suspended freely has period of oscillation of 2 s. Now, the bar magnet is broken into two equal halves and one-half is suspended to oscillate freely in the same magnetic field. Find the time period of oscillation of this half of the magnet.

62. Two bar magnets of same length and breadth having magnetic moments M and $2M$ are joined together in such a way that the north pole of one magnet coincides with the north pole of the other magnet. The combination is suspended freely in a magnetic field. If time period of oscillation of the system is $\sqrt{3}$ s, find the time period of oscillation of the system when the polarity of one of the magnets is changed and both the magnets are again made to oscillate in the same magnetic field.

63. Two equal magnetic poles placed 10 cm apart in air attract each other with a force of 0.4×10^{-4} N. What should be the distance of separation between them so that the force of attraction is 0.6×10^{-4} N?

64. The work done in turning a magnet of magnetic moment M by an angle of 90° from the magnetic meridian is n times the corresponding work done to turn through an angle of 60°. What is the value of n?

65. At a certain place, a magnet makes 30 oscillations per minute. At another place, the magnetic field is double. What is its time period?

66. A bar magnet has a magnetic moment of 2.5 J/T and is placed in a magnetic field of 0.2 T. Find the work done in turning the magnet from parallel to anti-parallel position relative to field direction.

ANSWER KEY

Checkpoints

1. Touch a bar to the center making a T. Iron bar will stick to the magnet if the end of the magnet touches the center of the bar but not otherwise.

2. (a) z direction; (b) negative x direction; (c) zero

3. (a) electron; (b) clockwise

4. (a) 2, then tie of 1 and 3 (zero); (b) 4
5. $\vec{v} \perp \vec{F}$, $\vec{B} \perp \vec{F}$ but $\vec{B}$ and $\vec{v}$ can be at an angle to each other.
6. $-Y$
7. (a) all tie; (b) 1 and 4 tie, then 2 and 3 tie

Problems

1. (a) 0.800 V/m; (b) 1.60 V; (c) the back surface in the xy plane becomes negatively charged
2. 1.87×10^{-9} s
3. (a) 79°; (b) no; (c) no; (d) 6.4 nm
4. $(-6.93 \times 10^{-3}\hat{j} + 1.49 \times 10^{-3}\hat{k})$ N
5. 25.7 N
6. 0.615 A
7. (a) 0.204 T; (b) 160 ns
8. (a) 6.91 $\text{A} \cdot \text{m}^2$; (b) 4.15 $\text{A} \cdot \text{m}^2$
9. 0.42 m
10. (a) 0.057 T; (b) 27°
11. 6.0×10^{-7} N
12. (a) 1.96×10^6 Hz; (b) 0.663 m
13. 143°
14. (a) 0.45 $\text{A} \cdot \text{m}^2$; (b) 5.4×10^{-2} $\text{N} \cdot \text{m}$
15. (a) 1.16×10^7 m/s; (b) 3.30×10^{-4} m
16. $\vec{\tau} = (-5.1 \times 10^{-2}\,\text{N} \cdot \text{m})\,\hat{j}$
17. (a) 0.226 $\text{A} \cdot \text{m}^2$; (b) 1.78 $\text{N} \cdot \text{m}$
18. (a) 1.00 A; (b) the current must be from left to right
19. 0.324 m/s
20. $1.24 \times 10^{-}$ $\text{N} \cdot \text{m}$
21. (a) 350 eV; (b) 35.0 keV; (c) 0.00499 or 0.499%
22. $(0.900\hat{j} - 0.80\hat{k})$ $\text{A} \cdot \text{m}^2$
23. (a) 3.35A; (b) 6.65×10^{-2} $\text{N} \cdot \text{m}$
24. (a) $\vec{B} = -(2.11 \times 10^{-4}\ \text{T})\hat{k}$; (b) the electron will move up toward the upper plate
25. (a) 4.36×10^{-2} m/s; (b) the direction is to the left (away from the generator)
26. (a) 39.5 N; (b) the force is to the west
27. (a) $(-6.4 \times 10^{-4}\hat{i} - 4.8 \times 10^{-4}\hat{j} + 1.6 \times 10^{-3}\hat{k})$ $\text{N} \cdot \text{m}$; (b) $-0.30\,\mu = -1.2 \times 10^{-3}$ J
28. 2.2×10^{-2} T
29. 0.40 T
30. (a) 0.787 T; (b) 6.56×10^6 eV; (c) 2.39×10^7 Hz; (d) 2.60×10^7 eV
31. (a) 1.48×10^{-5} T; (b) 2.42×10^{-6} s
32. 1.86×10^9 A
33. 238 km/s
34. $(-0.074\ \text{T})\hat{k}$
35. $(-11.4\hat{i} + 4.50\hat{j} + 3.60\hat{k})$ V/m
36. 2.30×10^{-22} J
37. 1.2×10^{-9} kg/C
38. (a) 83°; (b) 83°
39. 3.6×10^{-5} V
40. (a) 7.0×10^{-18} N; (b) 1.06×10^9 m/s²; (c) remain unchanged
41. (a) $(1.68 \times 10^{-18}\ \text{N})\hat{k}$; (b) $(4.00 \times 10^{-19}\ \text{N})\hat{k}$; (c) $(6.41 \times 10^{-19}\ \text{N})\hat{i} + (10.4 \times 10^{-19}\ \text{N})\hat{k}$
42. (a) 4.20×10^4 m/s; (b) 9.22 eV; (c) 7.02×10^{-23} $\text{kg} \cdot \text{m/s}$
43. (a) 4.00×10^6 m/s; (b) 0.00456 m; (c) 7.15×10^{-9} s
44. (a) 1.45×10^7 m/s; (b) 6.61×10^{-4} T; (c) 1.85×10^7 Hz; (d) 5.41×10^{-8} s; (e) 600 V
45. (a) 1.53×10^7 Hz; (b) 1.20×10^7 eV
46. (a) -4.2×10^3 m/s; (b) 8.4×10^3 m/s
47. (a) there is no force exerted on the hypotenuse of length 130 cm; (b) 0.277 N on the 50.0 cm side; (c) 0.277 N on the 120 cm side; (d) zero
48. 3.4×10^{-5} $\text{A} \cdot \text{m}^2$
49. 4.29×10^3 m/s
50. 1.1×10^6 V/m
51. (a) $\vec{F}_B$ is 1.1×10^{-13} N and points in the negative z direction; (b) $\vec{F}_B = +(1.1 \times 10^{-13}\ \text{N})\hat{k}$
52. 8.1 ns
53. (a) 2.60×10^6 m/s; (b) 1.09×10^{-7} s; (c) 140×10^5 eV; (d) 7.00×10^4 V; (e) 4

Practice Questions

Single Correct Choice Type

1. (c)
2. (a)
3. (d)
4. (b)
5. (c)
6. (a)
7. (a)
8. (d)
9. (d)
10. (c)
11. (b)
12. (b)
13. (c)
14. (b)
15. (d)
16. (b)
17. (d)
18. (a)
19. (d)
20. (c)
21. (c)
22. (a)
23. (b)
24. (d)
25. (b)
26. (b)
27. (b)
28. (a)
29. (d)
30. (b)
31. (d)
32. (c)
33. (b)
34. (a)
35. (a)
36. (b)

More than One Correct Choice Type

37. (a), (c)
38. (a), (b)
39. (a), (c)
40. (b), (c), (d)
41. (b), (d)
42. (a), (c), (d)
43. (a), (b), (c)
44. (b), (c), (d)
45. (c), (d)
46. (c), (d)
47. (a), (c)
48. (a), (b), (c)

Linked CompREHENSION

49. (a)
50. (b)
51. (d)
52. (b)
53. (d)
54. (d)
55. (d)

Matrix-Match

56. (a) → (s); (b) → (p); (c) → (q); (d) → (r)
57. (a) → (s); (b) → (r); (c) → (q); (d) → (p)
58. (1) → (c); (2) → (b); (3) → (c)
59. (1) → (a); (2) → (a); (3) → (d)

Integer Type

60. 1
61. 1
62. 1
63. 8
64. 2
65. 1.414
66. 1

CHAPTER

29

Magnetic Fields Due to Currents

Contents

29.1 What is Physics?
29.2 Calculating the Magnetic Field Due to a Current
29.3 Magnetic Field of Moving Charged Particles
29.4 Force Between Two Parallel Currents
29.5 Ampere's Law
29.6 Solenoids and Toroids
29.7 Limitations of Ampere's Law
29.8 A Note About the Frames of Reference

29.1 | WHAT IS PHYSICS?

One basic observation of physics is that a moving charged particle produces a magnetic field around itself. Thus a current of moving charged particles produces a magnetic field around the current. This feature of *electromagnetism*, which is the combined study of electric and magnetic effects, came as a surprise to the people who discovered it. Surprise or not, this feature has become enormously important in everyday life because it is the basis of countless electromagnetic devices. For example, a magnetic field is produced in maglev trains and other devices used to lift heavy loads.

Our first step in this chapter is to find the magnetic field due to the current in a very small section of current-carrying wire. Then we shall find the magnetic field due to the entire wire for several different arrangements of the wire.

29.2 | CALCULATING THE MAGNETIC FIELD DUE TO A CURRENT

Key Concepts

- The magnetic field set up by a current-carrying conductor can be found from the Biot–Savart law. This law asserts that the contribution $d\vec{B}$ to the field produced by a current-length element $i\,d\vec{s}$ at a point P located a distance r from the current element is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i d\vec{s}\times\hat{\mathrm{r}}}{r^2} \quad \text{(Biot–Savart law).}$$

Here $\hat{\mathrm{r}}$ is a unit vector that points from the element toward P. The quantity μ_0, called the permeability constant, has the value

$$4\pi\times10^{-7}\,\mathrm{T\cdot m/A} \approx 1.26\times10^{-6}\,\mathrm{T\cdot m/A}.$$

◆ For a long straight wire carrying a current i, the Biot–Savart law gives, for the magnitude of the magnetic field at a perpendicular distance R from the wire,

$$B = \frac{\mu_0 i}{2\pi R} \quad \text{(long straight wire).}$$

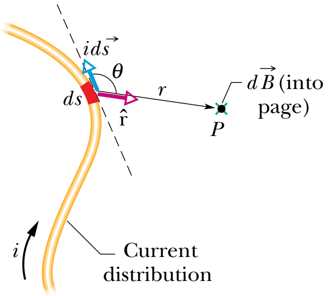
◆ The magnitude of the magnetic field at the center of a circular arc, of radius R and central angle ϕ (in radians), carrying current i, is

$$B = \frac{\mu_0 i\phi}{4\pi R} \quad \text{(at center of circular arc).}$$

◆ The magnetic field produced by a current-carrying coil, which is a magnetic dipole, at a point P located a distance z along the coil's perpendicular central axis is parallel to the axis and is given by

$$\vec{B}(z) = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3},$$

where $\vec{\mu}$ is the dipole moment of the coil. This equation applies only when z is much greater than the dimensions of the coil.



This element of current creates a magnetic field at P, into the page.

Figure 29-1 A current-length element $i\,d\vec{s}$ produces a differential magnetic field $d\vec{B}$ at point P. The green × (the tail of an arrow) at the dot for point P indicates that $d\vec{B}$ is directed *into* the page there.

Figure 29-1 shows a wire of arbitrary shape carrying a current i. We want to find the magnetic field $\vec{B}$ at a nearby point P. We first mentally divide the wire into differential elements ds and then define for each element a length vector $d\vec{s}$ that has length ds and whose direction is the direction of the current in ds. We can then define a differential *current-length element* to be $i\,d\vec{s}$; we wish to calculate the field $d\vec{B}$ produced at P by a typical current-length element. From experiment we find that magnetic fields, like electric fields, can be superimposed to find a net field. Thus, we can calculate the net field $\vec{B}$ at P by summing, via integration, the contributions $d\vec{B}$ from all the current-length elements. However, this summation is more challenging than the process associated with electric fields because of a complexity; whereas a charge element dq producing an electric field is a scalar, a current-length element $i\,d\vec{s}$ producing a magnetic field is a vector, being the product of a scalar and a vector.

Magnitude. The magnitude of the field $d\vec{B}$ produced at point P at distance r by a current-length element $i\,d\vec{s}$ turns out to be

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}, \tag{29-1}$$

where θ is the angle between the directions of $d\vec{s}$ and $\hat{r}$, a unit vector that points from ds toward P. Symbol μ_0 is a constant, called the *permeability constant*, whose value is defined to be exactly

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T\cdot m/A} \approx 1.26 \times 10^{-6}\ \mathrm{T\cdot m/A}. \tag{29-2}$$

Direction. The direction of $d\vec{B}$, shown as being into the page in Fig. 29-1, is that of the cross product $d\vec{s} \times \hat{r}$. We can therefore write Eq. 29-1 in vector form as

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\hat{r}}{r^2} \quad \text{(Biot–Savart law).} \tag{29-3}$$

This vector equation and its scalar form, Eq. 29-1, are known as the **law of Biot and Savart** (rhymes with "Leo and bazaar"). The law, which is experimentally deduced, is an inverse-square law. We shall use this law to calculate the net magnetic field $\vec{B}$ produced at a point by various distributions of current.

Here is one easy distribution: If current in a wire is either directly toward or directly away from a point P of measurement, can you see from Eq. 29-1 that the magnetic field at P from the current is simply zero (the angle θ is either 0° for *toward* or 180° for *away*, and both result in $\sin\theta = 0$)?

Magnetic Field Due to a Current in a Long Straight Wire

Shortly we shall use the law of Biot and Savart to prove that the magnitude of the magnetic field at a perpendicular distance R from a long (infinite) straight wire carrying a current i is given by

$$B = \frac{\mu_0 i}{2\pi R} \quad \text{(long straight wire).} \tag{29-4}$$

The field magnitude B in Eq. 29-4 depends only on the current and the perpendicular distance R of the point from the wire. We shall show in our derivation that the field lines of $\vec{B}$ form concentric circles around the wire, as Fig. 29-2 shows and as the iron filings in Fig. 29-3 suggest. The increase in the spacing of the lines in Fig. 29-2 with increasing distance from the wire represents the $1/R$ decrease in the magnitude of $\vec{B}$ predicted by Eq. 29-4. The lengths of the two vectors $\vec{B}$ in the figure also show the $1/R$ decrease.

Directions. Plugging values into Eq. 29-4 to find the field magnitude B at a given radius is easy. What is difficult for many students is finding the direction of a field vector $\vec{B}$ at a given point. The field lines form circles around a long straight wire, and the field vector at any point on a circle must be tangent to the circle. That means it must be perpendicular to a radial line extending to the point from the wire. But there are two possible directions for that perpendicular vector, as shown in Fig. 29-4. One is correct for current into the figure, and the other is correct for current out of the figure. How can you tell which is which? Here is a simple right-hand rule for telling which vector is correct:

Curled–straight right-hand rule: Grasp the element in your right hand with your extended thumb pointing in the direction of the current. Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.

The result of applying this right-hand rule to the current in the straight wire of Fig. 29-2 is shown in a side view in Fig. 29-5*a*. To determine the direction of the magnetic field $\vec{B}$ set up at any particular point by this current, mentally wrap your right hand around the wire with your thumb in the direction of the current. Let your fingertips pass through the point; their direction is then the direction of the magnetic field at that point. In the view of Fig. 29-2, $\vec{B}$ at any point is *tangent to a magnetic field line*; in the view of Fig. 29-5, it is *perpendicular to a dashed radial line connecting the point and the current.*

Proof of Eq. 29-4

Figure 29-6, which is just like Fig. 29-1 except that now the wire is straight and of infinite length, illustrates the task at hand. We seek the field $\vec{B}$ at point P, a perpendicular distance R from the wire.

The magnetic field vector at any point is tangent to a circle.

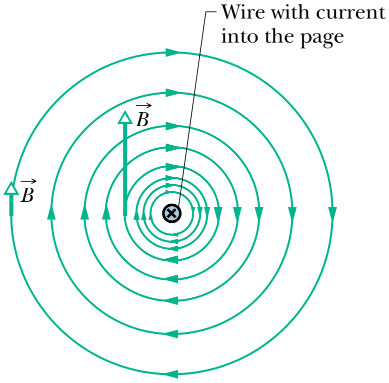
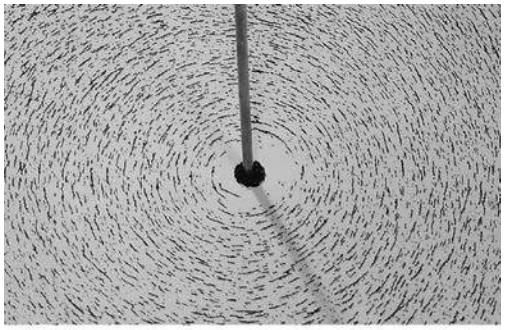


Figure 29-2 The magnetic field lines produced by a current in a long straight wire form concentric circles around the wire. Here the current is into the page, as indicated by the ×.



Courtesy Education Development Center

Figure 29-3 Iron filings that have been sprinkled onto cardboard collect in concentric circles when current is sent through the central wire. The alignment, which is along magnetic field lines, is caused by the magnetic field produced by the current.

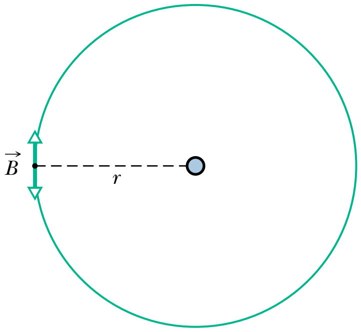
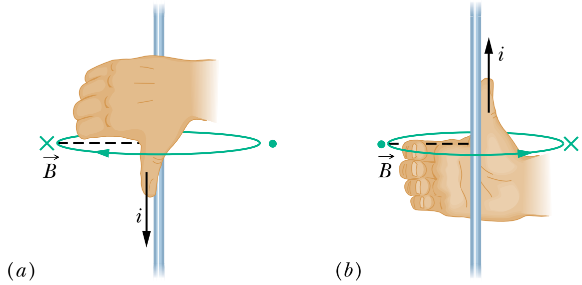


Figure 29-4 The magnetic field vector $\vec{B}$ is perpendicular to the radial line extending from a long straight wire with current, but which of the two perpendicular vectors is it?



The thumb is in the current's direction. The fingers reveal the field vector's direction, which is tangent to a circle.

Figure 29-5 A right-hand rule gives the direction of the magnetic field due to a current in a wire. (*a*) The situation of Fig. 29-2, seen from the side. The magnetic field $\vec{B}$ at any point to the left of the wire is perpendicular to the dashed radial line and directed into the page, in the direction of the fingertips, as indicated by the ×. (*b*) If the current is reversed, $\vec{B}$ at any point to the left is still perpendicular to the dashed radial line but now is directed out of the page, as indicated by the dot.

The magnitude of the differential magnetic field produced at P by the current-length element $i\,d\vec{s}$ located a distance r from P is given by Eq. 29-1:

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}.$$

The direction of $d\vec{B}$ in Fig. 29-6 is that of the vector $d\vec{s}\times\hat{r}$—namely, directly into the page.

Note that $d\vec{B}$ at point P has this same direction for all the current-length elements into which the wire can be divided. Thus, we can find the magnitude of the magnetic field produced at P by the current-length elements in the upper half of the infinitely long wire by integrating dB in Eq. 29-1 from 0 to ∞.

Since the vector $d\vec{s}$ points in the direction of the current flow, we have

$$d\vec{s} = ds(\hat{j}).$$

As can be seen from Fig. 29-6, we have

$$\vec{r} = r\cos(\pi-\theta)(-\hat{j}) + r\sin(\pi-\theta)(\hat{i}) = r\sin\theta(\hat{i}) + r\cos\theta(\hat{j}).$$

Note that the current element is chosen as the origin. Therefore

$$d\vec{s}\times\vec{r} = r\,ds\sin\theta(-\hat{k}).$$

Also

$$\tan(\pi-\theta) = R/s,\ \sin(\pi-\theta) = R/r \Rightarrow r = R\operatorname{cosec}\theta.$$

Therefore,

$$s = -R\cot\theta \Rightarrow ds = R\operatorname{cosec}^2\theta\,d\theta$$

$$\left|d\vec{B}\right| = \frac{\mu_0}{4\pi}i\frac{ds\,r\sin\theta}{r^3} = \frac{\mu_0 i}{4\pi}\sin\theta\frac{R\operatorname{cosec}^2\theta}{R^2\operatorname{cosec}^2\theta}d\theta,$$

$$B = \int_{-\alpha}^{\pi-\beta}\frac{\mu_0 i}{4\pi R}\sin\theta\,d\theta = \frac{\mu_0 i}{4\pi R}(\cos\alpha + \cos\beta). \tag{29-5}$$

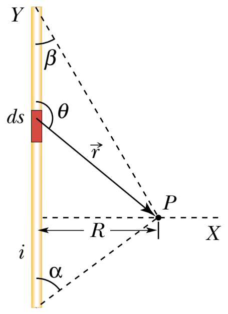


Figure 29-6 The magnetic field due to a finite current-carrying wire. Note that according to the right-hand rule, the magnetic field at P is inside the page.

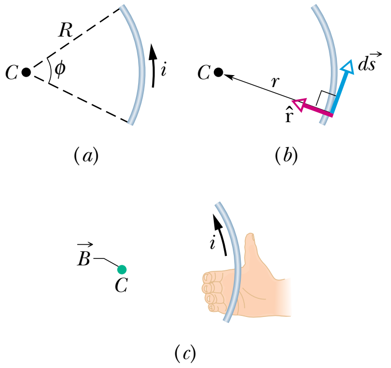
This can be easily extended to the case of an infinitely long wire, for which we can see that $\alpha = \beta = 0°$. So, the magnetic field due to an infinite wire at a distance R from it is

$$B = \frac{\mu_0 i}{2\pi R}, \tag{29-6}$$

which is the relation we set out to prove. Note that the magnetic field at P due to either the lower half or the upper half of the infinite wire in Fig. 29-5 is half this value; that is,

$$B = \frac{\mu_0 i}{4\pi R} \quad \text{(semi-infinite straight wire).} \tag{29-7}$$

Magnetic Field Due to a Current in a Circular Arc of Wire



The right-hand rule reveals the field's direction at the center.

Figure 29-7 (*a*) A wire in the shape of a circular arc with center C carries current i. (*b*) For any element of wire along the arc, the angle between the directions of $\vec{ds}$ and $\hat{r}$ is 90°. (*c*) Determining the direction of the magnetic field at the center C due to the current in the wire; the field is out of the page, in the direction of the fingertips, as indicated by the colored dot at C.

To find the magnetic field produced at a point by a current in a curved wire, we would again use Eq. 29-1 to write the magnitude of the field produced by a single current-length element, and we would again integrate to find the net field produced by all the current-length elements. That integration can be difficult, depending on the shape of the wire; it is fairly straightforward, however, when the wire is a circular arc and the point is the center of curvature.

Figure 29-7*a* shows such an arc-shaped wire with central angle ϕ, radius R, and center C, carrying current i. At C, each current-length element $i\,\vec{ds}$ of the wire produces a magnetic field of magnitude dB given by Eq. 29-1. Moreover, as Fig. 29-7*b* shows, no matter where the element is located on the wire, the angle θ between the vectors $\vec{ds}$ and $\hat{r}$ is 90°; also, $r = R$. Thus, by substituting R for r and 90° for θ in Eq. 29-1, we obtain

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 90^\circ}{R^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{R^2}. \tag{29-8}$$

The field at C due to each current-length element in the arc has this magnitude.

Directions. How about the direction of the differential field $d\vec{B}$ set up by an element? From above we know that the vector must be perpendicular to a radial line extending through point C from the element, either into the plane of Fig. 29-7*a* or out of it. To tell which direction is correct, we use the right-hand rule for any of the elements, as shown in Fig. 29-7*c*. Grasping the wire with the thumb in the direction of the current and bringing the fingers into the region near C, we see that the vector $d\vec{B}$ due to any of the differential elements is out of the plane of the figure, not into it.

Total Field. To find the total field at C due to all the elements on the arc, we need to add all the differential field vectors $d\vec{B}$. However, because the vectors are all in the same direction, we do not need to find components. We just sum the magnitudes dB as given by Eq. 29-8. Since we have a vast number of those magnitudes, we sum via integration. We want the result to indicate how the total field depends on the angle ϕ of the arc (rather than the arc length). So, in Eq. 29-8 we switch from ds to $d\phi$ by using the identity $ds = R\,d\phi$. The summation by integration then becomes

$$B = \int dB = \int_0^\phi \frac{\mu_0}{4\pi}\frac{iR\,d\phi}{R^2} = \frac{\mu_0 i}{4\pi R}\int_0^\phi d\phi.$$

Integrating, we find that

$$B = \frac{\mu_0 i\phi}{4\pi R} \quad \text{(at center of circular arc).} \tag{29-9}$$

Heads Up. Note that this equation gives us the magnetic field *only* at the center of curvature of a circular arc of current. When you insert data into the equation, you must be careful to express ϕ in radians rather than degrees.

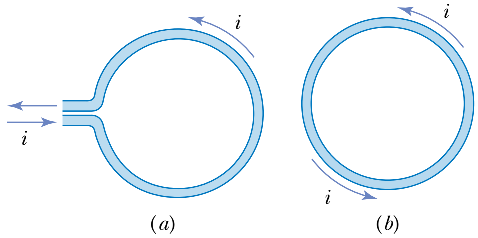


Figure 29-8 A current loop. (*a*) A practical current loop; (*b*) an ideal current loop.

For example, to find the magnitude of the magnetic field at the center of a full circle of current, you would substitute 2π rad for ϕ in Eq. 29-9, finding

$$B = \frac{\mu_0 i(2\pi)}{4\pi R} = \frac{\mu_0 i}{2R} \quad \text{(at center of full circle).} \tag{29-10}$$

Figure 29-8*a* shows a current loop, a circular loop of wire with radius R, that carries current I. Let us find the magnetic field of the current loop at distance z from the center on the axis of the loop.

Real coils need wires to bring 29-8*b*. You can easily see that if the wires are of small radius, their contribution to the magnetic field at any point in space will cancel.

Figure 29-9 shows a loop for which we've assumed that the current is circulating counterclockwise. We've chosen a coordinate system in which the loop lies in the xy plane, the center being the origin. Let k be the segment at the top of the loop. Vector $\Delta\vec{s}_k$ is parallel to the x axis and unit vector $\hat{r}$ is in the yz plane; thus θ_k, the angle between $\Delta\vec{s}_k$ and $\hat{r}$, is 90°.

The direction of $\vec{B}_k$, the magnetic field due to the current in segment k, is given by the cross-product $\Delta\vec{s}_k \times \hat{r} \cdot \vec{B}_k$ must be perpendicular to both $\Delta\vec{s}_k$ and $\hat{r}$. You should convince yourself that $\vec{B}_k$ in Fig. 29-9 points in the correct direction. Notice that the y component of $\vec{B}_k$ is canceled by the y component of magnetic field $\vec{B}_j$ due to the current segment at the bottom of the loop, on the opposite side of the loop, such that the x and y components of $\vec{B}$ cancel and the components of $\vec{B}$ parallel the z axis add. In other words, the symmetry of the loop requires the on-axis magnetic field to point along the z axis. Knowing that we need to sum only the z components will simplify our calculation.

We can use the Biot–Savart law to find the z component $[(B_k)_z = B_k \cos\phi]$ of the magnetic field of segment k. The cross-product $\Delta\vec{s}_k \times \hat{r}$ has magnitude $(\Delta s)(1)\sin 90° = \Delta s$. Thus,

$$(B_k)_z = \frac{\mu_0}{4\pi}\frac{i\Delta s}{r^2}\cos\phi = \frac{\mu_0 i\cos\phi}{4\pi(z^2+R^2)}\Delta s, \tag{29-11}$$

where we used $r = (z^2 + R^2)^{1/2}$. You can see, because $\phi + \gamma = 90°$, that angle ϕ is also the angle between $\hat{r}$ and the radius of the loop. Hence, $\cos\phi = R/r$, and

$$(B_k)_z = \frac{\mu_0 iR}{4\pi(z^2+R^2)^{3/2}}\Delta s. \tag{29-12}$$

The final step is to sum the magnetic fields due to all the segments:

$$B_{\text{loop}} = \sum_k (B_k)_z = \frac{\mu_0 iR}{4\pi(z^2+R^2)^{3/2}}\sum_k \Delta s. \tag{29-13}$$

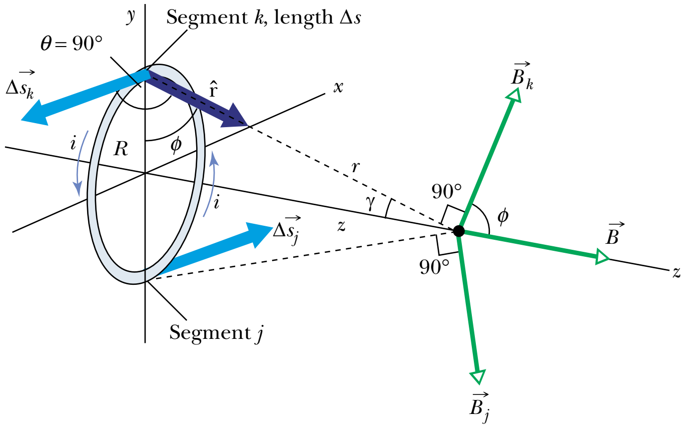


Figure 29-9 Calculating the magnetic field of a current loop.

In this case, unlike the straight wire, none of the terms multiplying with Δs depend on the position of segment k; therefore, all these terms can be factored out during the summation. We're left with summation that adds up the lengths of all the small segments. But this is just the circumference $2\pi R$. Thus the on-axis magnetic field of a current loop is

$$B_{\text{loop}} = \frac{\mu_0 iR}{4\pi(z^2+R^2)^{3/2}}2\pi R = \frac{\mu_0}{2}\frac{iR^2}{(z^2+R^2)^{3/2}}. \qquad (29\text{-}14)$$

If instead of a single loop, we had N coils overlapped over each other, the magnetic field would have been

$$B_{\text{loop}} = \frac{\mu_0}{2}\frac{NiR^2}{(z^2+R^2)^{3/2}}. \qquad (29\text{-}15)$$

SAMPLE PROBLEM 29.01

Magnetic field due to a solenoid

Find the magnetic field due to a solenoid having n turns per unit length at a point on its axis (Fig. 29-10).

KEY IDEAS

(1) A solenoid consists of a large number of circular coils placed close to each other. Each of these coils is not in a single plane, but if the turns are close we can assume them to be in a single plane.

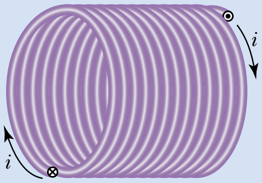


Figure 29-10 A solenoid carrying current i.

(2) We can apply the formula for the magnetic field due to a single coil we derived just now (Eq. 29-15). Each of the coils has a different distance from the point, but we can find the magnetic field due to a short part of the solenoid, and then add up the contribution due to each.

Calculations: If we consider a coil of width dx at a distance x from the point P on the axis of the solenoid as shown in Fig. 29-11, we have

$$dB = \frac{\mu_0}{4\pi}\frac{2\pi(dN)iR^2}{(R^2+x^2)^{3/2}}. \qquad (29\text{-}16)$$

If n is the number of turns per unit length, $dN = n\,dx$.

As in Fig. 29-11, $x = R\tan\phi$, that is, $dx = R\sec^2\phi\, d\phi$, so

$$dB = \frac{\mu_0}{4\pi}\frac{2\pi(n\,dx)iR^2}{(R^2+R^2\tan^2\phi)^{3/2}} = \frac{\mu_0}{4\pi}(2\pi ni)\cos\phi\, d\phi \quad (29\text{-}17)$$

and hence

$$B = \frac{\mu_0}{4\pi}(2\pi ni)\int_{-\alpha}^{\beta}\cos\phi\, d\phi. \qquad (29\text{-}18)$$

That is

$$B = \frac{\mu_0}{4\pi}(2\pi ni)(\sin\alpha + \sin\beta). \qquad (29\text{-}19)$$

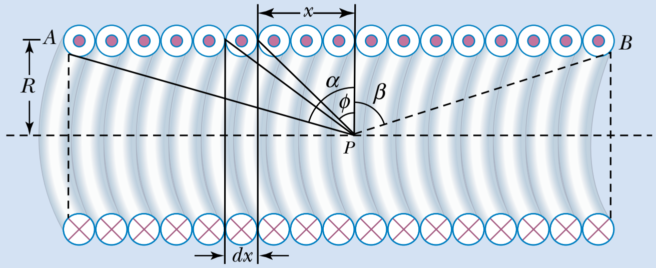


Figure 29-11 The magnetic field at point P on the axis of the solenoid due to the different coils of the solenoid.

This is the desired result and from this it is clear that

1. If the solenoid is of infinite length and the point is well inside the solenoid,

$$\alpha = \beta = \frac{\pi}{2}.$$

So,

$$B = \frac{\mu_0}{4\pi}(2\pi ni)(1+1)$$
$$B = \mu_0 ni. \qquad (29\text{-}20)$$

2. If the solenoid is of infinite length and the point is near one end,

$$\alpha = 0; \quad \beta = \frac{\pi}{2}.$$

So,

$$B = \frac{\mu_0}{4\pi}(2\pi ni)(1+0) = \frac{1}{2}(\mu_0 ni). \qquad (29\text{-}21)$$

The variation of the field with distance along the axis of a solenoid is shown in Fig. 29-12.

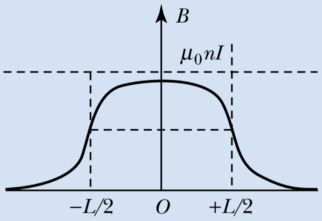


Figure 29-12 Variation of the magnetic field along the axis of a finite solenoid.

We turn our attention here to the field produced by a coil carrying a current. You saw in Section 28.10 that such a coil behaves as a magnetic dipole in that, if we place it in an external magnetic field $\vec{B}$, a torque $\vec{\tau}$ given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{29-22}$$

acts on it. Here $\vec{\mu}$ is the magnetic dipole moment of the coil and has the magnitude NiA, where N is the number of turns, i is the current in each turn, and A is the area enclosed by each turn. (*Caution:* Do not confuse the magnetic dipole moment $\vec{\mu}$ with the permeability constant μ_0.)

Recall that the direction of $\vec{\mu}$ is given by a curled–straight right-hand rule: Grasp the coil so that the fingers of your right hand curl around it in the direction of the current; your extended thumb then points in the direction of the dipole moment $\vec{\mu}$.

Magnetic Field of a Coil

We turn now to the other aspect of a current-carrying coil as a magnetic dipole. What magnetic field does *it* produce at a point in the surrounding space? The problem does not have enough symmetry to make Ampere's law useful; so we must turn to the law of Biot and Savart. For simplicity, we first consider only a coil with a single circular loop and only points on its perpendicular central axis, which we take to be a z axis. We shall show that the magnitude of the magnetic field at such points is

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}, \tag{29-23}$$

in which R is the radius of the circular loop and z is the distance of the point in question from the center of the loop. Furthermore, the direction of the magnetic field $\vec{B}$ is the same as the direction of the magnetic dipole moment $\vec{\mu}$ of the loop.

Large z. For axial points far from the loop, we have $z \gg R$ in Eq. 29-23. With that approximation, the equation reduces to

$$B(z) \approx \frac{\mu_0 i R^2}{2z^3}.$$

Recalling that πR^2 is the area A of the loop and extending our result to include a coil of N turns, we can write this equation as

$$B(z) = \frac{\mu_0}{2\pi} \frac{NiA}{z^3}.$$

Further, because $\vec{B}$ and $\vec{\mu}$ have the same direction, we can write the equation in vector form, substituting from the identity $\mu = NiA$:

$$\vec{B}(z) = \frac{\mu_0}{2\pi} \frac{\vec{\mu}}{z^3} \quad \text{(current-carrying coil).} \tag{29-24}$$

Thus, we have two ways in which we can regard a current-carrying coil as a magnetic dipole: (1) it experiences a torque when we place it in an external magnetic field; (2) it generates its own intrinsic magnetic field, given, for distant points along its axis, by Eq. 29-24. Figure 29-13 shows the magnetic field of a current loop; one side of the loop acts as a north pole (in the direction of $\vec{\mu}$) and the other side as a south pole, as suggested by the lightly drawn magnet in the figure. If we were to place a current-carrying coil in an external magnetic field, it would tend to rotate just like a bar magnet would.

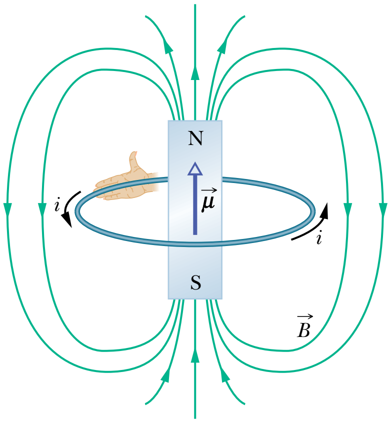
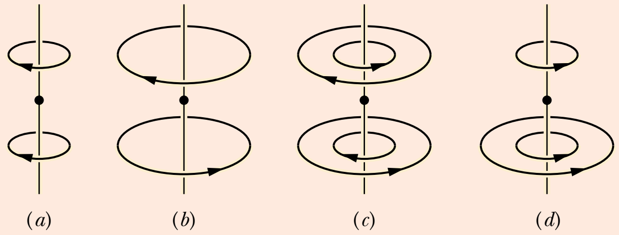


Figure 29-13 A current loop produces a magnetic field like that of a bar magnet and thus has associated north and south poles. The magnetic dipole moment $\vec{\mu}$ of the loop, its direction given by a curled–straight right-hand rule, points from the south pole to the north pole, in the direction of the field $\vec{B}$ within the loop.

CHECKPOINT 1

The figure here shows four arrangements of circular loops of radius r or $2r$, centered on vertical axes (perpendicular to the loops) and carrying identical currents in the directions indicated. Rank the arrangements according to the magnitude of the net magnetic field at the dot, midway between the loops on the central axis, greatest first.



SAMPLE PROBLEM 29.02

Magnetic field at the center of a circular arc of current

The wire in Fig. 29-14*a* carries a current i and consists of a circular arc of radius R and central angle $\pi/2$ rad, and two straight sections whose extensions intersect the center C of the arc. What magnetic field $\vec{B}$ (magnitude and direction) does the current produce at C?

KEY IDEAS

We can find the magnetic field $\vec{B}$ at point C by applying the Biot–Savart law of Eq. 29-3 to the wire, point by point along the full length of the wire. However, the application of Eq. 29-3 can be simplified by evaluating $\vec{B}$ separately for the three distinguishable sections of the wire—namely, (1) the straight section at the left, (2) the straight section at the right, and (3) the circular arc.

Straight sections: For any current-length element in section 1, the angle θ between $d\vec{s}$ and $\hat{r}$ is zero (Fig. 29-14*b*); so Eq. 29-1 gives us

$$dB_1 = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 0}{r^2} = 0.$$

Thus, the current along the entire length of straight section 1 contributes no magnetic field at C:

$$B_1 = 0.$$

The same situation prevails in straight section 2, where the angle θ between $d\vec{s}$ and $\hat{r}$ for any current-length element is 180°. Thus,

$$B_2 = 0.$$

Circular arc: Application of the Biot–Savart law to evaluate the magnetic field at the center of a circular arc leads to Eq. 29-9 ($B = \mu_0 i\phi/4\pi R$). Here the central angle ϕ of the arc is $\pi/2$ rad. Thus from Eq. 29-9, the magnitude of the magnetic field $\vec{B}_3$ at the arc's center C is

$$B_3 = \frac{\mu_0 i(\pi/2)}{4\pi R} = \frac{\mu_0 i}{8R}.$$

To find the direction of $\vec{B}_3$, we apply the right-hand rule displayed in Fig. 29-5. Mentally grasp the circular arc with your right hand as in Fig. 29-14*c*, with your thumb in the direction of the current. The direction in which your fingers curl around the wire indicates the direction of the magnetic field lines around the wire. They form circles around the wire, coming out of the page above the arc and going into the page inside the arc. In the region

Current directly toward or away from C does not create any field there.

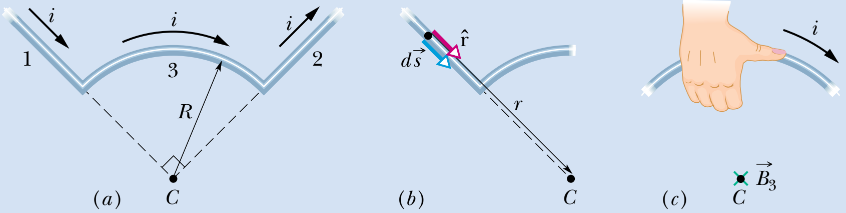


Figure 29-14 (*a*) A wire consists of two straight sections (1 and 2) and a circular arc (3), and carries current i. (*b*) For a current-length element in section 1, the angle between $d\vec{s}$ and $\hat{r}$ is zero. (*c*) Determining the direction of magnetic field $\vec{B}_3$ at C due to the current in the circular arc; the field is into the page there.

of point C (inside the arc), your fingertips point *into the plane* of the page. Thus, $\vec{B}_3$ is directed into that plane.

Net field: Generally, we combine multiple magnetic fields as vectors. Here, however, only the circular arc produces a magnetic field at point C. Thus, we can write the magnitude of the net field $\vec{B}$ as

$$B = B_1 + B_2 + B_3 = 0 + 0 + \frac{\mu_0 i}{8R} = \frac{\mu_0 i}{8R}. \qquad \text{(Answer)}$$

The direction of $\vec{B}$ is the direction of $\vec{B}_3$ —namely, into the plane of Fig. 29-14.

SAMPLE PROBLEM 29.03

Magnetic field off to the side of two long straight currents

Figure 29-15*a* shows two long parallel wires carrying currents i_1 and i_2 in opposite directions. What are the magnitude and direction of the net magnetic field at point P? Assume the following values: $i_1 = 15$ A, $i_2 = 32$ A, and $d = 5.3$ cm.

KEY IDEAS

(1) The net magnetic field $\vec{B}$ at point P is the vector sum of the magnetic fields due to the currents in the two wires. (2) We can find the magnetic field due to any current by applying the Biot–Savart law to the current. For points near the current in a long straight wire, that law leads to Eq. 29-4.

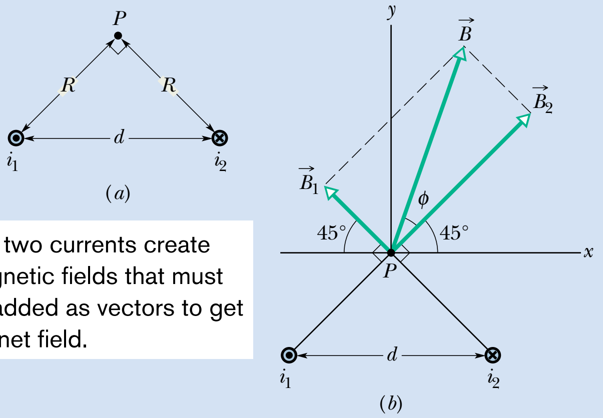
Finding the vectors: In Fig. 29-15*a*, point P is distance R from both currents i_1 and i_2. Thus, Eq. 29-4 tells us that at point P those currents produce magnetic fields $\vec{B}_1$ and $\vec{B}_2$ with magnitudes

$$B_1 = \frac{\mu_0 i_1}{2\pi R} \quad \text{and} \quad B_2 = \frac{\mu_0 i_2}{2\pi R}.$$

In the right triangle of Fig. 29-15*a*, note that the base angles (between sides R and d) are both 45°. This allows us to write cos 45° = R/d and replace R with d cos 45°. Then the field magnitudes B_1 and B_2 become

$$B_1 = \frac{\mu_0 i_1}{2\pi d \cos 45°} \quad \text{and} \quad B_2 = \frac{\mu_0 i_2}{2\pi d \cos 45°}.$$

We want to combine $\vec{B}_1$ and $\vec{B}_2$ to find their vector sum, which is the net field $\vec{B}$ at P. To find the directions of $\vec{B}_1$ and $\vec{B}_2$, we apply the right-hand rule of Fig. 29-5 to each current in Fig. 29-15*a*. For wire 1, with current out of the page, we mentally grasp the wire with the right hand, with the thumb pointing out of the page. Then the curled fingers indicate that the field lines run counterclockwise. In particular, in the region of point P, they are directed upward to the left. Recall that the magnetic field at a point near a long, straight current-carrying wire must be directed perpendicular to a radial line between the point and the current. Thus, $\vec{B}_1$ must be directed upward to the left as drawn in Fig. 29-15*b*. (Note carefully the perpendicular symbol between vector B_1 and the line connecting point P and wire 1.)



The two currents create magnetic fields that must be added as vectors to get the net field.

Figure 29-15 (*a*) Two wires carry currents i_1 and i_2 in opposite directions (out of and into the page). Note the right angle at P. (*b*) The separate fields $\vec{B}_1$ and $\vec{B}_2$ are combined vectorially to yield the net field $\vec{B}$.

Repeating this analysis for the current in wire 2, we find that $\vec{B}_2$ is directed upward to the right as drawn in Fig. 29-15*b*.

Adding the vectors: We can now vectorially add $\vec{B}_1$ and $\vec{B}_2$ to find the net magnetic field $\vec{B}$ at point P, either by using a vector-capable calculator or by resolving the vectors into components and then combining the components of $\vec{B}$. However, in Fig. 29-15*b*, there is a third method: Because $\vec{B}_1$ and $\vec{B}_2$ are perpendicular to each other, they form the legs of a right triangle, with $\vec{B}$ as the hypotenuse. So,

$$B = \sqrt{B_1^2 + B_2^2} = \frac{\mu_0}{2\pi d(\cos 45°)}\sqrt{i_1^2 + i_2^2}$$

$$= \frac{(4\pi \times 10^{-7}\,\mathrm{T\cdot m/A})\sqrt{(15\,\mathrm{A})^2 + (32\,\mathrm{A})^2}}{(2\pi)(5.3 \times 10^{-2}\,\mathrm{m})(\cos 45°)}$$

$$= 1.89 \times 10^{-4}\,\mathrm{T} \approx 190\,\mu\mathrm{T}. \qquad \text{(Answer)}$$

The angle ϕ between the directions of $\vec{B}$ and $\vec{B}_2$ in Fig. 29-15*b* follows from

$$\phi = \tan^{-1}\frac{B_1}{B_2},$$

which, with B_1 and B_2 as given above, yields

$$\phi = \tan^{-1}\frac{i_1}{i_2} = \tan^{-1}\frac{15\text{ A}}{32\text{ A}} = 25°.$$

The angle between $\vec{B}$ and the *x* axis shown in Fig. 29-15*b* is then

$$\phi + 45° = 25° + 45° = 70°. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 29.04

Magnetic field between straight current carrying wire and positively charged particle

A long, straight wire carries a current *i*. A particle having a positive charge *q* and mass *m*, kept at a distance x_0 from the wire, is projected toward it with a speed *v*. Find the minimum separation between the wire and the particle.

KEY IDEA

Let the particle be initially at *P* (Fig. 29-16). Take the wire as the *y* axis and the foot of perpendicular from *P* to the wire as the origin. Take the line *OP* as the *x* axis. We have, $OP = x_0$. The magnetic field *B* at any point to the right of the wire is along the negative *z* axis. The magnetic force on the particle is, therefore, in the *xy* plane. As there is no initial velocity along the *z* axis, the motion will be in the *xy* plane. We know that the magnetic field does not do any work. So, the speed of the charged particle does not change as it moves through the magnetic field. Since the magnetic field is not uniform, the particle does not go along a circle but rather a curved path in the *xy* plane.

Calculations: The force at time *t* is $\vec{F} = q\vec{v} \times \vec{B}$. Here we use the basic definition of force and velocity.

$$\vec{F} = q(v_x\hat{\text{i}} + v_y\hat{\text{j}}) \times \left(-\frac{\mu_0 i}{2\pi x}\hat{\text{k}}\right)$$

$$= qv_x\frac{\mu_0 i}{2\pi x}\hat{\text{j}} - qv_y\frac{\mu_0 i}{2\pi x}\hat{\text{i}}.$$

We can resolve this force along *x* and *y* directions. Thus

$$a_x = \frac{F_x}{m} = -\frac{\mu_0 qi}{2\pi m}\frac{v_y}{x} = -\lambda\frac{v_y}{x}, \qquad (29\text{-}25)$$

where

$$\lambda = \frac{\mu_0 qi}{2\pi m}.$$

Also,

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = \frac{v_x dv_x}{dx}. \qquad (29\text{-}26)$$

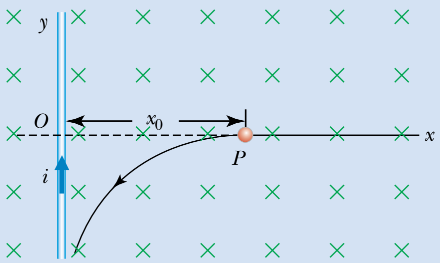


Figure 29-16 A positively charged particle at *P* is projected toward a current-carrying wire.

Since the speed remains constant,

$$v_x^2 + v_y^2 = v^2.$$

Differentiating this expression, we get

$$2v_x dv_x + 2v_y dv_y = 0$$

$$v_x dv_x = -v_y dv_y. \qquad (29\text{-}27)$$

From Eqs. 29-26 and 29-27, we have

$$\frac{v_y dv_y}{dx} = \frac{\lambda v_y}{x}; \quad \frac{dx}{x} = \frac{dv_y}{\lambda}.$$

Initially $x = x_0$ and $v_y = 0$. When *x* is minimum, by the concept of minima, $v_x = 0$ so that $v_y = -v$. Thus

$$\int_{x_0}^{x}\frac{dx}{x} = \int_0^{-v}\frac{dv_y}{\lambda}.$$

Solving, we get

$$\ln\frac{x}{x_0} = -\frac{v}{\lambda}$$

$$x = x_0 e^{-v/\lambda} = x_0 e^{-2\pi mv/\mu_0 qi}.$$

The charged particle will turn back without changing its speed. Even if a charged particle, say a stray electron, is emitted from the wire, it will move in a curved path and turn back at some point. However, in that case also, the path of the particle will not be circular.

PROBLEM-SOLVING TACTIC

Tactic 1: Right-Hand Rules To help you sort out the right-hand rules you have now seen (and the ones coming up), here is a review.

Right-Hand Rule for Cross Products. This is a way to determine the direction of the vector that results from a cross product. You point the fingers of your right hand so as to sweep the first vector expressed in the product into the second vector, through the smaller angle between the two vectors. Your outstretched thumb gives you the direction of the vector resulting from the cross product. In Chapter 11, we used this right-hand rule to find the directions of torque and angular momentum vectors; in Chapter 28, we used it to find the direction of the force on a current-carrying wire in a magnetic field.

Curled– Straight Right-Hand Rules for Magnetism. In many situations involving magnetism, you need to relate a "curled" element and a "straight" element. You can do so with the (curled) fingers and the (straight) thumb on your right hand. You have already seen an example in Chapter 28, where we related the current around a loop (curled element) to the normal vector $\vec{n}$ (straight element) of the loop: You curl the fingers of your right hand around in the direction of the current along the loop; your outstretched thumb then gives the direction of $\vec{n}$. This is also the direction of the magnetic dipole moment $\vec{\mu}$ of the loop.

In this section you were introduced to a second curled–straight right-hand rule for magnetism. To determine the direction of the magnetic field lines around a current-length element, you point the outstretched thumb of your right hand in the direction of the current. The fingers then naturally curl around the current-length element in the direction of the field lines around the element.

29.3 | MAGNETIC FIELD OF MOVING CHARGED PARTICLES

Key Concepts

- The magnetic field of a single moving charged particles is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{\mathrm{r}}}{r^2}$$

- The magnetic force exerted by one moving charge (q_1) on another (q_2) is given by

$$\vec{F}_{1,2} = q_2\vec{v}_2\times\vec{B}_1 = +\frac{\mu_0}{4\pi}\frac{q_1q_2v_1v_2y}{R^3}\hat{\mathrm{j}}$$

Let us now use the Biot–Savart law to find an expression for the magnetic field caused by uniformly moving charged particles. Consider first a small wire carrying a current i, as illustrated in Fig. 29-1. The magnetic field due to a small segment of length dx is given by Eq. 29-3:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\hat{\mathrm{r}}}{r^2}.$$

Suppose the segment contains an amount of charge dq; let this charge take an amount of time dt to move a distance dx due to the current i. According to the definition of the current, we have

$$i = \frac{dq}{dt}.$$

Therefore,

$$i\,d\vec{s} = \frac{dq}{dt}d\vec{s} = dq\frac{d\vec{s}}{dt} = \vec{v}\,dq,$$

where v is the velocity of the charge carriers along the wire. In the limiting case that the segment contains just a single carrier carrying a charge q, dq becomes simply q and so

$$i\,d\vec{s} = \vec{v}\,q.$$

Thus, we obtain an expression for the magnetic field of a single moving charged particle:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{\mathrm{r}}}{r^2}. \tag{29-28}$$

We would like to emphasize that the Biot–Savart law was valid only for steady currents; and obviously the motion of a single charge cannot constitute a steady current by any stretch of imagination. This equation is not strictly true; it is valid only when the charge moves with a velocity much smaller than the velocity of light.

SAMPLE PROBLEM 29.05

The magnetic field of a proton

A proton moves along the x axis with velocity $v_x = 1.0 \times 10^7$ m/s. As it passes the origin, what is the magnetic field at the (x, y, z) position given by (1 mm, 0 mm, 0 mm), (0 mm, 1 mm, 0 mm), and (1 mm, 1 mm, 0 mm)?

KEY IDEA

Figure 29-17 shows the geometry. The first point is on the x axis, directly in front of the proton, with $\theta_1 = 0°$. The second point is on the y axis, with $\theta_2 = 90°$, and the third is in the xy plane. The magnetic field is that of a moving charged particle.

Calculate: Position 1, which is along the line of motion, has $\theta_1 = 0°$. Thus, $\vec{B}_1 = \vec{0}$.

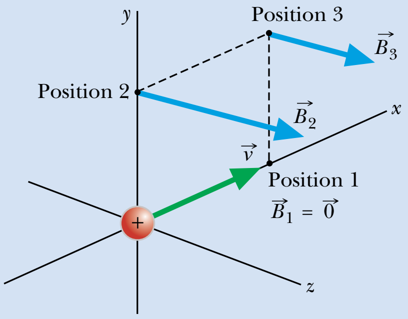


Figure 29-17 The magnetic field of Problem 29.05.

Position 2 (at 0 mm, 1 mm, 0 mm) is at distance $r_2 = 1$ mm $= 0.001$ m. The magnetic field strength at this point is

$$\begin{aligned} B &= \frac{\mu_0}{4\pi} \frac{qv \sin\theta_2}{r_2^2} \\ &= \frac{4\pi \times 10^{-7}\ \mathrm{Tm/A}}{4\pi} \frac{(1.60 \times 10^{-19}\ \mathrm{C})(1.0 \times 10^7\ \mathrm{m/s}) \sin 90°}{(0.0010\ \mathrm{m})^2} \\ &= 1.60 \times 10^{-13}\ \mathrm{T}. \end{aligned}$$

According to the right-hand rule, the field points in the positive z direction. Thus

$$\vec{B}_2 = 1.60 \times 10^{-13}\, \hat{\mathrm{k}}\ \mathrm{T},$$

where $\hat{\mathrm{k}}$ is the unit vector in the positive z direction. The field at position 3, at (1 mm, 1 mm, 0 mm), also points in the z direction, but it is weaker than at position 2 both because r is larger and because q is smaller. From the geometry we know that $r_3 = \sqrt{2}$ mm and $\theta_3 = 45°$. So

$$B_3 = 0.57 \times 10^{-13}\, \hat{\mathrm{k}}\ \mathrm{T}.$$

Note: These magnetic fields are indeed very small. Although Faraday asserted in 1838 that a moving charge should produce the magnetic field in its vicinity as a current-carrying wire does, it took almost 37 years to demonstrate this.

The Magnetic Force and Conservation of Momentum

The magnetic force exerted by one moving charge on another is found by combining Eq. 29-2 for the force on a moving charge in a magnetic field and Eq. 29-28 for the magnetic field of a charge. The force exerted by a charge q_1 moving with velocity v_1 on a charge q_2 moving with velocity v_2 is given by

$$\vec{F}_{1,2} = q_2\vec{v}_2 \times \vec{B}_1 = q_2\vec{v}_2 \times \left(\frac{\mu_0}{4\pi} \frac{q_1\vec{v}_1 \times \hat{\mathrm{r}}}{r_{1,2}^2} \right),$$

where $\vec{B}_1$ is the magnetic field at the position of charge q_2 due to charge q_1, and $\hat{\mathrm{r}}$ is the unit vector pointing from q_1 to q_2. Similarly, the force exerted by a charge q_2 moving with velocity v_2 on a charge q_1 moving with velocity v_1 is given by

$$\vec{F}_{2,1} = q_1\vec{v}_1 \times \vec{B}_2 = q_1\vec{v}_1 \times \left(\frac{\mu_0}{4\pi} \frac{q_2\vec{v}_2 \times (-\hat{\mathrm{r}})}{r_{2,1}^2} \right).$$

Note that the position of 2 as seen by 1 is opposite to the position of 1 as seen by 2. These relations are remarkable in that the force exerted by charge q_1 on charge q_2 is generally not equal and opposite to that exerted by charge q_2 on charge q_1. That is, these forces do not obey Newton's third law. Let us see this in light of a situation illustrated in Fig. 29-18.

Here, the magnetic field due to charge q_1 at the location of the charge q_2 is in the negative z direction, and the force on q_2 is to the left in the negative x direction. However, the magnetic field due to q_2 at q_1 is zero because q_1 lies along the line of motion of q_2. Thus, there is no magnetic force exerted by q_2 on q_1. So, the internal forces in this two-charge system will not cancel out. The system will accelerate in the direction of this force, and linear momentum will not be conserved. It seems that the law of conservation of momentum is violated.

However, the true picture is this: The charges interact with each other with the help of the magnetic field and electrostatic field. These fields carry energy as well as momentum. We have learnt that there is energy associated with an electric field, and we will see later that there is also energy associated with a magnetic field. Similarly, these fields also carry momentum. When the charges move, as in Fig. 29-18, the linear momentum produced when the system accelerates to the left is balanced by momentum in the opposite direction carried by the fields.

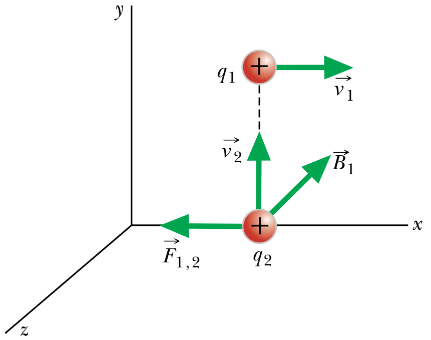


Figure 29-18 The forces exerted by moving charges on each other are not equal and opposite. q_1 is moving in positive x direction and q_2 is moving in the y direction. At this instant q_2 moves toward q_1.

Thus, when we include the momentum of the fields, the total momentum of the system is conserved.

Angular momentum is also carried by the electromagnetic fields produced by moving charges. Consider Fig. 29-19, which shows a point charge q_1 in the xy plane moving parallel to the x axis with velocity $\vec v_1 = v_1\hat{i}$, and a second point charge q_2 at the origin and moving along the x axis with velocity $\vec v_2 = v_2\hat{i}$. Let us calculate the magnetic force exerted by each charge on the other, assuming both charges to be positive.

We first find the force on charge q_1. We note that the vector $\vec r_{2,1}$ from charge q_2 to charge q_1 is just $\vec R$. Using $\vec r_{2,1} = \vec R/R$ we have

$$\frac{\vec v_2\times\hat{r}_{2,1}}{r_{2,1}^2}=\frac{\vec v_2\times\vec R}{R^3}=\frac{v_2\hat{i}\times(x\hat{i}+y\hat{j})}{R^3}=\frac{yv_2}{R^3}\hat{k}.$$

So, the magnetic field produced by charge q_2 at the position of charge q_1 is

$$\vec B_2=\frac{\mu_0}{4\pi}\frac{q_2\vec v_2\times\hat{r}_{2,1}}{r_{2,1}^2}=\frac{\mu_0}{4\pi}\frac{q_2yv_2}{R^3}\hat{k}.$$

The magnetic force exerted by charge q_2 on charge q_1 is then

$$\vec F_{2,1}=q_1\vec v_1\times\vec B_2=q_1v_1\hat{i}\times\frac{\mu_0}{4\pi}\frac{q_2yv_2}{R^3}\hat{k}=-\frac{\mu_0}{4\pi}\frac{q_1q_2v_1v_2y}{R^3}\hat{j}.$$

This force is downward and parallel to the y axis, as shown.

To find the magnetic force exerted by charge q_1 on charge q_2, we note that the vector $\vec r_{1,2}$ from charge q_1 to charge q_2 is $-\vec R$. Then the magnetic field produced by charge q_1 at the position of charge q_2 is

$$\vec B_1=\frac{\mu_0}{4\pi}\frac{q_1\vec v_1\times\hat{r}_{1,2}}{r_{1,2}^2}=\frac{\mu_0}{4\pi}\frac{q_1\vec v_1\times(-\vec R)}{R^3}=-\frac{\mu_0}{4\pi}\frac{q_1yv_1}{R^3}\hat{k}.$$

The magnetic force exerted by charge q_1 on charge q_2 is then

$$\vec F_{1,2}=q_2\vec v_2\times\vec B_1=q_2v_2\hat{i}\times\left(-\frac{\mu_0}{4\pi}\frac{q_1yv_1}{R^3}\hat{k}\right)=+\frac{\mu_0}{4\pi}\frac{q_1q_2v_1v_2y}{R^3}\hat{j}.$$

In this case, the forces are equal and opposite, as shown in Fig. 29-19b, but they are not along the line joining the two particles. The magnetic forces thus exert a torque on the two-particle system. Here also it seems that the angular momentum of the two-particle system is not conserved. But in actual the angular momentum is carried by the field.

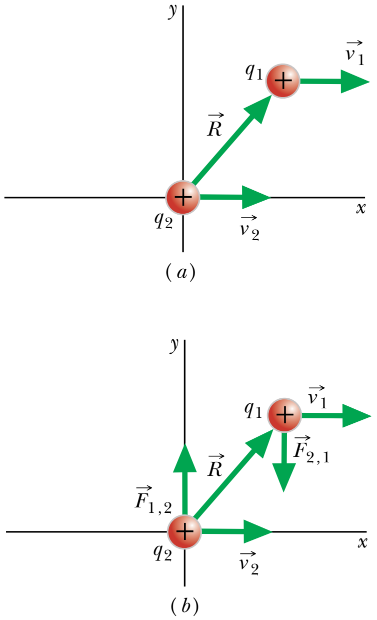


Figure 29-19 (a) Two charges moving in parallel directions. (b) The magnetic forces exerted by the charges on each other are equal and opposite, but they are not along the line joining the charges.

29.4 | FORCE BETWEEN TWO PARALLEL CURRENTS

Key Concept

- Parallel wires carrying currents in the same direction attract each other, whereas parallel wires carrying currents in opposite directions repel each other. The magnitude of the force on a length L of either wire is

$$F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d},$$

where d is the wire separation, and i_a and i_b are the currents in the wires.

Two long parallel wires carrying currents exert forces on each other. Figure 29-20 shows two such wires, separated by a distance d and carrying currents i_a and i_b. Let us analyze the forces on these wires due to each other.

We seek first the force on wire b in Fig. 29-20 due to the current in wire a. That current produces a magnetic field $\vec{B}_a$, and it is this magnetic field that actually causes the force we seek. To find the force, then, we need the magnitude and direction of the field $\vec{B}_a$ *at the site of wire b*. The magnitude of $\vec{B}_a$ at every point of wire b is, from Eq. 29-4,

$$B_a = \frac{\mu_0 i_a}{2\pi d}. \tag{29-29}$$

The curled–straight right-hand rule tells us that the direction of $\vec{B}_a$ at wire b is down, as Fig. 29-20 shows. Now that we have the field, we can find the force it produces on wire b. Equation 28-26 tells us that the force $\vec{F}_{ba}$ on a length L of wire b due to the external magnetic field $\vec{B}_a$ is

$$\vec{F}_{ba} = i_b \vec{L} \times \vec{B}_a, \tag{29-30}$$

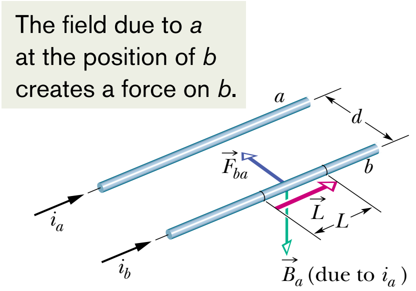

The field due to a at the position of b creates a force on b.

Figure 29-20 Two parallel wires carrying currents in the same direction attract each other. $\vec{B}_a$ is the magnetic field at wire b produced by the current in wire a. $\vec{F}_{ba}$ is the resulting force acting on wire b because it carries current in $\vec{B}_a$.

where $\vec{L}$ is the length vector of the wire. In Fig. 29-20, vectors $\vec{L}$ and $\vec{B}_a$ are perpendicular to each other, and so with Eq. 29-28, we can write

$$F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d}. \tag{29-31}$$

The direction of $\vec{F}_{ba}$ is the direction of the cross product $\vec{L} \times \vec{B}_a$. Applying the right-hand rule for cross products to $\vec{L}$ and $\vec{B}_a$ in Fig. 29-20, we see that $\vec{F}_{ba}$ is directly toward wire a, as shown.

The general procedure for finding the force on a current-carrying wire is this:

> To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

We could now use this procedure to compute the force on wire a due to the current in wire b. We would find that the force is directly toward wire b; hence, the two wires with parallel currents attract each other. Similarly, if the two currents were antiparallel, we could show that the two wires repel each other. Thus,

> Parallel currents attract each other, and antiparallel currents repel each other.

The force acting between currents in parallel wires is the basis for the definition of the ampere, which is one of the seven SI base units. The definition, adopted in 1946, is this: The ampere is that constant current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in vacuum, would produce on each of these conductors a force of magnitude 2×10^{-7} newton per meter of wire length.

Rail Gun

The basics of a rail gun are shown in Fig. 29-21*a*. A large current is sent out along one of two parallel conducting rails, across a conducting "fuse" (such as a narrow piece of copper) between the rails, and then back to the current source along the second rail. The projectile to be fired lies on the far side of the fuse and fits loosely between the rails. Immediately after the current begins, the fuse element melts and vaporizes, creating a conducting gas between the rails where the fuse had been.

The curled–straight right-hand rule of Fig. 29-5 reveals that the currents in the rails of Fig. 29-21*a* produce magnetic fields that are directed downward between the rails. The net magnetic field $\vec{B}$ exerts a force $\vec{F}$ on the gas due to the current i through the gas (Fig. 29-21*b*). With Eq. 29-30 and the right-hand rule for cross products, we find that $\vec{F}$ points outward along the rails. As the gas is forced outward along the rails, it pushes the projectile, accelerating it by as much as $5 \times 10^6 g$, and then launches it with a speed of 10 km/s, all within 1 ms. Someday rail guns may be used to launch materials into space from mining operations on the Moon or an asteroid.

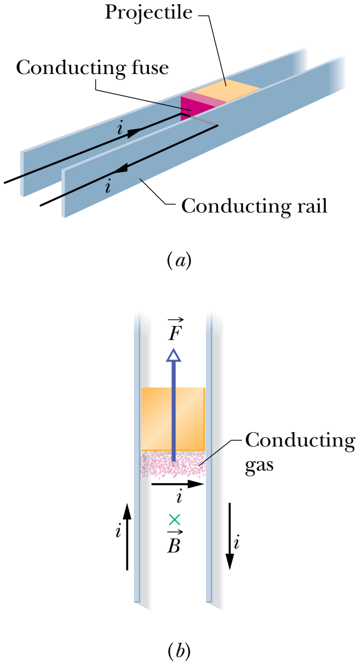
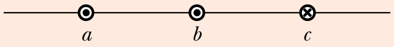


Figure 29-21 (*a*) A rail gun, as a current i is set up in it. The current rapidly causes the conducting fuse to vaporize. (*b*) The current produces a magnetic field $\vec{B}_a$ between the rails, and the field causes a force $\vec{F}$ to act on the conducting gas, which is part of the current path. The gas propels the projectile along the rails, launching it.

CHECKPOINT 2

The figure here shows three long, straight, parallel, equally spaced wires with identical currents either into or out of the page. Rank the wires according to the magnitude of the force on each due to the currents in the other two wires, greatest first.



SAMPLE PROBLEM 29.06

Force between two parallel current carrying wires

Two long parallel wires carry currents of equal magnitude but in opposite directions. These wires are suspended from rod PQ by four chords of same length L as shown in Fig. 29-22*a*. The mass per unit length of wires is λ. Determine the value of θ at equilibrium (assuming it to be small).

KEY IDEA

We know that current-carrying wires carrying antiparallel currents repel each other. Due to this repulsion, they acquire the configuration as shown in the figure. The force between two parallel current-carrying wires is given by

$$F = \frac{\mu_0}{4\pi}\frac{2i_1 i_2 L}{d}.$$

The force is repulsive if the currents are in opposite direction. The *free body diagram* of the problem is as shown in Fig. 29-22*b*.

Calculations: If L_0 is the length of each wire, for its vertical and horizontal equilibrium we have, respectively,

$$2T\cos\theta = L_0\lambda g,$$

$$2T\sin\theta = F.$$

Therefore,

$$\tan\theta = \frac{F}{L_0\lambda g}. \qquad (29\text{-}32)$$

But as,

$$F = \frac{\mu_0}{4\pi}\left[\frac{2I \times I}{2L\sin\theta}L_0\right] \quad (\text{as } d = 2L\sin\theta).$$

So,

$$\tan\theta = \frac{\mu_0}{4\pi}\frac{I^2}{\lambda g L\sin\theta}.$$

Since as for small θ, $\tan\theta = \sin\theta = \theta$

$$\theta = I\sqrt{\frac{\mu_0}{4\pi\lambda g L}}.$$

This force is quite small in magnitude and this is the reason why we do not observe its effect in our daily life.

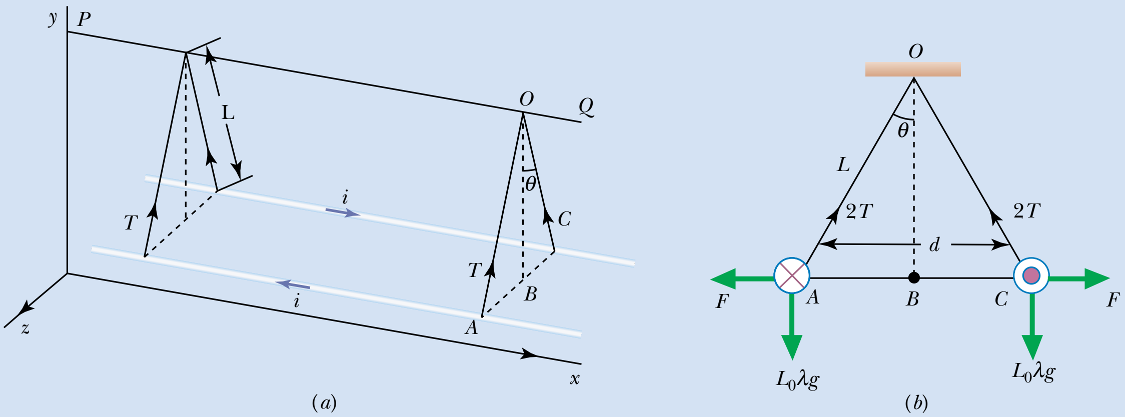


Figure 29-22 (*a*) Two parallel wires carrying current in the opposite direction; (*b*) free body diagram of the wires.

29.5 | AMPERE'S LAW

Key Concept

- Ampere's law states that

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} \quad \text{(Ampere's law).}$$

The line integral in this equation is evaluated around a closed loop called an Amperian loop. The current i on the right side is the *net* current encircled by the loop.

We can find the net electric field due to *any* distribution of charges by first writing the differential electric field $d\vec{E}$ due to a charge element and then summing the contributions of $d\vec{E}$ from all the elements. However, if the distribution is complicated, we may have to use a computer. Recall, however, that if the distribution has planar, cylindrical, or spherical symmetry, we can apply Gauss' law to find the net electric field with considerably less effort.

Similarly, we can find the net magnetic field due to *any* distribution of currents by first writing the differential magnetic field $d\vec{B}$ (Eq. 29-3) due to a current-length element and then summing the contributions of $d\vec{B}$ from all the elements. Again we may have to use a computer for a complicated distribution. However, if the distribution has some symmetry, we may be able to apply **Ampere's law** to find the magnetic field with considerably less effort. This law, which can be derived from the Biot–Savart law, has traditionally been credited to André-Marie Ampère (1775–1836), for whom the SI unit of current is named. However, the law actually was advanced by English physicist James Clerk Maxwell. Ampere's law is

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} \quad \text{(Ampere's law).} \qquad (29\text{-}33)$$

The loop on the integral sign means that the scalar (dot) product $\vec{B} \cdot d\vec{s}$ is to be integrated around a *closed* loop, called an *Amperian loop*. The current i_{enc} is the *net* current encircled by that closed loop.

To see the meaning of the scalar product $\vec{B} \cdot d\vec{s}$ and its integral, let us first apply Ampere's law to the general situation of Fig. 29-23. The figure shows cross sections of three long straight wires that carry currents i_1, i_2, and i_3 either directly into or directly out of the page. An arbitrary Amperian loop

Only the currents encircled by the loop are used in Ampere's law.

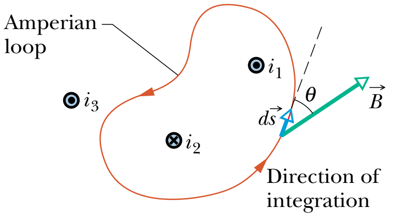


Figure 29-23 Ampere's law applied to an arbitrary Amperian loop that encircles two long straight wires but excludes a third wire. Note the directions of the currents.

lying in the plane of the page encircles two of the currents but not the third. The counterclockwise direction marked on the loop indicates the arbitrarily chosen direction of integration for Eq. 29-33.

To apply Ampere's law, we mentally divide the loop into differential vector elements $d\vec{s}$ that are everywhere directed along the tangent to the loop in the direction of integration. Assume that at the location of the element $d\vec{s}$ shown in Fig. 29-23, the net magnetic field due to the three currents is $\vec{B}$. Because the wires are perpendicular to the page, we know that the magnetic field at $\vec{B}$ due to each current is in the plane of Fig. 29-23; thus, their net magnetic field $\vec{B}$ at $d\vec{s}$ must also be in that plane. However, we do not know the orientation of $\vec{B}$ within the plane. In Fig. 29-23, $\vec{B}$ is arbitrarily drawn at an angle θ to the direction of $d\vec{s}$. The scalar product $\vec{B}\cdot d\vec{s}$ on the left side of Eq. 29-33 is equal to $B\cos\theta\, ds$. Thus, Ampere's law can be written as

$$\oint \vec{B}\cdot d\vec{s} = \oint B\cos\theta\; ds = \mu_0 i_{\text{enc}}. \tag{29-34}$$

We can now interpret the scalar product $\vec{B}\cdot d\vec{s}$ as being the product of a length ds of the Amperian loop and the field component $B\cos\theta$ tangent to the loop. Then we can interpret the integration as being the summation of all such products around the entire loop.

Signs. When we can actually perform this integration, we do not need to know the direction of $\vec{B}$ before integrating. Instead, we arbitrarily assume $\vec{B}$ to be generally in the direction of integration (as in Fig. 29-23). Then we use the following curled–straight right-hand rule to assign a plus sign or a minus sign to each of the currents that make up the net encircled current i_{enc}:

> Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

Finally, we solve Eq. 29-34 for the magnitude of $\vec{B}$. If B turns out positive, then the direction we assumed for $\vec{B}$ is correct. If it turns out negative, we neglect the minus sign and redraw $\vec{B}$ in the opposite direction.

Net Current. In Fig. 29-24 we apply the curled–straight right-hand rule for Ampere's law to the situation of Fig. 29-23. With the indicated counterclockwise direction of integration, the net current encircled by the loop is

$$i_{\text{enc}} = i_1 - i_2.$$

(Current i_3 is not encircled by the loop.) We can then rewrite Eq. 29-34 as

$$\oint B\cos\theta\; ds = \mu_0(i_1 - i_2). \tag{29-35}$$

You might wonder why, since current i_3 contributes to the magnetic-field magnitude B on the left side of Eq. 29-35, it is not needed on the right side. The answer is that the contributions of current i_3 to the magnetic field cancel out because the integration in Eq. 29-35 is made around the full loop. In contrast, the contributions of an encircled current to the magnetic field do not cancel out.

We need to explain more precisely just what is meant by the term "current encircled by the loop." The term means that the current pierces any surface that has the loop as a boundary (see Fig. 29-25). The loop need not be planar. Another way to think of the loop is as the perimeter of various hat-shaped surfaces; there is a wide variety of hat surfaces for a given hat size (perimeter), as a visit to a hat department in a department store will easily confirm! To thread the loop, the current must pierce the bubble surface or hat. The same path can bound many different hat-shaped surfaces.

Strictly speaking, the current must pierce the hat-shaped surface an odd number of times (see Fig. 29-26).

We cannot solve Eq. 29-35 for the magnitude B of the magnetic field because for the situation of Fig. 29-23 we do not have enough information to

This is how to assign a sign to a current used in Ampere's law.

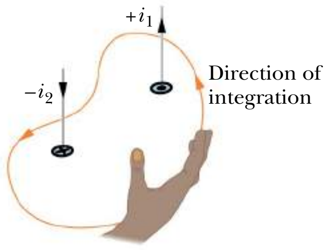


Figure 29-24 A right-hand rule for Ampere's law, to determine the signs for currents encircled by an Amperian loop. The situation is that of Fig. 29-23.

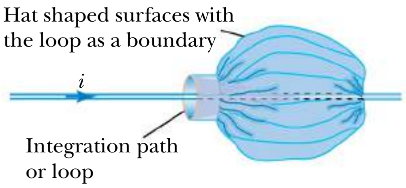


Figure 29-25 Hat-shaped surfaces with the loop as boundary.

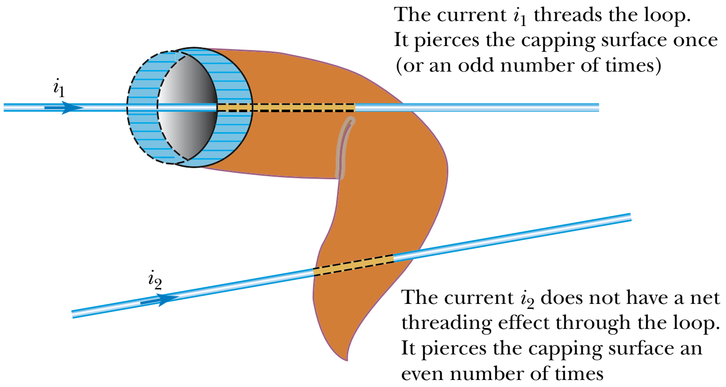

The current i_1 threads the loop. It pierces the capping surface once (or an odd number of times)
i_1
i_2
The current i_2 does not have a net threading effect through the loop. It pierces the capping surface an even number of times

Figure 29-26 The current must pierce the surface an odd number of times.

simplify and solve the integral. However, we do know the outcome of the integration; it must be equal to $\mu_0(i_1 - i_2)$, the value of which is set by the net current passing through the loop.

We shall now apply Ampere's law to two situations in which symmetry does allow us to simplify and solve the integral, hence to find the magnetic field.

Magnetic Field Outside a Long Straight Wire with Current

Figure 29-27 shows a long straight wire that carries current i directly out of the page. Equation 29-4 tells us that the magnetic field $\vec{B}$ produced by the current has the same magnitude at all points that are the same distance r from the wire; that is, the field $\vec{B}$ has cylindrical symmetry about the wire. We can take advantage of that symmetry to simplify the integral in Ampere's law (Eqs. 29-33 and 29-34) if we encircle the wire with a concentric circular Amperian loop of radius r, as in Fig. 29-27. The magnetic field then has the same magnitude B at every point on the loop. We shall integrate counterclockwise, so that $d\vec{s}$ has the direction shown in Fig. 29-27.

We can further simplify the quantity $B \cos\theta$ in Eq. 29-34 by noting that $\vec{B}$ is tangent to the loop at every point along the loop, as is $d\vec{s}$. Thus, $\vec{B}$ and $d\vec{s}$ are either parallel or antiparallel at each point of the loop, and we shall arbitrarily assume the former. Then at every point the angle θ between $d\vec{s}$ and $\vec{B}$ is 0°, so $\cos\theta = \cos 0° = 1$. The integral in Eq. 29-34 then becomes

All of the current is encircled and thus all is used in Ampere's law.

$$\oint \vec{B}\cdot d\vec{s} = \oint B\cos\theta\, ds = B\oint ds = B(2\pi r).$$

Note that $\oint ds$ is the summation of all the line segment lengths ds around the circular loop; that is, it simply gives the circumference $2\pi r$ of the loop.

Our right-hand rule gives us a plus sign for the current of Fig. 29-27. The right side of Ampere's law becomes $+\mu_0 i$, and we then have

$$B(2\pi r) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi r} \quad \text{(outside straight wire).} \tag{29-36}$$

With a slight change in notation, this is Eq. 29-4, which we derived earlier—with considerably more effort—using the law of Biot and Savart. In addition, because the magnitude B turned out positive, we know that the correct direction of $\vec{B}$ must be the one shown in Fig. 29-27.

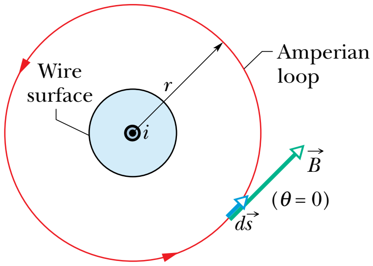


Figure 29-27 Using Ampere's law to find the magnetic field that a current i produces outside a long straight wire of circular cross section. The Amperian loop is a concentric circle that lies outside the wire.

Magnetic Field Inside a Long Straight Wire with Current

Figure 29-28 shows the cross section of a long straight wire of radius R that carries a uniformly distributed current i directly out of the page. Because the current is uniformly distributed over a cross section of the wire, the magnetic field $\vec{B}$ produced by the current must be cylindrically symmetrical. Thus, to find the magnetic field at points inside the wire, we can again use an Amperian loop of radius r, as shown in Fig. 29-28, where now $r < R$. Symmetry again suggests that $\vec{B}$ is tangent to the loop, as shown; so the left side of Ampere's law again yields

$$\oint \vec{B}\cdot d\vec{s} = B\oint ds = B(2\pi r). \tag{29-37}$$

Because the current is uniformly distributed, the current i_{enc} encircled by the loop is proportional to the area encircled by the loop; that is,

$$i_{\text{enc}} = i\frac{\pi r^2}{\pi R^2}. \tag{29-38}$$

Our right-hand rule tells us that i_{enc} gets a plus sign. Then Ampere's law gives us

$$B(2\pi r) = \mu_0 i\frac{\pi r^2}{\pi R^2}$$

or

$$B = \left(\frac{\mu_0 i}{2\pi R^2}\right)r \quad \text{(inside straight wire).} \tag{29-39}$$

Thus, inside the wire, the magnitude B of the magnetic field is proportional to r, is zero at the center, and is maximum at $r = R$ (the surface). Note that Eqs. 29-36 and 29-39 give the same value for B at the surface.

Only the current encircled by the loop is used in Ampere's law.

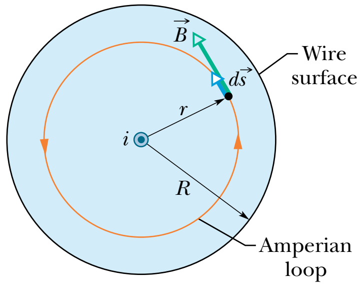
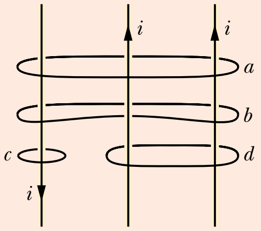


Figure 29-28 Using Ampere's law to find the magnetic field that a current i produces inside a long straight wire of circular cross section. The current is uniformly distributed over the cross section of the wire and emerges from the page. An Amperian loop is drawn inside the wire.

CHECKPOINT 3

The figure here shows three equal currents i (two parallel and one antiparallel) and four Amperian loops. Rank the loops according to the magnitude of $\oint \vec{B}\cdot d\vec{s}$ along each, greatest first.



SAMPLE PROBLEM 29.07

Ampere's law to find the field inside a long cylinder of current

Figure 29-29a shows the cross section of a long conducting cylinder with inner radius $a = 2.0$ cm and outer radius $b = 4.0$ cm. The cylinder carries a current out of the page, and the magnitude of the current density in the cross section is given by $J = cr^2$, with $c = 3.0 \times 10^6$ A/m^4 and r in meters. What is the magnetic field $\vec{B}$ at the dot in Fig. 29-29a, which is at radius $r = 3.0$ cm from the central axis of the cylinder?

KEY IDEAS

The point at which we want to evaluate $\vec{B}$ is inside the material of the conducting cylinder, between its inner and outer radii. We note that the current distribution has cylindrical symmetry (it is the same all around the cross section for any given radius). Thus, the symmetry allows us to use Ampere's law to find $\vec{B}$ at the point. We first draw the Amperian loop shown in Fig. 29-29b. The loop is concentric with the cylinder and has radius $r = 3.0$ cm because we want to evaluate $\vec{B}$ at that distance from the cylinder's central axis.

Next, we must compute the current i_{enc} that is encircled by the Amperian loop. However, we *cannot* set up a proportionality as in Eq. 29-38, because here the current is not uniformly distributed. Instead, we must integrate the current density magnitude from the cylinder's inner radius a to the loop radius r, using the steps shown in Figs. 29-29c through h.

Calculations: We write the integral as

$$i_{\text{enc}} = \int J\, dA = \int_a^r cr^2(2\pi r\, dr)$$
$$= 2\pi c\int_a^r r^3 dr = 2\pi c\left[\frac{r^4}{4}\right]_a^r$$
$$= \frac{\pi c(r^4 - a^4)}{2}.$$

Note that in these steps we took the differential area dA to be the area of the thin ring in Figs. 29-29*d*–*f* and then replaced it with its equivalent, the product of the ring's circumference $2\pi r$ and its thickness dr.

For the Amperian loop, the direction of integration indicated in Fig. 29-29*b* is (arbitrarily) clockwise. Applying the right-hand rule for Ampere's law to that loop, we find that we should take i_{enc} as negative because the current is directed out of the page but our thumb is directed into the page.

We next evaluate the left side of Ampere's law as we did in Fig. 29-28, and we again obtain Eq. 29-37. Then Ampere's law,

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i_{\text{enc}},$$

gives us

$$B(2\pi r) = -\frac{\mu_0 \pi c}{2}(r^4 - a^4).$$

Solving for B and substituting known data yield

$$B = -\frac{\mu_0 c}{4r}(r^4 - a^4)$$
$$= -\frac{(4\pi\times10^{-7}\,\text{T}\cdot\text{m/A})(3.0\times10^6\,\text{A/m}^4)}{4(0.030\text{ m})}$$
$$\times[(0.030\text{ m})^4 - (0.020\text{ m})^4]$$
$$= -2.0\times10^{-5}\,\text{T}.$$

Thus, the magnetic field $\vec{B}$ at a point 3.0 cm from the central axis has magnitude

$$B = 2.0\times10^{-5}\,\text{T} \qquad \text{(Answer)}$$

and forms magnetic field lines that are directed opposite our direction of integration, hence counterclockwise in Fig. 29-29*b*.

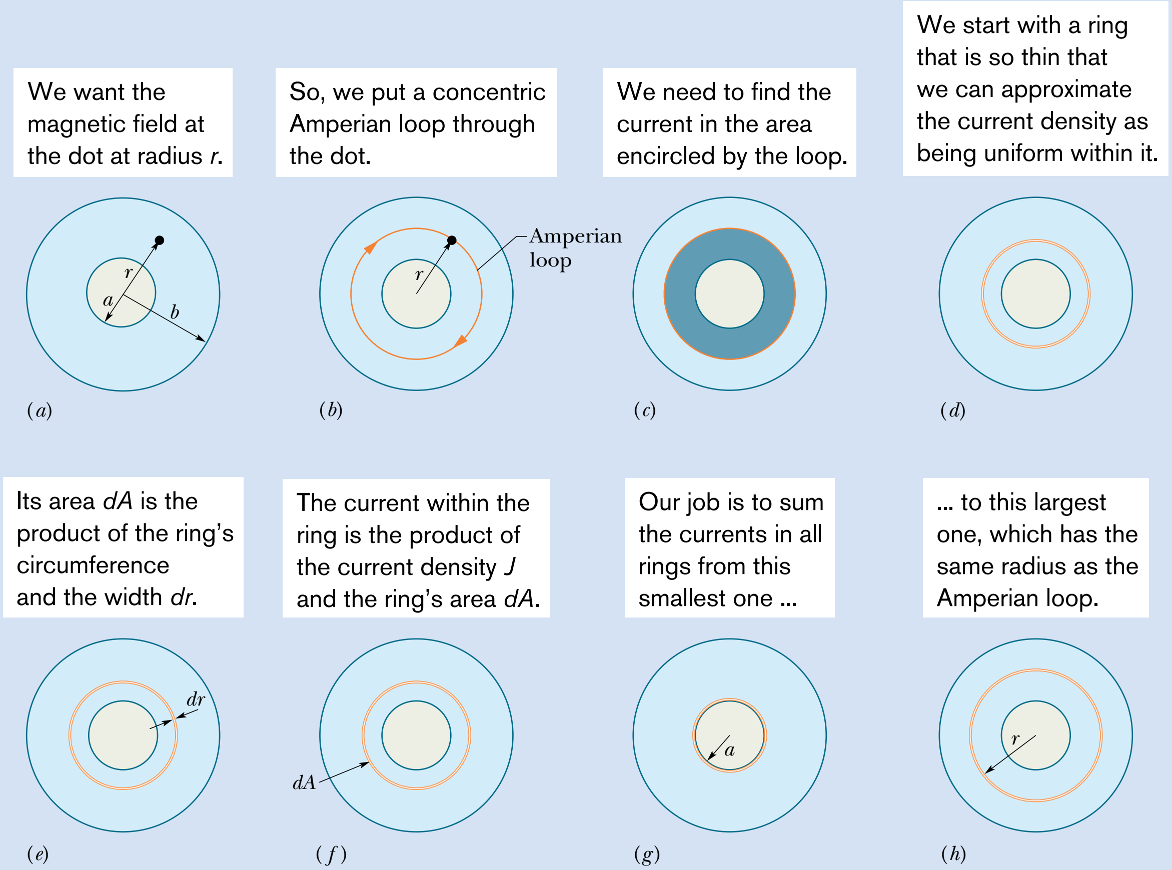

We want the magnetic field at the dot at radius r.
So, we put a concentric Amperian loop through the dot.
We need to find the current in the area encircled by the loop.
We start with a ring that is so thin that we can approximate the current density as being uniform within it.
r
a
b
r
Amperian loop
(a)
(b)
(c)
(d)
Its area dA is the product of the ring's circumference and the width dr.
The current within the ring is the product of the current density J and the ring's area dA.
Our job is to sum the currents in all rings from this smallest one ...
... to this largest one, which has the same radius as the Amperian loop.
dr
dA
a
r
(e)
(f)
(g)
(h)

Figure 29-29 (*a*)–(*b*) To find the magnetic field at a point within this conducting cylinder, we use a concentric Amperian loop through the point. We then need the current encircled by the loop. (*c*)–(*h*) Because the current density is nonuniform, we start with a thin ring and then sum (via integration) the currents in all such rings in the encircled area.

29.6 | SOLENOIDS AND TOROIDS

Key Concepts

- Inside a long solenoid carrying current i, at points not near its ends, the magnitude B of the magnetic field is

$$B = \mu_0 in \quad \text{(ideal solenoid)},$$

where n is the number of turns per unit length.

- At a point inside a toroid, the magnitude B of the magnetic field is

$$B = \frac{\mu_0 iN}{2\pi}\frac{1}{r} \quad \text{(toroid)},$$

where r is the distance from the center of the toroid to the point.

Magnetic Field of a Solenoid

We now turn our attention to another situation in which Ampere's law proves useful. It concerns the magnetic field produced by the current in a long, tightly wound helical coil of wire. Such a coil is called a **solenoid** (Fig. 29-30). Recall that in Sample Problem 29.01, we found the magnetic field due to the solenoid at points in the axis. Now we want to find the magnetic field at off-axis points as well. It will be defficult to find, since we do not know the magnetic field at off-axis points.

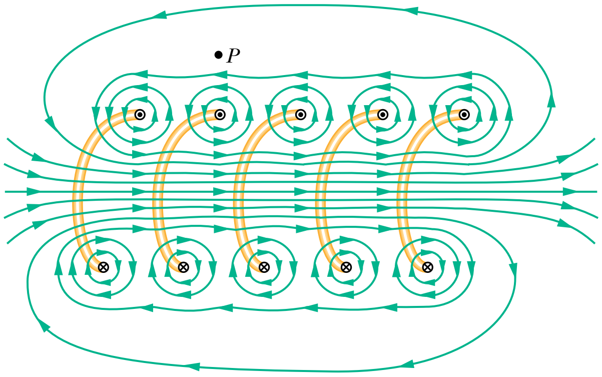


Figure 29-30 A vertical cross section through the central axis of a "stretched-out" solenoid. The back portions of five turns are shown, as are the magnetic field lines due to a current through the solenoid. Each turn produces circular magnetic field lines near itself. Near the solenoid's axis, the field lines combine into a net magnetic field that is directed along the axis. The closely spaced field lines there indicate a strong magnetic field. Outside the solenoid the field lines are widely spaced; the field there is very weak.

We assume that the length of the solenoid is much greater than the diameter. Figure 29-31 shows a section through a portion of a "stretched-out" solenoid. The solenoid's magnetic field is the vector sum of the fields produced by the individual turns (*windings*) that make up the solenoid. For points very close to a turn, the wire behaves magnetically almost like a long straight wire, and the lines of $\vec{B}$ there are almost concentric circles. Figure 29-31 suggests that the field tends to cancel between adjacent turns. It also suggests that, at points inside the solenoid and reasonably far from the wire, $\vec{B}$ is approximately parallel to the (central) solenoid axis. In the limiting case of an *ideal solenoid*, which is infinitely long and consists of tightly packed (*close-packed*) turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis.

At points above the solenoid, such as P in Fig. 29-31, the magnetic field set up by the upper parts of the solenoid turns (these upper turns are marked ⊙) is directed to the left (as drawn near P) and tends to cancel the field set up at P by the lower parts of the turns (these lower turns are marked ⊗), which is directed to the right (not drawn). In the limiting case of an ideal solenoid, the magnetic field outside the solenoid is zero. Taking the external field to be zero is an excellent assumption for a real solenoid if its length is much greater than its diameter and if we consider external points such as point P that are not at either end of the solenoid. The direction of the magnetic field along the solenoid axis is given by a curled – straight right-hand rule: Grasp the solenoid with your right hand so that your fingers follow the direction of the current in the windings; your extended right thumb then points in the direction of the axial magnetic field.

Figure 29-32 shows the lines of $\vec{B}$ for a real solenoid. The spacing of these lines in the central region shows that the field inside the coil is fairly strong and uniform over the cross section of the coil. The external field, however, is relatively weak.

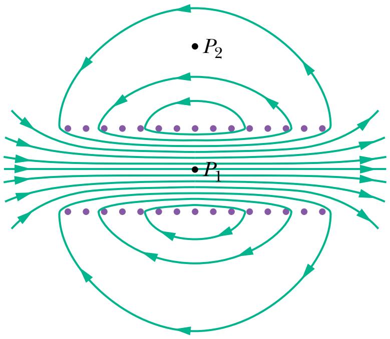


Figure 29-31 Magnetic field lines for a real solenoid of finite length. The field is strong and uniform at interior points such as P_1 but relatively weak at external points such as P_2.

As we stated in Sample Problem 29.01, the magnetic field at the axis of such an infinite solenoid is $B = \mu_0 ni$. To find the magnetic field at the off-axis points, we take a rectangular Amperian loop as shown in the Fig. 29-32*a* by loop *abefa*.

We can apply Ampere's law to find the magnetic field:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}.$$

As we can see there are four parts of the loop. We write $\oint \vec{B} \cdot d\vec{s}$ as the sum of four integrals, one for each loop segment

$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^e \vec{B} \cdot d\vec{s} + \int_e^f \vec{B} \cdot d\vec{s} + \int_f^a \vec{B} \cdot d\vec{s}.$$

Along the line segment *ab*, $\vec{B}$ is parallel to $d\vec{s}$, whereas along the line segment *be* and *af*, $\vec{B}$ and $d\vec{s}$ are mutually perpendicular. Along the line segment *ef*, $\vec{B}$ is antiparallel to $d\vec{s}$.

So, the first integral is $\mu_0 nih$, the second and fourth are zero, whereas the third is $-\mathrm{B}h$. Here, B is the magnetic field at the off-axis points (at point *e* or *f*). The current enclosed by the loop is zero. So, we obtain

$$\mu_0 nih + 0 + Bh + 0 = 0. \tag{29-40}$$

For a point outside the solenoid, we use the rectangular Amperian loop *abcda* as shown in Fig. 29-32*b*. We again apply Ampere's law to find the magnetic field:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}.$$

Again, we can see that there are four parts of the loop. We write $\oint \vec{B} \cdot d\vec{s}$ as the sum of four integrals, one for each loop segment

$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^e \vec{B} \cdot d\vec{s} + \int_e^d \vec{B} \cdot d\vec{s} + \int_d^a \vec{B} \cdot d\vec{s}.$$

Along the line segment *ab*, $\vec{B}$ is parallel to $d\vec{s}$, whereas along the line segment *bc* and *ad*, $\vec{B}$ and $d\vec{s}$ are mutually perpendicular. So, the first integral is $\mu_0\, nih$ as before, the second and fourth are zero, whereas the third is $-Bh$. In this situation, the current enclosed by the loop is not zero. The net current i_{enc} enclosed by the rectangular Amperian loop in Fig. 29-32*b* is not the same as the current *i* in the solenoid windings because the windings pass more than once through this loop. Let *n* be the number of turns per unit length of the solenoid; then the loop encloses *nh* turns and $i_{\text{enc}} = i(nh)$. So, we obtain)

$$B = 0.$$

So, the magnetic field outside the loop is zero.

Although we derived Eq. 29-41 for an infinitely long ideal solenoid, it holds quite well for actual solenoids if we apply it only at interior points and well away from the solenoid ends. Equation 29-41 is consistent with the experimental fact that the magnetic field magnitude *B* within a solenoid does not depend on the diameter or the length of the solenoid and that *B* is uniform over the solenoidal cross section. A solenoid thus provides a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor provides a practical way to set up a known uniform electric field.

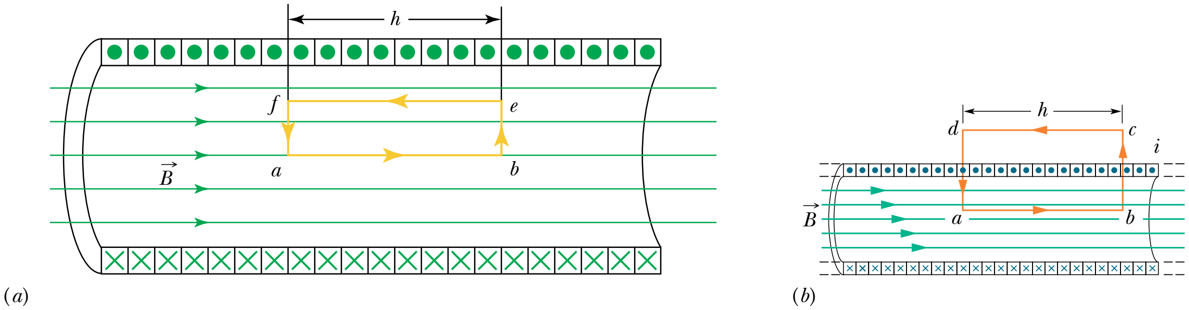


Figure 29-32 Ampere's law applied to find the magnetic field (*a*) inside the solenoid at off-axis points; (*b*) outside the solenoid.

Magnetic Field of a Toroid

Figure 29-33*a* shows a **toroid**, which we may describe as a (hollow) solenoid that has been curved until its two ends meet, forming a sort of hollow bracelet. What magnetic field $\vec{B}$ is set up inside the toroid (inside the hollow of the bracelet)? We can find out from Ampere's law and the symmetry of the bracelet.

From the symmetry, we see that the lines of $\vec{B}$ form concentric circles inside the toroid, directed as shown in Fig. 29-33*b*. Let us choose a concentric circle of radius r as an Amperian loop and traverse it in the clockwise direction. Ampere's law (Eq. 29-33) yields

$$(B)(2\pi r) = \mu_0 i N,$$

where i is the current in the toroid windings (and is positive for those windings enclosed by the Amperian loop) and N is the total number of turns. This gives

$$B = \frac{\mu_0 i N}{2\pi}\frac{1}{r} \quad \text{(toroid).} \tag{29-41}$$

In contrast to the situation for a solenoid, B is not constant over the cross section of a toroid.

It is easy to show, with Ampere's law, that $B = 0$ for points outside an ideal toroid (as if the toroid were made from an ideal solenoid). The direction of the magnetic field within a toroid follows from our curled–straight right-hand rule: Grasp the toroid with the fingers of your right hand curled in the direction of the current in the windings; your extended right thumb points in the direction of the magnetic field.

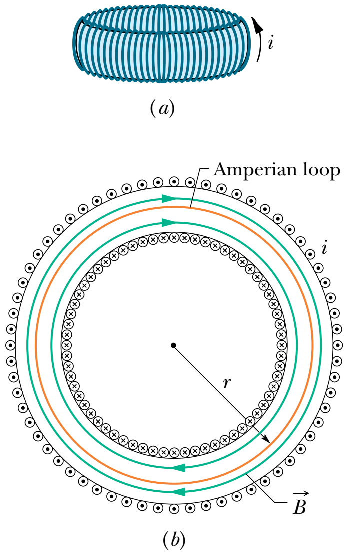


Figure 29-33 (*a*) A toroid carrying a current *i*. (*b*) A horizontal cross section of the toroid. The interior magnetic field (inside the bracelet-shaped tube) can be found by applying Ampere's law with the Amperian loop shown.

SAMPLE PROBLEM 29.08

The field inside a solenoid (a long coil of current)

A solenoid has length $L = 1.23$ m and inner diameter $d = 3.55$ cm, and it carries a current $i = 5.57$ A. It consists of five close-packed layers, each with 850 turns along length L. What is B at its center?

KEY IDEA

The magnitude B of the magnetic field along the solenoid's central axis is related to the solenoid's current i and number of turns per unit length n by Eq. 29-20 ($B = \mu_0 in$).

Calculation: Because B does not depend on the diameter of the windings, the value of n for five identical layers is simply five times the value for each layer. Equation 29-20 then tells us

$$B = \mu_0 in = (4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(5.57\ \text{A})\frac{5 \times 850\ \text{turns}}{1.23\ \text{m}}$$
$$= 2.42 \times 10^{-2}\ \text{T} = 24.2\ \text{mT}. \qquad \text{(Answer)}$$

To a good approximation, this is the field magnitude throughout most of the solenoid.

SAMPLE PROBLEM 29.09

Magnetic field of a large current-carrying sheet

A large flat current-carrying sheet carries a current K per unit length as shown in Figure 29-34*a*. (That is, a strip of width w of the sheet carries a current Kw.) What is the magnitude of the magnetic field at a distance d from the sheet?

KEY IDEAS

(1) This situation has *planar symmetry*, so we expect that, in analogy to the electric field of a flat charged sheet, the field on either side of the plate is uniform. That is to say,

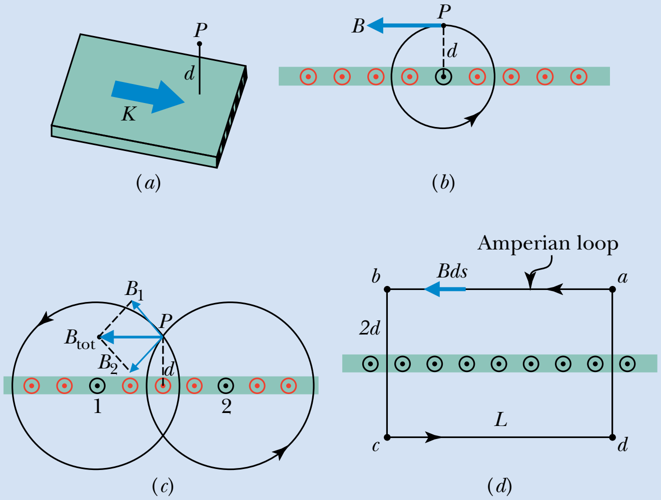


Figure 29-34 (*a*) Current-carrying sheet. (*b*) Contributions from a strip right underneath point *P*. (*c*) Contributions from two strips to the magnetic field at point *P*. (*d*) Amperian loop.

the magnitude and direction of the magnetic field does not depend on the distance from the plane. Before we can use Ampere's law to verify this dependence, we must determine the direction of the magnetic field at an arbitrary point *P* above the charged sheet.

(2) We divide the sheet into thin parallel strips in the direction of the current. We then treat each strip as a current-carrying wire. A cross-sectional view of the sheet, looking into a direction opposite that of the current through the sheet, is shown in Fig. 29-34*b*. Using the right-hand rule, we find that the strip right underneath point *P* contributes a magnetic field that points parallel to the plate and to the left. Next, we look at the contributions from two strips, 1 and 2, equidistant on either side of *P* (see Fig. 29-34*c*). The two circles show the field lines from either strip that go through *P*. The magnitudes of the contributions from the two strips are equal and so we see that the sum of the contributions from these two strips also point parallel to the plate and to the left. Arguing this way, we can say that the entire magnetic field due to the sheet is parallel to the sheet and points to the left.

Calculations: To use this fact in Ampere's law, we choose the rectangular loop *abcd* as shown in Fig. 29-34*d* as the Amperian loop. The width of this rectangular loop is *L* and its height is 2*d*. The line integral around the loop can then be written as the sum of four line integrals over each side of the rectangle:

$$\oint \vec{B}\cdot d\vec{s} = \int_a^b \vec{B}\cdot d\vec{s} + \int_b^c \vec{B}\cdot d\vec{s} + \int_c^d \vec{B}\cdot d\vec{s} + \int_d^a \vec{B}\cdot d\vec{s}.$$

Along the vertical sides *bc* and *da*, the magnetic field is perpendicular to the path and so the dot product $\vec{B}\cdot d\vec{s}$ is zero. Over the other two sides the magnetic field is parallel to the path and so $\vec{B}\cdot d\vec{s} = Bds$.

Symmetry dictates that the magnitude of the magnetic field a distance *d* below the sheet is the same as that a distance *d* above it (because flipping the plane upside-down should not alter the physical situation) and so we can take this magnitude, *B* out of the integral, that is

$$\oint \vec{B}\cdot d\vec{s} = \int_{ab} B\,ds + \int_{cd} B\,ds = B\int_{ab} ds + B\int_{cd} ds.$$

The two remaining line integrals simply yield the lengths of the two sides and so the left-hand side of Ampere's law becomes

$$\oint \vec{B}\cdot d\vec{s} = B(2L) = 2BL. \qquad (29\text{-}42)$$

Substituting for the right-hand side of Eq. (29-42) into Ampere's law, we get

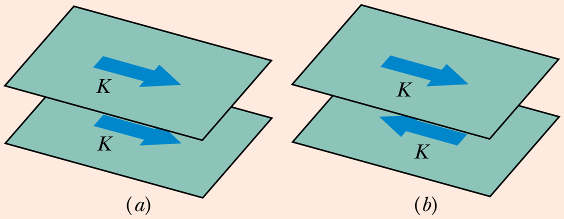
$$2BL = \mu_0 KL$$

$$B = \frac{1}{2}\mu_0 K.$$

Conclude: So we see that the magnetic field does not depend on the distance to the plane, just like the electric field of a large uniformly charged sheet.

CHECKPOINT 4

What are the directions of the magnetic field between and outside a pair of two parallel sheets carrying currents in (*a*) the same and (*b*) opposite directions, as illustrated in the adjoining figure?



29.7 | LIMITATIONS OF AMPERE'S LAW

Ampere's law is useful for calculating the magnetic field only when there is a high degree of symmetry. Consider the current loop shown in Fig. 29-35. According to Ampere's law, the line integral around a loop such as curve *C* in the figure equals μ_0 times the current *I* in the loop. Although Ampere's law is valid for this curve, the magnetic field is not constant along any curve encircling the current, nor is it everywhere tangent to any such loop. Thus, there is not enough symmetry in this situation to allow us to calculate using Ampere's law.

Figure 29-36 shows a finite current segment of length. We wish to find the magnetic field at point *P*, which is equidistant from the ends of the segment and at a distance *r* from the center of the segment. A direct application of Ampere's law gives

$$B = \frac{\mu_0}{2\pi}\frac{I}{r}.$$

This result is the same as for an infinitely long wire, since the same symmetry arguments apply. It does not agree with the result obtained from the Biot–Savart law, which depends on the length of the current segment and also agrees with experiment. Actually, there can never be just a finite current-carrying wire as shown in Fig. 29-36. There must be some leading wires which connect this current-carrying wire to the battery. As can be seen in Fig. 29-37, Ampere's law for the curve *C* is valid, but there is not enough symmetry to use it to find the magnetic field at point *P*.

In Fig. 29-38, the current in the segment arises from a small spherical conductor with initial charge $+Q$ at the left of the segment and another one at the right with charge $-Q$. When they are connected, a current $I = -dQ/dt$ exists in the segment for a short time, until the spheres are discharged. For this case, we do have the symmetry needed to assume that magnetic field is tangential to the curve and constant in magnitude along the curve. For a situation like this, in which the electric field is varying with time, Ampere's law is not valid. Maxwell was able to modify Ampere's law so that it holds for all currents. When Maxwell's generalized form of Ampere's law is used to calculate the magnetic field for a current segment, such as that shown in Fig. 29-38, the result agrees with that found from the Biot–Savart law.

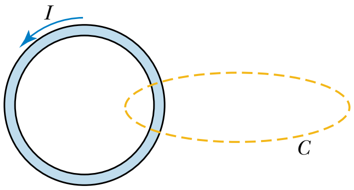


Figure 29-35 Ampere's law holds for curve *C* encircling the current in the circular loop, but it is not useful for finding the magnetic field because it is neither constant along the curve nor tangent to it.

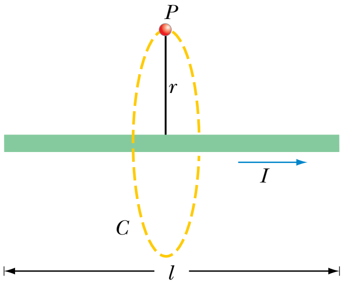


Figure 29-36 The application of Ampere's law to find the magnetic field on the bisector of a finite current segment gives an incorrect result.

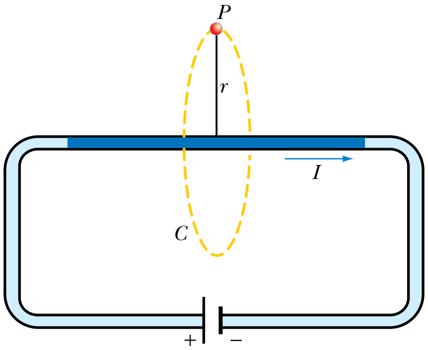


Figure 29-37 The current segment in Fig. 29-35 is part of a complete circuit.

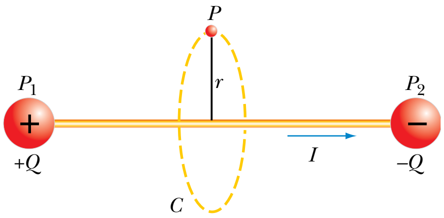


Figure 29-38 If the current segment in Fig. 29-36 is due to a momentary flow of charge from a small conductor on the left to the one at the right, there is enough symmetry to use Ampere's law to compute the magnetic field at *P*, but Ampere's law is not valid because the Electric field in the space is varying with time.

29.8 | A NOTE ABOUT THE FRAMES OF REFERENCE

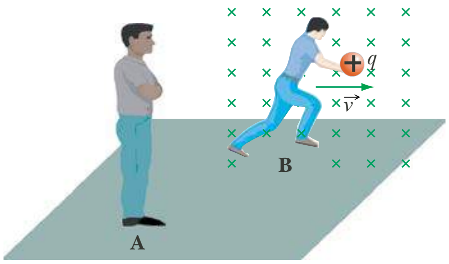


Figure 29-39 **B** carries a charge past **A**. Charge q moves with a velocity v relative to **A**. The charge q moves through a magnetic field created by **A.**

Now it seems clear that charges create an electric field $\vec{E}$ and that moving charges, or currents, create a magnetic field $\vec{B}$ in addition to it. Charges other than the source charges always respond to $\vec{E}$, but only moving charges respond to $\vec{B}$. There seems to be a problem, however.

Assume that there are two friends **A** and **B** as seen in Fig. 29-39. **A** is at rest and is carrying a magnet that produces a magnetic field $\vec{B}$ at the location of **B**. **B** is running with a velocity $\vec{v}$ as shown in the figure. **A** will claim that the charge carried by **B** will experience a force of $q\vec{v}\times\vec{B}$.

Using the right-hand rule, **A** determines that the force points straight up. But for **B** the charge is still at rest. Stationary charges don't experience magnetic forces, so **B** claims that $\vec{F}=\vec{0}$.

Figure 29-39 shows **B** running past **A** with velocity $\vec{v}$ while carrying charge q. Also, A sees a moving charge, and he knows that this charge creates a magnetic field. But from **B**'s perspective, the charge is at rest. Stationary charges do not create magnetic fields, so **B** claims that the magnetic field is zero. Is there or is there not a magnetic field?

Now, we may be a bit uncertain about magnetic fields because they are an abstract concept, but surely there can be no disagreement over forces. After all, either the charge is going to accelerate upward or it is not, and **A** and **B** should be able to agree on the outcome.

Here we have a genuine paradox, not merely faulty reasoning. This paradox has arisen because we have fields and forces that depend on velocity. The difficulty is that we have not looked at the issue of *velocity with respect to what or velocity as measured by whom*. A closer look at how electromagnetic fields are viewed by two experimenters moving relative to each other will lead us to conclude that $\vec{E}$ and $\vec{B}$ are not as we've been assuming: separate and independent entities. They are closely intertwined. We will explore this relationship in the chapter on relativity.

B and **A** may measure different positions and velocities for a particle, but they agree on its acceleration. This agreement is important because acceleration is directly related to force. An experimenter in frame S would find that a force $\vec{F}=m\vec{a}$ is acting on the particle. Similarly, the force measured in frame S' is $\vec{F}'=m\vec{a}$. But $\vec{a}'=\vec{a}$, hence

$$\vec{F}'=\vec{F}. \tag{29-43}$$

Experiments in all inertial reference frames agree about the force acting on a particle. This conclusion is the key to understanding how different experimenters see electric and magnetic fields. We are not going to study this in any more detail, but we need to understand that **the magnetic field and the electric field seen by observers in different inertial frames will be different**. So, the conclusion is that do not change the frame of reference: it may seem to simplify the problem, but the electric field and magnetic field will not remain the same as before.

REVIEW AND SUMMARY

The Biot–Savart Law The magnetic field set up by a currentcarrying conductor can be found from the *Biot–Savart law*. This law asserts that the contribution $d\vec{B}$ to the field produced by a current-length element $i\,d\vec{s}$ at a point P located a distance r from the current element is

$$d\vec{B}=\frac{\mu_0}{4\pi}\frac{i d\vec{s}\times\hat{\mathrm{r}}}{r^2} \quad \text{(Biot-Savart law).} \tag{29-3}$$

Here $\hat{\mathrm{r}}$ is a unit vector that points from the element toward P. The quantity μ_0, called the permeability constant, has the value

$$4\pi\times10^{-7}\,\mathrm{T\cdot m/A}\approx1.26\times10^{-6}\,\mathrm{T\cdot m/A}.$$

Magnetic Field of a Long Straight Wire For a long straight wire carrying a current i, the Biot–Savart law gives, for the magnitude of the magnetic field at a perpendicular distance R from the wire,

$$B=\frac{\mu_0 i}{2\pi R} \quad \text{(long straight wire).} \tag{29-4}$$

Magnetic Field of a Circular Arc The magnitude of the magnetic field at the center of a circular arc, of radius R and central angle ϕ (in radians), carrying current i, is

$$B=\frac{\mu_0 i\phi}{4\pi R} \quad \text{(at center of circular arc).} \tag{29-9}$$

Field of a Magnetic Dipole The magnetic field produced by a current-carrying coil, which is a *magnetic dipole*, at a point P located a distance z along the coil's perpendicular central axis is parallel to the axis and is given by

$$\vec{B}(z) = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3}, \tag{29-24}$$

where $\vec{\mu}$ is the dipole moment of the coil. This equation applies only when z is much greater than the dimensions of the coil.

Force Between Parallel Currents Parallel wires carrying currents in the same direction attract each other, whereas parallel wires carrying currents in opposite directions repel each other. The magnitude of the force on a length L of either wire is

$$F_{ba} = i_b L B_a \sin 90° = \frac{\mu_0 L i_a i_b}{2\pi d}, \tag{29-31}$$

where d is the wire separation, and i_a and i_b are the currents in the wires.

Ampere's Law **Ampere's law** states that

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} \quad \text{(Ampere's law).} \tag{29-33}$$

The line integral in this equation is evaluated around a closed loop called an *Amperian loop*. The current i on the right side is the *net* current encircled by the loop. For some current distributions, Eq. 29-33 is easier to use than Eq. 29-3 to calculate the magnetic field due to the currents.

Fields of a Solenoid and a Toroid Inside a *long solenoid* carrying current i, at points not near its ends, the magnitude B of the magnetic field is

$$B = \mu_0 i n \quad \text{(ideal solenoid),} \tag{29-20}$$

where n is the number of turns per unit length. Thus the internal magnetic field is uniform. Outside the solenoid, the magnetic field is approximately zero.

At a point inside a *toroid*, the magnitude B of the magnetic field is

$$B = \frac{\mu_0 i N}{2\pi}\frac{1}{r} \quad \text{(toroid),} \tag{29-42}$$

where r is the distance from the center of the toroid to the point.

PROBLEMS

1. In Fig. 29-40, point P_1 is at distance $R = 24.0$ cm on the perpendicular bisector of a straight wire of length $L = 18.0$ cm carrying current $i = 58.2$ mA. (Note that the wire is *not* long.) What are the (a) magnitude and (b) direction of the magnetic field at P_1 due to i? (c) If R is increased, what happens to the magnitude of the field?

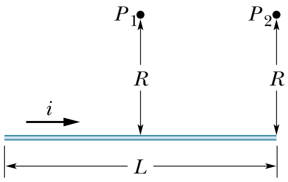


Figure 29-40 Problems 1 and 2.

2. In Fig. 29-40, point P_2 is at perpendicular distance $R = 25.1$ cm from one end of a straight wire of length $L = 13.6$ cm carrying current $i = 0.500$ A. (Note that the wire is *not* long.) (a) What is the magnitude of the magnetic field at P_2? (b) If the point of measurement is moved from P_2 to P_1, does the field magnitude increase, decrease, or remain the same?

3. In Fig. 29-41, four long straight wires are perpendicular to the page, and their cross sections form a square of edge length $a = 40$ cm. The currents are out of the page in wires 1 and 4 and into the page in wires 2 and 3, and each wire carries 12 A. In unit-vector notation, what is the net magnetic field at the square's center?

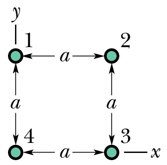


Figure 29-41 Problems 3, 5, and 6.

4. In Fig. 29-42, part of a long insulated wire carrying current $i = 5.78$ mA is bent into a circular section of radius $R = 1.89$ cm. In unit-vector notation, what is the magnetic field at the center of curvature C if the circular section (a) lies in the plane of the page as shown and (b) is perpendicular to the plane of the page after being rotated 90° counterclockwise as indicated?

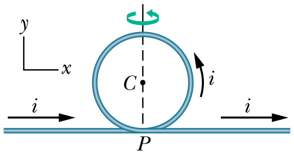


Figure 29-42 Problem 4.

5. In Fig. 29-41, four long straight wires are perpendicular to the page, and their cross sections form a square of edge length $a = 13.5$ cm. Each wire carries 7.50 A, and the currents are out of the page in wires 1, 3, and 4 and into the page in wire 2. In unit-vector notation, what is the net magnetic force *per meter of wire length* on wire 4?

6. In Fig. 29-41, four long straight wires are perpendicular to the page, and their cross sections form a square of edge length $a = 8.50$ cm. Each wire carries 15.0 A, and all the currents are out of the page. In unit-vector notation, what is the net magnetic force *per meter of wire length* on wire 1?

7. In Fig. 29-43, length a is 2.3 cm (short) and current i is 18 A. What are the (a) magnitude and (b) direction (into or out of the page) of the magnetic field at point P?

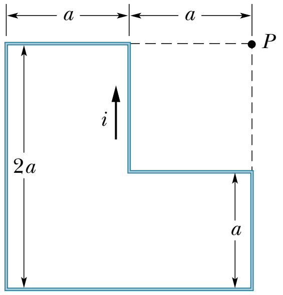


Figure 29-43 Problem 7.

8. In Fig. 29-44, five long parallel wires in an *xy* plane are separated by distance $d = 50.0$ cm. The currents into the page are $i_1 = 2.00$ A, $i_3 = 0.250$ A, $i_4 = 6.00$ A, and $i_5 = 2.00$ A; the current out of the page is $i_2 = 4.00$ A. What is the magnitude of the net force *per unit length* acting on wire 3 due to the currents in the other wires?

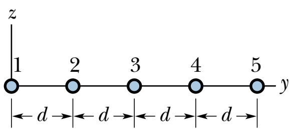
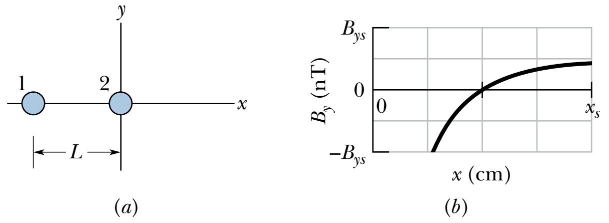


Figure 29-44 Problem 8.

9. Figure 29-45*a* shows, in cross section, two long, parallel wires carrying current and separated by distance *L*. The ratio i_1/i_2 of their currents is 4.00; the directions of the currents are not indicated. Figure 29-45*b* shows the *y* component B_y of their net magnetic field along the *x* axis to the right of wire 2. The vertical scale is set by $B_{ys} = 4.0$ nT, and the horizontal scale is set by $x_s = 20.0$ cm. (a) At what value of $x > 0$ is B_y maximum? (b) If $i_2 = 3$ mA, what is the value of that maximum? What is the direction (into or out of the page) of (c) i_1 and (d) i_2?



(*a*) (*b*)

Figure 29-45 Problem 9.

10. Figure 29-46 shows two closed paths wrapped around two conducting loops carrying currents $i_1 = 5.0$ A and $i_2 = 3.0$ A. What is the value of the integral $\oint \vec{B}\cdot d\vec{s}$ for (a) path 1 and (b) path 2?

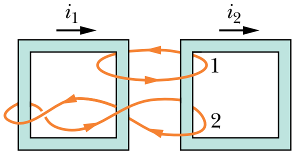


Figure 29-46 Problem 10.

11. Figure 29-47 shows wire 1 in cross section; the wire is long and straight, carries a current of 2.50 mA out of the page, and is at distance $d_1 = 4.00$ cm from a surface. Wire 2, which is parallel to wire 1 and also long, is at horizontal distance $d_2 = 5.00$ cm from wire 1 and carries a current of 6.80 mA into the page. What is the *x* component of the magnetic force *per unit length* on wire 2 due to wire 1?

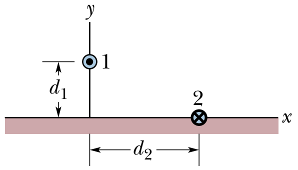


Figure 29-47 Problem 11.

12. An electron is shot into one end of a solenoid. As it enters the uniform magnetic field within the solenoid, its speed is 800 m/s and its velocity vector makes an angle of 30° with the central axis of the solenoid. The solenoid carries 4.0 A and has 8000 turns along its length. How many revolutions does the electron make along its helical path within the solenoid by the time it emerges from the solenoid's opposite end? (In a real solenoid, where the field is not uniform at the two ends, the number of revolutions would be slightly less than the answer here.)

13. A toroid having a square cross section, 5.00 cm on a side, and an inner radius of 19.0 cm has 460 turns and carries a current of 0.400 A. (It is made up of a square solenoid—instead of a round one as in Fig. 29-30—bent into a doughnut shape.) What is the magnetic field inside the toroid at (a) the inner radius and (b) the outer radius?

14. Figure 29-48 shows a cross section of a long thin ribbon of width $w = 6.20$ cm that is carrying a uniformly distributed total current $i = 4.61$ μA into the page. In unit-vector notation, what is the magnetic field $\vec{B}$ at a point *P* in the plane of the ribbon at a distance $d = 1.61$ cm from its edge? (*Hint:* Imagine the ribbon as being constructed from many long, thin, parallel wires.)

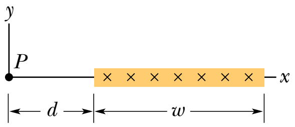


Figure 29-48 Problem 14.

15. Figure 29-49 shows a cross section across a diameter of a long cylindrical conductor of radius $a = 2.00$ cm carrying uniform current 170 A. What is the magnitude of the current's magnetic field at radial distance (a) 0, (b) 6.00 mm, (c) 2.00 cm (wire's surface), and (d) 5.90 cm?

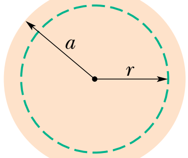


Figure 29-49 Problem 15.

16. Eight wires cut the page perpendicularly at the points shown in Fig. 29-50. A wire labeled with the integer *k* ($k = 1, 2, \ldots, 8$) carries the current *ki*, where $i = 4.50$ mA. For those wires with odd *k*, the current is out of the page; for those with

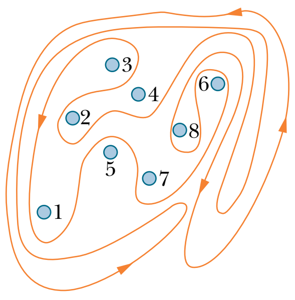


Figure 29-50 Problem 16.

even k, it is into the page. Evaluate $\oint \vec{B} \cdot d\vec{s}$ along the closed path indicated and in the direction shown.

17. Each of the eight conductors in Fig. 29-51 carries 5.0 A of current into or out of the page. Two paths are indicated for the line integral $\oint \vec{B} \cdot d\vec{s}$. What is the value of the integral for (a) path 1 and (b) path 2?

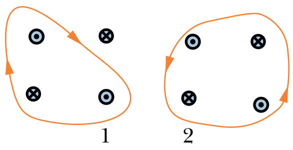


Figure 29-51 Problem 17.

18. A long solenoid with 10.0 turns/cm and a radius of 7.00 cm carries a current of 35.0 mA. A current of 5.00 A exists in a straight conductor located along the central axis of the solenoid. (a) At what radial distance from the axis will the direction of the resulting magnetic field be at 45.0° to the axial direction? (b) What is the magnitude of the magnetic field there?

19. In Fig. 29-52, two semicircular arcs have radii $R_2 = 7.80$ cm and $R_1 = 3.15$ cm, carry current $i = 0.281$ A, and have the same center of curvature C. What are the (a) magnitude and (b) direction (into or out of the page) of the net magnetic field at C?

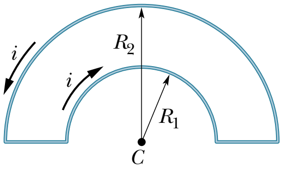


Figure 29-52 Problem 19.

20. A 470-turn solenoid having a length of 25 cm and a diameter of 10 cm carries a current of 0.29 A. Calculate the magnitude of the magnetic field $\vec{B}$ inside the solenoid.

21. What is the magnitude of the magnetic dipole moment $\vec{\mu}$ of the solenoid described in Problem 20?

22. Figure 29-53 shows two very long straight wires (in cross section) that each carry a current of 4.00 A directly out of the page. Distance d_1 = 6.00 m and distance d_2 = 8.00 m. What is the magnitude of the net magnetic field at point P, which lies on a perpendicular bisector to the wires?

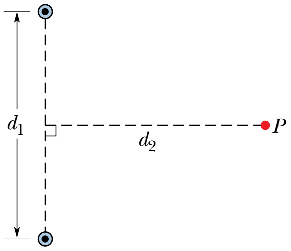


Figure 29-53 Problem 22.

23. A long solenoid has 123 turns/cm and carries current i. An electron moves within the solenoid in a circle of radius 2.30 cm perpendicular to the solenoid axis. The speed of the electron is 0.0187c (c = speed of light). Find the current i in the solenoid.

24. The current density $\vec{J}$ inside a long, solid, cylindrical wire of radius a = 4.5 mm is in the direction of the central axis, and its magnitude varies linearly with radial distance r from the axis according to $J = J_0 r/a$, where $J_0 = 420$ A/m^2. Find the magnitude of the magnetic field at (a) $r = 0$, (b) $r = a/2$, and (c) $r = a$.

25. A circular loop of radius 12 cm carries a current of 7.2 A. A flat coil of radius 0.82 cm, having 50 turns and a current of 1.3 A, is concentric with the loop. The plane of the loop is perpendicular to the plane of the coil. Assume the loop's magnetic field is uniform across the coil. What is the magnitude of (a) the magnetic field produced by the loop at its center and (b) the torque on the coil due to the loop?

26. A surveyor is using a magnetic compass 12.2 m below a power line in which there is a steady current of 200 A. (a) What is the magnetic field at the site of the compass due to the power line? (b) Will this field interfere seriously with the compass reading? The horizontal component of Earth's magnetic field at the site is 20 μT.

27. A student makes a short electromagnet by winding 280 turns of wire around a wooden cylinder of diameter d = 5.0 cm. The coil is connected to a battery producing a current of 3.8 A in the wire. (a) What is the magnitude of the magnetic dipole moment of this device? (b) At what axial distance $z \gg d$ will the magnetic field have the magnitude 5.0 μT (approximately one-tenth that of Earth's magnetic field)?

28. In Fig. 29-54, two long straight wires are perpendicular to the page and separated by distance d_1 = 0.75 cm. Wire 1 carries 6.5 A into the page. What are the (a) magnitude and (b) direction (into or out of the page) of the current in wire 2 if the net magnetic field due to the two currents is zero at point P located at distance d_2 = 2.50 cm from wire 2? If the current in wire 2 is then reversed, what are the (c) size and (d) direction of the net field at point P?

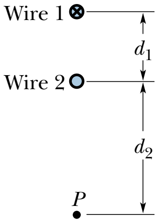


Figure 29-54 Problem 28.

29. In Fig. 29-55, a current i = 2.2 A is set up in a long hairpin conductor formed by bending a wire into a semicircle of radius R = 8.5 mm. Point b is midway between the straight sections and so distant from the semicircle that each straight section can be approximated as being an infinite wire. What are the (a) magnitude and (b) direction (into or out of the page) of $\vec{B}$ at a and the (c) magnitude and (d) direction of $\vec{B}$ at b?

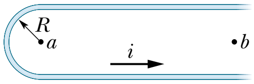


Figure 29-55 Problem 29.

30. Figure 29-56 shows two current segments. The lower segment carries a current of i_1 = 0.40 A and includes a semicircular arc with radius 5.0 cm, angle 180°, and center point P. The upper segment carries current $i_2 = 3i_1$ and includes a circular arc with radius 4.0 cm, angle 120°, and the same center point P. What are the (a) magnitude and (b) direction of the net magnetic field $\vec{B}$ at P for the indicated current directions? What are the (c) magnitude and (d) direction of $\vec{B}$ if i_1 is reversed?

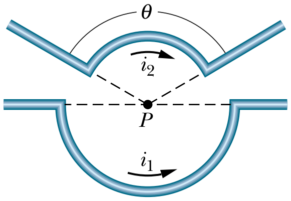


Figure 29-56 Problem 30.

31. In Fig. 29-57, two long straight wires (shown in cross section) carry the currents $i_1 = 30.0$ mA and $i_2 = 50.0$ mA directly out of the page. They are equal distances from the origin, where they set up a magnetic field $\vec{B}$. To what value must current i_1 be changed in order to rotate $\vec{B}$ 25° clockwise?

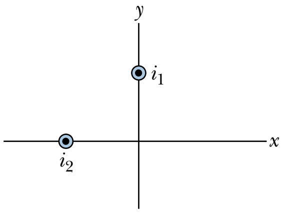


Figure 29-57 Problem 31.

32. Figure 29-58 shows a snapshot of a proton moving at velocity $\vec{v} = (-380 \text{ m/s})\hat{\text{j}}$ toward a long straight wire with current $i = 470$ mA. At the instant shown, the proton's distance from the wire is $d = 2.89$ cm. In unit-vector notation, what is the magnetic force on the proton due to the current?

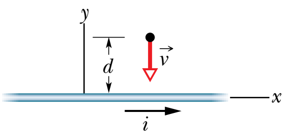


Figure 29-58 Problem 32.

33. A straight conductor carrying current $i = 5.0$ A splits into identical semicircular arcs as shown in Fig. 29-59. What is the magnetic field at the center C of the resulting circular loop?

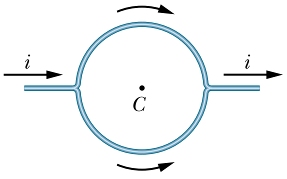


Figure 29-59 Problem 33.

34. One long wire lies along an x axis and carries a current of 60 A in the positive x direction. A second long wire is perpendicular to the xy plane, passes through the point (0, 4.0 m, 0), and carries a current of 40 A in the positive z direction. What is the magnitude of the resulting magnetic field at (a) the point (0, 2.0 m, 0) and (b) the point (2.0 m, 4.0 m, 0)?

35. In Fig. 29-60, two concentric circular loops of wire carrying current in the same direction lie in the same plane. Loop 1 has radius 1.50 cm and carries 4.00 mA. Loop 2 has radius 2.50 cm and carries 6.00 mA. Loop 2 is to be rotated about a diameter while the net magnetic field $\vec{B}$ set up by the two loops at their common center is measured. Through what angle must loop 2 be rotated so that the magnitude of that net field is 100 nT?

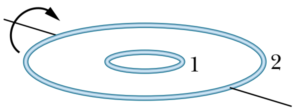


Figure 29-60 Problem 35.

36. In Fig. 29-61, two circular arcs have radii $a = 18.9$ cm and $b = 10.7$ cm, subtend angle $\theta = 74.0°$, carry current $i = 0.411$ A, and share the same center of curvature P. What are the (a) magnitude and (b) direction (into or out of the page) of the net magnetic field at P?

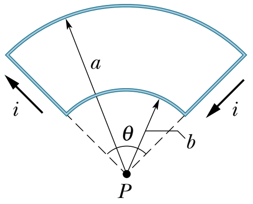


Figure 29-61 Problem 36.

37. Two long straight wires are parallel and 16 cm apart. They are to carry equal currents such that the magnetic field at a point halfway between them has magnitude 450 μT. (a) Should the currents be in the same or opposite directions? (b) How much current is needed?

38. Figure 29-62 shows, in cross section, four thin wires that are parallel, straight, and very long. They carry identical currents in the directions indicated. Initially all four wires are at distance $d = 15.0$ cm from the origin of the coordinate system, where they create a net magnetic field $\vec{B}$. (a) To what value of x must you move wire 1 along the x axis in order to rotate $\vec{B}$ counterclockwise by 30°? (b) With wire 1 in that new position, to what value of x must you move wire 3 along the x axis to rotate $\vec{B}$ by 30° back to its initial orientation?

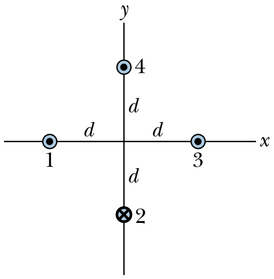


Figure 29-62 Problem 38.

39. Find the intensity of a magnetic field on the axis of a circular ring at a distance of 3 cm from its plane. The ring radius is 4 cm and the current in the ring is 2 A.

40. An electron moves in a straight line at a speed of 6 × 107 m/s. Calculate the magnitude and direction of the magnetic field at (a) point 5 mm behind the electron and (b) 5 mm below its line of motion.

41. In a homogeneous magnetic field B there is an electron moving in a circular orbit with a speed v. Find the ratio of the magnetic field generated by the moving electron at the middle of the circle and the magnetic field making it revolve.

42. Two circular coils shown in Fig. 29-63 have the same number of turns and carry equal current in the same sense. They have different diameters but subtend the same angle at P. (a) Which coil makes the larger contribution to the magnetic field at P? (b) If the smaller coil is midway between P and the larger one, and the smaller coil has radius R whereas the larger coil has a radius $2R$ and the distance of the smaller coil to the point P is also R, what is the ratio of the magnetic field due to larger coil at P to the magnetic field due to the smaller coil at P?

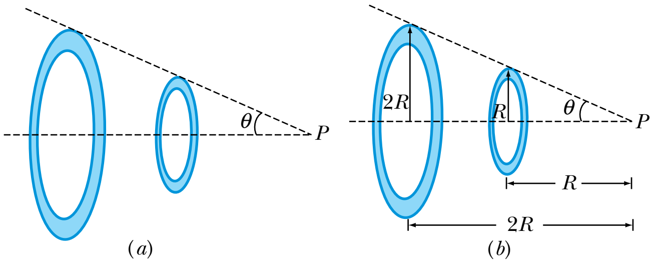


Figure 29-63 Problem 42.

43. Two protons move parallel to each other with an equal velocity $v = 300$ km/s. Find the ratio of forces of magnetic and electrical interaction of the protons.

44. A long solenoid is fabricated by closely winding a wire of radius 0.5 mm over a cylindrical nonmagnetic frame so that the successive turns nearly touch each other. What would be the magnetic field B at the center of the solenoid if it carries a current of 5 A?

45. Figure 29-64 shows a cross section of a long cylindrical conductor of radius $a = 4.00$ cm containing a long cylindrical hole of radius $b = 1.50$ cm. The central axes of the cylinder and hole are parallel and are distance $d = 2.00$ cm apart; current $i = 5.25$ A is uniformly distributed over the tinted area. (a) What is the magnitude of the magnetic field at the center of the hole? (b) Discuss the two special cases $b = 0$ and $d = 0$.

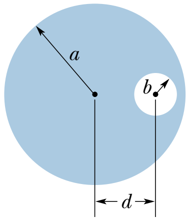


Figure 29-64 Problem 45.

PRACTICE QUESTIONS

Single Correct Choice Type

1. Each second, 1.25×10^{19} electrons in a narrow beam pass through a small hole in a wall. The beam is perpendicular to the wall. Using Ampere's law, determine the magnitude of the magnetic field in the wall at a radius of 0.750 m from the center of the beam.

(a) 1.18×10^{-9} T
(b) 2.62×10^{-3} T
(c) 5.33×10^{-7} T
(d) 4.14×10^{8} T

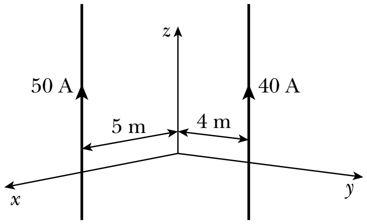
2. Which one of the following statements concerning the magnetic field inside (far from the surface) a long, current-carrying solenoid is true?

(a) The magnetic field is zero
(b) The magnetic field is non-zero and nearly uniform
(c) The magnetic field is independent of the number of windings
(d) The magnetic field is independent of the current in the solenoid

3. A solenoid of length 0.250 m and radius 0.0200 m is comprised of 120 turns of wire. Determine the magnitude of the magnetic field at the center of the solenoid when it carries a current of 15.0 A.

(a) 2.26×10^{-3} T **(b)** 9.05×10^{-3} T
(c) 4.52×10^{-3} T **(d)** 7.50×10^{-3} T

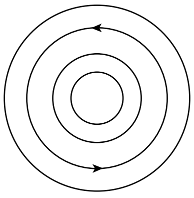
4. The drawing shows two long, thin wires that carry currents in the positive z direction. Both wires are parallel to the z axis. The 50-A wire is in the x-z plane and is 5 m from the z axis. The 40-A wire is in the y-z plane and is 4 m from the z axis. What is the magnitude of the magnetic field at the origin?



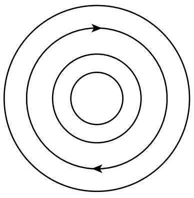
(a) zero tesla **(b)** 1×10^{-6} T
(c) 3×10^{-6} T **(d)** 5×10^{-6} T

5. A coil of wire carries current I as shown in the figure. If the observer could "see" the magnetic field inside this arrangement of loops, how would it appear?

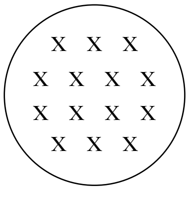
(a)



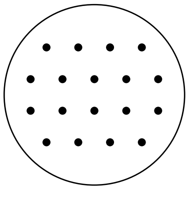
(b)



(c)



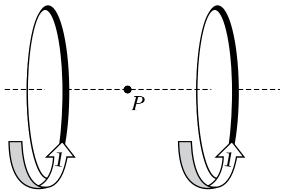
(d)

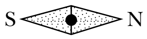


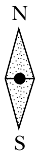
6. The figure shows two concentric metal loops, each carrying a current. The larger loop carries a current of 8.0 A and has a radius of 0.060 m. The smaller loop has a radius of 0.040 m. What is the value of a current in the smaller loop that will result in zero total magnetic field at the center of the system?

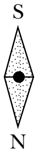
(a) 5.3 A
(b) 6.0 A
(c) 8.8 A
(d) 12 A

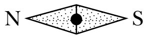
7. Two loops carry equal currents I in the same direction. They are held in the positions shown in the figure and project above and below the plane of the paper. The point P lies exactly halfway between them on the line that joins their centers. The centers of the loops and the point P lie in the plane of the paper. Which one of the figures below shows the position of a compass needle if the compass were placed in the plane of the paper at P?



(a)


(b)


(c)


(d)


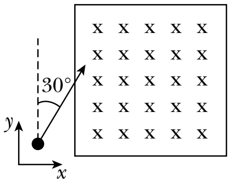
8. An electron is moving with a speed of 3.5×10^5 m/s when it encounters a magnetic field of 0.60 T. The direction of the magnetic field makes an angle of 60.0° with respect to the velocity of the electron. What is the magnitude of the magnetic force on the electron?

(a) 4.9×10^{-13} N
(b) 2.9×10^{-14} N
(c) 3.2×10^{-13} N
(d) 3.4×10^{-14} N

9. An electron travels through a region of space with no acceleration. Which one of the following statements is the best conclusion?

(a) Both E and B must be zero in that region
(b) E must be zero, but B might be non-zero in that region
(c) E and B might both be non-zero, but they must be mutually perpendicular
(d) B must be zero, but E might be non-zero in that region

10. An electron enters a region that contains a magnetic field directed into the page as shown. The velocity vector of the electron makes an angle of 30° with the $+y$ axis. What is the direction of the magnetic force on the electron when it enters the field?

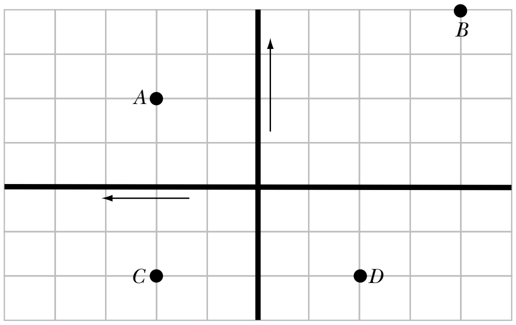


(a) Up, out of the page
(b) At an angle of 30° below the positive x axis
(c) At an angle of 30° above the positive x axis
(d) At an angle of 60° below the positive x axis

11. A wire carries current toward the north. In the region beneath this wire, what is the direction of the magnetic field due to the current?

(a) Northward
(b) Southward
(c) Eastward
(d) Westward

12. Two infinitely long perpendicular wires carry equal currents in the directions indicated in the following figure. At which of the labeled points is the magnitude of the magnetic field created by the currents the greatest? Note: All gridlines are equally spaced.

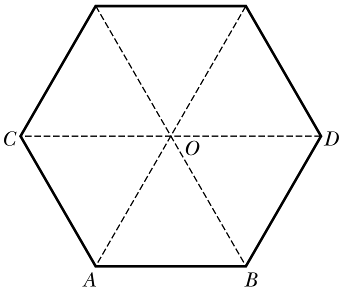


(a) A
(b) B
(c) C
(d) D

13. A circular loop is kept in a vertical plane with its axis in the East–West direction. It carries a current that is toward north at the top most point. Let A be the point of the axis of the circle to the east of it and B a point on the axis to the west of it. The magnetic field due to the loop

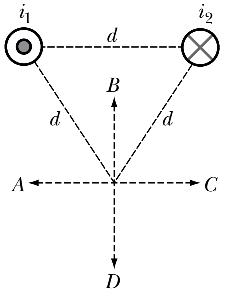
(a) Is toward east at A and toward west at B
(b) Is toward east at both A and B
(c) Is toward west at A and toward east at B
(d) Is toward west at both A and B

14. The given figure shows a regular hexagon having uniform resistance per unit length. If a battery is connected across the ends A and B then the magnetic field at the center is B_0. If the same battery is connected across A and D then magnetic field at the center would be



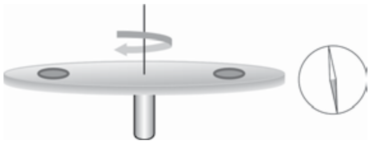
(a) Equal to B_0
(b) Greater than B_0
(c) Less than B_0
(d) Less than B_0

15. The given figure shows two long wires carrying equal currents i_1 and i_2 flowing in opposite direction. Which of the arrows labeled A–D correctly represents the direction of the magnetic field due to the wires at a point located at an equal distance d from each wire?



(a) A **(b)** B
(c) C **(d)** D

16. Rowland (1876) placed some charges (in a fixed location) on a nonconducting disk, which he then rotated at high speed about its central axis near a delicate compass (as shown in the following figure). Rowland observed that:



(a) There was no magnetic field, so nothing happened to the compass
(b) The current created a magnetic field that deflected the compass
(c) The charges repelled the compass by Coulomb's Law.
(d) None of the above

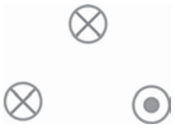
17. Consider a long, straight wire of cross-sectional area A carrying a current i. Let there be n free electrons per unit volume. An observer is opposite to the current with a speed $v = i/nAe$ and separated from the wire by a distance r. The magnetic field seen by the observer is very nearly

(a) $\frac{\mu_0 i}{2\pi r}$ **(b)** Zero
(c) $\frac{\mu_0 i}{\pi r}$ **(d)** $\frac{2\mu_0 i}{\pi r}$

18. Four very long straight wires carry equal electric currents in the $+z$ direction. They intersect the xy plane at $(x, y) = (-a, 0), (0, a), (a, 0)$, and $(0, -a)$. The magnetic force exerted on the wire at position $(-a, 0)$ is along

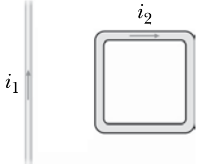
(a) $+y$ **(b)** $-y$
(c) $+x$ **(d)** $-x$

19. Cross section of three wires is shown in the following figure. If the current in the bottom two wires is opposite in direction to each other as indicated by ⊗ for into the page and ⊙ for out of the page, and the top wire has current flowing into the page ⊗, what is the direction of net force on the top wire?



(a) Left **(b)** Right
(c) Up **(d)** Down

20. Consider the situation shown in the following figure. The straight wire is fixed but the loop can move under magnetic force. The loop will



(a) Remain stationary
(b) Move toward the wire
(c) Move away from the wire
(d) Rotate about the wire

21. Two long straight thin wires carrying steady currents pass near each other at right angles to each other. As time passes

(a) They will move away from each other, parallel to their original positions
(b) They will move toward each other parallel to their original position
(c) They will rotate about the line of the shortest distance between them and tend to be parallel to each other
(d) They will rotate about the line of the shortest distance between them and tend to be anti-parallel to each other

22. In a coaxial, straight cable, the central conductor and the outer conductor carry equal currents in opposite directions. The magnetic field is zero

(a) Outside the cable
(b) Inside the inner conductor
(c) Inside the outer conductor
(d) In between the two conductors

23. A solenoid exerts a force of attraction on an iron plate when a current flow through it. This force can be increased by

(a) Putting an iron core in the solenoid
(b) Cooling the solenoid to reduce the resistance of the coil
(c) Increasing the current flow through the coil
(d) All of the above

24. A proton (mass $= 1.67 \times 10^{-27}$ kg and charge $= 1.67 \times 10^{-19}$ C) enters perpendicular to a magnetic field of intensity 2 Wb/m^2 with a velocity $= 3.4 \times 10^7$ m/s. The acceleration of the proton should be

(a) 6.5×10^{15} m/s^2 **(b)** 6.5×10^{13} m/s^2
(c) 6.5×10^{11} m/s^2 **(d)** 6.5×10^{9} m/s^2

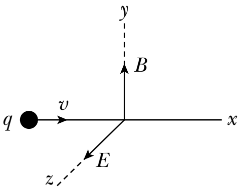
25. If only 2% of the main current is to be passed through a galvanometer of resistance G, then the resistance of the shunt will be

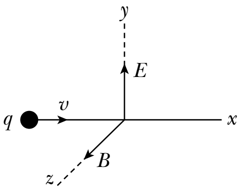
(a) $\frac{G}{50}$ **(b)** $\frac{G}{49}$
(c) $50G$ **(d)** $49G$

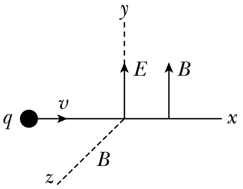
26. A deuteron of kinetic energy 50 keV is describing a circular orbit of radius 0.5 m in a plane perpendicular to magnetic field $\vec{B}$. The kinetic energy of the protons that describes a circular orbits of radius 0.5 min the same plane with the same $\vec{B}$ is

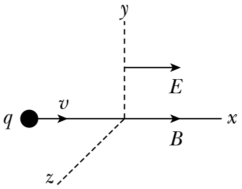
(a) 25 keV **(b)** 50 keV
(c) 200 keV **(d)** 100 keV

27. A particle of charge q and mass m is moving along the x-axis with a velocity v and enters a region of electric field E and magnetic field B as shown in the below figure. For which among the four graphs shown in the below figure, the net force on the charge may be zero?

(a) 

(b) 

(c) 

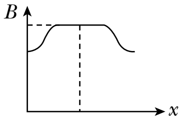
(d) 

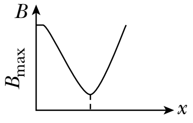
28. The ratio of the magnetic field at the center of a current-carrying circular wire and the magnetic field at the center of a square coil made from the same length of wire will be

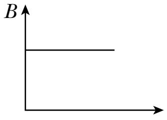
(a) $\frac{\pi^2}{4\sqrt{2}}$ **(b)** $\frac{\pi^2}{8\sqrt{2}}$

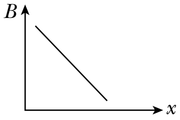
(c) $\frac{\pi}{2\sqrt{2}}$ **(d)** $\frac{\pi}{4\sqrt{2}}$

29. The correct curve between the magnetic induction B along the axis of a long solenoid due to current flow I in it and distance x from one end is shown by which among the four graphs shown in the below figure?

(a) 

(b) 

(c) 

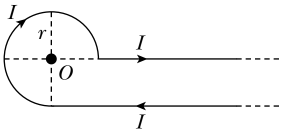
(d) 

30. Currents of 10 A and 2 A are passed through two parallel wires A and B, respectively, in opposite directions. The wire A is infinitely long, and the length of the wire B is 2 m. The force acting on the conductor B, which is situated at 10 cm distance from A will be

(a) 8×10^{-5} N **(b)** 5×10^{-5} N

(c) $8\pi \times 10^{-5}$ N **(d)** $4\pi \times 10^{-5}$ N

31. Current I is flowing in a conductor shaped as shown in the below figure. The radius of the curved part is r and the length of straight portion is very large. The value of the magnetic field at the center O will be



(a) $\frac{\mu_0}{4\pi r}\left(\frac{\pi}{2}+1\right)$ **(b)** $\frac{\mu_0 I}{4\pi r}\left(\frac{\pi}{2}-1\right)$

(c) $\frac{\mu_0 I}{4\pi r}\left(\frac{3\pi}{2}+1\right)$ **(d)** $\frac{\mu_0 I}{4\pi r}\left(\frac{3\pi}{2}-1\right)$

32. A proton of mass m and charge $+e$ is moving in a circular orbit in a magnetic field with energy 1 MeV. What should be the energy of α-particle (mass = 4 and charge = $+2e$), so that it can revolve in the path of same radius

(a) 1 MeV **(b)** 4 MeV

(c) 2 MeV **(d)** 0.5 MeV

33. A proton and an α-particle enter a uniform magnetic field perpendicularly with the same speed. If proton takes 25 μs to make five revolutions, then the periodic time for the α-particle would be

(a) 50 μs **(b)** 25 μs

(c) 10 μs **(d)** 5 μs

34. A strong magnetic field is applied on a stationary electron, then

(a) The electron moves in the direction of the field

(b) The electron moves in an opposite direction

(c) The electron remains stationary

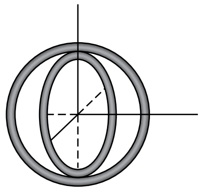
(d) The electron starts spinning

35. Two concentric coils of 10 turns each are situated in the same plane. Their radii are 20 and 40 cm and they carry, respectively, 0.2 and 0.3 A current in opposite direction. The magnetic field in Wb/m² at the center is

(a) $\frac{35}{4}\mu_0$ **(b)** $\frac{5}{4}\mu_0$

(c) $\frac{7}{80}\mu_0$ **(d)** $\frac{\mu_0}{80}$

36. Two insulated rings, one of slightly smaller diameter than the other are suspended along their common diameter as shown in the below figure. Initially, the planes of the rings are mutually perpendicular. When as steady current is set up in each of them:



(a) The two rings rotate into a common plane.

(b) The inner ring oscillates about its initial position.

(c) The inner ring stays stationary while the outer one moves into the plane of the inner ring.

(d) The outer ring stays stationary while the inner one moves into the plane of the outer ring.

More than One Correct Choice Type

37. A proton moving with a constant velocity passes through a region of space without any change in its velocity. If $\vec{E}$ and $\vec{B}$ represent the electric and magnetic fields, respectively, then this region of space may have

(a) $E = 0, B = 0$ **(b)** $E = 0, B \neq 0$

(c) $E \neq 0, B = 0$ **(d)** $E \neq 0, B \neq 0$

38. The radius of curvature of the path of a charged particle moving in a static uniform magnetic field is

(a) Directly proportional to the magnitude of the charge on the particle

(b) Directly proportional to the magnitude of the linear momentum of the particle

(c) Directly proportional to the kinetic energy of the particle

(d) Inversely proportional to the magnitude of the magnetic field

39. The current sensitivity of a moving coil galvanometer can be increased by

(a) Increasing the magnetic field of the permanent magnet

(b) Increasing the area of the deflecting coil

(c) Increasing the number of turns in the coil

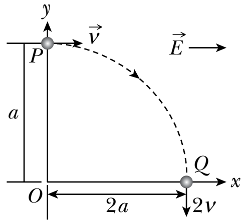
(d) Increasing the restoring couple of the coil

40. Two coaxial solenoids 1 and 2 of the same length are set so that one is inside the other. The number of turns per unit length are n_1 and n_2. The currents I_1 and I_2 are flowing in opposite directions. The magnetic field inside the inner coil is zero. This is possible when

(a) $I_1 \neq I_2$ and $n_1 = n_2$ **(b)** $I_1 = I_2$ and $n_1 \neq n_2$

(c) $I_1 = I_2$ and $n_1 = n_2$ **(d)** $I_1 n_1 = I_2 n_2$

41. A particle of charge $+q$ and mass m moving under the influence of a uniform electric field $E\hat{\imath}$ and a uniform magnetic field $B\hat{k}$ follows trajectory from P to Q as shown in the below figure. The velocities at P and Q are $v\hat{\imath}$ and $-2v\hat{\jmath}$, respectively. Which of the following statement(s) is/are correct?



(a) $E = \dfrac{3}{4}\dfrac{mv^2}{qa}$

(b) Rate of work done by electric field at P is $\dfrac{3}{4}\dfrac{mv^3}{a}$

(c) Rate of work done by electric field at P is zero

(d) Rate of work done by both the fields at Q is zero

42. H^+, He^+, and O^{++} ions having same kinetic energy pass through a region of space filled with uniform magnetic field B directed perpendicular to the velocity of ions. The masses of the ions H^+, He^+, and O^{++} are, respectively, in the ratio 1:4:16. As a result

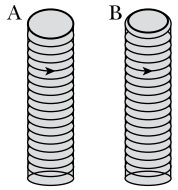
(a) H^+ ions will be deflected most

(b) O^{++} ions will be deflected least

(c) He^+ and O^{++} ions will suffer same deflection

(d) All ions will suffer the same deflection

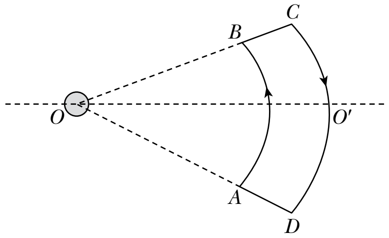
43. Two metallic rings A and B, identical in shape and size but having different resistivity ρ_A and ρ_B are kept on top of two identical solenoids as shown in the below figure. When current I is switched on in both the solenoids in identical manner, the rings A and B jump to heights h_A and h_B, respectively, with $h_A > h_B$. The possible relation(s) between their resistivities and their masses m_A and m_B is (are)



(a) $\rho_A > \rho_B$ and $m_A = m_B$ **(b)** $\rho_A < \rho_B$ and $m_A = m_B$

(c) $\rho_A > \rho_B$ and $m_A > m_B$ **(d)** $\rho_A < \rho_B$ and $m_A < m_B$

44. An infinite current-carrying wire passes through point O and in perpendicular to the plane containing a current-carrying loop $ABCD$ as shown in the below figure. Choose the correct option(s):



(a) Net force on the loop is zero

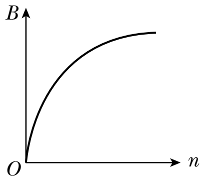
(b) Net torque on the loop is zero

(c) As seen from O, the loop rotates clockwise

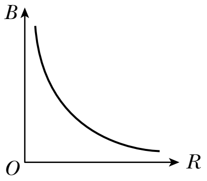
(d) As seen from O, the loop rotates anticlockwise

45. A thin wire of length l is carrying a constant current. The wire is bent to from a circular coil. If radius of the coil, thus formed, is equal to R and number of turns in it is equal to n, then which of the following graphs from the below figure represents variation of magnetic field induction B at center of the coil

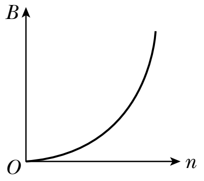
(a)



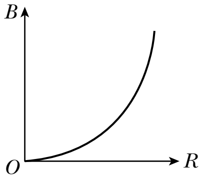
(b)



(c)



(d)



46. A long, straight wire carries a current along the z axis. One can find two points in the xy plane such that

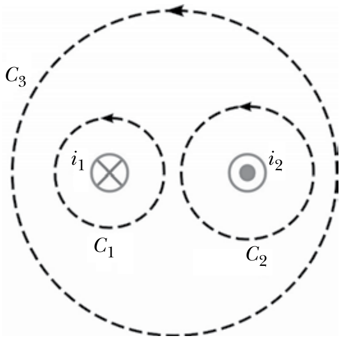
(a) The magnetic field are equal

(b) The directions of the magnetic field are the same

(c) The magnitudes of the magnetic fields are equal

(d) The field at one point is opposite to that at the other point

47. Which one of the following line integrals is correct? Note: The direction of the loops orientation is shown in the following figure.



(a) $\oint_{C_1} \vec{B}\cdot d\vec{l} = \mu_0 i_1$

(b) $\oint_{C_2} \vec{B}\cdot d\vec{l} = \mu_0 i_2$

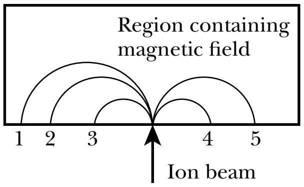
(c) $\oint_{C_1} \vec{B}\cdot d\vec{l} = -\mu_0 i_2$

(d) $\oint_{C_3} \vec{B}\cdot d\vec{l} = -\mu_0 (i_1 - i_2)$

48. A steady electric current is flowing through a cylindrical conductor.
(a) The electric field at the axis of the conductor is zero
(b) The magnetic field at the axis of the conductor is zero
(c) The electric field in the vicinity of the conductor is zero
(d) The magnetic field in the vicinity of the conductor is zero

Linked Comprehension

Paragraph for Questions 49–51: A beam consisting of five types of ions labeled **A**, **B**, **C**, **D**, and **E** enters a region that contains a uniform magnetic field as shown in the figure below. The field is perpendicular to the plane of the paper, but its precise direction is not given. All ions in the beam travel with the same speed. The table below gives the masses and charges of the ions. Note: 1 mass unit = 1.67×10^{-27} kg and $e = 1.6 \times 10^{-19}$ C



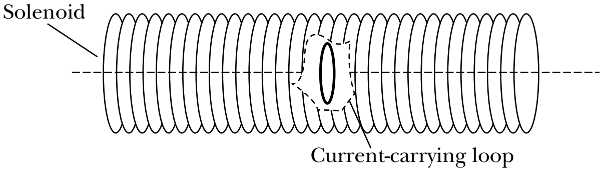
Ion	Mass	Charge
A	2 units	$+e$
B	4 units	$+e$
C	6 units	$+e$
D	2 units	$-e$
E	4 units	$-e$

49. Which ion falls at position **2**?
(a) **A** **(b)** **C**
(c) **B** **(d)** **D**

50. What is the direction of the magnetic field?
(a) Toward the right **(b)** Into the page
(c) Toward the bottom **(d)** Out of the page

51. Determine the magnitude of the magnetic field if ion **A** travels in a semicircular path of radius 0.50 m at a speed of 5.0×10^6 m/s.
(a) 1.0 T **(b)** 0.84 T
(c) 0.42 T **(d)** 0.21 T

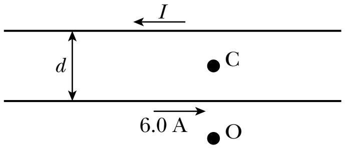
Paragraph for Questions 52 and 53: A single circular loop of wire with radius 0.020 m carries a current of 8.0 A. It is placed at the center of a solenoid that has length 0.65 m, radius 0.080 m, and 1400 turns.



52. Determine the value of the current in the solenoid so that the magnetic field at the center of the loop is zero tesla.
(a) 1.4×10^{-1} A **(b)** 4.4×10^{-2} A
(c) 9.3×10^{-2} A **(d)** 2.5×10^{-4} A

53. Determine the magnitude of the total magnetic field at the center of the loop (due both to the loop and the solenoid) if the current in the loop is reversed in direction from that needed to make the total field equal to zero tesla.
(a) 2.5×10^{-4} T **(b)** 6.4×10^{-4} T
(c) 5.0×10^{-4} T **(d)** 8.7×10^{-4} T

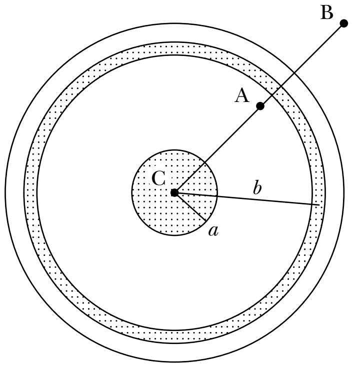
Paragraph for Questions 54 and 55: Two long, straight, parallel wires separated by a distance d carry currents in opposite directions as shown in the figure. The bottom wire carries a current of 6.0 A. Point **C** is at the midpoint between the wires and point **O** is a distance 0.50d below the 6-A wire as suggested in the figure. The total magnetic field at point **O** is zero tesla.



54. Determine the value of the current, I, in the top wire.
(a) 2 A **(b)** 3 A
(c) 6 A **(d)** 18 A

55. Determine the magnitude of the magnetic field at point **C** if $d = 0.10$ m.
(a) 2.4×10^{-5} T **(b)** 9.6×10^{-5} T
(c) 4.8×10^{-5} T **(d)** 1.1×10^{-4} T

Paragraph for Questions 56 and 57: A long, coaxial cable, shown in cross-section in the drawing, is made using two conductors that share a common central axis, labeled **C**. The conductors are separated by an electrically insulating material that is also used as the outer cover of the cable. The current in the *inner* conductor is 2.0 A directed *into the page* and that in the *outer* conductor is 2.5 A directed *out of the page*. The distance from point **C** to point **A** is 0.0015 m; and the distance from **C** to **B** is 0.0030 m. The radii a and b of the conductors are 6.0×10^{-4} m and 1.9×10^{-3} m, respectively.



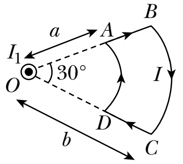
56. What is the magnitude and direction of the magnetic field at point **A**?

(a) 3.3×10^{-5} T, clockwise
(b) 6.8×10^{-5} T, counterclockwise
(c) 3.3×10^{-5} T, counterclockwise
(d) 2.7×10^{-4} T, clockwise

57. What is the magnitude and direction of the magnetic field at point **B**?

(a) 3.3×10^{-5} T, clockwise
(b) 6.8×10^{-5} T, counterclockwise
(c) 3.3×10^{-5} T, counterclockwise
(d) 2.7×10^{-4} T, clockwise

Paragraph for Questions 58 and 59: A current loop $ABCD$ is held fixed on the plane of the paper as shown in the below figure. The arcs BC (radius $= b$) and DA (radius $= a$) of the loop are joined by two straight wires AB and CD as shown in the below figure. A steady current I is flowing in the loop. Angle made by AB and CD at the origin O is 30°. Another straight thin wire with steady current I_1 flowing out of the plane of the paper is kept at the origin.



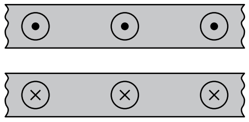
58. The magnitude of the magnetic field B due to the loop $ABCD$ at origin (O) is

(a) Zero
(b) $\dfrac{\mu_0 I(b-a)}{2\pi ab}$.
(c) $\dfrac{\mu_0 I(b-a)}{4\pi ab}$.
(d) $\dfrac{\mu_0 I}{4\pi}\left[2(b-a)+\dfrac{\pi}{3}(a+b)\right]$.

59. Due to the presence of the current I_1 at the origin

(a) The forces on AB and DC are zero.
(b) The forces on AD and BC are zero.
(c) The magnitude of the net force on the loop is given by $\dfrac{I_1 I}{4\pi}\mu_0\left[2(b-a)+\dfrac{\pi}{3}(a+b)\right]$.
(d) The magnitude of the net force on the loop is given by $\dfrac{\mu_0 I_1 I}{24ab}(b-a)$.

Paragraph for Questions 60 and 61: System shown in the below figure consists of two large parallel metallic plates carrying current in opposite directions. Current density in each plate is j per unit width.



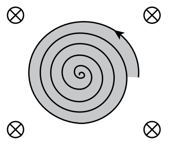
60. Magnetic field in space between the plates is

(a) $\dfrac{1}{2}\mu_0 j^2$ **(b)** Zero
(c) $\mu_0 j$ **(d)** $2\,\mu_0 j$

61. Force acting per unit area of each plate

(a) $\dfrac{1}{2}\mu_0 j^2$ **(b)** Zero
(c) $m_0 j$ **(d)** $2\,m_0$

Paragraph for Questions 62–64: The below figure shows the trace of the path of a charged particle in a bubble chamber. Assume that the magnetic field is into the plane of the paper, with magnitude 0.4 T. The smooth spiral path occurs because the particle loses energy in ionizing molecules along the path.



B is into the page

62. Choose the correct statement:

(a) Kinetic energy of the particle is maximum at outer part of the spiral
(b) Kinetic energy of the particle is maximum at inner part of the spiral
(c) Kinetic energy of the particle first decreases then increases during motion
(d) Kinetic energy of the particle remains constant during motion along spiral path

63. Regarding the nature of the charge, we can conclude that:

(a) The charge is negative
(b) The charge is positive
(c) The particle has no charge
(d) No conclusion can be made regarding nature of charge

64. The radius of curvature ranges from 70 to 10 mm. What is the range of values of the magnitude of momentum (p) if the magnitude of the charge is e?

(a) $8e \times 10^{-2}$ kg m/s $\le p \le 28e \times 10^{-3}$ kg m/s
(b) $4e \times 10^{-2}$ kg m/s $\le p \le 28e \times 10^{-3}$ kg m/s
(c) $10e \times 10^{-2}$ kg m/s $\le p \le 32e \times 10^{-3}$ kg m/s
(d) $5e \times 10^{-2}$ kg m/s $\le p \le 20e \times 10^{-3}$ kg m/s

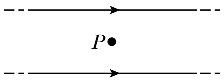
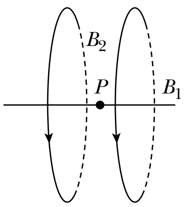
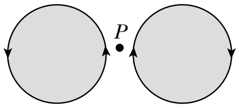
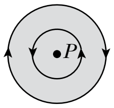
Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

65.

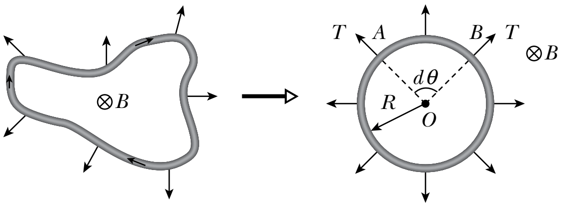
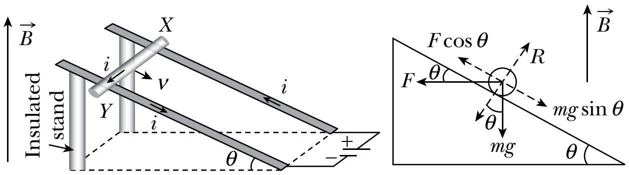
Column I	Column II
(a) Stationary dielectric ring having uniform charge.	**(p)** Electric field.
(b) Dielectric ring having uniform charge is rotating with constant angular speed.	**(q)** Magnetostatic field.
(c) A constant current I_0 in the loop.	**(r)** Time dependent induced electric field outside the loop.
(d) Time varying sinusoidal current in the loop $I = I_0 \sin \omega t$.	**(s)** Magnetic moment in the loop.

66.

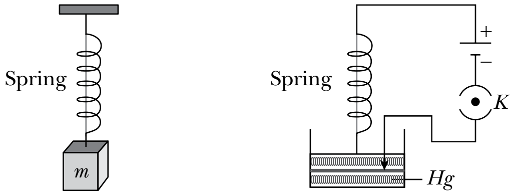
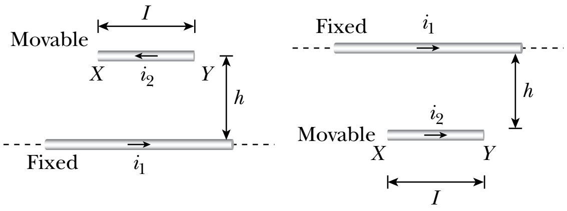
Column I		Column II
(a) Point P is situated midway between the wires.		**(p)** The magnetic field (B) at P due to the currents in the wires are in the same direction.
(b) Point P is situated at the mid-point of the line joining the centers of the circular wires, which have same radii.		**(q)** The magnetic field (B) at P due to the currents in the wires is in opposite directions.
(c) Point P is situated at the mid-point of the line joining the centers of the circular wires, which have same radii.		**(r)** There is no magnetic field at P.
(d) Point P is situated at the common center of the wires.		**(s)** The wires repel each other.

Directions for Questions 67 and 68: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

67. Consider standard cases for force on current carrying conductors. In the given table, Column I shows the action of current on the element, Column II shows the effect of the current in the element and Column III shows the figure of the element under force of current and magnetic field and its equivalent figure in general mechanical form.

Column I	Column II	Column III
(I) Each element of loop experiences a magnetic force	**(i)** turns in the same direction.	**(J)** 
(II) When a conducting rod	**(ii)** loop stretches and open into circular loop and tension developed in its each part	**(K)**  $\vec{B}$ X i v Y i i Insulated stand θ + − F cos θ R F θ θ mg sin θ mg θ $\vec{B}$

(Contd.)

Column I	Column II	Column III
(III) Wire is suspended freely in air	**(iii)** Equilibrium of a current carrying conductor	**(L)** 
(IV) Current will flow through	**(iv)** slides on conducting rails	**(M)**  I Movable X i_2 Y h Fixed i_1 Fixed i_1 h i_2 Movable X Y I

(1) What happens when a finite length current carrying wire is kept parallel to another infinite length current carrying wire?

(a) (I) (ii) (J)
(b) (IV) (ii) (M)
(c) (II) (i) (M)
(d) (III) (iii) (M)

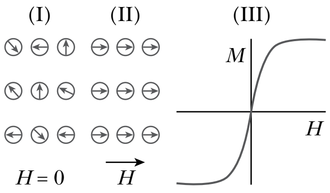
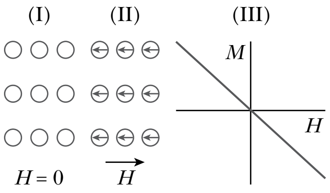
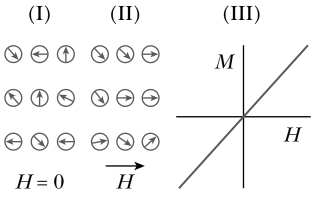
(2) What happens when an arbitrary current carrying loop is placed in a magnetic field (perpendicular to the plane of loop)?

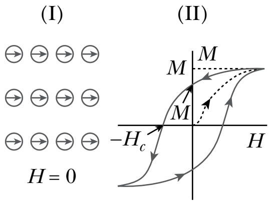
(a) (I) (ii) (J)
(b) (IV) (iii) (L)
(c) (II) (iii) (L)
(d) (I) (i) (M)

(3) What happens when current is passed through a spring?

(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (IV) (i) (L) **(d)** (II) (iii) (M)

68. We know that magnetic substances follow Curie-Weiss Law. In the given table, Column I shows the type of attraction with magnets of magnetic substances, Column II shows the examples of magnetic substances and Column III shows the three figures – figure (I) shows direction of magnetic momentum of each electron when there is no magnetic field inside magnetic substance; figure (II) shows the direction of magnetic momentum of electron when there is magnetic field inside magnetic substance and figure (III) shows the curve between M and H inside magnetic substance.

Column I	Column II	Column III
(I) Substances that are weakly attracted by the magnets	**(i)** antimony, bismuth	**(J)** 
(II) Substances that are repelled by the magnets	**(ii)** oxygen, manganese,	**(K)** 
(III) Substances that get magnetized in the direction opposite to that of magnetic field	**(iii)** chromium, nickel	**(L)** 

Column I	Column II	Column III
(IV) substances which are strongly attracted by the magnets	**(iv)** iron, cobalt	**(M)** 

(1) Which combination is characteristic of ferromagnetic materials?

(a) (I) (iii) (L) **(b)** (IV) (iv) (M)
(c) (II) (i) (M) **(d)** (I) (iii) (J)

(2) Which combination is characteristic of paramagnetic materials?

(a) (I) (ii) (L) **(b)** (IV) (iii) (L)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) Which combination is characteristic of diamagnetic materials?

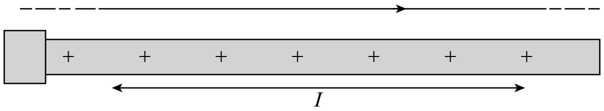
(a) (III) (i) (L) **(b)** (I) (i) (J)
(c) (III) (iii) (J) **(d)** (II) (i) (K)

Integer Type

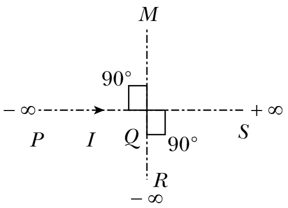
69. A steady current I goes through a wire loop PQR having shape of a right-angled triangle with PQ = 3, PR = $4x$, and QR = $5x$. If the magnitude of the magnetic field at P due to this loop is $k[(\mu_0 I)/(48\pi x)]$, find the value of k.

70. A circular ring of radius a (of 2 m length) and resistance R = 10 Ω is placed in xy plane with its center as origin. A magnetic field $\vec{B} = 2x\hat{\mathrm{i}} + 3y\hat{\mathrm{j}} + 2i\hat{\mathrm{k}}$ is switched on at $t = 0$. If the force (in newton) acting on the ring is found as $y \times 10$ N, find y.

71. A non-conducting rod of length $l = 1$ m and linear charge density $\lambda = 2 \times 10^{-6}$ C/m is hinged at one end. It is kept in horizontal position in vertical plane in front of a current-carrying wire as shown in the below figure. The rod is released from horizontal position. The magnetic force on the rod when it makes an angle 60° with downward vertical is given by $n \times 4 \times \sqrt{15} \times 10^{12}$. Find the value of n.



72. An infinitely long conductor PQR is bent to form a right angle as shown. A current I flows through PQR. The magnetic field due to this current at the point M is H_1. Now, another infinitely long straight conductor QS is connected to Q so that the current is I/2 in QR as well as in QS, the current in PQ remaining unchanged. The magnetic field at M is now H_2. The ratio of H_1/H_2 is given by



ANSWER KEY

Checkpoints

1. d, a, tie of b and c (zero) **2.** b, c, a **3.** d, tie of a and c, then b
4. (a) zero in between and $\mu_0 K$ outside; (b) $\mu_0 K$ in between and zero outside

Problems

1. (a) 17.0 nT; (b) out of figure; (c) increasing R decreases B **2.** (a) 94.9 nT; (b) increase
3. $\vec{B}_{\text{net}} = (2.4 \times 10^{-5}\ \text{T})\hat{\mathrm{j}} = (24\mu\text{T})\hat{\mathrm{j}}$ **4.** (a) $(2.53 \times 10^{-7}\,\text{T})\hat{\mathrm{k}}$; (b) $(1.92 \times 10^{-7}\,\text{T})\hat{\mathrm{i}}$; (c) $(6.12 \times 10^{-8}\,\text{T})\hat{\mathrm{k}}$
5. $(41.7\,\mu\text{N/m})\hat{\mathrm{i}} + (41.7\,\mu\text{N/m})\hat{\mathrm{j}}$ **6.** $(7.94 \times 10^{-4}\,\text{N/m})\hat{\mathrm{i}} + (-7.94 \times 10^{-4}\,\text{N/m})\hat{\mathrm{j}}$
7. (a) 55 μT; (b) into **8.** 1.00 μN/m **9.** (a) 30 cm; (b) 2.0 nT; (c) out of page; (d) into the page
10. (a) -2.5×10^{-6} T·m; (b) -1.6×10^{-5} T·m **11.** 41.5 pN/m **12.** ≈1.62×10^{6}
13. (a) 1.94×10^{-4} T; (b) 1.53×10^{-4} T **14.** $(23.5\ \text{pT})\hat{\mathrm{j}}$ **15.** (a) 0; (b) 0.510 mT; (c) 1.70 mT; (d) 0.576 mT
16. 2.83×10^{-8} T·m **17.** (a) -6×10^{-6} T·m; (b) 0 **18.** (a) 2.27 cm; (b) 6.22×10^{-5} T

Chapter 29 | Magnetic Fields Due to Currents

19. (a) 1.67×10^{-6} T; (b) into the page
20. 0.69 mT
21. 1.1 $\text{A} \cdot \text{m}^2$
22. 175 nT
23. 89.9 mA
24. (a) 0; (b) 0.20 μT; (c) 0.79 μT
25. (a) 38 μT; (b) 0.52 $\mu\text{N} \cdot \text{m}$
26. (a) 3.3 μT; (b) yes
27. (a) 2.1 $\text{A} \cdot \text{m}^2$; (b) 44 cm
28. (a) 3.0 A; (b) out; (c) 32 μT; (d) net field points to the left
29. (a) 0.13 mT; (b) out; (c) 0.10 mT; (d) out
30. (a) 3.8 μT; (b) into; (c) 8.8 μT; (d) into
31. 74.0 mA
32. $(-1.98 \times 10^{-22}\,\text{N})\hat{\text{i}}$
33. 0
34. (a) 7.2 μT; (b) 5.0 μT
35. 144°
36. (a) 2.15×10^{-7} T; (b) out
37. (a) opposite; (b) 90 A
38. (a) −7.0 cm; (b) +7.0 cm
39. $\dfrac{64\pi}{125} \times 10^{-5}\,\text{T}$
40. (a) 0; (b) 3.84×10^{-14} T
41. $\dfrac{\mu_0 e^3 B}{4\pi m v^2}$
42. (a) the smaller coil; (b)1/2
43. $\dfrac{F_m}{F_e} = \mu_0 \varepsilon_0 v^2 = \dfrac{v}{c^2} = 1.00 \times 10^{-6}$
44. $2\pi \times 10^{-3}$ T
45. (a) 15.3 μT; (b) if $b = 0$, then $B = \dfrac{\mu_0 i d}{2\pi a^2}$
Now for $d = 0$, the formula gives $B = 0$.

Practice Questions

Single Correct Choice Type

1. (c)
2. (b)
3. (b)
4. (c)
5. (c)
6. (a)
7. (d)
8. (b)
9. (c)
10. (b)
11. (d)
12. (c)
13. (d)
14. (a)
15. (b)
16. (b)
17. (a)
18. (c)
19. (c)
20. (b)
21. (c)
22. (a)
23. (d)
24. (a)
25. (b)
26. (d)
27. (b)
28. (c)
29. (a)
30. (a)
31. (c)
32. (a)
33. (c)
34. (c)
35. (b)
36. (a)

More than One Correct Choice Type

37. (a), (b), (d)
38. (b), (d)
39. (a), (b), (c)
40. (c), (d)
41. (a), (b), (d)
42. (a), (c)
43. (b), (d)
44. (a), (c)
45. (b), (c)
46. (b), (c), (d)
47. (b), (d)
48. (b), (c)

Linked Comprehension

49. (c)
50. (b)
51. (d)
52. (c)
53. (c)
54. (d)
55. (b)
56. (d)
57. (c)
58. (b)
59. (b)
60. (c)
61. (a)
62. (a)
63. (b)
64. (b)

Matrix-Match

65. (a) → (p); (b) → (p), (q), (s); (c) → (q), (s); (d) → (q), (r), (s)
66. (a) → (p), (s); (b) → (q), (r); (c) → (p), (q); (d) → (s)
67. (1) → (d); (2) → (a); (3) → (c)
68. (1) → (b); (2) → (a); (3) → (d)

Integer Type

69. 7
70. 4
71. 4
72. 0.67

CHAPTER

30

Electromagnetic Induction

Contents

30.1 What is Physics?
30.2 Gauss' Law of Magnetic Fields
30.3 Two Experiments
30.4 Faraday's Law of Induction
30.5 Lenz's Law
30.6 Motional EMF
30.7 Induction and Energy Transfers
30.8 Induced Electric Fields
30.9 Inductors and Inductance
30.10 Self-Induction
30.11 *RL* Circuits
30.12 Energy Stored in a Magnetic Field
30.13 Mutual Induction

30.1 | WHAT IS PHYSICS?

In Chapter 29 we discussed the fact that a current produces a magnetic field. That fact came as a surprise to the scientists who discovered the effect. Perhaps even more surprising was the discovery of the reverse effect: A magnetic field can produce an electric field that can drive a current. This link between a magnetic field and the electric field it produces (*induces*) is now called *Faraday's law of induction*.

The observations by Michael Faraday and other scientists that led to this law were at first just basic science. Today, however, applications of that basic science are almost everywhere. For example, induction is the basis of the electric guitars that revolutionized early rock and still drive heavy metal and punk today. It is also the basis of the electric generators that power cities and transportation lines and of the huge induction furnaces that are commonplace in foundries where large amounts of metal must be melted rapidly.

Before we get to applications like the electric guitar, we must examine two simple experiments about Faraday's law of induction. We will begin with the concept of magnetic flux and then go on to the experiments of Faraday.

30.2 | GAUSS' LAW OF MAGNETIC FIELDS

Key Concepts

- The magnetic flux Φ_B through an area A in a magnetic field $\vec{B}$ is defined as

$$\Phi_B = \int \vec{B} \cdot d\vec{A},$$

where the integral is taken over the area. The SI unit of magnetic flux is the weber, where 1 Wb = 1 T · m^2.

- If $\vec{B}$ is perpendicular to the area and uniform over it, the flux is

$$\Phi_B = BA \quad (\vec{B} \perp A, \vec{B} \text{ uniform}).$$

Before, we understood the process of electromagnetic induction, we would define an associated concept of magnetic flux. In Chapter 23, in a similar situation, we needed to calculate the amount of electric field that passes through a surface. There we defined an electric flux $\Phi_E = \int \vec{E} \cdot d\vec{A}$. Here we define a *magnetic flux:* Suppose a loop enclosing an area A is placed in a magnetic field $\vec{B}$. Then the **magnetic flux** through the loop is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \quad \text{(magnetic flux through area } A\text{).} \tag{30-1}$$

As in Chapter 23, $d\vec{A}$ is a vector of magnitude dA that is perpendicular to a differential area dA. As with electric flux, we want the component of the field that *pierces* the surface (not skims along it). The dot product of the field and the area vector automatically gives us that piercing component.

Special Case. As a special case of Eq. 30-1, suppose that the loop lies in a plane and that the magnetic field is perpendicular to the plane of the loop. Then we can write the dot product in Eq. 30-1 as $B\,dA \cos 0° = B\,dA$. If the magnetic field is also uniform, then B can be brought out in front of the integral sign. The remaining $\int dA$ then gives just the area A of the loop. Thus, Eq. 30-1 reduces to

$$\Phi_B = BA \quad (\vec{B} \perp \text{area } A, \vec{B} \text{ uniform}). \tag{30-2}$$

Unit. From Eqs. 30-1 and 30-2, we see that the SI unit for magnetic flux is the tesla–square meter, which is called the *weber* (abbreviated Wb):

$$1 \text{ weber} = 1 \text{ Wb} = 1 \text{ T} \cdot \text{m}^2. \tag{30-3}$$

Gauss' Law

In Chapter 28 we established that we cannot separate a magnet into its north and south poles. They are inseparable. We can conclude that the simplest magnetic structure that can exist is a magnetic dipole. Magnetic monopoles do not exist (as far as we know). Gauss' law for magnetic fields is a formal way of saying that magnetic monopoles do not exist. The law asserts that the net magnetic flux ϕ through any closed Gaussian surface is zero:

$$\Phi = \oint \vec{B} \cdot d\vec{A} = 0 \qquad \text{(Gauss' law for magnetic fields).}$$

Contrast this with Gauss' law for electric fields,

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \qquad \text{(Gauss' law for electric fields).}$$

In both the equations, the integral is taken over a *closed* Gaussian surface. Gauss' law for electric fields says that this integral (the net electric flux through the surface) is proportional to the net electric charge q_{enc} enclosed by the surface. Gauss' law for magnetic fields says that there can be no net magnetic flux through the surface because there can be no net "magnetic charge" (individual magnetic poles) enclosed by the surface. The simplest magnetic structure that can exist, and thus be enclosed, by a Gaussian surface is a dipole, which consists of both a source and a sink for the field lines. Thus, there must always be as much magnetic flux into the surface as out of it, and the net magnetic flux must always be zero.

Gauss' law for magnetic fields holds for structures more complicated than a magnetic dipole, and it holds even if the Gaussian surface does not enclose the entire structure. Gaussian surface II near the bar magnet of Fig. 30-1 encloses no poles, and we can easily conclude that the net magnetic flux through it is zero. Gaussian surface I is more difficult. It may seem to enclose only the north pole of the magnet because it encloses the label N and not the label S. However, a south pole must be associated with the lower boundary of the surface because magnetic field lines enter the surface there. (The enclosed section is like one piece of the broken bar magnet.) Thus, Gaussian surface I encloses a magnetic dipole, and the net flux through the surface is zero.

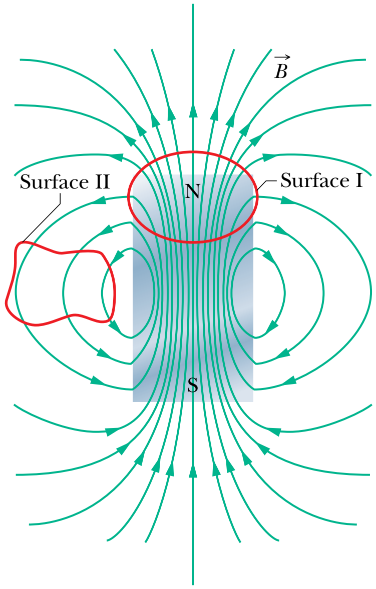


Figure 30-1 The field lines for the magnetic field of a short bar magnet. The black curves represent cross sections of closed, three-dimensional Gaussian surfaces.

30.3 | TWO EXPERIMENTS

Let us examine two simple experiments to prepare for our discussion of Faraday's law of induction.

First Experiment. Figure 30-2 shows a conducting loop connected to a sensitive ammeter. Because there is no battery or other source of emf included, there is no current in the circuit. However, if we move a bar magnet toward the loop, a current suddenly appears in the circuit. The current disappears when the magnet stops. If we then move the magnet away, a current again suddenly appears, but now in the opposite direction. If we experimented for a while, we would discover the following:

1. A current appears only if there is relative motion between the loop and the magnet (one must move relative to the other); the current disappears when the relative motion between them ceases.
2. Faster motion produces a greater current.
3. If moving the magnet's north pole toward the loop causes, say, clockwise current, then moving the north pole away causes counterclockwise current. Moving the south pole toward or away from the loop also causes currents, but in the reversed directions.

The current produced in the loop is called an **induced current**; the work done per unit charge to produce that current (to move the conduction electrons that constitute the current) is called an **induced emf**; and the process of producing the current and emf is called **induction**.

Second Experiment. For this experiment we use the apparatus of Fig. 30-3, with the two conducting loops close to each other but not touching. If we close switch S, to turn on a current in the right-hand loop, the meter suddenly and briefly registers a current—an induced current—in the left-hand loop. If we then open the switch, another sudden and brief induced current appears in the left-hand loop, but in the opposite direction. We get an induced current (and thus an induced emf) only when the current in the right-hand loop is changing (either turning on or turning off) and not when it is constant (even if it is large).

The induced emf and induced current in these experiments are apparently caused when something changes—but what is that "something"? Faraday knew.

The magnet's motion creates a current in the loop.

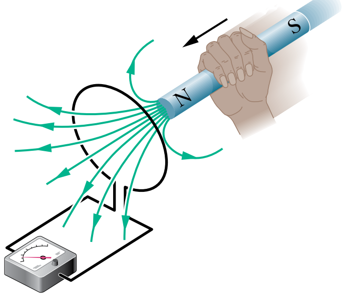
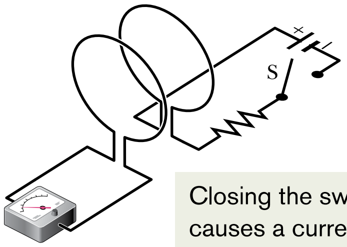


Figure 30-2 An ammeter registers a current in the wire loop when the magnet is moving with respect to the loop.



Closing the switch causes a current in the left-hand loop.

Figure 30-3 An ammeter registers a current in the left-hand wire loop just as switch S is closed (to turn on the current in the right-hand wire loop) or opened (to turn off the current in the right-hand loop). No motion of the coils is involved.

30.4 | FARADAY'S LAW OF INDUCTION

Key Concepts

- If the magnetic flux Φ_B through an area bounded by a closed conducting loop changes with time, a current and an emf are produced in the loop; this process is called induction. The induced emf is

$$\mathscr{E} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law).}$$

- If the loop is replaced by a closely packed coil of N turns, the induced emf is

$$\mathscr{E} = -N\frac{d\Phi_B}{dt}.$$

Faraday realized that an emf and a current can be induced in a loop, as in our two experiments, by changing the *amount of magnetic field* passing through the loop. He further realized that the "amount of magnetic field" can be visualized in terms of the magnetic field lines passing through the loop. **Faraday's law of induction**, stated in terms of our experiments, is this:

> An emf is induced in the loop at the left in Figs. 30-2 and 30-3 when the number of magnetic field lines that pass through the loop is changing.

The actual number of field lines passing through the loop does not matter; the values of the induced emf and induced current are determined by the *rate* at which that number changes.

In our first experiment (Fig. 30-2), the magnetic field lines spread out from the north pole of the magnet. Thus, as we move the north pole closer to the loop, the number of field lines passing through the loop increases. That increase apparently causes conduction electrons in the loop to move (the induced current) and provides energy (the induced emf) for their motion. When the magnet stops moving, the number of field lines through the loop no longer changes and the induced current and induced emf disappear.

In our second experiment (Fig. 30-3), when the switch is open (no current), there are no field lines. However, when we turn on the current in the right-hand loop, the increasing current builds up a magnetic field around that loop and at the left-hand loop. While the field builds, the number of magnetic field lines through the left-hand loop increases. As in the first experiment, the increase in field lines through that loop apparently induces a current and an emf there. When the current in the right-hand loop reaches a final, steady value, the number of field lines through the left-hand loop no longer changes, and the induced current and induced emf disappear.

A Quantitative Treatment

Faraday's Law. With the notion of magnetic flux, we can state Faraday's law in a more quantitative and useful way:

> The magnitude of the emf $\mathscr{E}$ induced in a conducting loop is equal to the rate at which the magnetic flux Φ_B through that loop changes with time.

As you will see below, the induced emf $\mathscr{E}$ tends to oppose the flux change, so Faraday's law is formally written as

$$\mathscr{E} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law),} \qquad (30\text{-}4)$$

with the minus sign indicating that opposition. We often neglect the minus sign in Eq. 30-4, seeking only the magnitude of the induced emf.

If we change the magnetic flux through a coil of N turns, an induced emf appears in every turn and the total emf induced in the coil is the sum of these individual induced emfs. If the coil is tightly wound (*closely packed*), so that the same magnetic flux Φ_B passes through all the turns, the total emf induced in the coil is

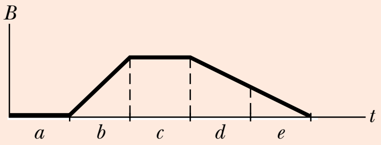
$$\mathscr{E} = -N\frac{d\Phi_B}{dt} \quad \text{(coil of } N \text{ turns).} \tag{30-5}$$

Here are the general means by which we can change the magnetic flux through a coil:

1. Change the magnitude B of the magnetic field within the coil.
2. Change either the total area of the coil or the portion of that area that lies within the magnetic field (for example, by expanding the coil or sliding it into or out of the field).
3. Change the angle between the direction of the magnetic field $\vec{B}$ and the plane of the coil (for example, by rotating the coil so that field $\vec{B}$ is first perpendicular to the plane of the coil and then is along that plane).

☑ CHECKPOINT 1

The graph gives the magnitude $B(t)$ of a uniform magnetic field that exists throughout a conducting loop, with the direction of the field perpendicular to the plane of the loop. Rank the five regions of the graph according to the magnitude of the emf induced in the loop, greatest first.



SAMPLE PROBLEM 30.01

Induced emf in coil due to a solenoid

The long solenoid S shown (in cross section) in Fig. 30-4 has 220 turns/cm and carries a current $i = 1.5$ A; its diameter D is 3.2 cm. At its center we place a 130-turn closely packed coil C of diameter $d = 2.1$ cm. The current in the solenoid is reduced to zero at a steady rate in 25 ms. What is the magnitude of the emf that is induced in coil C while the current in the solenoid is changing?

KEY IDEAS

(1) Because it is located in the interior of the solenoid, coil C lies within the magnetic field produced by current i in the solenoid; thus, there is a magnetic flux Φ_B through coil C.

(2) Because current i decreases, flux Φ_B also decreases.

(3) As Φ_B decreases, emf $\mathscr{E}$ is induced in coil C.

(4) The flux through each turn of coil C depends on the area A and orientation of that turn in the solenoid's magnetic field $\vec{B}$. Because $\vec{B}$ is uniform and directed perpendicular to area A, the flux is given by Eq. 30-2 ($\Phi_B = BA$).

(5) The magnitude B of the magnetic field in the interior of a solenoid depends on the solenoid's current i and its number n of turns per unit length, according to Eq. 29-20 ($B = \mu_0 in$).

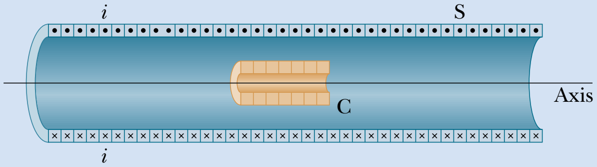


Figure 30-4 A coil C is located inside a solenoid S, which carries current i.

Calculations: Because coil C consists of more than one turn, we apply Faraday's law in the form of Eq. 30-5 ($\mathscr{E} = -N\,d\Phi_B/dt$), where the number of turns N is 130 and $d\Phi_B/dt$ is the rate at which the flux changes.

Because the current in the solenoid decreases at a steady rate, flux Φ_B also decreases at a steady rate, and so we can write $d\Phi_B/dt$ as $\Delta\Phi_B/\Delta t$. Then, to evaluate $\Delta\Phi_B$, we need the final and initial flux values. The final flux $\Delta\Phi_{B,f}$ is zero because the final current in the solenoid is zero. To find the initial flux $\Delta\Phi_{B,i}$, we note that area A is $1/4\pi d^2$ ($= 3.464 \times 10^{-4}$ m^2) and the number n is 220 turns/cm, or 22 000 turns/m. Substituting Eq. 29-20 into Eq. 30-2 then leads to

$$\begin{aligned}\Phi_{B,i} &= BA = (\mu_0 in)A\\ &= (4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(1.5\ \text{A})(2200\ \text{turns/m})\\ &\quad \times (3.464 \times 10^{-4}\ \text{m}^2)\\ &= 1.44 \times 10^{-5}\ \text{Wb.}\end{aligned}$$

Now we can write

$$\frac{d\Phi_B}{dt}=\frac{\Delta\Phi_B}{\Delta t}=\frac{\Phi_{B,f}-\Phi_{B,i}}{\Delta t}$$
$$=\frac{(0-1.44\times10^{-5}\ \text{Wb})}{25\times10^{-3}\,\text{s}}$$
$$=-5.76\times10^{-4}\ \text{Wb/s}=-5.76\times10^{-4}\ \text{V}.$$

We are interested only in magnitudes; so we ignore the minus signs here and in Eq. 30-5, writing

$$\mathscr{E}=N\frac{d\Phi_B}{dt}=(130\ \text{turns})(5.76\times10^{-4}\,\text{V})$$
$$=7.5\times10^{-2}\ \text{V}$$
$$=75\ \text{mV}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 30.02

Charge flowing through the Ballistic galvanometer

A small coil of N turns has its plane perpendicular to a uniform magnetic field as shown in Fig. 30-5. The coil is connected to a Ballistic galvanometer, a device designed to measure the total charge passing through it. Find the charge passing through the coil if the coil is rotated through 180º about its diameter.

KEY IDEA

When the coil in Fig. 30-5 is rotated, the magnetic flux through it changes, causing an induced emf, E. The emf in turn causes a current $I = E/R$, where R is the total resistance of the circuit. Since $I = dQ/dt$, we can find the charge passing through the coil by integrating I; that is,

$$Q=\int dQ=\int Idt. \qquad (30\text{-}6)$$

Calculations: As we stated just now, the total charge flown is the integral of the current:

$$Q=\int dQ=\int Idt.$$

The current is related to the emf by Ohm's law as

$$I=\frac{E}{R},$$

and the emf is only due to the changing magnetic flux, that is

$$E=\frac{d\Phi_B}{dt}.$$

Thus, the induced current is given by

$$I=\frac{E}{R}=\frac{1}{R}\frac{d\Phi_B}{dt}.$$

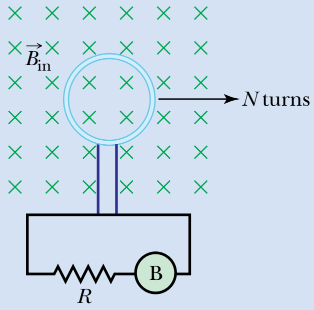


Figure 30-5 A coil of N turns is rotated inside a magnetic field.

Substituting it back in Eq. 30-6, we get

$$Q=\int Idt=\frac{\Delta\Phi_B}{R}.$$

The initial flux through the loop is given by

$$\Phi_i=NBA$$

and after rotating the loop through 180º, the flux would be

$$\Phi_f=-NBA.$$

Hence, the charge flowing through the Ballistic galvanometer is given by

$$Q=\frac{2NBA}{R}$$

Note that the charge Q is independent of the time involved in rotating the coil—all that matters is the change in magnetic flux. A coil used in this way is called a flip coil. It is used to measure magnetic fields. For example, if the Ballistic galvanometer measures a total charge Q passing through the coil when it is flipped, the magnetic field can be found from the equation above.

30.5 | LENZ'S LAW

Key Concept

- An induced current has a direction such that the magnetic field *due to the current* opposes the change in the magnetic flux that induces the current. The induced emf has the same direction as the induced current.

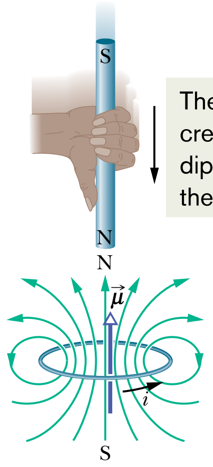
Soon after Faraday proposed his law of induction, Heinrich Friedrich Lenz devised a rule for determining the direction of an induced current in a loop:

> An induced current has a direction such that the magnetic field due to *the current* opposes the change in the magnetic flux that induces the current.

Furthermore, the direction of an induced emf is that of the induced current. The key word in Lenz's law is "opposition." Let's apply the law to the motion of the north pole toward the conducting loop in Fig. 30-6.

1. ***Opposition to Pole Movement*.** The approach of the magnet's north pole in Fig. 30-6 increases the magnetic flux through the loop and thereby induces a current in the loop. From Fig. 29-13, we know that the loop then acts as a magnetic dipole with a south pole and a north pole, and that its magnetic dipole moment $\vec{\mu}$ is directed from south to north. To *oppose* the magnetic flux increase being caused by the approaching magnet, the loop's north pole (and thus $\vec{\mu}$) must face *toward* the approaching north pole so as to repel it (Fig. 30-6). Then the curled – straight right-hand rule for $\vec{\mu}$ (Fig. 29-13) tells us that the current induced in the loop must be counterclockwise in Fig. 30-6.

   If we next pull the magnet away from the loop, a current will again be induced in the loop. Now, however, the loop will have a south pole facing the retreating north pole of the magnet, so as to oppose the retreat. Thus, the induced current will be clockwise.
2. ***Opposition to Flux Change*.** In Fig. 30-6, with the magnet initially distant, no magnetic flux passes through the loop. As the north pole of the magnet then nears the loop with its magnetic field $\vec{B}$ directed *downward*, the flux through the loop increases. To oppose this increase in flux, the induced current i must set up its own field $\vec{B}_{\text{ind}}$ directed *upward* inside the loop, as shown in Fig. 30-7*a*; then the upward flux of field $\vec{B}_{\text{ind}}$ opposes the increasing downward flux of field $\vec{B}$. The curled–straight right-hand rule of Fig. 29-13 then tells us that i must be counterclockwise in Fig. 30-7*a*.



The magnet's motion creates a magnetic dipole that opposes the motion.

Figure 30-6 Lenz's law at work. As the magnet is moved toward the loop, a current is induced in the loop. The current produces its own magnetic field, with magnetic dipole moment $\vec{\mu}$ oriented so as to oppose the motion of the magnet. Thus, the induced current must be counterclockwise as shown.

***Heads Up*.** The flux of $\vec{B}_{\text{ind}}$ always opposes the *change* in the flux of $\vec{B}$, but $\vec{B}_{\text{ind}}$ is not always opposite $\vec{B}$. For example, if we next pull the magnet away from the loop in Fig. 30-6, the magnet's flux Φ_B is still downward through the loop, but it is now decreasing. The flux of $\vec{B}_{\text{ind}}$ must now be downward inside the loop, to oppose that *decrease* (Fig. 30-7*b*). Thus, $\vec{B}_{\text{ind}}$ and $\vec{B}$ are now in the same direction. In Figs. 30-7*c* and *d*, the south pole of the magnet approaches and retreats from the loop, again with opposition to change.

Lenz's law follows from the law of conservation of energy. Let us say that the current was induced in the opposite direction. In other words, for the time being, let us assume that the current is induced in the loop so as to attract the magnet. Then after giving a slight push, magnet would move in the forward direction being attracted by the induced current. We need not supply any energy to produce the current. Why even pay for the electricity bill, then? Just buy a powerful magnet and voila, you have zero electricity bill. But that is not how it works, you need to pay for every joule of energy that you consume. So, what we assumed just now would be converse of the real world.

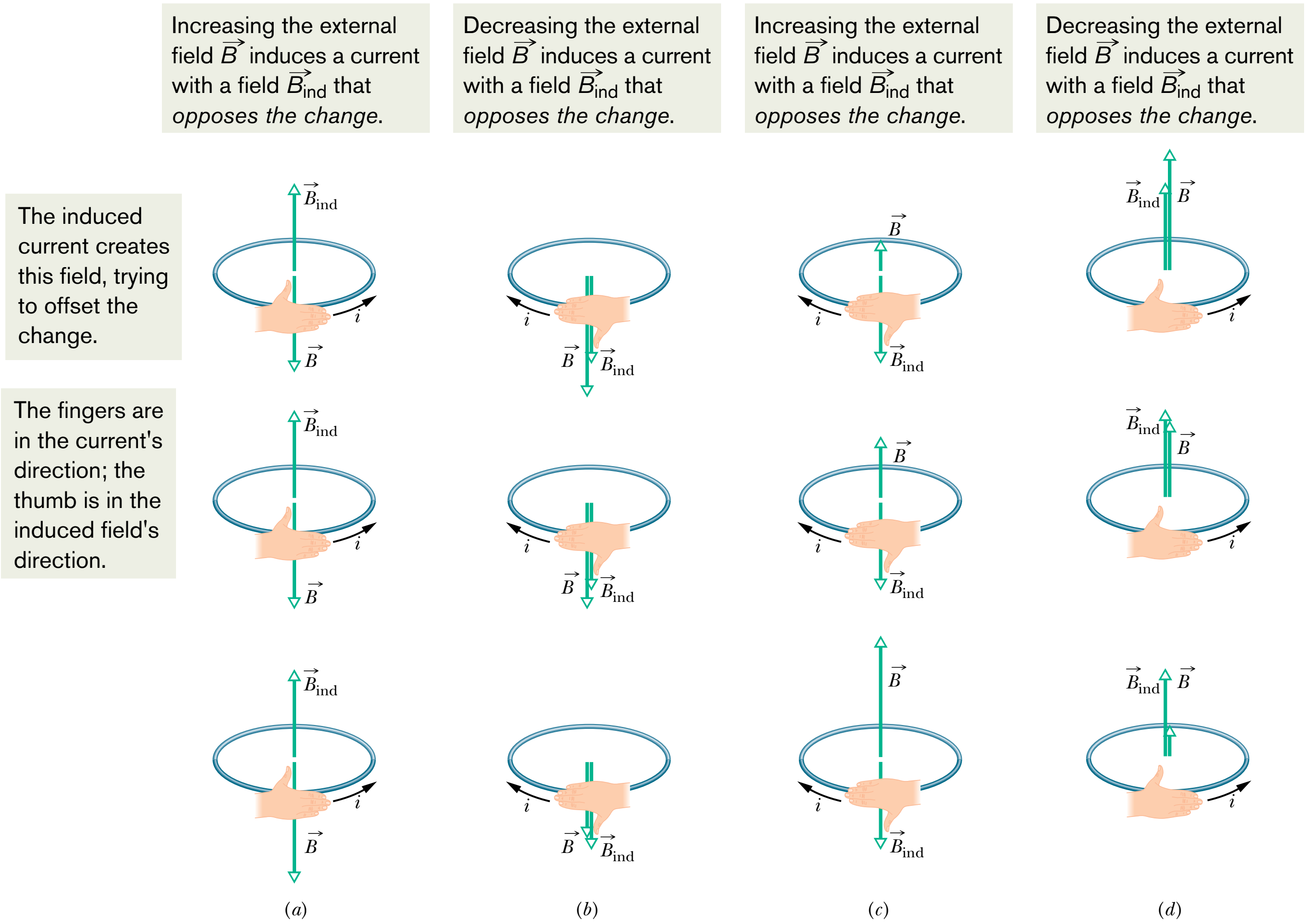
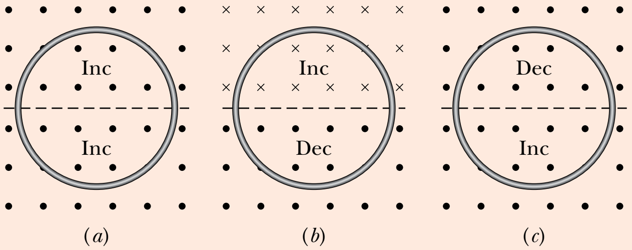

Increasing the external field $\vec{B}$ induces a current with a field $\vec{B}_{ind}$ that *opposes the change.*
Decreasing the external field $\vec{B}$ induces a current with a field $\vec{B}_{ind}$ that *opposes the change.*
Increasing the external field $\vec{B}$ induces a current with a field $\vec{B}_{ind}$ that *opposes the change.*
Decreasing the external field $\vec{B}$ induces a current with a field $\vec{B}_{ind}$ that *opposes the change.*
The induced current creates this field, trying to offset the change.
The fingers are in the current's direction; the thumb is in the induced field's direction.
$\vec{B}_{ind}$
$\vec{B}$
i
(a)
(b)
(c)
(d)

Figure 30-7 The direction of the current i induced in a loop is such that the current's magnetic field $\vec{B}_{ind}$ opposes the *change* in the magnetic field $\vec{B}$ inducing i. The field $\vec{B}_{ind}$ is always directed opposite an increasing field $\vec{B}$ (a, c) and in the same direction as a decreasing field $\vec{B}$ (b, d). The curled–straight right-hand rule gives the direction of the induced current based on the direction of the induced field.

CHECKPOINT 2

The figure shows three situations in which identical circular conducting loops are in uniform magnetic fields that are either increasing (Inc) or decreasing (Dec) in magnitude at identical rates. In each, the dashed line coincides with a diameter. Rank the situations according to the magnitude of the current induced in the loops, greatest first.



SAMPLE PROBLEM 30.03

Induced emf and current due to a changing uniform B field

Figure 30-8 shows a conducting loop consisting of a half-circle of radius $r = 0.20$ m and three straight sections. The halfcircle lies in a uniform magnetic field $\vec{B}$ that is directed out of the page; the field magnitude is given by $B = 4.0t^2 + 2.0t + 3.0$, with B in teslas and t in seconds. An ideal battery with emf $\mathscr{E}_{\text{bat}} = 2.0$ V is connected to the loop. The resistance of the loop is 2.0 Ω.

(a) What are the magnitude and direction of the emf $\mathscr{E}_{\text{ind}}$ induced around the loop by field $\vec{B}$ at $t = 10$ s?

KEY IDEAS

(1) According to Faraday's law, the magnitude of $\mathscr{E}_{\text{ind}}$ is equal to the rate $d\Phi_B/dt$ at which the magnetic flux through the loop changes.
(2) The flux through the loop depends on how much of the loop's area lies within the flux and how the area is oriented in the magnetic field $\vec{B}$.
(3) Because $\vec{B}$ is uniform and is perpendicular to the plane of the loop, the flux is given by Eq. 30-2 ($\Phi_B = BA$). (We do not need to integrate B over the area to get the flux.)
(4) The induced field B_{ind} (due to the induced current) must always oppose the *change* in the magnetic flux.

Magnitude: Using Eq. 30-2 and realizing that only the field magnitude B changes in time (not the area A), we rewrite Faraday's law, Eq. 30-4, as

$$\mathscr{E}_{\text{ind}} = \frac{d\Phi_B}{dt} = \frac{d(BA)}{dt} = A\frac{dB}{dt}.$$

Because the flux penetrates the loop only within the half-circle, the area A in this equation is $(1/2)\pi r^2$. Substituting this and the given expression for B yields

$$\mathscr{E}_{\text{ind}} = A\frac{dB}{dt} = \frac{\pi r^2}{2}\frac{d}{dt}(4.0t^2 + 2.0t + 3.0)$$
$$= \frac{\pi r^2}{2}(8.0t + 2.0).$$

At $t = 10$ s, then,

$$\mathscr{E}_{\text{ind}} = \frac{\pi(0.20\text{ m})^2}{2}[8.0(10) + 2.0]$$
$$= 5.152\text{ V} \approx 5.2\text{ V}. \qquad \text{(Answer)}$$

Direction: To find the direction of $\mathscr{E}_{\text{ind}}$, we first note that in Fig. 30-8 the flux through the loop is out of the page and increasing. Because the induced field B_{ind} (due to the induced current) must oppose that increase, it must be *into* the page. Using the curled–straight right-hand rule (Fig. 30-7*c*), we find that the induced current is clockwise around the loop, and thus so is the induced emf $\mathscr{E}_{\text{ind}}$.
(b) What is the current in the loop at $t = 10$ s?

KEY IDEA

The point here is that *two* emfs tend to move charges around the loop.

Calculation: The induced emf $\mathscr{E}_{\text{ind}}$ tends to drive a current clockwise around the loop; the battery's emf $\mathscr{E}_{\text{bat}}$ tends to drive a current counterclockwise. Because $\mathscr{E}_{\text{ind}}$ is greater than $\mathscr{E}_{\text{bat}}$, the net emf $\mathscr{E}_{\text{ind}}$ is clockwise, and thus so is the current. To find the current at $t = 10$ s, we use Eq. 27-2 ($i = \mathscr{E}/R$):

$$i = \frac{\mathscr{E}_{\text{net}}}{R} = \frac{\mathscr{E}_{\text{ind}} - \mathscr{E}_{\text{bat}}}{R}$$
$$= \frac{5.152\text{ V} - 2.0\text{ V}}{2.0\ \Omega} = 1.58\text{ A} \approx 1.6\text{ A}. \qquad \text{(Answer)}$$

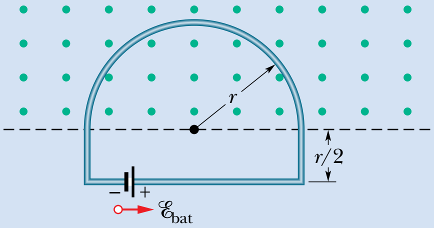


Figure 30-8 A battery is connected to a conducting loop that includes a half-circle of radius r lying in a uniform magnetic field. The field is directed out of the page; its magnitude is changing.

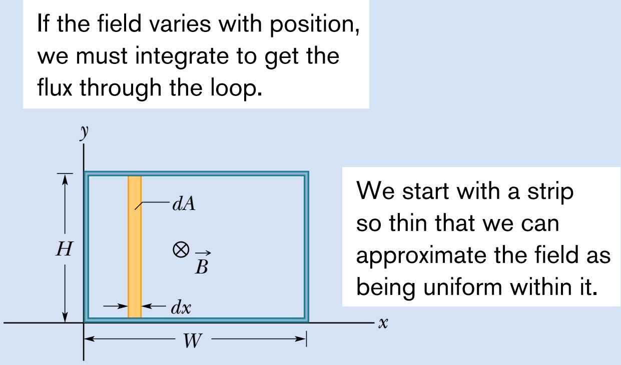
SAMPLE PROBLEM 30.04

Induced emf due to a changing nonuniform *B* field

Figure 30-9 shows a rectangular loop of wire immersed in a nonuniform and varying magnetic field $\vec{B}$ that is perpendicular to and directed into the page. The field's magnitude is given by $B = 4t^2x^2$, with B in teslas, t in seconds, and x in meters. (Note that the function depends on *both* time and position.) The loop has width $W = 3.0$ m and height $H = 2.0$ m. What are the magnitude and direction of the induced emf $\mathscr{E}$ around the loop at $t = 0.10$ s?

KEY IDEAS

(1) Because the magnitude of the magnetic field $\vec{B}$ is changing with time, the magnetic flux Φ_B through the loop is also changing.
(2) The changing flux induces an emf $\mathscr{E}$ in the loop according to Faraday's law, which we can write as $\mathscr{E} = d\Phi_B/dt$.
(3) To use that law, we need an expression for the flux Φ_B at any time t. However, because B is *not* uniform over the area enclosed by the loop, we *cannot* use Eq. 30-2 ($\Phi_B = BA$) to find that expression; instead we must use Eq. 30-1 ($\Phi_B = \int \vec{B} \cdot d\vec{A}$).


If the field varies with position, we must integrate to get the flux through the loop.

We start with a strip so thin that we can approximate the field as being uniform within it.

Figure 30-9 A closed conducting loop, of width W and height H, lies in a nonuniform, varying magnetic field that points directly into the page. To apply Faraday's law, we use the vertical strip of height H, width dx, and area dA.

Calculations: In Fig. 30-9, $\vec{B}$ is perpendicular to the plane of the loop (and hence parallel to the differential area vector $d\vec{A}$); so the dot product in Eq. 30-1 gives $B\,dA$. Because the magnetic field varies with the coordinate x but not with the coordinate y, we can take the differential area dA to be the area of a vertical strip of height H and width dx (as shown in Fig. 30-9). Then $dA = H\,dx$, and the flux through the loop is

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int B\,dA = \int BH\,dx = \int 4t^2x^2H\,dx.$$

Treating t as a constant for this integration and inserting the integration limits $x = 0$ and $x = 3.0$ m, we obtain

$$\Phi_B = 4t^2H\int_0^{3.0} x^2\,dx = 4t^2H\left[\frac{x^3}{3}\right]_0^{3.0} = 72t^2,$$

where we have substituted $H = 2.0$ m and Φ_B is in webers. Now, we can use Faraday's law to find the magnitude of $\mathscr{E}$ at any time t:

$$\mathscr{E} = \frac{d\Phi_B}{dt} = \frac{d(72t^2)}{dt} = 144t,$$

in which $\mathscr{E}$ is in volts. At $t = 0.10$ s,

$$\mathscr{E} = (144\text{ V/s})(0.10\text{ s}) \approx 14\text{ V}. \qquad \text{(Answer)}$$

The flux of $\vec{B}$ through the loop is into the page in Fig. 30-9 and is increasing in magnitude because B is increasing in magnitude with time. By Lenz's law, the field B_{ind} of the induced current opposes this increase and so is directed out of the page. The curled–straight right-hand rule in Fig. 30-7*a* then tells us that the induced current is counterclockwise around the loop, and thus so is the induced emf $\mathscr{E}$.

30.6 | MOTIONAL EMF

Key Concepts

- Motional emf is the emf induced in a conductor moving through a constant magnetic field.
- For a straight conductor (rod) of length l moving through a uniform magnetic field directed into the page, the electric field produced in the conductor is related to the potential difference across the ends of the conductor as

$$\Delta V = -\int \vec{E} \cdot d\vec{l} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}.$$

- If the magnetic field is constant and the velocity of all the points of the rod is the same, the emf induced in the rod can be given by

$$\Delta V = (\vec{B} \times \vec{v}) \cdot \vec{l}$$

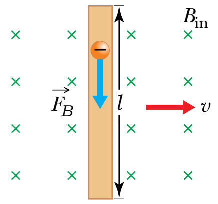


Figure 30-10 A straight electrical conductor of length l moving with a velocity v through a uniform magnetic field B directed perpendicular to v. A potential difference $\Delta V = Blv$ appears between the ends of the conductor.

In Sample Problems 30.1, 30.3, and 30.4, we considered cases in which an emf is induced in a stationary circuit placed in a magnetic field when the field changes with time. In this section, we describe what is called motional emf, which is the emf induced in a conductor moving through a constant magnetic field. The straight conductor of length l shown in Fig. 30-10 is moving through a uniform magnetic field directed into the page.

For simplicity, we assume that the conductor is moving in a direction perpendicular to the field with constant velocity under the influence of some external agent. The electrons in the conductor experience a force

$$\vec{F} = q(\vec{v}\times\vec{B})$$

that is directed along the length l, perpendicular to both v and B (Eq. 28-2). Under the influence of this force, the electrons move to the lower end of the conductor and accumulate there, leaving a net positive charge at the upper end. Because of this charge separation, an electric field is produced inside the conductor. The charges accumulate at both ends until the downward magnetic force qvB is balanced by the upward electric force qE. At this point, electrons stop moving. The condition for equilibrium requires that

$$q[\vec{E} + (\vec{v}\times\vec{B})] = 0,$$

or in other words,

$$\vec{E} = -(\vec{v}\times\vec{B}).$$

The electric field produced in the conductor (once the electrons stop moving and E is constant) is related to the potential difference across the ends of the conductor according to the relationship

$$\Delta V = -\int \vec{E}\cdot d\vec{l} = \int (\vec{v}\times\vec{B})\cdot d\vec{l}. \tag{30-7}$$

Obviously, that end will be at a higher potential toward which $\vec{v}\times\vec{B}$ points. Thus, a potential difference is maintained between the ends of the conductor as long as the conductor continues to move through the magnetic field. This moving conductor can be deemed to be a battery of emf given by the Eq. 30-7. If the direction of the motion is reversed, the polarity of the equivalent battery is also reversed.

A more interesting situation occurs when the moving conductor is part of a closed conducting path. This situation is particularly useful for illustrating how a changing magnetic flux causes an induced current in a closed circuit. Consider a circuit consisting of a conducting bar of length l sliding along two fixed parallel conducting rails, as shown in Fig. 30-11*a*.

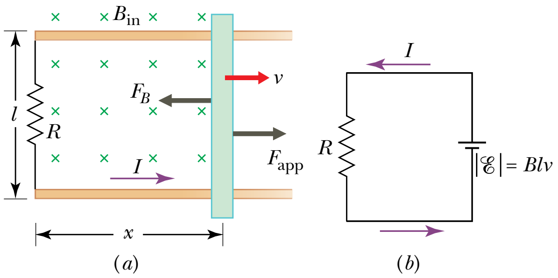


Figure 30-11 (*a*)A conducting rod is moving on two rails under the influence of a force F_{app}. This induces an emf in the rod. Since the circuit is complete, it produces a current and the rod experiences a force F_B due to the magnetic field. (*b*) The corresponding equivalent circuit diagram. The rod is resistance less, but if it had some resistance, it would form the internal resistance of the equivalent battery.

For simplicity, we assume that the bar has zero resistance and that the stationary part of the circuit has a resistance R. A uniform and constant magnetic field B is applied perpendicular to the plane of the circuit. As the bar is pulled to the right with a velocity v, free charges in the bar experience a magnetic force directed along the length of the bar. This force sets up an induced current because the charges are free to move in the closed conducting path.

Since the magnetic field is uniform here, and the velocity of the rod at any instant is perpendicular to the rod, we can simplify Eq. 30-7 as

$$\Delta V = Blv$$

You can see for yourself that the flux through the loop is changing, and Faraday's law also predicts the same value of the induced emf. This rod will act as a source of emf, and the current will flow in the circuit. In Fig. 30-10, the rod was not the part of a closed loop, so there was no possibility of charge flowing continuously. The interesting part is that due to this flow of current, the rod will experience a force which will tend to retard its motion. Check for yourself that the force due to the magnetic field on the rod is indeed given by $\vec{F}_B$. We will explore this point more in the next section.

If the magnetic field is constant and the velocity of all the points of the rod is the same, the emf induced in the rod can be given by

$$\Delta V = (\vec{B}\times\vec{v})\cdot\vec{l}$$

If you recollect from your study in mathematics classes, this is scalar triple product, and all that matters in scalar triple product is the components of the vectors that are mutually perpendicular to each other. Thus, if we take the velocity of the rod as the reference direction, then

$$\Delta V = B_\perp l_\perp v,$$

where $B_\perp$ is the component of the magnetic field perpendicular to the velocity and $l_\perp$ is the component of the length perpendicular to the velocity.

SAMPLE PROBLEM 30.05

Motional emf induced across the ends of the rod when rotated in a uniform magnetic field

A conducting rod of length l is rotated about its end with an angular velocity ω in a uniform magnetic field directed opposite to its axis of rotation, as shown in Fig. 30-12. Find the emf induced in the rod.

KEY IDEA

Although at a first glance, it seems that there should be no emf induced in the rod. After all, there is no flux associated with the rod. But when a conductor is moving across a magnetic field, a motional emf is induced.

We can use Eq. 30-7 here:

$$\Delta V = -\int \vec{E}\cdot d\vec{l} = \int(\vec{v}\times\vec{B})\cdot d\vec{l}.$$

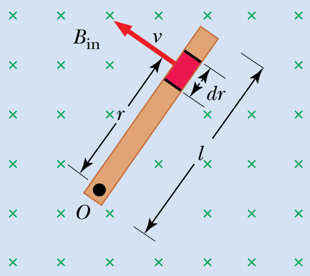


Figure 30-12 A conducting rod rotating around one end in a uniform magnetic field that is perpendicular to the plane of rotation. A motional emf is induced across the ends of the rod.

Calculation: Consider a segment of the bar of length dr having a velocity v. According to Eq. 30-7, the magnitude of the emf induced in the entire rod is

$$\Delta V = \int(\vec{v}\times\vec{B}) = \int B\omega\, dr = \frac{B\omega l^2}{2}. \qquad \text{(Answer)}$$

Learn: Even in the situation shown in Fig. 30-13, there is a current in the external circuit. This device is popularly known as Faraday's disk. It is wrong to say that the flux through the disk is changing, but the conducting disk is moving in the magnetic field and hence there is a motional emf across the radius of the disk.

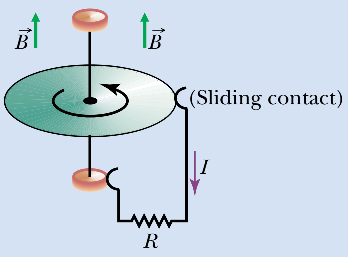


Figure 30-13 Faraday's disk.

CHECKPOINT 3

If the rod were translating as well as rotating, what will be its emf? Assume that the center of mass has a velocity v and the rod is rotating with an angular velocity ω about its center of mass.

SAMPLE PROBLEM 30.06

Generator

A coil containing N conducting loops of area A is rotated at constant angular speed ω in a uniform magnetic field B, as shown in Fig. 30-14. What is the emf induced in the coil?

KEY IDEA

Equation 30-4 tells us that the emf induced in the coil equals the rate of change of the magnetic flux through the coil with respect to time. So, let's write an expression for the magnetic flux through the entire coil as a function of time, and then differentiate it to obtain the emf.

$$\phi = NBA\cos\omega t$$

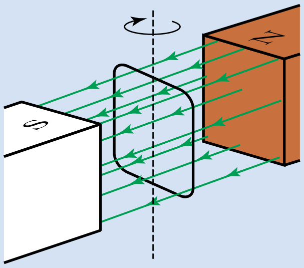


Figure 30-14 A sinusoidal emf is induced in a conducting coil rotating in a magnetic field. The figure depicts the initial position of the coil in the magnetic field.

Calculation: When the coil lies in the plane perpendicular to the magnetic field, the magnetic flux through the coil has its greatest possible value, NBA. As the coil rotates, the surface area vector associated with the coil is no longer perpendicular to the magnetic field, and thus changes with time. At any time, the angle between the surface area vector and the magnetic field is ωt, so the flux through the coil is

$$\phi = NBA \cos \omega t.$$

Therefore, the emf is

$$\text{emf} = -\frac{d}{dt}(NBA\cos\omega t) = NBA\omega\sin\omega t$$

Learn: Note that the emf starts at zero and then increases to a positive value as the enclosed flux decreases due to the rotation of the loop. The resulting induced emf oscillates sinusoidally in time and this emf can be used to establish an oscillating potential difference across two points. Oscillating sources of potential differences are usually referred to as "alternating" sources of potential difference. We will discuss these in more detail in Chapter 31.

30.7 | INDUCTION AND ENERGY TRANSFERS

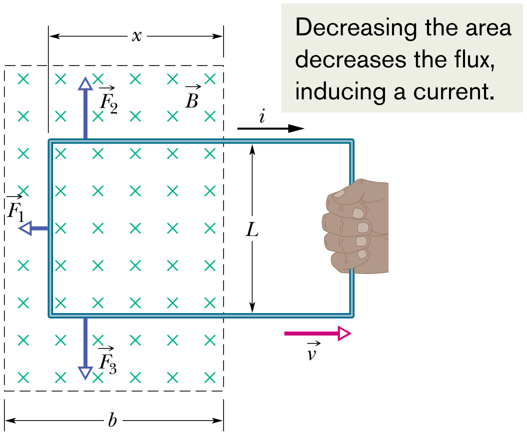
Key Concept

- The induction of a current by a changing flux means that energy is being transferred to that current. The energy can then be transferred to other forms, such as thermal energy.

By Lenz's law, whether you move the magnet toward or away from the loop in Fig. 30-2, a magnetic force resists the motion, requiring your applied force to do positive work. At the same time, thermal energy is produced in the material of the loop because of the material's electrical resistance to the current that is induced by the motion. The energy you transfer to the closed *loop* + *magnet* system via your applied force ends up in this thermal energy. (For now, we neglect energy that is radiated away from the loop as electromagnetic waves during the induction.) The faster you move the magnet, the more rapidly your applied force does work and the greater the rate at which your energy is transferred to thermal energy in the loop; that is, the power of the transfer is greater.

Regardless of how current is induced in a loop, energy is always transferred to thermal energy during the process because of the electrical resistance of the loop (unless the loop is superconducting). For example, in Fig. 30-3, when switch S is closed and a current is briefly induced in the left-hand loop, energy is transferred from the battery to thermal energy in that loop.

Figure 30-15 shows another situation involving induced current. A rectangular loop of wire of width L has one end in a uniform external magnetic field that is directed perpendicularly into the plane of the loop. This field may be produced, for example, by a large electromagnet. The dashed lines in Fig. 30-15 show the assumed limits of the



Decreasing the area decreases the flux, inducing a current.

Figure 30-15 You pull a closed conducting loop out of a magnetic field at constant velocity $\vec{v}$. While the loop is moving, a clockwise current i is induced in the loop, and the loop segments still within the magnetic field experience forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$.

magnetic field; the fringing of the field at its edges is neglected. You are to pull this loop to the right at a constant velocity $\vec{v}$.

Flux Change. The situation of Fig. 30-15 does not differ in any essential way from that of Fig. 30-2. In each case a magnetic field and a conducting loop are in relative motion; in each case the flux of the field through the loop is changing with time. It is true that in Fig. 30-2 the flux is changing because $\vec{B}$ is changing and in Fig. 30-15 the flux is changing because the area of the loop still in the magnetic field is changing, but that difference is not important. The important difference between the two arrangements is that the arrangement of Fig. 30-15 makes calculations easier. Let us now calculate the rate at which you do mechanical work as you pull steadily on the loop in Fig. 30-15.

Rate of Work. As you will see, to pull the loop at a constant velocity $\vec{v}$, you must apply a constant force $\vec{F}$ to the loop because a magnetic force of equal magnitude but opposite direction acts on the loop to oppose you. From Eq. 8-88, the rate at which you do work—that is, the power—is then

$$P = Fv, \tag{30-8}$$

where F is the magnitude of your force. We wish to find an expression for P in terms of the magnitude B of the magnetic field and the characteristics of the loop—namely, its resistance R to current and its dimension L.

As you move the loop to the right in Fig. 30-15, the portion of its area within the magnetic field decreases. Thus, the flux through the loop also decreases and, according to Faraday's law, a current is produced in the loop. It is the presence of this current that causes the force that opposes your pull.

Induced emf. To find the current, we first apply Faraday's law. When x is the length of the loop still in the magnetic field, the area of the loop still in the field is Lx. Then from Eq. 30-2, the magnitude of the flux through the loop is

$$\Phi_B = BA = BLx. \tag{30-9}$$

As x decreases, the flux decreases. Faraday's law tells us that with this flux decrease, an emf is induced in the loop. Dropping the minus sign in Eq. 30-4 and using Eq. 30-9, we can write the magnitude of this emf as

$$\mathscr{E} = \frac{d\Phi_B}{dt} = \frac{d}{dt}BLx = BL\frac{dx}{dt} = BLv, \tag{30-10}$$

in which we have replaced dx/dt with v, the speed at which the loop moves.

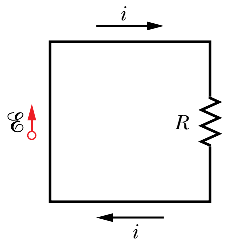


Figure 30-16 A circuit diagram for the loop of Fig. 30-15 while the loop is moving.

Figure 30-16 shows the loop as a circuit: induced emf $\mathscr{E}$ is represented on the left, and the collective resistance R of the loop is represented on the right. The direction of the induced current i is obtained with a right-hand rule as in Fig. 30-7*b* for decreasing flux; applying the rule tells us that the current must be clockwise, and $\mathscr{E}$ must have the same direction.

Induced Current. To find the magnitude of the induced current, we cannot apply the loop rule for potential differences in a circuit because, as you will see in Section 30.8, we cannot define a potential difference for an induced emf. However, we can apply the equation $i = \mathscr{E}/R$. With Eq. 30-10, this becomes

$$i = \frac{BLv}{R}. \tag{30-11}$$

Because three segments of the loop in Fig. 30-15 carry this current through the magnetic field, sideways deflecting forces act on those segments. From Eq. 28-29 we know that such a deflecting force is, in general notation,

$$\vec{F}_d = i\vec{L} \times \vec{B}. \tag{30-12}$$

In Fig. 30-15, the deflecting forces acting on the three segments of the loop are marked $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$. Note, however, that from the symmetry, forces $\vec{F}_2$ and $\vec{F}_3$ are equal in magnitude and cancel. This leaves only force $\vec{F}_1$, which is directed opposite your force $\vec{F}$ on the loop and thus is the force opposing you. So, $\vec{F} = -\vec{F}_1$.

Using Eq. 30-12 to obtain the magnitude of $\vec{F}_1$ and noting that the angle between $\vec{B}$ and the length vector $\vec{L}$ for the left segment is 90°, we write

$$F = F_1 = iLB \sin 90° = iLB. \tag{30-13}$$

Substituting Eq. 30-11 for i in Eq. 30-13 then gives us

$$F = \frac{B^2L^2v}{R}. \tag{30-14}$$

Because B, L, and R are constants, the speed v at which you move the loop is constant if the magnitude F of the force you apply to the loop is also constant.

Rate of Work. By substituting Eq. 30-14 into Eq. 30-8, we find the rate at which you do work on the loop as you pull it from the magnetic field:

$$P = Fv = \frac{B^2L^2v^2}{R} \quad \text{(rate of doing work).} \tag{30-15}$$

Thermal Energy. To complete our analysis, let us find the rate at which thermal energy appears in the loop as you pull it along at constant speed. We calculate it from Eq. 26-27,

$$P = i^2R. \tag{30-16}$$

Substituting for i from Eq. 30-11, we find

$$P = \left(\frac{BLv}{R}\right)^2 R = \frac{B^2L^2v^2}{R} \quad \text{(thermal energy rate),} \tag{30-17}$$

which is exactly equal to the rate at which you are doing work on the loop (Eq. 30-15). Thus, the work that you do in pulling the loop through the magnetic field appears as thermal energy in the loop.

If you recollect, we said in Chapter 28, Section 28.4, that the magnetic forces cannot change the speed of the charged particles and in that sense do not do any work on them. Here also we see that the magnetic forces do not do any work. All they do is transfer energy from mechanical to electrical one.

You might wonder that in Chapter 29, Section 29.4, we found that two current-carrying wires attract each other. Magnetic force is doing work after all. But if you look closely, note that the wires must be a part of a closed circuit to have a current. As the wires come close, the magnetic flux linked with the loops changes. So, an induced emf is generated in the loop which opposes the change in flux. This kinetic energy of the wire is due to the work done by the battery maintaining the current in the wire.

Burns During MRI Scans

A patient undergoing an MRI scan lies in an apparatus containing two magnetic fields: a large constant field $\vec{B}_{\text{con}}$ and a small sinusoidally varying field $\vec{B}(t)$.

Normally the scan requires the patient to lie motionless for a long time. Any patient unable to lie motionless, such as a child, say, is sedated. Because sedation, especially a general anesthetic, can be dangerous, a sedated patient must be carefully monitored, usually with a *pulse oximeter,* a device that measures the oxygen level in the patient's blood. This device includes a probe attached to one of the patient's fingers and a cable running from the probe to a monitor located outside the MRI apparatus.

MRI scans should be perfectly harmless to a patient. In a few cases, however, disregard of Faraday's law of induction led to a sedated patient receiving severe burns. In those cases, the oximeter cable was allowed to touch the patient's arm (Fig. 30-17). The cable and the lower part of the arm then formed a closed loop through which the varying magnetic field $\vec{B}(t)$ produced a varying flux. This flux variation induced an emf around the loop. Although the cable insulation and the skin both had high electrical resistance, the induced emf was large enough to drive a significant current around the loop. As with any other circuit in which there is resistance, the current transferred energy to thermal energy at the points of resistance. In this way, the finger and the skin where the cable touched the lower arm were burned. MRI staff are now trained to keep any monitor cable from touching a patient at more than one point.

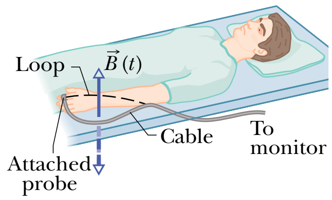


Figure 30-17 A probe attached to a finger of a patient undergoing an MRI scan, in which a vertical magnetic field $\vec{B}(t)$ varies sinusoidally. The probe cable touches the patient's skin along the arm, and the cable and the lower part of the arm form a closed loop.

Eddy Currents

Suppose we replace the conducting loop of Fig. 30-15 with a solid conducting plate. If we then move the plate out of the magnetic field as we did the loop (Fig. 30-18*a*), the relative motion of the field and the conductor again induces a current in the conductor. Thus, we again encounter an opposing force and must do work because of the induced current. With the plate, however, the conduction electrons making up the induced current do not follow one path as they do with the loop. Instead, the electrons swirl about within the plate as if they were caught in an eddy (whirlpool) of water. Such a current is called an *eddy current* and can be represented, as it is in Fig. 30-18*a*, *as if* it followed a single path.

As with the conducting loop of Fig. 30-15, the current induced in the plate results in mechanical energy being dissipated as thermal energy. The dissipation is more apparent in the arrangement of Fig. 30-18*b*; a conducting plate, free to rotate about a pivot, is allowed to swing down through a magnetic field like a pendulum. Each time the plate enters and leaves the field, a portion of its mechanical energy is transferred to its thermal energy. After several swings, no mechanical energy remains and the warmed-up plate just hangs from its pivot.

Similar dissipation of energy is the basis of an induction furnace. Figure 30-19 shows the basic design: Metal is held within a crucible around which insulated wires are wrapped. The (AC) current in the wires alternates in direction and magnitude. Thus, the magnetic field due to the current continuously varies in direction and magnitude. This changing field $\vec{B}(t)$ creates eddy currents within the metal, and electrical energy is dissipated as thermal energy at the rate given by Eq. 30-16 ($P = i^2R$). The dissipation increases the temperature of the metal to the melting point, and then the molten metal can be poured.

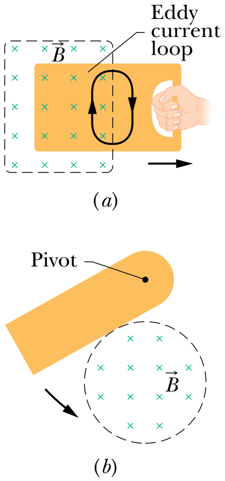
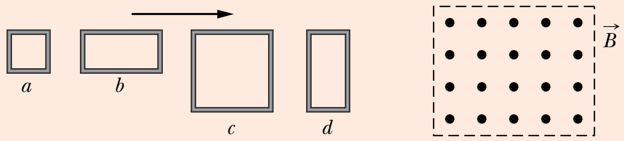


Figure 30-18 (*a*) As you pull a solid conducting plate out of a magnetic field, *eddy currents* are induced in the plate. A typical loop of eddy current is shown. (*b*) A conducting plate is allowed to swing like a pendulum about a pivot and into a region of magnetic field. As it enters and leaves the field, eddy currents are induced in the plate.

CHECKPOINT 4

The figure shows four wire loops, with edge lengths of either L or $2L$. All four loops will move through a region of uniform magnetic field $\vec{B}$ (directed out of the page) at the same constant velocity. Rank the four loops according to the maximum magnitude of the emf induced as they move through the field, greatest first.



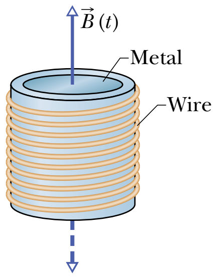


Figure 30-19 An induction furnace.

SAMPLE PROBLEM 30.07

Conducting rod kept on two parallel conducting wires

Assume that two conducting wires with negligible resistance are laid parallel to each other, as shown in Fig. 30-20, to make a smooth inclined plane of inclination θ with the horizontal. A conducting rod of mass m and negligible resistance is kept on the wires. A uniform magnetic field is directed in vertically upward direction. At the upper end, the wires are connected by a resistance R. Find the steady-state velocity of the rod.

KEY IDEAS

(1) As the wire slides downward, its velocity increases under the influence of gravitational force. But the flux passing through the loop also increases. Hence, an emf is induced in the loop which opposes the change in flux, or in other words, opposes the motion of the rod. Looking from the perspective of motional emf, an emf is induced in the rod making the end A at higher potential. A current will flow in the loop in the clockwise direction as seen from above. Due to this current, the rod experiences a force due to the magnetic field. A component of this force opposes the motion of the rod. In steady state, the force due to the magnetic field will balance the force due to gravity.
(2) We have two possible approaches to this problem.
(a) As stated just now, we can apply Newton's laws to find the acceleration of the rod. In steady state, its acceleration will be zero.
(b) We know that magnetic field does not do any work on the system. The gravitational force does work on the rod, increasing its kinetic energy, whereas the resistance of the resistor at the other end dissipates the energy. In steady state, when the kinetic energy becomes constant, the rate at which gravity is doing work will be balanced by the rate at which energy will be dissipated in the resistance.

Calculations: **(a) First method:** Let us assume that the rod has acquired a velocity v. By applying the concept of motional emf (Eq. 30-9), the emf of the rod is

$$\Delta V = (\vec{v}\times\vec{B})\cdot l = Blv\cos\theta.$$

This emf will be such that the current is flowing toward the observer as in Fig. 30-21. The resulting current at that instant is

$$i = \frac{\Delta V}{R} = \frac{Blv\cos\theta}{R}.$$

Applying Eq. 28-29, we get

$$|\vec{F}| = |i\vec{l}\times\vec{B}| = \frac{B^2l^2v\cos\theta}{R}.$$

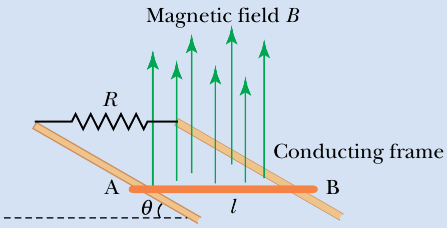


Figure 30-20 A conducting rod is released from rest on a smooth conducting frame inclined at an angle θ to the horizontal.

This force is directed horizontally. A component of this force will oppose the motion of the rod as can be seen in Fig. 30-21.

Hence the acceleration of the rod is given by

$$mg\sin\theta - \frac{B^2l^2v\cos^2\theta}{R} = ma.$$

It can be easily seen that the acceleration of the rod decreases as its velocity increases. After a long time, the acceleration of the rod will tend to zero. The velocity of the rod at this instant is known as the terminal velocity:

$$v_T = \frac{mgR\sin\theta}{B^2l^2\cos^2\theta}.$$

(b) Second method:
Let us find the rate of work done by gravity when the velocity of the rod is v.

We know that the power developed by gravity is

$$P = \vec{F}\cdot\vec{v} = mgv\sin\theta.$$

The rate at which power will be dissipated is

$$P_d = \frac{(\Delta V)^2}{R} = \frac{(Blv\cos\theta)^2}{R}.$$

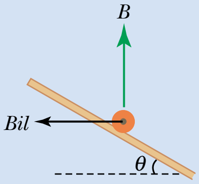


Figure 30-21 A figure showing the direction of current flow and the force on the rod.

In steady state,

$$mgv\sin\theta = \frac{(Blv\cos\theta)^2}{R}$$

$$v_T = \frac{mgR\sin\theta}{B^2 l^2 \cos^2\theta}.$$

which is the same as before.

We can also find the velocity of the rod as a function of time. Put

$$a = \frac{dv}{dt}.$$

We get

$$\frac{dv}{dt} = g\sin\theta\left(1 - \frac{v}{v_T}\right).$$

By integrating, we get

$$v = v_T(1 - e^{-g\sin\theta t/v_T})$$

The equation can be easily checked. By substituting $t = 0$, the velocity turns out to be zero, as expected. At $t = \infty$, the velocity turns out to be v_T.

30.8 | INDUCED ELECTRIC FIELDS

Key Concepts

- An emf is induced by a changing magnetic flux even if the loop through which the flux is changing is not a physical conductor but an imaginary line. The changing magnetic field induces an electric field $\vec{E}$ at every point of such a loop; the induced emf is related to $\vec{E}$ by

$$\mathscr{E} = \oint \vec{E} \cdot d\vec{s}.$$

- Using the induced electric field, we can write Faraday's law in its most general form as

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law).}$$

A changing magnetic field induces an electric field $\vec{E}$.

Let us place a copper ring of radius r in a uniform external magnetic field, as in Fig. 30-22*a*. The field—neglecting fringing—fills a cylindrical volume of radius R. Suppose that we increase the strength of this field at a steady rate,

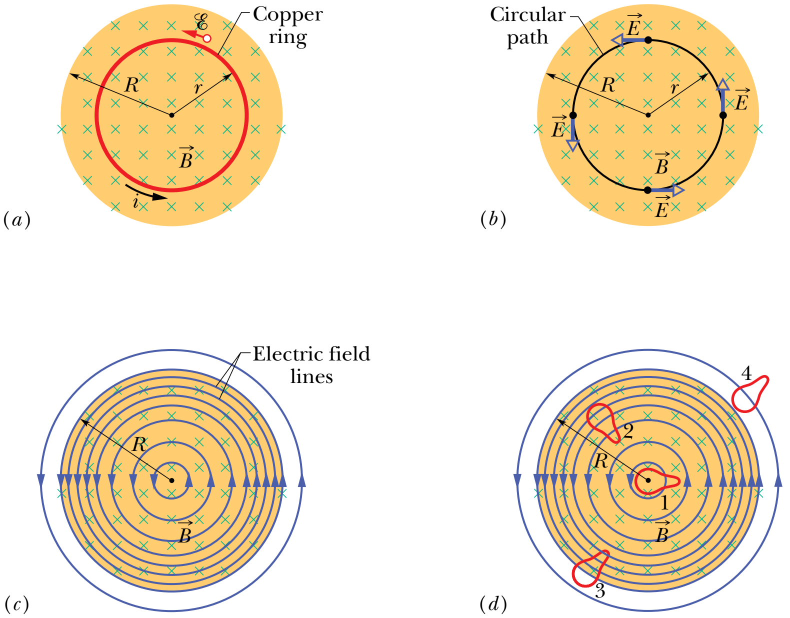


Figure 30-22 (*a*) If the magnetic field increases at a steady rate, a constant induced current appears, as shown, in the copper ring of radius r. (*b*) An induced electric field exists even when the ring is removed; the electric field is shown at four points. (*c*) The complete picture of the induced electric field, displayed as field lines. (*d*) Four similar closed paths that enclose identical areas. Equal emfs are induced around paths 1 and 2, which lie entirely within the region of changing magnetic field. A smaller emf is induced around path 3, which only partially lies in that region. No net emf is induced around path 4, which lies entirely outside the magnetic field.

perhaps by increasing—in an appropriate way—the current in the windings of the electromagnet that produces the field. The magnetic flux through the ring will then change at a steady rate and—by Faraday's law—an induced emf and thus an induced current will appear in the ring. From Lenz's law we can deduce that the direction of the induced current is counterclockwise in Fig. 30-22*a*.

If there is a current in the copper ring, an electric field must be present along the ring because an electric field is needed to do the work of moving the conduction electrons. Moreover, the electric field must have been produced by the changing magnetic flux. This **induced electric field** $\vec{E}$ is just as real as an electric field produced by static charges; either field will exert a force $q_0\vec{E}$ on a particle of charge q_0.

By this line of reasoning, we are led to a useful and informative restatement of Faraday's law of induction:

> A changing magnetic field produces an electric field.

The striking feature of this statement is that the electric field is induced even if there is no copper ring. Thus, the electric field would appear even if the changing magnetic field were in a vacuum.

To fix these ideas, consider Fig. 30-22*b*, which is just like Fig. 30-22*a* except the copper ring has been replaced by a hypothetical circular path of radius r. We assume, as previously, that the magnetic field $\vec{B}$ is increasing in magnitude at a constant rate dB/dt. The electric field induced at various points around the circular path must—from the symmetry—be tangent to the circle, as Fig. 30-22*b* shows.* Hence, the circular path is an electric field line. There is nothing special about the circle of radius r, so the electric field lines produced by the changing magnetic field must be a set of concentric circles, as in Fig. 30-22*c*.

As long as the magnetic field is *increasing* with time, the electric field represented by the circular field lines in Fig. 30-22*c* will be present. If the magnetic field remains *constant* with time, there will be no induced electric field and thus no electric field lines. If the magnetic field is *decreasing* with time (at a constant rate), the electric field lines will still be concentric circles as in Fig. 30-22*c*, but they will now have the opposite direction. All this is what we have in mind when we say "A changing magnetic field produces an electric field."

A Reformulation of Faraday's Law

Consider a particle of charge q_0 moving around the circular path of Fig. 30-22*b*. The work W done on it in one revolution by the induced electric field is $W = \mathscr{E}q_0$, where $\mathscr{E}$ is the induced emf—that is, the work done per unit charge in moving the test charge around the path. From another point of view, the work is

$$W = \int \vec{F} \cdot d\vec{s} = (q_0 E)(2\pi r), \qquad \text{(30-18)}$$

where q_0E is the magnitude of the force acting on the test charge and $2\pi r$ is the distance over which that force acts. Setting these two expressions for W equal to each other and canceling q_0, we find that

$$\mathscr{E} = 2\pi r E. \qquad \text{(30-19)}$$

Next we rewrite Eq. 30-18 to give a more general expression for the work done on a particle of charge q_0 moving along any closed path:

$$W = \oint \vec{F} \cdot d\vec{s} = q_0 \oint \vec{E} \cdot d\vec{s}. \qquad \text{(30-20)}$$

(The loop on each integral sign indicates that the integral is to be taken around the closed path.) Substituting $\mathscr{E}q_0$ for W, we find that

$$\mathscr{E} = \oint \vec{E} \cdot d\vec{s}. \qquad \text{(30-21)}$$

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*Arguments of symmetry would also permit the lines of $\vec{E}$ around the circular path to be *radial*, rather than tangential. However, such radial lines would imply that there are free charges, distributed symmetrically about the axis of symmetry, on which the electric field lines could begin or end; there are no such charges.

This integral reduces at once to Eq. 30-19 if we evaluate it for the special case of Fig. 30-22*b*.

Meaning of emf. With Eq. 30-21, we can expand the meaning of induced emf. Upto this point, induced emf has meant the work per unit charge done in maintaining current due to a changing magnetic flux, or it has meant the work done per unit charge on a charged particle that moves around a closed path in a changing magnetic flux. However, with Fig. 30-22*b* and Eq. 30-21, an induced emf can exist without the need of a current or particle: An induced emf is the sum—via integration—of quantities $\vec{E}\cdot d\vec{s}$ around a closed path, where $\vec{E}$ is the electric field induced by a changing magnetic flux and $d\vec{s}$ is a differential length vector along the path.

If we combine Eq. 30-21 with Faraday's law in Eq. 30-4 ($\mathscr{E} = -\,d\Phi_B/dt$), we can rewrite Faraday's law as

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law).} \tag{30-22}$$

This equation says simply that a changing magnetic field induces an electric field. The changing magnetic field appears on the right side of this equation, the electric field on the left.

Faraday's law in the form of Eq. 30-22 can be applied to *any* closed path that can be drawn in a changing magnetic field. Figure 30-22*d*, for example, shows four such paths, all having the same shape and area but located in different positions in the changing field. The induced emfs $\mathscr{E}\left(=\oint \vec{E}\cdot d\vec{s}\right)$ for paths 1 and 2 are equal because these paths lie entirely in the magnetic field and thus have the same value of $d\Phi_B/dt$. This is true even though the electric field vectors at points along these paths are different, as indicated by the patterns of electric field lines in the figure. For path 3 the induced emf is smaller because the enclosed flux Φ_B (hence $d\Phi_B/dt$) is smaller, and for path 4 the induced emf is zero even though the electric field is not zero at any point on the path.

It is important to understand here that Faraday's law just sums up the two reasons for induced emf. It is an experimental law formulated by Faraday which was found to be true by him in all the situations that he experimented upon. But it is not a universal principle, nor is the changing flux the cause of induced emf. The cause is motion of a conductor or induced electric field. We saw an example of Faraday's disk in Sample Problem 30.05 where the Faraday's flux law is violated. Let us see another example which brings out the point clearly.

The circuit consists of a pair of spring clips and moves across the body of a magnet (see Fig. 30-23*a*). When the loop escapes the magnet, the clips rub over the magnet and the body of the magnet becomes a part of the circuit. The use of Faraday's law in the form of Eq. 30-3 is misleading and predicts a nonzero emf, because the magnetic flux through the loop decreases. However, no emf is created. The magnetic field B does not change in time and no charge carrier moves through the magnetic field. The segment of the magnet closing the circuit, which changes at each instant, is at rest relative to the magnet. Note that if the loop were pulled out of the magnet between the magnetic poles (Fig. 30-23*b*), a motional emf would be created in the loop.

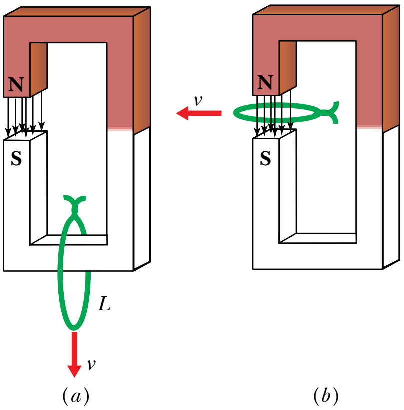


Figure 30-23 (*a*) Clips allow a conducting loop L to escape the magnet. During the escape, the body of magnet (a conductor) becomes a segment of the closed circuit. Although the flux of the magnetic field through the circuit L decreases, no induction emf is created. (*b*) The conducting loop L escapes the magnet between the magnet's poles. Induced emf is created in the loop.

A New Look at Electric Potential

Induced electric fields are produced not by static charges but by a changing magnetic flux. Although electric fields produced in either way exert forces on charged particles, there is an important difference between them. The simplest evidence of this difference is that the field lines of induced electric fields form closed loops, as in Fig. 30-22*c*. Field lines produced by static charges never do so but must start on positive charges and end on negative charges. Thus, a field line from a charge can never loop around and back onto itself as we see for each of the field lines in Fig. 30-22*c*.

In a more formal sense, we can state the difference between electric fields produced by induction and those produced by static charges in these words:

Electric potential has meaning only for electric fields that are produced by static charges; it has no meaning for electric fields that are produced by induction.

You can understand this statement qualitatively by considering what happens to a charged particle that makes a single journey around the circular path in Fig. 30-22*b*. It starts at a certain point and, on its return to that same point, has experienced an emf $\mathscr{E}$ of, let us say, 5 V; that is, work of 5 J/C has been done on the particle by the electric field, and thus the particle should then be at a point that is 5 V greater in potential. However, that is impossible because the particle is back at the same point, which cannot have two different values of potential. Thus, potential has no meaning for electric fields that are set up by changing magnetic fields.

We can take a more formal look by recalling Eq. 24-20, which defines the potential difference between two points i and f in an electric field $\vec{E}$ in terms of an integration between those points:

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}. \tag{30-23}$$

In Chapter 24 we had not yet encountered Faraday's law of induction; so the electric fields involved in the derivation of Eq. 24-20 were those due to static charges. If i and f in Eq. 30-23 are the same point, the path connecting them is a closed loop, V_i and V_f are identical, and Eq. 30-23 reduces to

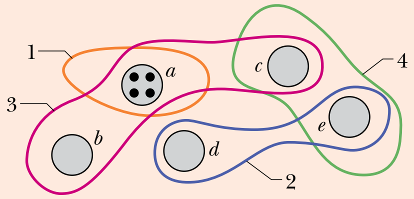
$$\oint \vec{E}\cdot d\vec{s} = 0. \tag{30-24}$$

However, when a changing magnetic flux is present, this integral is *not* zero but is $-d\Phi_B/dt$, as Eq. 30-22 asserts. Thus, assigning electric potential to an induced electric field leads us to a contradiction. We must conclude that electric potential has no meaning for electric fields associated with induction.

CHECKPOINT 5

The figure shows five lettered regions in which a uniform magnetic field extends either directly out of the page or into the page, with the direction indicated only for region a. The field is increasing in magnitude at the same steady rate in all five regions; the regions are identical in area. Also shown are four numbered paths along which $\oint \vec{E}\cdot d\vec{s}$ has the magnitudes given below in terms of a quantity "mag." Determine whether the magnetic field is directed into or out of the page for regions b through e.

Path	1	2	3	4
$\oint \vec{E}\cdot d\vec{s}$	mag	2(mag)	3(mag)	0



CHECKPOINT 6

A uniform copper ring is threaded by a steadily increasing magnetic field so that the magnetic flux through the ring is changing by a constant rate. Use a voltmeter to measure the voltage across the metal ring. What do you predict?

SAMPLE PROBLEM 30.08

Induced electric field due to changing *B* field, inside and outside

In Fig. 30-22*b*, take $R = 8.5$ cm and $dB/dt = 0.13$ T/s.

(a) Find an expression for the magnitude E of the induced electric field at points within the magnetic field, at radius r from the center of the magnetic field. Evaluate the expression for $r = 5.2$ cm.

KEY IDEA

An electric field is induced by the changing magnetic field, according to Faraday's law.

Calculations: To calculate the field magnitude E, we apply Faraday's law in the form of Eq. 30-22. We use a circular path of integration with radius $r \le R$ because we want E for points within the magnetic field. We assume from the symmetry that $\vec{E}$ in Fig. 30-22*b* is tangent to the circular path at all points. The path vector $d\vec{s}$ is also always tangent to the circular path; so the dot product $\vec{E} \cdot d\vec{s}$ in Eq. 30-22 must have the magnitude $E\,ds$ at all points on the path. We can also assume from the symmetry that E has the same value at all points along the circular path. Then the left side of Eq. 30-22 becomes

$$\oint \vec{E} \cdot d\vec{s} = \oint E\,ds = E\oint ds = E(2\pi r). \qquad (30\text{-}25)$$

(The integral $\oint ds$ is the circumference $2\pi r$ of the circular path.)

Next, we need to evaluate the right side of Eq. 30-22. Because $\vec{B}$ is uniform over the area A encircled by the path of integration and is directed perpendicular to that area, the magnetic flux is given by Eq. 30-2:

$$\Phi_B = BA = B(\pi r^2). \qquad (30\text{-}26)$$

Substituting this and Eq. 30-25 into Eq. 30-22 and dropping the minus sign, we find that

$$E(2\pi r) = (\pi r^2)\frac{dB}{dt}$$

or

$$E = \frac{r}{2}\frac{dB}{dt}. \qquad \text{(Answer)} \qquad (30\text{-}27)$$

Equation 30-27 gives the magnitude of the electric field at any point for which $r \le R$ (that is, within the magnetic field). Substituting given values yields, for the magnitude of $\vec{E}$ at $r = 5.2$ cm,

$$E = \frac{(5.2 \times 10^{-2}\,\text{m})}{2}(0.13\,\text{T/s})$$
$$= 0.0034\,\text{V/m} = 3.4\,\text{mV/m}. \qquad \text{(Answer)}$$

(b) Find an expression for the magnitude E of the induced electric field at points that are outside the magnetic field, at radius r from the center of the magnetic field. Evaluate the expression for $r = 12.5$ cm.

KEY IDEAS

Here again an electric field is induced by the changing magnetic field, according to Faraday's law, except that now we use a circular path of integration with radius $r \ge R$ because we want to evaluate E for points outside the magnetic field. Proceeding as in (a), we again obtain Eq. 30-25. However, we do not then obtain Eq. 30-26 because the new path of integration is now outside the magnetic field, and so the magnetic flux encircled by the new path is only that in the area πR^2 of the magnetic field region.

Calculations: We can now write

$$\Phi_B = BA = B(\pi R^2). \qquad (30\text{-}28)$$

Substituting this and Eq. 30-25 into Eq. 30-22 (without the minus sign) and solving for E yield

$$E = \frac{R^2}{2r}\frac{dB}{dt}. \qquad \text{(Answer)} \qquad (30\text{-}29)$$

Because E is not zero here, we know that an electric field is induced even at points that are outside the changing magnetic field, an important result that (as you will see in Section 31.9) makes transformers possible.

With the given data, Eq. 30-29 yields the magnitude of $\vec{E}$ at $r = 12.5$ cm:

$$E = \frac{(8.5 \times 10^{-2}\,\text{m})^2}{(2)(12.5 \times 10^{-2}\,\text{m})}(0.13\ \text{T/s})$$
$$= 3.8 \times 10^{-3}\,\text{V/m} = 3.8\ \text{mV/m}. \qquad \text{(Answer)}$$

Equations 30-27 and 30-29 give the same result for $r = R$. Figure 30-24 shows a plot of $E(r)$. Note that the inside and outside plots meet at $r = R$.

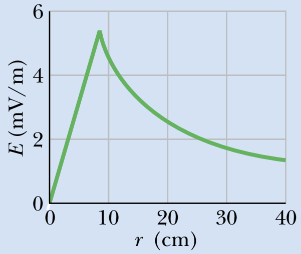


Figure 30-24 A plot of the induced electric field $E(r)$.

30.9 | INDUCTORS AND INDUCTANCE

Key Concepts

- An inductor is a device that can be used to produce a known magnetic field in a specified region. If a current i is established through each of the N windings of an inductor, a magnetic flux Φ_B links those windings. The inductance L of the inductor is

$$L = \frac{N\Phi_B}{i} \quad \text{(inductance defined).}$$

- The SI unit of inductance is the henry (H), where $1 \text{ henry} = 1 \text{ H} = 1 \text{ T} \cdot \text{m}^2/\text{A}$.
- The inductance per unit length near the middle of a long solenoid of cross-sectional area A and n turns per unit length is

$$\frac{L}{l} = \mu_0 n^2 A \quad \text{(solenoid).}$$

We found in Chapter 25 that a capacitor can be used to produce a desired electric field. We considered the parallel-plate arrangement as a basic type of capacitor. Similarly, an **inductor** (symbol ℓℓℓℓ) can be used to produce a desired magnetic field. We shall consider a long solenoid (more specifically, a short length near the middle of a long solenoid, to avoid any fringing effects) as our basic type of inductor.

If we establish a current i in the windings (turns) of the solenoid we are taking as our inductor, the current produces a magnetic flux Φ_B through the central region of the inductor. The **inductance** of the inductor is then defined in terms of that flux as

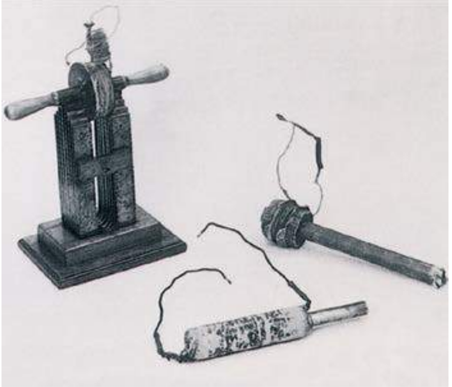
$$L = \frac{N\Phi_B}{i} \quad \text{(inductance defined),} \tag{30-30}$$

in which N is the number of turns. The windings of the inductor are said to be *linked* by the shared flux, and the product $N\Phi_B$ is called the *magnetic flux linkage*. The inductance L is thus a measure of the flux linkage produced by the inductor per unit of current.

Because the SI unit of magnetic flux is the tesla–square meter, the SI unit of inductance is the tesla–square meter per ampere ($\text{T} \cdot \text{m}^2/\text{A}$). We call this the **henry** (H), after American physicist Joseph Henry, the codiscoverer of the law of induction and a contemporary of Faraday. Thus,

$$1 \text{ henry} = 1 \text{ H} = 1 \text{ T} \cdot \text{m}^2/\text{A}. \tag{30-31}$$

Through the rest of this chapter we assume that all inductors, no matter what their geometric arrangement, have no magnetic materials such as iron in their vicinity. Such materials would distort the magnetic field of an inductor.



The Royal Institution/Bridgeman Art Library/NY

The crude inductors with which Michael Faraday discovered the law of induction. In those days amenities such as insulated wire were not commercially available. It is said that Faraday insulated his wires by wrapping them with strips cut from one of his wife's petticoats.

Inductance of a Solenoid

Consider a long solenoid of cross-sectional area A. What is the inductance per unit length near its middle? To use the defining equation for inductance (Eq. 30-30), we must calculate the flux linkage set up by a given current in the solenoid windings. Consider a length l near the middle of this solenoid. The flux linkage there is

$$N\Phi_B = (nl)(BA),$$

in which n is the number of turns per unit length of the solenoid and B is the magnitude of the magnetic field within the solenoid.

The magnitude B is given by Eq. 29-20,

$$B = \mu_0 in,$$

and so from Eq. 30-30,

$$\begin{aligned} L &= \frac{N\Phi_B}{i} = \frac{(nl)(BA)}{i} = \frac{(nl)(\mu_0 in)(A)}{i} \\ &= \mu_0 n^2 lA. \end{aligned} \tag{30-32}$$

Thus, the inductance per unit length near the center of a long solenoid is

$$\frac{L}{l} = \mu_0 n^2 A \quad \text{(solenoid).} \tag{30-33}$$

Inductance—like capacitance—depends only on the geometry of the device. The dependence on the square of the number of turns per unit length is to be expected. If you, say, triple n, you not only triple the number of turns (N) but you also triple the flux ($\Phi_B = BA = \mu_0 inA$) through each turn, multiplying the flux linkage $N\Phi_B$ and thus the inductance L by a factor of 9.

If the solenoid is very much longer than its radius, then Eq. 30-32 gives its inductance to a good approximation. This approximation neglects the spreading of the magnetic field lines near the ends of the solenoid, just as the parallel-plate capacitor formula ($C = \varepsilon_0 A/d$) neglects the fringing of the electric field lines near the edges of the capacitor plates.

From Eq. 30-32, and recalling that n is a number per unit length, we can see that an inductance can be written as a product of the permeability constant μ_0 and a quantity with the dimensions of a length. This means that μ_0 can be expressed in the unit henry per meter:

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\,\mathrm{T \cdot m/A} \\ &= 4\pi \times 10^{-7}\,\mathrm{H/m}. \end{aligned} \tag{30-34}$$

The latter is the more common unit for the permeability constant.

The inductance can be increased substantially by inserting an iron rod inside the solenoid. For a given current, the magnetic flux is now much greater because of the increase in the field originating from the magnetization of the ferromagnetic material. For example, the inductance would increase by a factor of 500 for material with magnetic permeability of 500 μ_0.

30.10 | SELF-INDUCTION

Key Concepts

- If a current i in a coil changes with time, an emf is induced in the coil. This self-induced emf is

$$\mathscr{E}_L = -L\frac{di}{dt}.$$

- The direction of $\mathscr{E}_L$ is found from Lenz's law: The self-induced emf acts to oppose the change that produces it.

If two coils—which we can now call inductors—are near each other, a current i in one coil produces a magnetic flux Φ_B through the second coil. We have seen that if we change this flux by changing the current, an induced emf appears in the second coil according to Faraday's law. An induced emf appears in the first coil as well.

An induced emf $\mathscr{E}_L$ appears in any coil in which the current is changing.

This process (see Fig. 30-25) is called **self-induction**, and the emf that appears is called a **self-induced emf**. It obeys Faraday's law of induction just as other induced emfs do.

For any inductor, Eq. 30-30 tells us that

$$N\Phi_B = Li. \tag{30-35}$$

Faraday's law tells us that

$$\mathscr{E}_L = -\frac{d(N\Phi_B)}{dt}. \tag{30-36}$$

By combining Eqs. 30-35 and 30-36 we can write

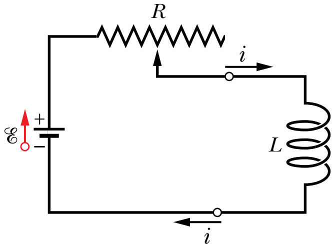


Figure 30-25 If the current in a coil is changed by varying the contact position on a variable resistor, a self-induced emf $\mathscr{E}_L$ will appear in the coil *while the current is changing*.

$$\mathscr{E}_L = -L\frac{di}{dt} \quad \text{(self-induced emf).} \tag{30-37}$$

Thus, in any inductor (such as a coil, a solenoid, or a toroid) a self-induced emf appears whenever the current changes with time. The magnitude of the current has no influence on the magnitude of the induced emf; only the rate of change of the current counts.

Direction. You can find the *direction* of a self-induced emf from Lenz's law. The minus sign in Eq. 30-37 indicates that—as the law states—the self-induced emf $\mathscr{E}_L$ has the orientation such that it opposes the change in current i. We can drop the minus sign when we want only the magnitude of $\mathscr{E}_L$.

Suppose that you set up a current i in a coil and arrange to have the current increase with time at a rate di/dt. In the language of Lenz's law, this increase in the current in the coil is the "change" that the self-induction must oppose. Thus, a self-induced emf must appear in the coil, pointing so as to oppose the increase in the current, trying (but failing) to maintain the initial condition, as shown in Fig. 30-26*a*. If, instead, the current decreases with time, the self-induced emf must point in a direction that tends to oppose the decrease (Fig. 30-26*b*), again trying to maintain the initial condition.

Electric Potential. In Section 30.9 we saw that we cannot define an electric potential for an electric field (and thus for an emf) that is induced by a changing magnetic flux. This means that when a self-induced emf is produced in the inductor of Fig. 30-25, we cannot define an electric potential within the inductor itself, where the flux is changing. However, potentials can still be defined at points of the circuit that are not within the inductor—points where the electric fields are due to charge distributions and their associated electric potentials.

Moreover, we can define a self-induced potential difference V_L *across an inductor* (between its terminals, which we assume to be outside the region of changing flux). For an *ideal inductor* (its wire has negligible resistance), the magnitude of V_L is equal to the magnitude of the self-induced emf $\mathscr{E}_L$.

If, instead, the wire in the inductor has resistance r, we mentally separate the inductor into a resistance r (which we take to be outside the region of changing flux) and an ideal inductor of self-induced emf $\mathscr{E}_L$. As with a real battery of emf $\mathscr{E}$ and internal resistance r, the potential difference across the terminals of a real inductor then differs from the emf. Unless otherwise indicated, we assume here that inductors are ideal.

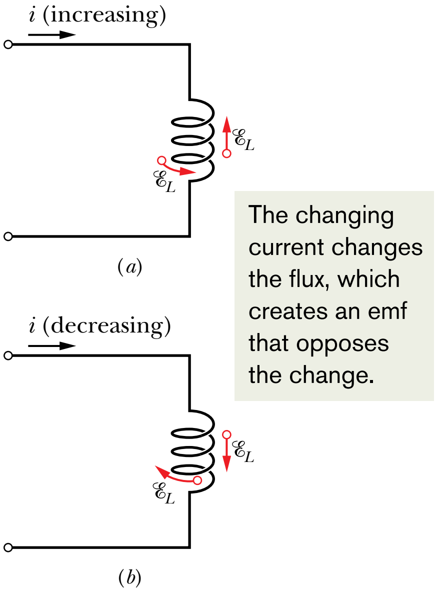


Figure 30-26 (*a*) The current i is increasing, and the self-induced emf $\mathscr{E}_L$ appears along the coil in a direction such that it opposes the increase. The arrow representing $\mathscr{E}_L$ can be drawn along a turn of the coil or alongside the coil. Both are shown. (*b*) The current i is decreasing, and the self-induced emf appears in a direction such that it opposes the decrease.

Let us now find the equivalent inductance for two inductors when placed in (a) series as in Fig. 30-27*a* and (b) parallel as in Fig. 30-27*b*. Let us assume that the inductors are well separated so that their effect on each other can be ignored.

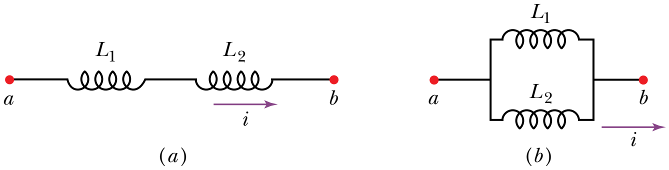


Figure 30-27 (*a*) Inductance in series; (*b*) inductance in parallel.

1. From Fig. 30-27*a*, since the inductors are in series, the current through them will be the same. Hence the total induced emf across both of them will be a sum of the induced emf across the individual inductors. We have already stated that the inductors are well separated so that the current in one does not affect the magnetic field and hence the induced emf of the other:

$$E_{ab} = -L_1\frac{di}{dt} - L_2\frac{di}{dt}.$$

For equivalent inductance,

$$E_{ab} = -L\frac{di}{dt}.$$

Thus $L = L_1 + L_2$.

2. From Fig. 30-27*b*, the current in each of the inductors will not be the same. But since the inductors are placed in parallel, the induced emf across both of them will be equal

$$E_{ab} = -L_1\frac{di}{dt} = -L_2\frac{di}{dt}.$$

By Kirchhoff's current law, we get

$$i = i_1 + i_2.$$

By differentiating this expression, we get

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}.$$

For equivalent inductance:

$$E_{ab} = -L\frac{di}{dt}.$$

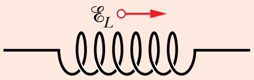
Then

$$\frac{E_{ab}}{L} = \frac{E_{ab}}{L_1} + \frac{E_{ab}}{L_2} \quad \text{or} \quad \frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}.$$

Thus, inductors in series and parallel add like resistance, as long as their mutual inductance can be ignored.

CHECKPOINT 7

The figure shows an emf $\mathscr{E}_L$ induced in a coil. Which of the following can describe the current through the coil: (a) constant and rightward, (b) constant and leftward, (c) increasing and rightward, (d) decreasing and rightward, (e) increasing and leftward, (f) decreasing and leftward?



30.11 | *RL* CIRCUITS

Key Concepts

- If a constant emf $\mathscr{E}$ is introduced into a single-loop circuit containing a resistance R and an inductance L, the current rises to an equilibrium value of $\mathscr{E}/R$ according to

$$i = \frac{\mathscr{E}}{R}(1 - e^{-t/\tau_L}) \quad \text{(rise of current).}$$

Here τ_L $(= L/R)$ governs the rate of rise of the current and is called the inductive time constant of the circuit.

- When the source of constant emf is removed, the current decays from a value i_0 according to

$$i = i_0 e^{-t/\tau_L} \quad \text{(decay of current).}$$

In Section 27.7 we saw that if we suddenly introduce an emf $\mathscr{E}$ into a single-loop circuit containing a resistor R and a capacitor C, the charge on the capacitor does not build up immediately to its final equilibrium value $C\mathscr{E}$ but approaches it in an exponential fashion:

$$q = C\mathscr{E}\,(1 - e^{-t/\tau_C}). \tag{30-38}$$

The rate at which the charge builds up is determined by the capacitive time constant τ_C, defined in Eq. 27-46 as

$$\tau_C = RC. \tag{30-39}$$

If we suddenly remove the emf from this same circuit, the charge does not immediately fall to zero but approaches zero in an exponential fashion

$$q = q_0 e^{-t/\tau_C}. \tag{30-40}$$

The time constant τ_C describes the fall of the charge as well as its rise.

An analogous slowing of the rise (or fall) of the current occurs if we introduce an emf $\mathscr{E}$ into (or remove it from) a single-loop circuit containing a resistor R and an inductor L. When the switch S in Fig. 30-28 is closed on a, for example, the current in the resistor starts to rise. If the inductor were not present, the current would rise rapidly to a steady value $\mathscr{E}/R$. Because of the inductor, however, a self-induced emf $\mathscr{E}_L$ appears in the circuit; from Lenz's law, this emf opposes the rise of the current, which means that it opposes the battery emf $\mathscr{E}$ in polarity. Thus, the current in the resistor responds to the difference between two emfs, a constant $\mathscr{E}$ due to the battery and a variable $\mathscr{E}_L$ $(=-L\,di/dt)$ due to self-induction. As long as this $\mathscr{E}_L$ is present, the current will be less than $\mathscr{E}/R$.

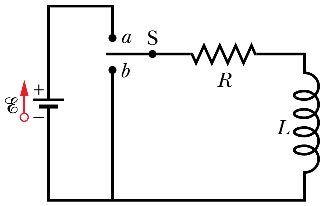


Figure 30-28 An *RL* circuit. When switch S is closed on *a*, the current rises and approaches a limiting value $\mathscr{E}/R$.

As time goes on, the rate at which the current increases becomes less rapid and the magnitude of the self-induced emf, which is proportional to di/dt, becomes smaller. Thus, the current in the circuit approaches $\mathscr{E}/R$ asymptotically.

We can generalize these results as follows:

> Initially, an inductor acts to oppose changes in the current through it. A long time later, it acts like ordinary connecting wire.

Now let us analyze the situation quantitatively. With the switch S in Fig. 30-28 thrown to a, the circuit is equivalent to that of Fig. 30-29. Let us apply the loop rule, starting at point x in this figure and moving clockwise around the loop along with current i.

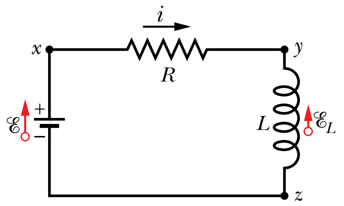


Figure 30-29 The circuit of Fig. 30-28 with the switch closed on *a*. We apply the loop rule for the circuit clockwise, starting at *x*.

1. *Resistor*. Because we move through the resistor in the direction of current i, the electric potential decreases by iR. Thus, as we move from point x to point y, we encounter a potential change of $-iR$.
2. *Inductor*. Because current i is changing, there is a self-induced emf $\mathscr{E}_L$ in the inductor. The magnitude of $\mathscr{E}_L$ is given by Eq. 30-37 as $L\,di/dt$.

The direction of $\mathscr{E}_L$ is upward in Fig. 30-29 because current i is downward through the inductor *and* increasing. Thus, as we move from point y to point z, opposite the direction of $\mathscr{E}_L$, we encounter a potential change of $-L\,di/dt$.

3. *Battery*. As we move from point z back to starting point x, we encounter a potential change of $+\mathscr{E}$ due to the battery's emf.

Thus, the loop rule gives us

$$-iR - L\frac{di}{dt} + \mathscr{E} = 0$$

or

$$L\frac{di}{dt} + Ri = \mathscr{E} \quad (RL \text{ circuit}). \tag{30-41}$$

Equation 30-41 is a differential equation involving the variable i and its first derivative di/dt. To solve it, we seek the function $i(t)$ such that when $i(t)$ and its first derivative are substituted in Eq. 30-41, the equation is satisfied and the initial condition $i(0) = 0$ is satisfied.

Equation 30-41 and its initial condition are of exactly the form of Eq. 27-42 for an RC circuit, with i replacing q, L replacing R, and R replacing $1/C$. The solution of Eq. 30-41 must then be of exactly the form of Eq. 27-43 with the same replacements. That solution is

$$i = \frac{\mathscr{E}}{R}(1 - e^{-Rt/L}), \tag{30-42}$$

which we can rewrite as

$$i = \frac{\mathscr{E}}{R}(1 - e^{-t/\tau_L}) \quad (\text{rise of current}). \tag{30-43}$$

Here τ_L, the **inductive time constant**, is given by

$$\tau_L = \frac{L}{R} \quad (\text{time constant}). \tag{30-44}$$

Let's examine Eq. 30-43 for just after the switch is closed (at time $t = 0$) and for a time long after the switch is closed ($t \to \infty$). If we substitute $t = 0$ into Eq. 30-43, the exponential becomes $e^{-0} = 1$. Thus, Eq. 30-43 tells us that the current is initially $i = 0$, as we expected. Next, if we let t go to ∞, then the exponential goes to $e^{-\infty} = 0$. Thus, Eq. 30-43 tells us that the current goes to its equilibrium value of $\mathscr{E}/R$.

We can also examine the potential differences in the circuit. For example, Fig. 30-30 shows how the potential differences V_R $(= iR)$ across the resistor and V_L $(= L\,di/dt)$ across the inductor vary with time for particular values of $\mathscr{E}$, L, and R. Compare this figure carefully with the corresponding figure for an RC circuit (Fig. 27-34).

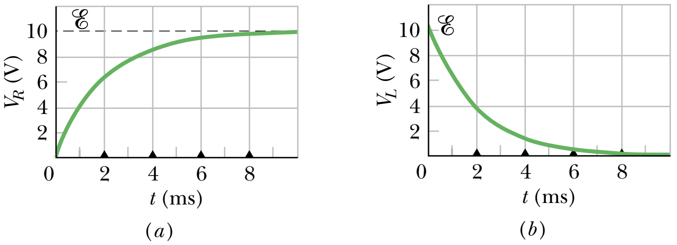
To show that the quantity τ_L $(= L/R)$ has the dimension of time (as it must, because the argument of the exponential function in Eq. 30-43 must be dimensionless), we convert from henries per ohm as follows

$$1\frac{\text{H}}{\Omega} = 1\frac{\text{H}}{\Omega}\left(\frac{1\ \text{V}\cdot\text{s}}{1\ \text{H}\cdot\text{A}}\right)\left(\frac{1\ \Omega\cdot\text{A}}{1\ \text{V}}\right) = 1\ \text{s}.$$

The first quantity in parentheses is a conversion factor based on Eq. 30-37, and the second one is a conversion factor based on the relation $V = iR$.

Time Constant. The physical significance of the time constant follows from Eq. 30-43. If we put $t = \tau_L = L/R$ in this equation, it reduces to

$$i = \frac{\mathscr{E}}{R}(1 - e^{-1}) = 0.63\frac{\mathscr{E}}{R}. \tag{30-45}$$



The resistor's potential difference turns on. The inductor's potential difference turns off.

Figure 30-30 The variation with time of (*a*) V_R, the potential difference across the resistor in the circuit of Fig. 30-29, and (*b*) V_L, the potential difference across the inductor in that circuit. The small triangles represent successive intervals of one inductive time constant $\tau_L = L/R$. The figure is plotted for $R = 2000\ \Omega$, $L = 4.0$ H, and $\mathscr{E} = 10$ V.

Thus, the time constant τ_L is the time it takes the current in the circuit to reach about 63% of its final equilibrium value $\mathscr{E}/R$. Since the potential difference V_R across the resistor is proportional to the current i, a graph of the increasing current versus time has the same shape as that of V_R in Fig. 30-30*a*.

Current Decay. If the switch S in Fig. 30-28 is closed on *a* long enough for the equilibrium current $\mathscr{E}/R$ to be established and then is thrown to *b*, the effect will be to remove the battery from the circuit. (The connection to *b* must actually be made an instant before the connection to *a* is broken. A switch that does this is called a *make-before-break* switch.) With the battery gone, the current through the resistor will decrease. However, it cannot drop immediately to zero but must decay to zero over time. The differential equation that governs the decay can be found by putting $\mathscr{E} = 0$ in Eq. 30-41:

$$L\frac{di}{dt} + iR = 0. \tag{30-46}$$

By analogy with Eqs. 27-48 and 27-49, the solution of this differential equation that satisfies the initial condition $i(0) = i_0 = \mathscr{E}/R$ is

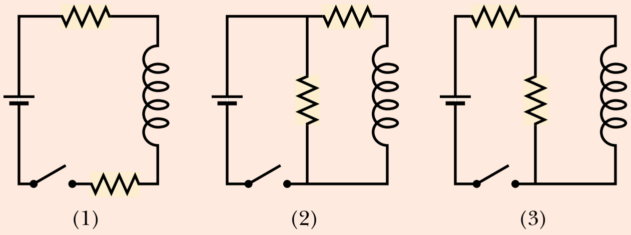
$$i = \frac{\mathscr{E}}{R}e^{-t/\tau_L} = i_0 e^{-t/\tau_L} \quad \text{(decay of current).} \tag{30-47}$$

We see that both current rise (Eq. 30-43) and current decay (Eq. 30-47) in an *RL* circuit are governed by the same inductive time constant, τ_L.

We have used i_0 in Eq. 30-47 to represent the current at time $t = 0$. In our case that happened to be $\mathscr{E}/R$, but it could be any other initial value.

CHECKPOINT 8

The figure shows three circuits with identical batteries, inductors, and resistors. Rank the circuits according to the current through the battery (a) just after the switch is closed and (b) a long time later, greatest first. (If you have trouble here, work through the next sample problem and then try again.)



SAMPLE PROBLEM 30.09

RL circuit, immediately after switching and after a long time

Figure 30-31*a* shows a circuit that contains three identical resistors with resistance $R = 9.0\ \Omega$, two identical inductors with inductance $L = 2.0$ mH, and an ideal battery with emf $\mathscr{E} = 18$ V.

(a) What is the current i through the battery just after the switch is closed?

KEY IDEA

Just after the switch is closed, the inductor acts to oppose a change in the current through it.

Calculations: Because the current through each inductor is zero before the switch is closed, it will also be zero just afterward. Thus, immediately after the switch is closed, the inductors act as broken wires, as indicated in Fig. 30-31*b*. We then have a single-loop circuit for which the loop rule gives us

$$\mathscr{E} - iR = 0.$$

Substituting given data, we find that

$$i = \frac{\mathscr{E}}{R} = \frac{18\ \text{V}}{9.0\ \Omega} = 2.0\ \text{A}. \qquad \text{(Answer)}$$

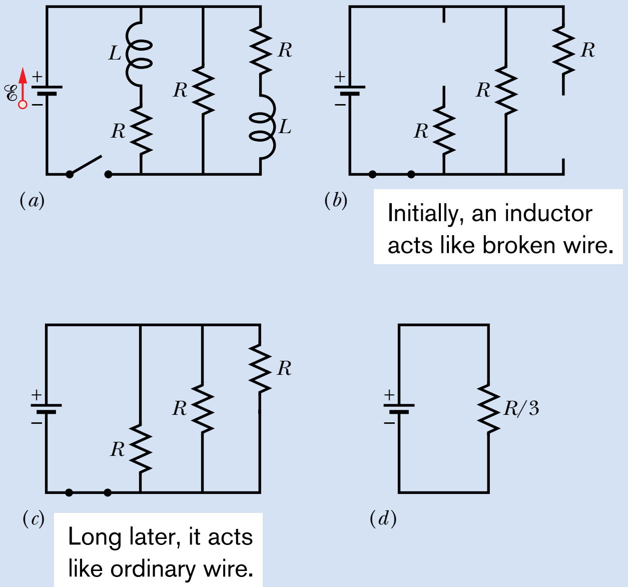
(b) What is the current i through the battery long after the switch has been closed?

KEY IDEA

Long after the switch has been closed, the currents in the circuit have reached their equilibrium values, and the inductors act as simple connecting wires, as indicated in Fig. 30-31*c*.

Calculations: We now have a circuit with three identical resistors in parallel; from Eq. 27-23, their equivalent resistance is $R_{eq} = R/3 = (9.0\ \Omega)/3 = 3.0\ \Omega$. The equivalent circuit shown in Fig. 30-31*d* then yields the loop equation $\mathscr{E} - iR_{eq} = 0$, or

$$i = \frac{\mathscr{E}}{R_{eq}} = \frac{18\ \text{V}}{3.0\ \Omega} = 6.0\ \text{A}. \qquad \text{(Answer)}$$



$\mathscr{E}$
+
–
L
R
R
R
L
(a)
+
–
R
R
R
(b)
Initially, an inductor acts like broken wire.
+
–
R
R
R
(c)
Long later, it acts like ordinary wire.
+
–
R/3
(d)

Figure 30-31 (*a*) A multiloop *RL* circuit with an open switch. (*b*) The equivalent circuit just after the switch has been closed. (*c*) The equivalent circuit a long time later. (*d*) The single-loop circuit that is equivalent to circuit (*c*).

SAMPLE PROBLEM 30.10

RL circuit, current during the transition

A solenoid has an inductance of 53 mH and a resistance of 0.37 Ω. If the solenoid is connected to a battery, how long will the current take to reach half its final equilibrium value? (This is a *real solenoid* because we are considering its small, but nonzero, internal resistance.)

KEY IDEA

We can mentally separate the solenoid into a resistance and an inductance that are wired in series with a battery, as in Fig. 30-29. Then application of the loop rule leads to Eq. 30-41, which has the solution of Eq. 30-43 for the current i in the circuit.

Calculations: According to that solution, current i increases exponentially from zero to its final equilibrium value of $\mathscr{E}/R$. Let t_0 be the time that current i takes to reach half its equilibrium value. Then Eq. 30-43 gives us

$$\frac{1}{2}\frac{\mathscr{E}}{R} = \frac{\mathscr{E}}{R}(1 - e^{-t_0/\tau_L}).$$

We solve for t_0 by canceling $\mathscr{E}/R$, isolating the exponential, and taking the natural logarithm of each side. We find

$$\begin{aligned} t_0 &= \tau_L \ln 2 = \frac{L}{R}\ln 2 = \frac{53 \times 10^{-3}\,\mathrm{H}}{0.37\;\Omega}\ln 2 \\ &= 0.10\ \mathrm{s}. \end{aligned} \qquad \text{(Answer)}$$

30.12 | ENERGY STORED IN A MAGNETIC FIELD

Key Concepts

- If an inductor L carries a current i, the inductor's magnetic field stores an energy given by

$$U_B = \frac{1}{2}Li^2 \quad \text{(magnetic energy).}$$

- If B is the magnitude of a magnetic field at any point (in an inductor or anywhere else), the density of stored magnetic energy at that point is

$$u_B = \frac{B^2}{2\mu_0} \quad \text{(magnetic energy density).}$$

When we pull two charged particles of opposite signs away from each other, we say that the resulting electric potential energy is stored in the electric field of the particles. We get it back from the field by letting the particles move closer together again. In the same way we say energy is stored in a magnetic field, but now we deal with current instead of electric charges.

To derive a quantitative expression for that stored energy, consider again Fig. 30-29, which shows a source of emf $\mathscr{E}$ connected to a resistor R and an inductor L. Equation 30-41, restated here for convenience,

$$\mathscr{E} = L\frac{di}{dt} + iR, \tag{30-48}$$

is the differential equation that describes the growth of current in this circuit. Recall that this equation follows immediately from the loop rule and that the loop rule in turn is an expression of the principle of conservation of energy for single-loop circuits. If we multiply each side of Eq. 30-48 by i, we obtain

$$\mathscr{E}i = Li\frac{di}{dt} + i^2R, \tag{30-49}$$

which has the following physical interpretation in terms of the work done by the battery and the resulting energy transfers:

1. If a differential amount of charge dq passes through the battery of emf $\mathscr{E}$ in Fig. 30-29 in time dt, the battery does work on it in the amount $\mathscr{E}\,dq$. The rate at which the battery does work is $(\mathscr{E}\,dq)/dt$, or $\mathscr{E}i$. Thus, the left side of Eq. 30-49 represents the rate at which the emf device delivers energy to the rest of the circuit.
2. The rightmost term in Eq. 30-49 represents the rate at which energy appears as thermal energy in the resistor.
3. Energy that is delivered to the circuit but does not appear as thermal energy must, by the conservation-of-energy hypothesis, be stored in the magnetic field of the inductor. Because Eq. 30-49 represents the principle of conservation of energy for RL circuits, the middle term must represent the rate dU_B/dt at which magnetic potential energy U_B is stored in the magnetic field.

Thus

$$\frac{dU_B}{dt} = Li\frac{di}{dt}. \tag{30-50}$$

We can write this as

$$dU_B = Li\,di.$$

Integrating yields

$$\int_0^{U_B} dU_B = \int_0^i Li\,di$$

or

$$U_B = \frac{1}{2}Li^2 \quad \text{(magnetic energy)}, \tag{30-51}$$

which represents the total energy stored by an inductor L carrying a current i. Note the similarity in form between this expression for the energy stored in a magnetic field and the expression for the energy stored in an electric field by a capacitor with capacitance C and charge q; namely,

$$U_E = \frac{q^2}{2C}. \tag{30-52}$$

(The variable i^2 corresponds to q^2, and the constant L corresponds to $1/C$.)

SAMPLE PROBLEM 30.11

Energy stored in a magnetic field

A coil has an inductance of 53 mH and a resistance of 0.35 Ω.

(a) If a 12 V emf is applied across the coil, how much energy is stored in the magnetic field after the current has built up to its equilibrium value?

KEY IDEA

The energy stored in the magnetic field of a coil at any time depends on the current through the coil at that time, according to Eq. 30-51 ($U_B = 1/2Li^2$).

Calculations: Thus, to find the energy $U_{B\infty}$ stored at equilibrium, we must first find the equilibrium current. From Eq. 30-43, the equilibrium current is

$$i_\infty = \frac{\mathscr{E}}{R} = \frac{12\text{ V}}{0.35\ \Omega} = 34.3\text{ A}. \tag{30-53}$$

Then substitution yields

$$U_{B\infty} = \frac{1}{2}Li_\infty^2 = \left(\frac{1}{2}\right)(53\times10^{-3}\text{ H})(34.3\text{ A})^2$$
$$= 31\text{ J}. \qquad \text{(Answer)}$$

(b) After how many time constants will half this equilibrium energy be stored in the magnetic field?

Calculations: Now we are being asked: At what time t will the relation

$$U_B = \frac{1}{2}U_{B\infty}$$

be satisfied? Using Eq. 30-51 twice allows us to rewrite this energy condition as

$$\frac{1}{2}Li^2 = \left(\frac{1}{2}\right)\frac{1}{2}Li_\infty^2$$

or

$$i = \left(\frac{1}{\sqrt{2}}\right)i_\infty. \tag{30-54}$$

This equation tells us that, as the current increases from its initial value of 0 to its final value of i_∞, the magnetic field will have half its final stored energy when the current has increased to this value. In general, we know that i is given by Eq. 30-43, and here i_∞ (see Eq. 30-53) is $\mathscr{E}/R$; so Eq. 30-54 becomes

$$\frac{\mathscr{E}}{R}(1-e^{-t/\tau_L}) = \frac{\mathscr{E}}{\sqrt{2}R}.$$

By canceling $\mathscr{E}/R$ and rearranging, we can write this as

$$e^{-t/\tau_L} = 1 - \frac{1}{\sqrt{2}} = 0.293,$$

which yields

$$\frac{t}{\tau_L} = -\ln 0.293 = 1.23$$

or $$t \approx 1.2\tau_L. \qquad \text{(Answer)}$$

Thus, the energy stored in the magnetic field of the coil by the current will reach half its equilibrium value 1.2 time constants after the emf is applied.

Energy Density of a Magnetic Field

Consider a length l near the middle of a long solenoid of cross-sectional area A carrying current i; the volume associated with this length is Al. The energy U_B stored by the length l of the solenoid must lie entirely within this volume because the magnetic field outside such a solenoid is approximately zero. Moreover, the stored energy must be uniformly distributed within the solenoid because the magnetic field is (approximately) uniform everywhere inside.

Thus, the energy stored per unit volume of the field is

$$u_B = \frac{U_B}{Al}$$

or, since

$$U_B = \frac{1}{2}Li^2,$$

we have

$$u_B = \frac{Li^2}{2Al} = \frac{L}{l}\frac{i^2}{2A}. \tag{30-55}$$

Here L is the inductance of length l of the solenoid.

Substituting for L/l from Eq. 30-33, we find

$$u_B = \frac{1}{2}\mu_0 n^2 i^2, \tag{30-56}$$

where n is the number of turns per unit length. From Eq. 29-20 ($B = \mu_0 in$) we can write this *energy density* as

$$u_B = \frac{B^2}{2\mu_0} \quad \text{(magnetic energy density).} \tag{30-57}$$

This equation gives the density of stored energy at any point where the magnitude of the magnetic field is B. Even though we derived it by considering the special case of a solenoid, Eq. 30-57 holds for all magnetic fields, no matter how they are generated. The equation is comparable to Eq. 25-25,

$$u_E = \frac{1}{2}\varepsilon_0 E^2, \tag{30-58}$$

which gives the energy density (in a vacuum) at any point in an electric field. Note that both u_B and u_E are proportional to the square of the appropriate field magnitude, B or E.

CHECKPOINT 9

The table lists the number of turns per unit length, current, and cross-sectional area for three solenoids. Rank the solenoids according to the magnetic energy density within them, greatest first.

Solenoid	Turns per Unit Length	Current	Area
a	$2n_1$	i_1	$2A_1$
b	n_1	$2i_1$	A_1
c	n_1	i_1	$6A_1$

SAMPLE PROBLEM 30.12

Finding total energy stored in a magnetic field for a length of coaxial cable from the energy density

A long coaxial cable (Fig. 30-32) consists of two thin-walled concentric conducting cylinders with radii a and b. The inner cylinder carries a steady current i, and the outer cylinder provides the return path for that current. The current sets up a magnetic field between the two cylinders.

(a) Calculate the energy stored in the magnetic field for a length l of the cable.

KEY IDEA

We can calculate the (total) energy U_B stored in the magnetic field from the energy density u_B of the field. That energy density depends on the magnitude B of the magnetic field according to Eq. 30-57 ($u_B = B^2/2\mu_0$). Because of the circular symmetry of the cable, we can find B by using Ampere's law with the given current i.

Calculations: Finding B: To apply these ideas, we begin with Ampere's law, using a circular path of integration with radius r such that $a < r < b$ (between the two cylinders, as indicated by the dashed line in Fig. 30-32). The only current enclosed by this path is current i on the inner cylinder. Thus, we can write Ampere's law as

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i. \tag{30-59}$$

Next, we simplify the integral: Because of the circular symmetry, at all points along the circular path, $\vec{B}$ is tangent to the path and has the same magnitude B. Let us take the direction of integration along the path as the direction of the magnetic field around the path. Then we can replace $\vec{B} \cdot d\vec{s}$ with $B\,ds \cos\theta = B\,ds$ and then move magnitude B in front of the integration symbol. The integral that remains is $\oint ds$, which just gives the circumference $2\pi r$ of the path. Thus, Eq. 30-59 simplifies to

$$B(2\pi r) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi r} \tag{30-60}$$

Finding u_B: Next, to obtain the energy density, we substitute Eq. 30-60 into Eq. 30-57:

$$u_B = \frac{B^2}{2\mu_0} = \frac{\mu_0 i^2}{8\pi^2 r^2}. \tag{30-61}$$

Finding u_B: Note that u_B is not uniform in the volume between the two cylinders, but instead depends on the radial distance r. Thus, to find the total energy U_B stored between the cylinders, we must integrate u_B over that volume.

Because the volume between the cylinders has circular symmetry about the cable's central axis, we consider the volume dV of a cylindrical shell located between the cylinders; the shell has inner radius r, outer radius $r + dr$ (Fig. 30-32), and length l. The shell's cross-sectional area is the product of its circumference $2\pi r$ and thickness dr. Thus, the shell's volume dV is $(2\pi r)(dr)(l)$; that is, $dV = 2\pi rl\,dr$.

Because points within this shell are all at approximately the same radial distance r, they all have approximately the same energy density u_B. Thus, the total energy dU_B contained in the shell of volume dV is given by

$$\text{energy} = \begin{pmatrix}\text{energy per} \\ \text{unit volume}\end{pmatrix}(\text{volume})$$

or

$$dU_B = u_B dV$$

Substituting Eq. 30-61 for u_B and $2\pi rldr$ for dV, we obtain

$$dU_B = \frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi rl)dr = \frac{\mu_0 i^2 l}{4\pi}\frac{dr}{r}.$$

To find the total energy contained between the two cylinders, we integrate this equation over the volume between the two cylinders:

$$U_B = \int dU_B = \frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{dr}{r}$$

$$= \frac{\mu_0 i^2 l}{4\pi}\ln\frac{b}{a} \qquad \text{(Answer)} \tag{30-62}$$

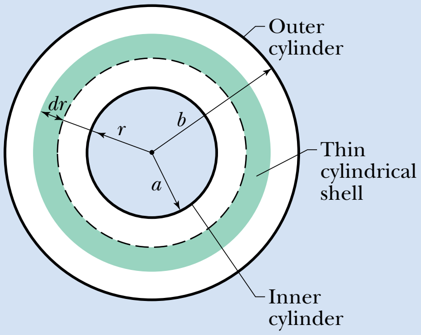


Figure 30-32 A cross section of a long coaxial cable consisting of two thin-walled conducting cylinders, the inner cylinder of radius a and the outer cylinder of radius b.

No energy is stored outside the outer cylinder or inside the inner cylinder because the magnetic field is zero in both locations, as you can show with Ampere's law.

(b) What is the stored energy per unit length of the cable if $a = 1.2$ mm, $b = 3.5$ mm, and $i = 2.7$ A?

Calculation: From Eq. 30-62 we have

$$\frac{U_B}{l} = \frac{\mu_0 i^2}{4\pi}\ln\frac{b}{a}$$

$$= \frac{(4\pi\times10^{-7}\,\text{J/m})(2.7\,\text{A})^2}{4\pi}\ln\frac{3.5\text{ mm}}{1.2\text{ mm}}$$

$$= 7.8\times10^{-7}\,\text{J/m} = 780\,\text{J/m} \qquad \text{(Answer)}$$

Learn: By the way, this suggests a very simple way to calculate the self-inductance of the cable. According to Eq. 30-56, the energy can also be written as $(1/2)Li^2$. Comparing the two expressions, we get

$$L = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}.$$

This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume. In such cases different parts of the current may circle different amounts of flux, and it can be very tricky to get L directly from Eq. 30-33.

Magnetic Pressure

The idea of energy in a magnetic field gives a very important and useful result. Let us try to find the pressure due to magnetic field on a surface through which current flows. As we discussed in the previous chapter, a current-carrying wire experiences a force in a magnetic field. This force is perpendicular to the wire. Similarly, we can note that any current-carrying strip placed in a magnetic field will experience a force. If there is a component of the force perpendicular to the surface, there is a pressure due to magnetic field.

Consider the situation depicted in Fig. 30-33. The gray lines show the direction of the magnetic field above the strip and below the strip. The strip is a large one.

In space, the magnetic field will be due to the current flowing on the surface and the external magnetic field as well. So, one way for finding the force is to find the external magnetic field, the current flowing on the surface and then use Amperian force formula to find the pressure. However, the situation will be very simplified if we use the concept of pressure and energy.

Imagine that the magnetic field above the strip under consideration is B_1 and below the strip is B_2. In general, the magnetic fields above and below will be different due to the contribution of the strip. Suppose that under the influence of this magnetic field, the strip moves in the upward direction by a distance ds.

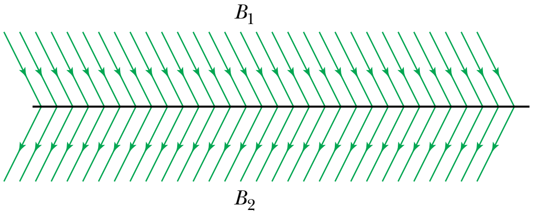


Figure 30-33 Magnetic field pattern near an infinite current-carrying sheet. Here an external uniform magnetic field superimposes with the magnetic field of the sheet to produce a magnetic field B_1 in the space above the sheet and B_2 in a space below the sheet.

The energy related to the magnetic field will change. In the region above, the energy will decrease by

$$\Delta U_1 = \frac{B_1^2}{2\mu_0}A\,ds.$$

In the region above, the energy will increase by

$$\Delta U_2 = \frac{B_2^2}{2\mu_0}A\,ds.$$

We are assuming that the magnetic fields above and below are remaining the same. Actually, the current-carrying strip should be connected to a source of emf which will maintain the current in the strip. That source will do the work. We also know that work done by the external agent is the change in the potential energy of the system:

$$dW = F\,ds = \Delta U_2 - \Delta U_1 = \frac{B_2^2 - B_1^2}{2\mu_0}A\,ds.$$

By definition of pressure,

$$p = \frac{F}{A} = \frac{B_2^2 - B_1^2}{2\mu_0}. \tag{30-63}$$

30.13 | MUTUAL INDUCTION

Key Concept

- If coils 1 and 2 are near each other, a changing current in either coil can induce an emf in the other. This mutual induction is described by

$$\mathscr{E}_2 = -M\frac{di_1}{dt}$$

and

$$\mathscr{E}_1 = -M\frac{di_2}{dt},$$

where M (measured in henries) is the mutual inductance.

In this section we return to the case of two interacting coils, which we first discussed in Section 30.2, and we treat it in a somewhat more formal manner. We saw earlier that if two coils are close together as in Fig. 30-3, a steady current i in one coil will set up a magnetic flux Φ through the other coil (*linking* the other coil). If we change i with time, an emf $\mathscr{E}$ given by Faraday's law appears in the second coil; we called this process *induction*. We could better have called it **mutual induction**, to suggest the mutual interaction of the two coils and to distinguish it from *self-induction*, in which only one coil is involved.

Let us look a little more quantitatively at mutual induction. Figure 30-34*a* shows two circular close-packed coils near each other and sharing a common central axis. With the variable resistor set at a particular resistance R, the

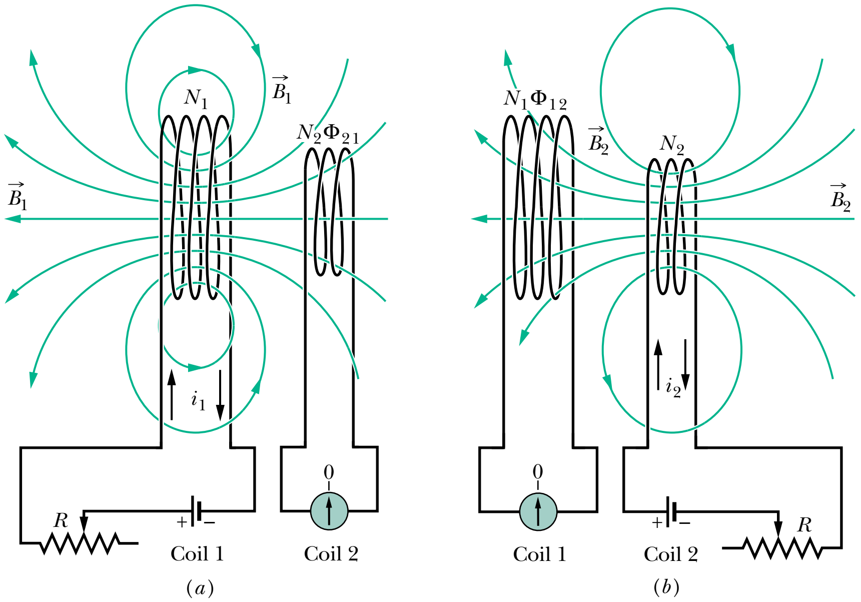


Figure 30-34 Mutual induction. (*a*) The magnetic field $\vec{B}_1$ produced by current i_1 in coil 1 extends through coil 2. If i_1 is varied (by varying resistance R), an emf is induced in coil 2 and current registers on the meter connected to coil 2. (*b*) The roles of the coils interchanged.

battery produces a steady current i_1 in coil 1. This current creates a magnetic field represented by the lines of $\vec{B}_1$ in the figure. Coil 2 is connected to a sensitive meter but contains no battery; a magnetic flux Φ_{21} (the flux through coil 2 associated with the current in coil 1) links the N_2 turns of coil 2.

We define the mutual inductance M_{21} of coil 2 with respect to coil 1 as

$$M_{21} = \frac{N_2\Phi_{21}}{i_1}, \tag{30-64}$$

which has the same form as Eq. 30-30,

$$L = N\Phi/i, \tag{30-65}$$

the definition of inductance. We can recast Eq. 30-64 as

$$M_{21}i_1 = N_2\Phi_{21}. \tag{30-66}$$

If we cause i_1 to vary with time by varying R, we have

$$M_{21}\frac{di_1}{dt} = N_2\frac{d\Phi_{21}}{dt}. \tag{30-67}$$

The right side of this equation is, according to Faraday's law, just the magnitude of the emf $\mathscr{E}_2$ appearing in coil 2 due to the changing current in coil 1. Thus, with a minus sign to indicate direction,

$$\mathscr{E}_2 = -M_{21}\frac{di_1}{dt}, \tag{30-68}$$

which you should compare with Eq. 30-37 for self-induction ($\mathscr{E} = -L\ di/dt$).

Interchange. Let us now interchange the roles of coils 1 and 2, as in Fig. 30-34*b*; that is, we set up a current i_2 in coil 2 by means of a battery, and this produces a magnetic flux Φ_{12} that links coil 1. If we change i_2 with time by varying R, we then have, by the argument given above,

$$\mathscr{E}_1 = -M_{12}\frac{di_2}{dt}. \tag{30-69}$$

Thus, we see that the emf induced in either coil is proportional to the rate of change of current in the other coil. The proportionality constants M_{21} and M_{12} seem to be different. However, they turn out to be the same, although we cannot prove that fact here. Thus, we have

$$M_{21} = M_{12} = M, \tag{30-70}$$

and we can rewrite Eqs. 30-68 and 30-69 as

$$\mathscr{E}_2 = -M\frac{di_1}{dt} \tag{30-71}$$

and

$$\mathscr{E}_1 = -M\frac{di_2}{dt}. \tag{30-72}$$

SAMPLE PROBLEM 30.13

Mutual inductance of two parallel coils

Figure 30-35 shows two circular close-packed coils, the smaller (radius R_2, with N_2 turns) being coaxial with the larger (radius R_1, with N_1 turns) and in the same plane.

(a) Derive an expression for the mutual inductance M for this arrangement of these two coils, assuming that $R_1 \gg R_2$.

KEY IDEA

The mutual inductance M for these coils is the ratio of the flux linkage ($N\Phi$) through one coil to the current i in the other coil, which produces that flux linkage. Thus, we need to assume that currents exist in the coils; then we need to calculate the flux linkage in one of the coils.

Calculations: The magnetic field through the larger coil due to the smaller coil is nonuniform in both magnitude and direction; so the flux through the larger coil due to the smaller coil is nonuniform and difficult to calculate. However, the smaller coil is small enough for us to assume that the magnetic field through it due to the larger coil is approximately uniform. Thus, the flux through it due to the larger coil is also approximately uniform. Hence, to find M we shall assume a current i_1 in the larger coil and calculate the flux linkage $N_2\Phi_{21}$ in the smaller coil:

$$M = \frac{N_2\Phi_{21}}{i_1}. \tag{30-73}$$

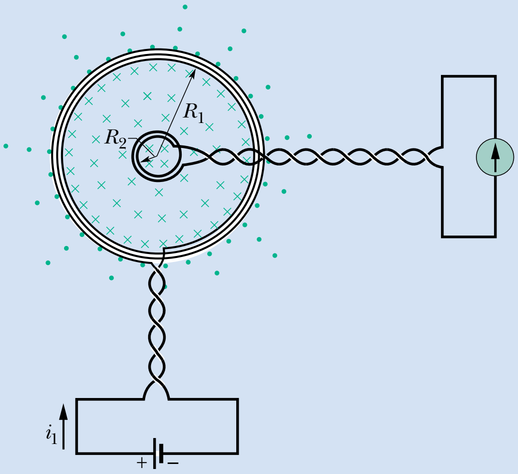


Figure 30-35 A small coil is located at the center of a large coil. The mutual inductance of the coils can be determined by sending current i_1 through the large coil.

The flux Φ_{21} through each turn of the smaller coil is, from Eq. 30-2,

$$\Phi_{21} = B_1A_2,$$

where B_1 is the magnitude of the magnetic field at points within the small coil due to the larger coil and $A_2 (= \pi R_2^2)$ is the area enclosed by the turn. Thus, the flux linkage in the smaller coil (with its N_2 turns) is

$$N_2\Phi_{21} = N_2B_1A_2. \tag{30-74}$$

To find B_1 at points within the smaller coil, we can use Eq. 29-23,

$$B(z) = \frac{\mu_0 iR^2}{2(R^2 + z^2)^{3/2}}.$$

with z set to 0 because the smaller coil is in the plane of the larger coil. That equation tells us that each turn of the larger coil produces a magnetic field of magnitude $\mu_0 i_1/2R_1$ at points within the smaller coil. Thus, the larger coil (with its N_1 turns) produces a total magnetic field of magnitude

$$B_1 = N_1\frac{\mu_0 i_1}{2R_1} \tag{30-75}$$

at points within the smaller coil.

Substituting Eq. 30-75 for B_1 and πR_2^2 for A_2 in Eq. 30-74 yields

$$N_2\Phi_{21} = \frac{\pi\mu_0 N_1N_2R_2^2 i_1}{2R_1}$$

Substituting this result into Eq. 30-73, we find

$$M = \frac{N_2\Phi_{21}}{i_1} = \frac{\pi\mu_0 N_1N_2R_2^2}{2R_1}. \quad \text{(Answer)} \tag{30-76}$$

(b) What is the value of M for $N_1 = N_2 = 1200$ turns, $R_2 = 1.1$ cm, and $R_1 = 15$ cm?

Calculations: Equation 30-76 yields

$$M = \frac{(\pi)(4\pi \times 10^{-7}\,\text{H/m})(1200)(1200)(0.011\,\text{m})^2}{(2)(0.15\,\text{m})}$$
$$= 2.29 \times 10^{-3}\,\text{H} \approx 2.3\,\text{mH}. \quad \text{(Answer)}$$

Consider the situation if we reverse the roles of the two coils—that is, if we produce a current i_2 in the smaller coil and try to calculate M from Eq. 30-64 in the form

$$M = \frac{N_1\Phi_{12}}{i_2}.$$

The calculation of Φ_{12} (the nonuniform flux of the smaller coil's magnetic field encompassed by the larger coil) is not simple. If we were to do the calculation numerically using a computer, we would find M to be 2.3 mH, as above! This emphasizes that Eq. 30-70 ($M_{21} = M_{12} = M$) is not obvious.

SAMPLE PROBLEM 30.14

Induced emf due to two coils wound in the same sense and connected

In Fig. 30-36, coil AA′ and coil BB′ are wound on a long plastic tube in the same sense. Ends A and B are joined together and a source of current I is connected to A′ and B′. Show that the induced emf is equal to $-(L_A + L_B - 2M_{AB})\,(di/dt)$, where L_A and L_B are self-inductances of coils AA′ and BB′, and M is mutual inductance.

KEY IDEA

The self-inductance of each coil is independent of the method of connection. If terminals A and B are connected, the sense of the current flow around coil AA′ is opposite to that around coil BB′. So, we can imagine that inside the plastic tube, the magnetic field will be due to coils AA′ and BB′ both and both the magnetic fields will be in the opposite direction.

We can consider the induced emf as the sum of the induced emf due to the self-inductance and due to mutual inductance.

Calculation: The emf induced across coil BB′ must be

$$V_B = -L_B\frac{di_B}{dt} + M\frac{di_A}{dt}.$$

Note that the magnetic field due to the coil itself and due to coil AA′ are in opposite directions. So, their induced emf will also have opposite signs. Here M is the (magnitude of the) mutual inductance between the coils. Since the coils are connected in series $i_A = i_B = i$, so that

$$V_B = -(L_B - M)\frac{di}{dt}.$$

Similarly, we find that the emf induced across coil AA′ is

$$V_A = -(L_A - M)\frac{di}{dt}.$$

The emf across the entire series coil assembly is the sum of V_A and V_B:

$$V = V_A + V_B = -(L_A + L_B - 2M)\frac{di}{dt}. \qquad \text{(Answer)}$$

Note: The interesting point to note is that if both the coils have the same number of turns, $L_A = L_B$, and since no flux escapes from the sides, $M = \sqrt{L_A L_B}$, so the induced emf in this case turns out to be zero. This is the way we wind the resistances in a resistance box so that the effect of induced emf is cancelled. In the language of electrical engineers, the coil has non inductive winding.

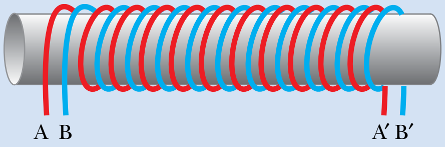


Figure 30-36 Two coils of wire are wound over a plastic tube. The coils are wound in the same sense.

CHECKPOINT 10

Suppose the situation in Fig. 30-36 is changed so that ends A and B′ are joined together and the current source is connected to A′ and B. Show that the induced emf is

$$V = V_A + V_B = -(L_A + L_B + 2M)\frac{di}{dt}.$$

Energy Stored in Two Coils with Mutual Inductance M

Let us now find an expression for the total magnetic energy stored in two coils with inductances L_1 and L_2 and mutual inductance M, when the currents in the coils are I_1 and I_2, respectively. While the currents are building up we have for the emfs:

$$V_1 = -L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \quad \text{and} \quad V_2 = -L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt} \tag{30-77}$$

where the ± appear consistently in both equations, and depends on the coil geometry and circuit sense. Then the external work done in time dt to push charge dq_1 and dq_2 through each circuit, respectively, is

$$\begin{aligned} dW &= -V_1\, dq_1 - V_2\, dq_2 \\ &= L_1 \frac{di_1}{dt} dq_1 \mp M \frac{di_2}{dt} dq_1 + L_2 \frac{di_2}{dt} dq_2 \mp M \frac{di_1}{dt} dq_2 . \end{aligned}$$

Noting that $i_1 = dq_1/dt$ and $i_2 = dq_2/dt$, we have

$$\begin{aligned} dW &= L_1 i_1 di_1 + L_2 i_2 di_2 \mp (M i_1 di_1 + M i_2 di_2) \\ &= L_1 i_1 di_1 + L_2 i_2 di_2 \mp M d(i_1 i_2). \end{aligned}$$

Integrating from 0 to final current, we get total magnetic energy stored as

$$\begin{aligned} U = \int dW &= L_1 \int_0^{I_1} i_1 di_1 + L_2 \int_0^{I_2} i_2 di_2 \mp M \int_0^{I_1 I_2} d(i_1 i_2) \\ &= \tfrac{1}{2} L_1 I_1^{\,2} + \tfrac{1}{2} L_1 I_2^{\,2} \mp M I_1 I_2 . \end{aligned} \tag{30-78}$$

REVIEW AND SUMMARY

Magnetic Flux The *magnetic flux* Φ_B through an area A in a magnetic field $\vec{B}$ is defined as

$$\Phi_B = \int \vec{B} \cdot d\vec{A}, \tag{30-1}$$

where the integral is taken over the area. The SI unit of magnetic flux is the weber, where 1 Wb = 1 T·m². If $\vec{B}$ is perpendicular to the area and uniform over it, Eq. 30-1 becomes

$$\Phi_B = BA \quad (\vec{B} \perp A, \vec{B} \text{ uniform}). \tag{30-2}$$

Faraday's Law of Induction If the magnetic flux Φ_B through an area bounded by a closed conducting loop changes with time, a current and an emf are produced in the loop; this process is called *induction*. The induced emf is

$$\mathscr{E} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law).} \tag{30-4}$$

If the loop is replaced by a closely packed coil of N turns, the induced emf is

$$\mathscr{E} = -N \frac{d\Phi_B}{dt}. \tag{30-5}$$

Lenz's Law An induced current has a direction such that the magnetic field *due to the current* opposes the change in the magnetic flux that induces the current. The induced emf has the same direction as the induced current.

Emf and the Induced Electric Field An emf is induced by a changing magnetic flux even if the loop through which the flux is changing is not a physical conductor but an imaginary line. The changing magnetic field induces an electric field $\vec{E}$ at every point of such a loop; the induced emf is related to $\vec{E}$ by

$$\mathscr{E} = \oint \vec{E} \cdot d\vec{s}, \tag{30-21}$$

where the integration is taken around the loop. From Eq. 30-21 we can write Faraday's law in its most general form,

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law).} \tag{30-22}$$

A changing magnetic field induces an electric field $\vec{E}$.

Inductors An **inductor** is a device that can be used to produce a known magnetic field in a specified region. If a current i is established through each of the N windings of an inductor, a magnetic flux Φ_B links those windings. The **inductance** L of the inductor is

$$L = \frac{N\Phi_B}{i} \quad \text{(inductance decfined).} \qquad (30\text{-}30)$$

The SI unit of inductance is the **henry** (H), where 1 henry = 1 H = 1 T · m²/A. The inductance per unit length near the middle of a long solenoid of cross-sectional area A and n turns per unit length is

$$\frac{L}{l} = \mu_0 n^2 A \quad \text{(solenoid).} \qquad (30\text{-}33)$$

Self-Induction If a current i in a coil changes with time, an emf is induced in the coil. This self-induced emf is

$$\mathscr{E}_L = -L\frac{di}{dt}. \qquad (30\text{-}37)$$

The direction of $\mathscr{E}_L$ is found from Lenz's law: The self-induced emf acts to oppose the change that produces it.

Series *RL* Circuits If a constant emf $\mathscr{E}$ is introduced into a single-loop circuit containing a resistance R and an inductance L, the current rises to an equilibrium value of $\mathscr{E}/R$:

$$i = \frac{\mathscr{E}}{R}(1 - e^{-t/\tau_L}) \quad \text{(rise of current).} \qquad (30\text{-}43)$$

Here τ_L (= L/R) is the **inductive time constant**. When the source of constant emf is removed, the current decays from a value i_0 according to

$$i = i_0 e^{-t/\tau_L} \quad \text{(decay of current).} \qquad (30\text{-}47)$$

Magnetic Energy If an inductor L carries a current i, the inductor's magnetic field stores an energy given by

$$U_B = \frac{1}{2}Li^2 \quad \text{(magnetic energy).} \qquad (30\text{-}51)$$

If B is the magnitude of a magnetic field at any point (in an inductor or anywhere else), the density of stored magnetic energy at that point is

$$u_B = \frac{B^2}{2\mu_0} \quad \text{(magnetic energy density).} \qquad (30\text{-}57)$$

Mutual Induction If coils 1 and 2 are near each other, a changing current in either coil can induce an emf in the other. This mutual induction is described by

$$\mathscr{E}_2 = -M\frac{di_1}{dt} \qquad (30\text{-}71)$$

and

$$\mathscr{E}_1 = -M\frac{di_2}{dt}, \qquad (30\text{-}72)$$

where M (measured in henries) is the mutual inductance.

PROBLEMS

1. In Fig. 30-37, a metal rod is forced to move with constant velocity $\vec{v}$ along two parallel metal rails, connected with a strip of metal at one end. A magnetic field of magnitude B = 0.125 T points out of the page. (a) If the rails are separated by L = 25.0 cm and the speed of the rod is 38.0 cm/s, what emf is generated? (b) If the rod has a resistance of 18.0 Ω and the rails and connector have negligible resistance, what is the current in the rod? (c) At what rate is energy being transferred to thermal energy? (d) What is the magnitude of the leftward force that causes the rod to move?

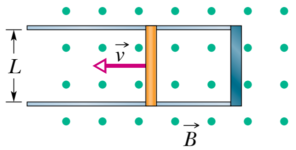


Figure 30-37 Problems 1 and 3.

2. Coil 1 has L_1 = 25 mH and N_1 = 100 turns. Coil 2 has L_2 = 40 mH and N_2 = 200 turns. The coils are fixed in place; their mutual inductance M is 9.0 mH. A 6.0 mA current in coil 1 is changing at the rate of 4.0 A/s. (a) What magnetic flux Φ_{12} links coil 1, and (b) what self-induced emf appears in that coil? (c) What magnetic flux Φ_{21} links coil 2, and (d) what mutually induced emf appears in that coil?

3. The conducting rod shown in Fig. 30-37 has length L and is being pulled along horizontal, frictionless conducting rails at a constant velocity $\vec{v}$. The rails are connected at one end with a metal strip. A uniform magnetic field $\vec{B}$, directed out of the page, fills the region in which the rod moves. Assume that L = 10 cm, v = 5.0 m/s, and B = 1.2 T. What are the (a) magnitude and (b) direction (up or down the page) of the emf induced in the rod? What are the (c) size and (d) direction of the current in the conducting loop? Assume that the resistance of the rod is 0.40 Ω and that the resistance of the rails and metal strip is negligibly small. (e) At what rate is thermal energy being generated in the rod? (f) What external force on the rod is needed to maintain $\vec{v}$? (g) At what rate does this force do work on the rod?

4. A coil C of N turns is placed around a long solenoid S of radius R and n turns per unit length, as in Fig. 30-38. (a) Show that the mutual inductance for the coil–solenoid combination is given by $M = \mu_0 \pi R^2 nN$. (b) Explain why M does not depend on the shape, size, or possible lack of close packing of the coil.

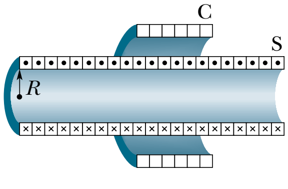


Figure 30-38 Problem 4.

5. If 50.0 cm of copper wire (diameter = 3.00 mm) is formed into a circular loop and placed perpendicular to a uniform magnetic field that is increasing at the constant rate of 30.0 mT/s, at what rate is thermal energy generated in the loop?

6. Figure 30-39 shows two parallel loops of wire having a common axis. The smaller loop (radius r) is above the larger loop (radius R) by a distance $x \gg R$. Consequently, the magnetic field due to the counterclockwise current i in the larger loop is nearly uniform throughout the smaller loop. Suppose that x is increasing at the constant rate $dx/dt = v$. (a) Find an expression for the magnetic flux through the area of the smaller loop as a function of x. In the smaller loop, find (b) an expression for the induced emf and (c) the direction of the induced current.

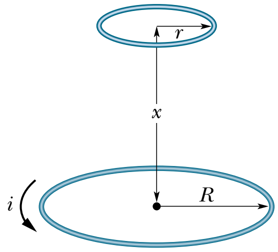


Figure 30-39 Problem 6.

7. Suppose the emf of the battery in the circuit shown in Fig. 30-31 varies with time t so that the current is given by $i(t) = 3.0 + 5.0t$, where i is in amperes and t is in seconds. Take $R = 4.0\ \Omega$ and $L = 6.0$ H, and find an expression for the battery emf as a function of t. (*Hint:* Apply the loop rule.)

8. The magnetic field of a cylindrical magnet that has a pole-face diameter of 3.3 cm can be varied sinusoidally between 29.6 T and 31.2 T at a frequency of 15 Hz. (The current in a wire wrapped around a permanent magnet is varied to give this variation in the net field.) At a radial distance of 1.6 cm, what is the amplitude of the electric field induced by the variation?

9. At $t = 0$, a battery is connected to a series arrangement of a resistor and an inductor. At what multiple of the inductive time constant will the energy stored in the inductor's magnetic field be 0.500 of its steady-state value?

10. A 12 H inductor carries a current of 2.0 A. At what rate must the current be changed to produce a 60 V emf in the inductor?

11. A rectangular loop of N closely packed turns is positioned near a long straight wire as shown in Fig. 30-40. What is the mutual inductance M for the loop–wire combination if $N = 150$, $a = 1.0$ cm, $b = 9.5$ cm, and $l = 30$ cm?

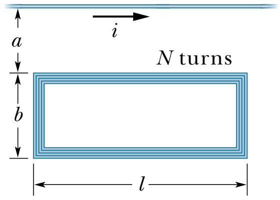


Figure 30-40 Problem 11.

12. In Fig. 30-41, $\mathscr{E} = 100$ V, $R_1 = 10.0\ \Omega$, $R_2 = 20.0\ \Omega$, $R_3 = 31.0\ \Omega$, and $L = 2.00$ H. Immediately after switch S is closed, what are (a) i_1 and (b) i_2? (Let currents in the indicated directions have positive values and currents in the opposite directions have negative values.) A long time later, what are (c) i_1 and (d) i_2? The switch is then reopened. Just then, what are (e) i_1 and (f) i_2? A long time later, what are (g) i_1 and (h) i_2?

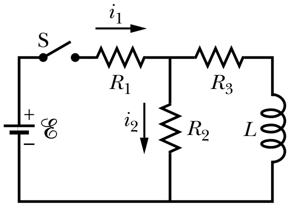


Figure 30-41 Problem 12.

13. A coil is connected in series with a 10.0 kΩ resistor. An ideal 50.0 V battery is applied across the two devices, and the current reaches a value of 2.00 mA after 5.00 ms. (a) Find the inductance of the coil. (b) How much energy is stored in the coil at this same moment?

14. Two coils are at fixed locations. When coil 1 has no current and the current in coil 2 increases at the rate 21.0 A/s, the emf in coil 1 is 25.0 mV. (a) What is their mutual inductance? (b) When coil 2 has no current and coil 1 has a current of 1.35 A, what is the flux linkage in coil 2?

15. The switch in Fig. 30-28 is closed on a at time $t = 0$. What is the ratio $\mathscr{E}_L/\mathscr{E}$ of the inductor's self-induced emf to the battery's emf (a) just after $t = 0$ and (b) at $t = 2.00\tau_L$? (c) At what multiple of τ_L will $\mathscr{E}_L/\mathscr{E} = 0.500$?

16. Two identical long wires of radius $a = 0.530$ mm are parallel and carry identical currents in opposite directions. Their center-to-center separation is $d = 20.0$ cm. Neglect the flux within the wires but consider the flux in the region between the wires. What is the inductance per unit length of the wires?

17. In Fig. 30-42, a wire forms a closed circular loop, of radius $R = 0.32$ m and resistance $0.056\ \Omega$. The circle is centered on a long straight wire; at time $t = 0$, the current in the long straight wire is 5.0 A rightward. Thereafter, the current changes according to $i = 5.0\text{ A} - (2.0\text{ A/s}^2)t^2$. (The straight wire is insulated; so there is no electrical contact between it and the wire of the loop.) What is the magnitude of the current induced in the loop at times $t > 0$?

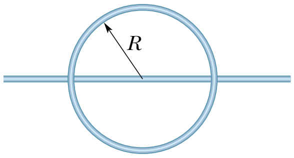


Figure 30-42 Problem 17.

18. The current in an RL circuit drops from 2.30 A to 3.40 mA in 0.025 s following removal of the battery from the circuit. If L is 10 H, find the resistance R in the circuit.

19. A coil with an inductance of 2.0 H and a resistance of 10 Ω is suddenly connected to an ideal battery with $\mathscr{E} = 100$ V. At 0.10 s after the connection is made, what is the rate at which (a) energy is being stored in the magnetic field, (b) thermal energy is appearing in the resistance, and (c) energy is being delivered by the battery?

20. A length of copper wire carries a current of 3.5 A uniformly distributed through its cross section. Calculate the energy density of (a) the magnetic field and (b) the electric field at the surface of the wire. The wire diameter is 2.5 mm, and its resistance per unit length is 3.3 Ω/km.

21. In Fig. 30-43, two straight conducting rails form a right angle. A conducting bar in contact with the rails starts at the vertex at time $t = 0$ and moves with a constant velocity of 5.20 m/s along them. A magnetic field with $B = 0.350$ T is directed out of the page. Calculate (a) the flux through the triangle formed by the rails

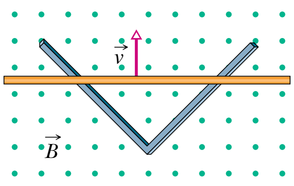


Figure 30-43 Problem 21.

and bar at $t = 3.00$ s and (b) the emf around the triangle at that time. (c) If the emf is $\mathscr{E} = at^n$, where a and n are constants, what is the value of n?

22. A solenoid having an inductance of 9.70 μH is connected in series with a 1.20 kΩ resistor. (a) If a 14.0 V battery is connected across the pair, how long will it take for the current through the resistor to reach 40.0% of its final value? (b) What is the current through the resistor at time $t = 0.50\tau_L$?

23. A solenoid that is 85.0 cm long has a cross-sectional area of 17.0 cm^2. There are 1210 turns of wire carrying a current of 6.60 A. (a) Calculate the energy density of the magnetic field inside the solenoid. (b) Find the total energy stored in the magnetic field there (neglect end effects).

24. A rectangular loop (area = 0.15 m^2) turns in a uniform magnetic field, $B = 0.20$ T. When the angle between the field and the normal to the plane of the loop is $\pi/2$ rad and increasing at 0.60 rad/s, what emf is induced in the loop?

25. For the circuit of Fig. 30-29, assume that $\mathscr{E} = 10.0$ V, $R = 112\ \Omega$, and $L = 5.50$ H. The ideal battery is connected at time $t = 0$. (a) How much energy is delivered by the battery during the first 2.00 s? (b) How much of this energy is stored in the magnetic field of the inductor? (c) How much of this energy is dissipated in the resistor?

26. In Fig. 30-44, after switch S is closed at time $t = 0$, the emf of the source is automatically adjusted to maintain a constant current i through S. (a) Find the current through the inductor as a function of time. (b) At what time is the current through the resistor equal to the current through the inductor?

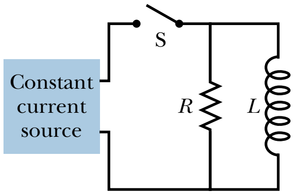


Figure 30-44 Problem 26.

27. A toroidal inductor with an inductance of 90 mH encloses a volume of 0.0200 m^3. If the average energy density in the toroid is 70.0 J/m^3, what is the current through the inductor?

28. At $t = 0$, a battery is connected to a series arrangement of a resistor and an inductor. If the inductive time constant is 60.0 ms, at what time is the rate at which energy is dissipated in the resistor equal to the rate at which energy is stored in the inductor's magnetic field?

29. Two solenoids are part of the spark coil of an automobile. When the current in one solenoid falls from 6.0 A to zero in 2.5 ms, an emf of 31 kV is induced in the other solenoid. What is the mutual inductance M of the solenoids?

30. A small circular loop of area 2.00 cm^2 is placed in the plane of, and concentric with, a large circular loop of radius 1.00 m. The current in the large loop is changed at a constant rate from 50.0 A to −50.0 A (a change in direction) in a time of 1.00 s, starting at $t = 0$. What is the magnitude of the magnetic field $\vec{B}$ at the center of the small loop due to the current in the large loop at (a) $t = 0$, (b) $t = 0.500$ s, and (c) $t = 1.00$ s? (d) From $t = 0$ to $t = 1.00$ s, is $\vec{B}$ reversed? Because the inner loop is small, assume $\vec{B}$ is uniform over its area. (e) What emf is induced in the small loop at $t = 0.500$ s?

31. In Fig. 30-45, a rectangular loop of wire with length $a = 2.2$ cm, width $b = 0.80$ cm, and resistance $R = 0.40$ mΩ is placed near an infinitely long wire carrying current $i = 4.7$ A. The loop is then moved away from the wire at constant speed $v = 3.2$ mm/s. When the center of the loop is at distance $r = 1.5b$, what are (a) the magnitude of the magnetic flux through the loop and (b) the current induced in the loop?

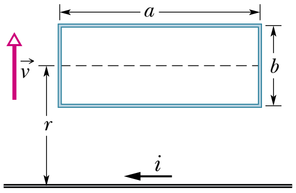


Figure 30-45 Problem 31.

32. In Fig. 30-46, a stiff wire bent into a semicircle of radius $a = 1.4$ cm is rotated at constant angular speed 30 rev/s in a uniform 20 mT magnetic field. What are the (a) frequency and (b) amplitude of the emf induced in the loop?

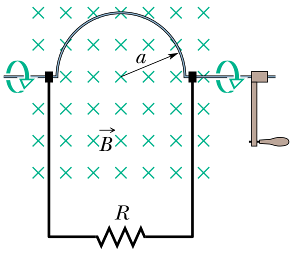


Figure 30-46 Problem 32.

33. The current in an RL circuit builds up to one-third of its steady-state value in 5.00 s. Find the inductive time constant.

34. As seen in Fig. 30-47, a square loop of wire has sides of length 3.0 cm. A magnetic field is directed out of the page; its magnitude is given by $B = 5.0t^2y$, where B is in teslas, t is in seconds, and y is in meters. At $t = 2.5$ s, what are the (a) magnitude and (b) direction of the emf induced in the loop?

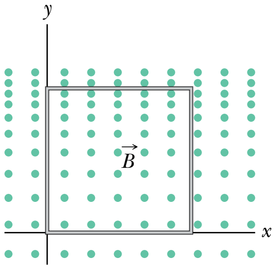


Figure 30-47 Problem 34.

35. What must be the magnitude of a uniform electric field if it is to have the same energy density as that possessed by a 20 mT magnetic field?

36. A long solenoid has a diameter of 12.0 cm. When a current i exists in its windings, a uniform magnetic field of magnitude $B = 30.0$ mT is produced in its interior. By decreasing i, the field is caused to decrease at the rate of 6.50 mT/s. Calculate the magnitude of the induced electric field (a) 4.20 cm and (b) 10.3 cm from the axis of the solenoid.

37. *Inductors in series.* Two inductors L_1 and L_2 are connected in series and are separated by a large distance so that the magnetic field of one cannot affect the other. (a) Show that the equivalent inductance is given by

$$L_{eq} = L_1 + L_2.$$

(*Hint:* Review the derivations for resistors in series and capacitors in series. Which is similar here?) (b) What is the generalization of (a) for N inductors in series?

38. Two coils connected as shown in Fig. 30-48 separately have inductances L_1 and L_2. Their mutual inductance is M. (a) Show that this combination can be replaced by a single coil of equivalent inductance given by

$$L_{eq} = L_1 + L_2 + 2M.$$

(b) How could the coils in Fig. 30-48 be reconnected to yield an equivalent inductance of

$$L_{eq} = L_1 + L_2 - 2M?$$

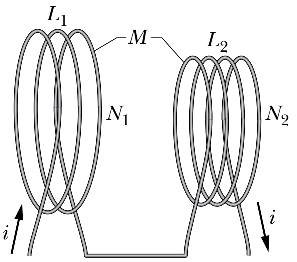


Figure 30-48 Problem 38.

39. *Inductors in parallel.* Two inductors L_1 and L_2 are connected in parallel and separated by a large distance so that the magnetic field of one cannot affect the other. (a) Show that the equivalent inductance is given by

$$\frac{1}{L_{eq}} = \frac{1}{L_1} = \frac{1}{L_2}.$$

(*Hint:* Review the derivations for resistors in parallel and capacitors in parallel. Which is similar here?) (b) What is the generalization of (a) for N inductors in parallel?

40. A small loop of area 3.1 mm^2 is placed inside a long solenoid that has 672 turns/cm and carries a sinusoidally varying current i of amplitude 1.28 A and angular frequency 212 rad/s. The central axes of the loop and solenoid coincide. What is the amplitude of the emf induced in the loop?

41. At a certain place, Earth's magnetic field has magnitude $B = 0.590$ gauss and is inclined downward at an angle of 70.0° to the horizontal. A flat horizontal circular coil of wire with a radius of 10.0 cm has 1000 turns and a total resistance of 85.0 Ω. It is connected in series to a meter with 140 Ω resistance. The coil is flipped through a half-revolution about a diameter, so that it is again horizontal. How much charge flows through the meter during the flip?

42. A square wire loop with 2.00 m sides is perpendicular to a uniform magnetic field, with half the area of the loop in the field as shown in Fig. 30-49. The loop contains an ideal battery with emf $\mathscr{E} = 12.0$ V. If the magnitude of the field varies with time according to $B = 0.603 - 1.25t$, with B in teslas and t in seconds, what are (a) the net emf in the circuit and (b) the direction of the (net) current around the loop?

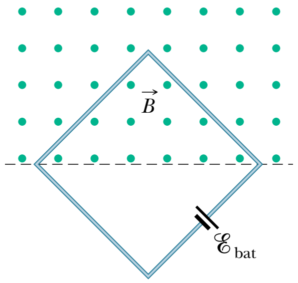


Figure 30-49 Problem 42.

43. Figure 30-50 shows a rod of length $L = 12.0$ cm that is forced to move at constant speed $v = 5.00$ m/s along horizontal rails. The rod, rails, and connecting strip at the right form a conducting loop. The rod has resistance 0.400 Ω; the rest of the loop has negligible resistance. A current $i = 100$ A through the long straight wire at distance $a = 5.00$ mm from the loop sets up a (nonuniform) magnetic field through the loop. Find the (a) emf and (b) current induced in the loop. (c) At what rate is thermal energy generated in the rod? (d) What is the magnitude of the force that must be applied to the rod to make it move at constant speed? (e) At what rate does this force do work on the rod?

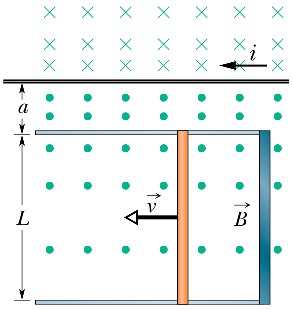


Figure 30-50 Problem 43.

44. A wooden toroidal core with a square cross section has an inner radius of 10 cm and an outer radius of 12 cm. It is wound with one layer of wire (of diameter 1.0 mm and resistance per meter 0.020 Ω/m). What are (a) the inductance and (b) the inductive time constant of the resulting toroid? Ignore the thickness of the insulation on the wire.

45. At a given instant the current and self-induced emf in an inductor are directed as indicated in Fig. 30-51. (a) Is the current increasing or decreasing? (b) The induced emf is 23 V, and the rate of change of the current is 18 kA/s; find the inductance.

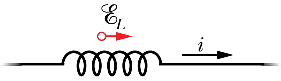


Figure 30-51 Problem 45.

46. A battery is connected to a series RL circuit at time $t = 0$. At what multiple of τ_L will the current be 1.00% less than its equilibrium value?

47. A circular region in an xy plane is penetrated by a uniform magnetic field in the positive direction of the z axis. The field's magnitude B (in teslas) increases with time t (in seconds) according to $B = at$, where a is a constant. The magnitude E of the electric field set up by that increase in the magnetic field is given by Fig. 30-52 versus radial distance r; the vertical axis scale is set by

E_s = 310 μN/C, and the horizontal axis scale is set by r_s = 4.00 cm. Find a.

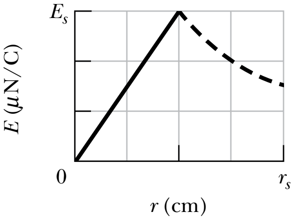


Figure 30-52 Problem 47.

48. A circular coil has a 15.0 cm radius and consists of 30.0 closely wound turns of wire. An externally produced magnetic field of magnitude 2.60 mT is perpendicular to the coil. (a) If no current is in the coil, what magnetic flux links its turns? (b) When the current in the coil is 2.80 A in a certain direction, the net flux through the coil is found to vanish. What is the inductance of the coil?

49. A certain elastic conducting material is stretched into a circular loop of 12.0 cm radius. It is placed with its plane perpendicular to a uniform 0.800 T magnetic field. When released, the radius of the loop starts to shrink at an instantaneous rate of 75.0 cm/s. What emf is induced in the loop at that instant?

50. The inductor arrangement of Fig. 30-53, with L_1 = 50.0 mH, L_2 = 80.0 mH, L_3 = 20.0 mH, and L_4 = 15.0 mH, is to be connected to a varying current source. What is the equivalent inductance of the arrangement?

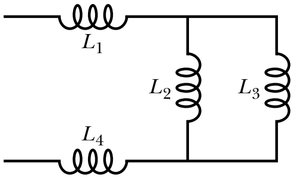


Figure 30-53 Problem 50.

51. Two long, parallel copper wires of diameter 4.0 mm carry currents of 7.0 A in opposite directions. (a) Assuming that their central axes are 20 mm apart, calculate the magnetic flux per meter of wire that exists in the space between those axes. (b) What percentage of this flux lies inside the wires? (c) Repeat part (a) for parallel currents.

52. In Fig. 30-54, a 120-turn coil of radius 1.8 cm and resistance 5.3 Ω is coaxial with a solenoid of 220 turns/cm and diameter 3.2 cm. The solenoid current drops from 0.42 A to zero in time interval Δt = 0.67 s. What current is induced in the coil during Δt?

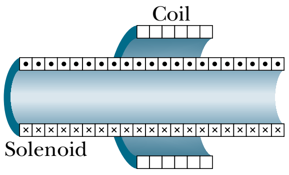


Figure 30-54 Problem 52.

53. A loop antenna of area 2.00 cm^2 and resistance 5.21 $\mu\Omega$ is perpendicular to a uniform magnetic field of magnitude 17.0 μT. The field magnitude drops to zero in 2.96 ms. How much thermal energy is produced in the loop by the change in field?

54. Figure 30-55 shows two circular regions R_1 and R_2 with radii r_1 = 20.0 cm and r_2 = 31.0 cm. In R_1 there is a uniform magnetic field of magnitude B_1 = 50.0 mT directed into the page, and in R_2 there is a uniform magnetic field of magnitude B_2 = 75.0 mT directed out of the page (ignore fringing). Both fields are decreasing at the rate of 8.50 mT/s. Calculate $\oint \vec{E} \cdot d\vec{s}$ for (a) path 1, (b) path 2, and (c) path 3.

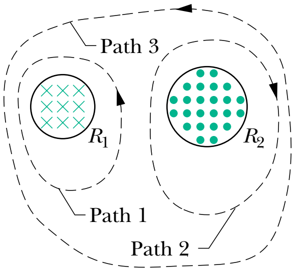


Figure 30-55 Problem 54.

55. In Fig. 30-56, a circular loop of wire 10 cm in diameter (seen edge-on) is placed with its normal $\vec{N}$ at an angle θ = 20° with the direction of a uniform magnetic field $\vec{B}$ of magnitude 1.5 T. The loop is then rotated such that $\vec{N}$ rotates in a cone about the field direction at the rate 100 rev/min; angle θ remains unchanged during the process. What is the emf induced in the loop?

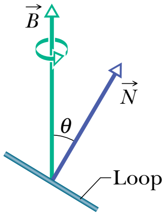


Figure 30-56 Problem 55.

56. The inductance of a closely packed coil of 400 turns is 8.0 mH. Calculate the magnetic flux through the coil when the current is 5.0 mA.

57. One hundred turns of (insulated) copper wire are wrapped around a wooden cylindrical core of cross-sectional area 1.90 × 10^{-3} m^2. The two ends of the wire are connected to a resistor. The total resistance in the circuit is 9.50 Ω. If an externally applied uniform longitudinal magnetic field in the core changes from 1.60 T in one direction to 1.60 T in the opposite direction, how much charge flows through a point in the circuit during the change?

58. A wire is bent into three circular segments, each of radius r = 10 cm, as shown in Fig. 30-57. Each segment is a quadrant of a circle, ab lying in the xy plane, bc lying in the yz plane, and ca lying in the zx plane. (a) If a uniform magnetic field $\vec{B}$ points in the positive x direction, what is the magnitude of the emf developed in the wire when B increases at the rate of 3.0 mT/s? (b) What is the direction of the current in segment bc?

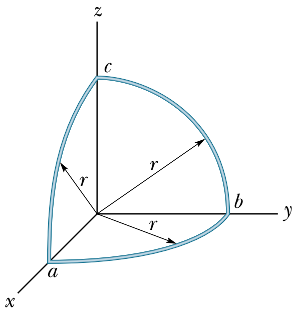


Figure 30-57 Problem 58.

59. In Fig. 30-58, R = 15 Ω, L = 15 H, the ideal battery has $\mathscr{E}$ = 10 V, and the fuse in the upper branch is an ideal 3.0 A fuse. It has zero resistance as long as the current through it remains less than 3.0 A. If the current reaches 3.0 A, the fuse "blows" and thereafter has infinite resistance. Switch S is closed at time $t = 0$. (a) When does the fuse blow? (b) Sketch a graph of the current i through the inductor as a function of time. Mark the time at which the fuse blows.

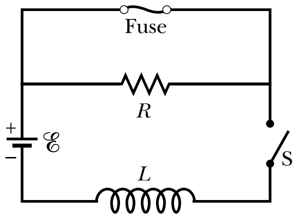


Figure 30-58 Problem 59.

60. In Fig. 30-59, a long rectangular conducting loop, of width L, resistance R, and mass m, is hung in a horizontal, uniform magnetic field $\vec{B}$ that is directed into the page and that exists only above line aa. The loop is then dropped; during its fall, it accelerates until it reaches a certain terminal speed v_t. Ignoring air drag, find an expression for v_t.

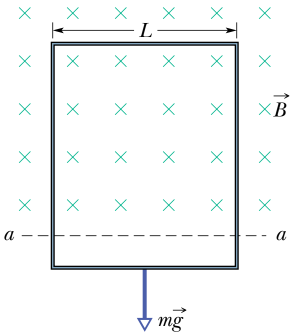


Figure 30-59 Problem 60.

61. In Fig. 30-60, the magnetic flux through the loop increases according to the relation $\Phi_B = 3.0t^2 = 7.0t$, where Φ_B is in milliwebers and t is in seconds. (a) What is the magnitude of the emf induced in the loop when $t = 1.5$ s? (b) Is the direction of the current through R to the right or left?

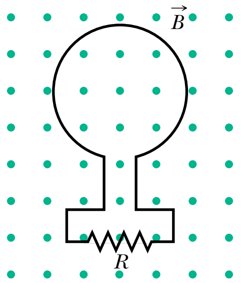


Figure 30-60 Problem 61.

62. Figure 30-61 shows a uniform magnetic field $\vec{B}$ confined to a cylindrical volume of radius R. The magnitude of $\vec{B}$ is decreasing at a constant rate of 10 mT/s. In unit-vector notation, what is the initial acceleration of an electron released at (a) point a (radial distance $r = 5.0$ cm), (b) point b ($r = 0$), and (c) point c ($r = 5.0$ cm)?

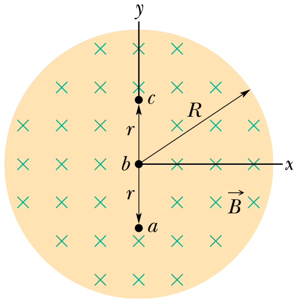


Figure 30-61 Problem 62.

63. A square loop 10 cm to a side is placed on a wooden table in a uniform magnetic field of magnitude 0.25 T. It is found that the biggest magnetic flux through the loop is measured when the loop is flat on the table. What is the flux through the loop when it is tilted such that the plane of the loop makes a 60° angle with the table?

64. The magnetic flux through each of five faces of a die (singular of "dice") is given by $\Phi = \pm N$ Wb, where N (= 1 to 5) is the number of spots on the face. The flux is positive (outward) for N even and negative (inward) for N odd. What is the flux through the sixth face of the die?

65. Initially there is no magnetic flux through a certain conducting loop. A magnetic field near the loop is then suddenly turned on, and 5 s later, the magnetic flux through the loop is 1.0 Tm². What is the average magnitude of the induced emf in the loop during those 5 s?

66. At the Earth's equator, B is horizontal and to the north, with a magnitude of approximately 7×10^{-5} T. (a) What is the magnetic flux through a loop of radius 0.1 m, lying flat on the ground? (b) If the same loop is balanced on its edge such that its axis points northwest, what is the magnetic flux through the loop? (c) If the loop is now rotated such that its axis points north, what is the magnetic flux through the loop? (d) If the loop is now rotated such that its axis points west, what is the magnetic flux through the loop?

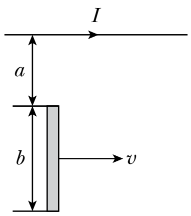
67. An airliner with a wingspan of 33.5 m travels toward the east at a speed of 280 m/s in a region where the Earth's magnetic field is 4.2×10^{-5} T directed toward the north and 53° below horizontal. (a) What is the induced emf between the plane's wingtips? (b) Will a bulb glow if we connect the tip of wings by a conducting wire with a bulb in between?

68. A cubical region of space with side lengths 0.05 m contains a uniform magnetic field. If the total magnetic energy stored in this region is π J, what is the magnitude of the magnetic field?

PRACTICE QUESTIONS

Single Correct Choice Type

1. A rod of length b moves with a constant velocity v in the magnetic field of a straight long conductor that carries a current I as shown in the below figure. The emf induced in the rod is



(a) $\dfrac{\mu_0 I v}{2\pi}\tan^{-1}\left(\dfrac{a}{b}\right)$

(b) $\dfrac{\mu_0 I v}{2\pi}\ln\left(1+\dfrac{a}{b}\right)$

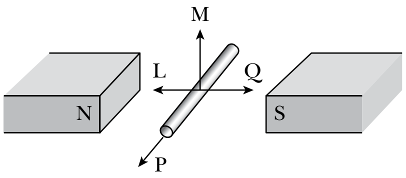
(c) $\dfrac{\mu_0 I v\sqrt{ab}}{2\pi(a+b)}$

(d) $\dfrac{\mu_0 I v(a+b)}{4\pi ab}$

2. A car moves on a plane road, induced emf produced across the axis is maximum when it
(a) Moves at the pole
(b) Moves at equator
(c) Remain stationary
(d) No emf induced at all

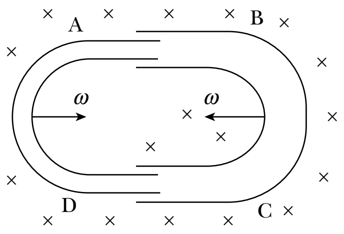
3. The current is flowing in two coaxial coils in the same direction. On increasing the distance between the two, the electric current will
(a) Increase
(b) Decrease
(c) Remain unchanged
(d) The information is incomplete

4. An electric potential difference will be induced between the ends of the conductor shown in the figure below when the conductor moves in the direction



(a) P
(b) Q
(c) L
(d) M

5. One conducting U tube can slide inside another as shown in the below figure, maintaining electrical contacts between the tubes. The magnetic field B is perpendicular to the plane of the figure. If each tube moves toward the other at a constant speed v then the emf induced in the circuit in terms of B, l, and v where l is the width of each tube, will be



(a) Zero
(b) $2B/v$
(c) B/v
(d) $-B/v$

6. Find the inductance L of a solenoid of length l whose windings are made of material of density D and resistivity r. The winding resistance is ρ

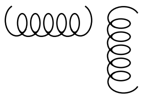
(a) $\dfrac{\mu_0}{4\pi l}\times\dfrac{Rm}{\rho D}$

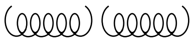
(b) $\dfrac{\mu_0}{4\pi R}\times\dfrac{lm}{\rho D}$

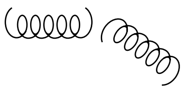
(c) $\dfrac{\mu_0}{4\pi l}\times\dfrac{R^2 m}{\rho D}$

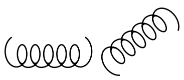
(d) $\dfrac{\mu_0}{2\pi R}\times\dfrac{lm}{\rho D}$

7. For which arrangement of two coils, shown in the below figure, coefficient of mutual inductance is maximum and for which it is minimum?

(a)


(b)


(c)


(d)


(a) For (a) and (d), respectively
(b) For (b) and (a), respectively
(c) For (c) and (a), respectively
(d) For (a) and (d), respectively

8. Find the inductance of a solenoid of length l whose winding is made of copper wire of mass m. The winding resistance is equal to R. The solenoid is considerably less than its length.

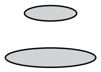
(a) $\dfrac{\mu_0 R}{4\pi m l \rho\rho_0}$

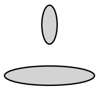
(b) $\dfrac{\mu_0 l \rho\rho_0}{4\pi m R}$

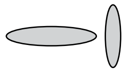
(c) $\dfrac{\mu_0}{4\pi}\times\dfrac{mR}{l\rho\rho_0}$

(d) None of these

9. Two circular coils can be arranged in any of the three situations shown in the following figure. Their mutual inductance will be

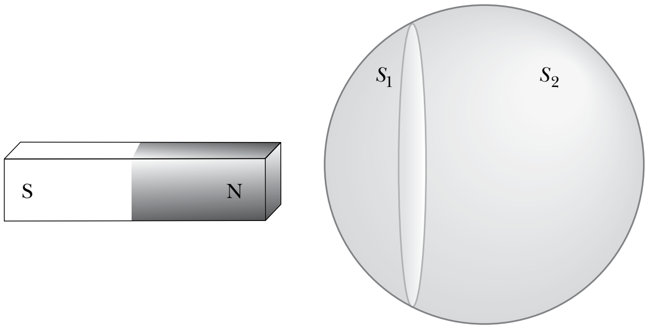
(i)


(ii)


(iii)


- **(a)** Maximum in situation depicted in figure (i)
- **(b)** Maximum in situation depicted in figure (ii)
- **(c)** Maximum in situation depicted in figure (iii)
- **(d)** The same in all situations

10. A spherical surface is split into two parts, S_1 and S_2, by a circular loop. The sphere is placed near a bar magnet, as shown in the following figure.



Through which of the two parts is the magnitude of the magnetic flux larger?

- **(a)** S_1
- **(b)** S_2
- **(c)** The magnitude of the flux is the same for both
- **(d)** Cannot tell without more information about the magnetic field

11. A sphere of radius R is placed near a long, straight wire that carries a steady current I. The magnetic field generated by the current is B. The total magnetic flux passing through the sphere is

- **(a)** $\mu_0 I$
- **(b)** $\mu_0 I/(4\pi R^2)$
- **(c)** Zero
- **(d)** Need more information

12. A coil of wire rotates about a horizontal north–south axis. Assume that at that point the magnetic field is along North direction. How does the Earth's magnetic field contribute to the emf induced in this coil?

- **(a)** Both the horizontal and vertical components contribute to the induced emf
- **(b)** Only the horizontal component induces an emf
- **(c)** Only the vertical component induces an emf
- **(d)** There is no emf induced in the coil

13. A magnet moves inside a coil. Which of the following factors can affect the emf induced in the coil?

I. The speed at which the magnet moves

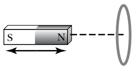
II. The magnetic field of the magnet

III. The number of turns in the coil

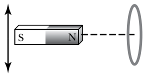
- **(a)** I only
- **(b)** I and II only
- **(c)** II and III only
- **(d)** I, II and III

14. A magnet is in form of a bar as shown in the figure. The line joining the center of the bar and the center of the coil (central axis) is perpendicular to the plane of the coil. Which of the following motions of the bar will not induce electric current in the coil?

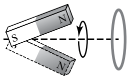
- **(a)** Translation motion back and forth



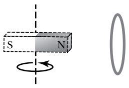
- **(b)** Translation motion up and down



- **(c)** The bar axis is at angle to the central axis and spins around it.



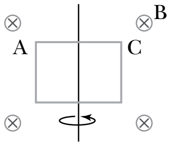
- **(d)** The bar spins around an axis in the paper plane.



15. A rectangular loop of wire is placed perpendicular to a uniform magnetic field and then spun around one of its sides at frequency f. The induced emf is a maximum when

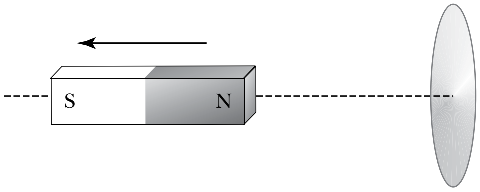
- **(a)** The flux is zero
- **(b)** The flux is a maximum
- **(c)** The flux is half its maximum value
- **(d)** The derivative of the flux with respect to time is zero

16. A loop of wire is turned round the vertical axis by 180° in a uniform magnetic field so that at the beginning of motion the point A moves toward the observer while the point C moves away from the observer. The direction of current, induced in the loop, is



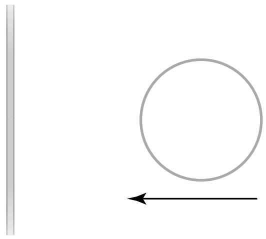
- **(a)** Permanently from A to C
- **(b)** Permanently from C to A
- **(c)** From A to C during the first half of the turn and then has the opposite direction
- **(d)** From C to A during the first half of the turn and then has the opposite direction

17. A bar magnet with its north (N) and south (S) poles as shown in the following figure is initially moving to the left, along the axis of, and away from a circular conducting loop. A current I is induced in the loop, with a being the acceleration of the magnet due to this current. As seen from the magnet looking in the direction of the loop,



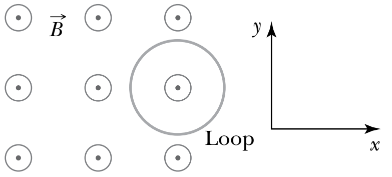
- **(a)** I runs clockwise and a points to the left
- **(b)** I runs counterclockwise and a points to the right
- **(c)** I runs clockwise and a points to the right
- **(d)** I runs counterclockwise and a points to the left

18. A coil is moving toward a straight long wire carrying a steady electric current. The wire and the motion are within the plane of the coil as shown in the following figure. The force exerted by the wire on the coil is in the direction



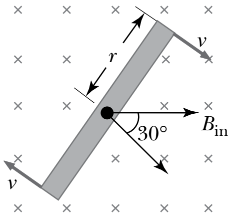
(a) Away from the wire
(b) Toward the wire
(c) Into the paper plane
(d) Out of the paper plane

19. A circular loop of wire is positioned half in and half out of a square region of uniform magnetic field directed in the $+z$ direction, out of the paper, as shown in the following figure. To induce a counterclockwise current in this loop



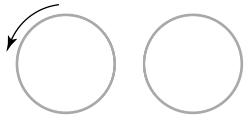
(a) Move it in the $+y$ direction
(b) Move it in the $-x$ direction
(c) Move it in the $+x$ direction
(d) Move it in the $-y$ direction

20. A rectangular loop, in edge view, as shown in the following figure, is rotating in a horizontal magnetic field. The axis of rotation is perpendicular to the page and is shown by a black dot in the middle of the figure. The parts of the loop perpendicular to the page are moving at a velocity v as shown in the figure. At the instant depicted, the normal to the loop makes an angle of 310° with the east direction. Which of the following statements, about the magnetic flux ϕ through the loop and the induced emf in the loop, are true, at the instant depicted?



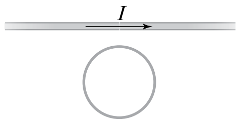
(a) ϕ is increasing and emf is increasing
(b) ϕ is increasing and emf is decreasing
(c) ϕ is decreasing and emf is increasing
(d) ϕ and emf are both zero

21. A constant current of 3.0 A flows counterclockwise in the circular coil in the following figure. What is the direction of the induced current in the coil on the right?



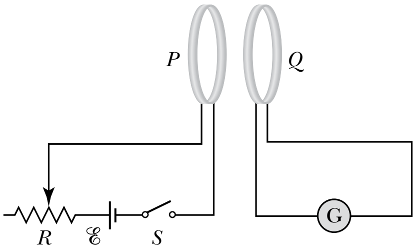
(a) Counterclockwise
(b) Clockwise
(c) Out of the page
(d) There is no induced current

22. A long straight wire carries a current of 10.05 A to the right. A circular loop of wire lies below the long straight wire, as shown in the following figure. The current in the long straight wire decreases to zero in a time of 4.652 s. What direction does the induced current flow around the circular loop of wire?



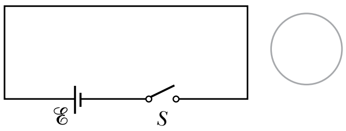
(a) Counterclockwise
(b) Clockwise
(c) No induced current flows around the loop
(d) The information given is insufficient for determining the direction of the induced current

23. Coils P and Q each have a large number of turns of insulated wire. When switch S is closed, the pointer of galvanometer G is deflected toward the left (as shown in the following figure). With S now closed, to make the pointer of G deflect toward the right one could



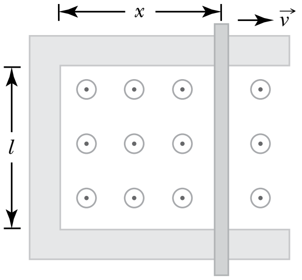
(a) Move the slide of the rheostat R quickly to the right
(b) Move coil P toward coil Q
(c) Move coil Q toward coil P
(d) Open S

24. Consider the situation as shown in the following figure. If the switch is closed and after some time it is opened again, the closed circular loop will show



(a) An anticlockwise current-pulse
(b) A clockwise current-pulse
(c) An anticlockwise current-pulse and then a clockwise current-pulse
(d) A clockwise current-pulse and then an anticlockwise current-pulse

25. A rod with resistance R lies across frictionless conducting rails in a constant uniform magnetic field $\vec{B}$, as shown in the following figure. Assume the rails have negligible resistance. The magnitude of the force that must be applied by a person to pull the rod to the right at constant speed v is



(a) 0 **(b)** Blv
(c) Blv/R **(d)** B^2l^2v/R

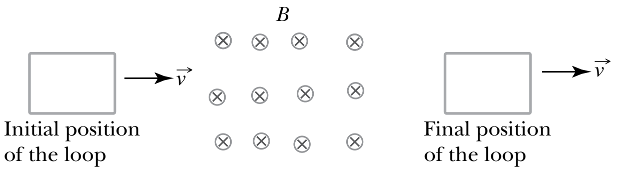
26. A rod of length l with resistance R lies in a constant uniform magnetic field rB perpendicular to the rod. The surface is frictionless. The magnitude of the force that must be applied by a person to pull the rod at constant speed v is

(a) 0 **(b)** Blv
(c) Blv/R **(d)** B^2l^2v/R

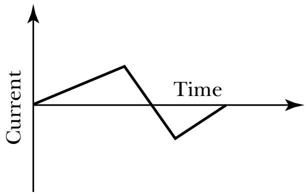
27. A metal rod is falling toward the surface of the Earth near the equator. As it falls, one end of the rod become positively charged due to the motional emf of the rod through the Earth's magnetic field. The rod is oriented so that

(a) The rod is horizontal with the positive end toward the west
(b) The rod is vertical with the positive end toward the north
(c) The rod is vertical with the positive end higher
(d) The rod is horizontal with the positive end toward the east

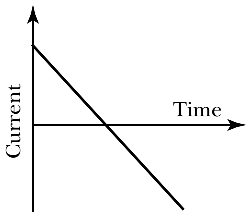
28. A rectangular loop of wire in the plane of the paper starts outside a perpendicular uniform magnetic field directed into the paper as shown in the following figure. The loop is pulled with a constant velocity through the magnetic field and out the other side. Which of the following graphs would best represent the variation of the electric current in the wire with time?



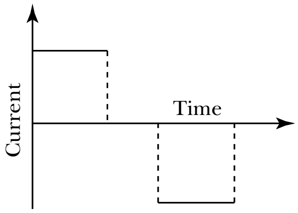
(a)



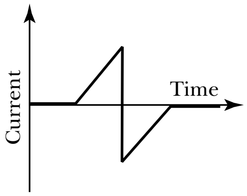
(b)



(c)



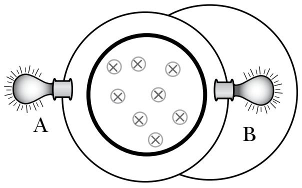
(d)



29. Which of the following is impossible in electromagnetism?

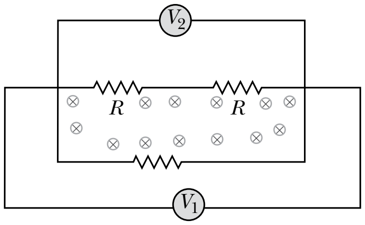
(a) Breaking a magnet in half, with one piece having only northpole and the other having only south pole
(b) Breaking a charged conductor in half with one piece carrying positive charge and the other carrying negative charge
(c) Detecting static electric charges with a compass
(d) Detecting steady electric currents with a compass
(e) Detecting static magnetic field with a voltmeter

30. A very long solenoid perpendicular to the page generates a downward magnetic field whose magnitude increases with time (as shown in the following figure). This induces an emf in a conducting wire loop inside the solenoid which lights two identical bulbs connected in series along the wire. Now two points diametrically opposed on the wire loop are shorted with another wire lying to the right of bulb B in the plane of the page. After the shorting wire is inserted,



(a) Bulb A goes out, and bulb B dims
(b) Bulb A goes out, and bulb B gets brighter
(c) Bulb B goes out, and bulb A dims
(d) bulb B goes out, and bulb A gets brighter

31. A changing magnetic field pierces the interior of a circuit containing three identical resistors. Two ideal voltmeters are connected to the same points, as shown in the following figure. If V_1 reads 1 mV, then V_2 reads

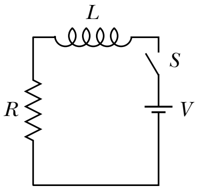


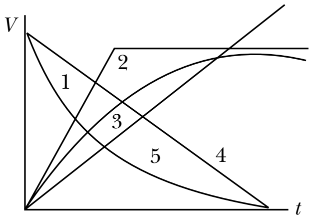
(a) 0
(b) 13 mV
(c) 12 mV
(d) 1 mV

32. What will happen to the inductance of a solenoid if it is stretched to twice its length while maintaining the same number of turns and cross-sectional area?

(a) Its inductance will double
(b) Its inductance will be reduced to half
(c) Its inductance will remain unchanged
(d) Its inductance will be reduced to one fourth

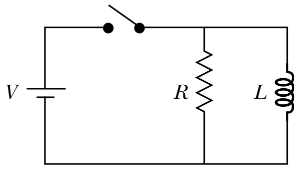
33. An open switch in an *RL* circuit is closed at time $t = 0$, as shown in the following figure. The curve that best illustrates the variation of potential difference across the resistor as a function of time is





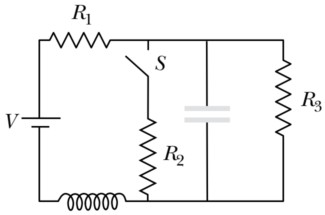
(a) 1
(b) 2
(c) 3
(d) 4

34. A 2 H coil, a 10 μF resistor, and a 5 V battery are all connected in parallel. After a current is well established through the resistor and inductor, the battery is disconnected from the circuit by opening a switch as shown in the following figure. Immediately after the battery is disconnected, Lenz's law would predict



(a) The voltage across *R* is zero
(b) The current through *R* reverses direction
(c) The current through the inductor reverses direction, but its magnitude tends to remain the same
(d) The magnitude and direction of the voltage across the inductor tend to remain the same

35. The circuit shown in the following figure is in a steady state with the switch open. When the switch is closed, which of the following will change immediately? (All of them eventually change, but three of them stay the same for an instant.)

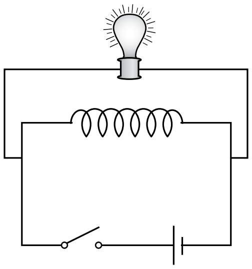


(a) The potential difference around the capacitor *C*
(b) The current through the inductor *L*
(c) The potential difference around the resistor R_1
(d) The current through the resistor R_3

36. In a circuit made up of inductor, resistance, ammeter, battery, and switch in series, at which of the following times after the switch is closed is the rate of increase of current greatest?

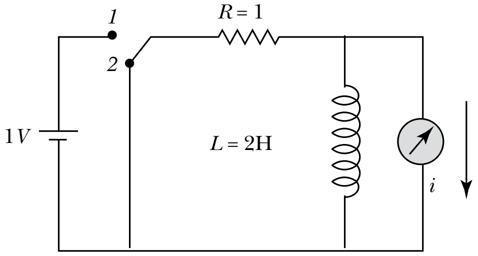
(a) The rate of current increase does not change in time
(b) One time constant
(c) Reciprocal of one time constant
(d) Zero

37. A battery is connected to an ideal solenoid and a light bulb in parallel (As shown in the following figure). When the switch is opened, the light bulb

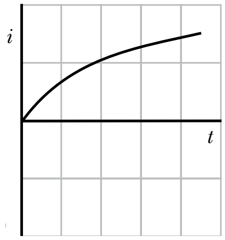


(a) Remains off
(b) Instantly goes off
(c) Slowly dims out
(d) Flares up brightly, then dims and goes off

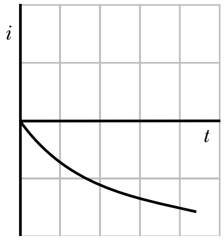
38. In the circuit shown in the following figure, the switch is moved to position 1 at $t = 0$. Which of the following most closely represents the current *I* as a function of time?



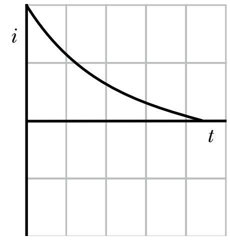
(a)



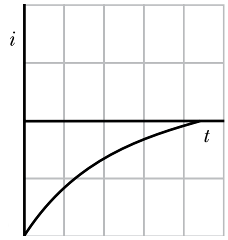
(b)



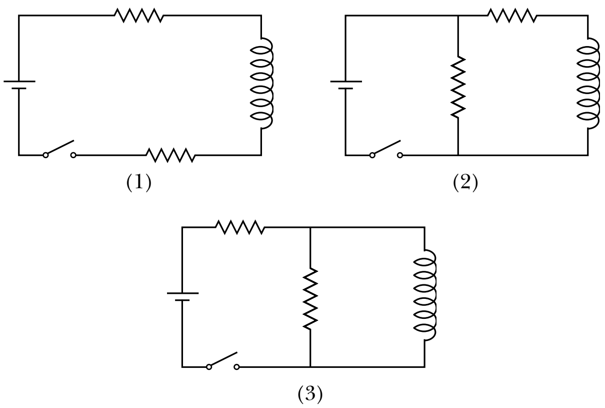
(c)



(d)



39. The following figure shows three circuits with identical batteries, identical inductors, and identical resistors. Rank them according to the current through the battery just after the switch is closed, from least to greatest.



(a) 3, 2, 1
(b) 1, 3, 2
(c) 1, 2, 3
(d) 3, 1, 2

40. The Earth's magnetic field passes through a square tabletop with a magnitude of 4.95×10^{-5} T and is directed at an angle of 165° relative to the normal of the tabletop. If the tabletop has 1.50 m sides, what is the magnitude of the magnetic flux through it?

(a) 1.08×10^{-4} Wb	**(c)** 2.88×10^{-5} Wb
(b) 7.11×10^{-5} Wb	**(d)** 1.92×10^{-5} Wb

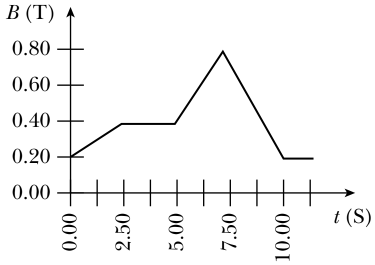
41. A circular copper loop is placed perpendicular to a uniform magnetic field of 0.50 T. Due to external forces, the area of the loop decreases at a rate of 1.26×10^{-3} m^2/s. Determine the induced emf in the loop.

(a) 3.1×10^{-4} V	**(b)** 1.2×10^{-3} V
(c) 6.3×10^{-4} V	**(d)** 7.9×10^{-3} V

42. A uniform magnetic field passes through two areas, A_1 and A_2. The angles between the magnetic field and the normals of areas A_1 and A_2 are 30.0° and 60.0°, respectively. If the magnetic flux through the two areas is the same, what is the ratio A_1/A_2?

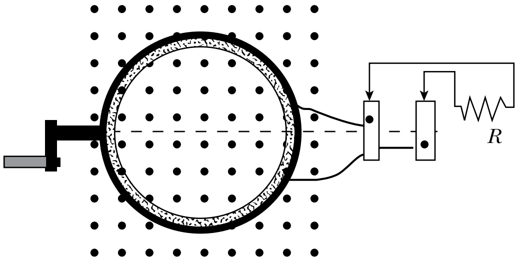
(a) 0.577	**(c)** 1.00
(b) 0.816	**(d)** 1.23

43. A circular coil of wire has 25 turns and has a radius of 0.075 m. The coil is located in a variable magnetic field whose behavior is shown on the graph. At all times, the magnetic field is directed at an angle of 75° relative to the normal to the plane of a loop. What is the average emf induced in the coil in the time interval from $t = 5.00$ s to 7.50 s?



(a) −18 mV	**(b)** −140 mV
(c) −49 mV	**(d)** −180 mV

44. A circular coil has 275 turns and a radius of 0.045 m. The coil is used as an ac generator by rotating it in a 0.500 T magnetic field, as shown in the figure. At what angular speed should the coil be rotated so that the maximum emf is 175 V?



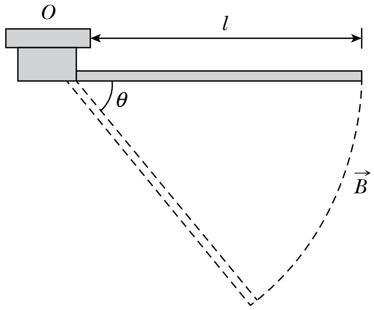
(a) 28 rad/s	**(b)** 200 rad/s
(c) 50 rad/s	**(d)** 490 rad/s

45. A transformer has 450 turns in its primary coil and 30 turns in its secondary coil. Which one of the following statements concerning this transformer is true?

(a) The *turns ratio* is 15 for this transformer.
(b) The ratio of the voltages V_s/V_p is 15 for this transformer.
(c) The ratio of the currents I_s/I_p is 0.067 for this transformer.
(d) The power delivered to the secondary must be the same as that delivered to the primary.

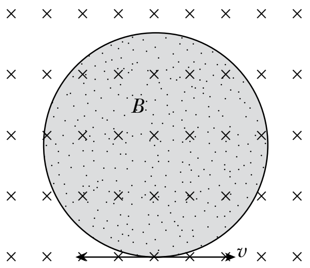
More than One Correct Choice Type

46. A conducting rod of length l is hinged at point O. It is free to rotate in a vertical plane. There exists a uniform magnetic field $\vec{B}$ in horizontal direction. The rod is released from the position shown in the following figure. The potential difference between the two ends of the rod is proportional to



(a) $l^{3/2}$ **(b)** l^2
(c) $\sin\theta$ **(d)** $(\sin\theta)^{1/2}$

47. A circular conducting loop of radius r_0 and having resistance per unit length λ as shown in the below figure is placed in a magnetic field B which is constant in space and time. The ends of the loop are crossed and pulled in opposite directions with a velocity v such that the loop always remains circular and the radius of the loop goes on decreasing, then

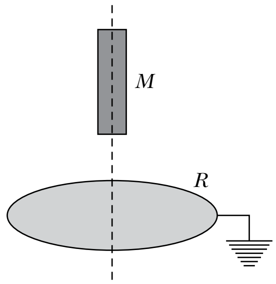


(a) Radius of the loop changes with $r = r_0 - vt/\pi$
(b) The emf induced in the loop as a function of time is $e = 2Bv(r_0 - vt/\pi)$
(c) The current induced in the loop is $I = Bv/2\pi\lambda$
(d) The current induced in the loop is $I = Bv/\pi\lambda$

48. A constant current I is maintained in a solenoid. Which of the following quantities will increase if an iron rod is inserted in the solenoid along its axis?
(a) Magnetic field at the center
(b) Magnetic flux linked with the solenoid
(c) Self-inductance of the solenoid
(d) Rate of Joule heating

49. Two different coils have self-inductance $L_1 = 8$ mH, $L_2 = 2$ mH. The current in one coil is increased at a constant rate. The current in the second coil is also increased at the same rate. At a certain instant of time, the power given to the two coils is the same. At that time the current the induced voltage and the energy stored in the first coil are I_1, V_1, and W_1, respectively. Corresponding values for the second coil at the same instant are I_2, V_2, and W_2, respectively. Then
(a) $I_1/I_2 = 1/4$ **(b)** $I_1/I_2 = 4$
(c) $W_2/W_1 = 4$ **(d)** $V_2/V_1 = 1/4$

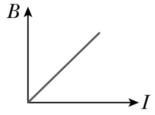
50. A small magnet M is allowed to fall through a fixed horizontal conducting ring R as shown in the following figure. Let g be the acceleration due to gravity. The acceleration of M will be

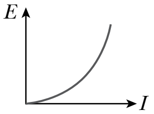


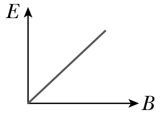
(a) $< g$ when it is above R and moving toward R
(b) $> g$ when it is above R and moving toward R
(c) $= g$ when it is below R and moving away from R
(d) $= g$ when it is below R and moving away from R

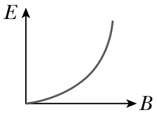
51. The magnetic flux (ϕ) linked with a coil depends on time t as $\phi = at^n$, where a is a constant. The induced emf in the coil is ε:
(a) If $0 < n$, then $\varepsilon = 0$
(b) If $0 < n < 1$, then $\varepsilon \neq 0$ and $|\varepsilon|$ decreases with time
(c) If $n = 1$, then ε is constant
(d) If $n > 1$, then $|\varepsilon|$ increases with time

52. If B and E denote induction of magnetic field and energy density at the midpoint of a long solenoid carrying a current I, then which of the graph/graphs, shown in the following figure, is/are correct?

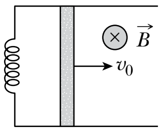
(a)


(b)


(c)


(d)


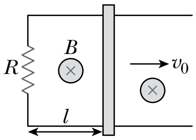
53. A loop is formed by two parallel conductors connected by a solenoid with inductance L and a conducting rod of mass M which can freely slide over the conductors. The conductors are located in a uniform magnetic field with induction B perpendicular to the plane of loop. The distance between conductors is l. At $t = 0$, the rod is given a velocity v_0 directed toward right and the current through the inductor is initially zero (below figure).



(a) The maximum current in circuit during the motion of rod is $v_0\sqrt{M/L}$
(b) The rod moves for some distance and comes to permanently rest
(c) The velocity of rod when current in the circuit is half of maximum is $(\sqrt{3}/2)v_0$
(d) The rod oscillates in SHM

54. A metal sheet is placed in front of a strong magnetic pole. A force is needed to
(a) Hold the sheet there if the metal is magnetic
(b) Hold the sheet there if the metal is non-magnetic
(c) Move the sheet away from the pole with uniform velocity if the metal is magnetic
(d) Move the sheet away from the pole with uniform velocity if the metal is nonmagnetic (Neglect any effect of paramagnetism, diamagnetism, and gravity.)

55. A conducting rod of length l is moved at constant velocity v_0 on two parallel, conducting, smooth, and fixed rails, that are placed in a uniform constant magnetic field B perpendicular to the plane of the rails as shown in the below figure. A resistance R is connected between the two ends of the rail. Then, which of the following is/are correct?



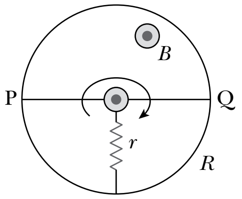
(a) The thermal power dissipated in the resistor is equal to rate of work done by external.

(b) If applied external force is doubled than a part of external power increases the velocity of rod.

(c) Lenz's law is not satisfied if the rod is accelerated by external force.

(d) If resistance R is doubled then power required to maintain the constant velocity v_0 becomes half.

56. In the following figure, R is a fixed conducting fixed ring of negligible resistance and radius a. PQ is a uniform rod of resistance r. It is hinged at the center of the ring and rotated about this point in clockwise direction with a uniform angular velocity ω. There is a uniform magnetic field of strength B pointing inward. Here r is a stationary resistance, then



(a) Current through r is zero

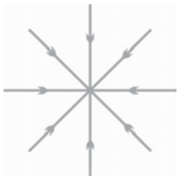
(b) Current through r is $2B\omega a^2/5r$

(c) Direction of current in external r is from center to circumference

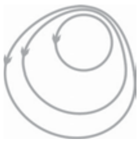
(d) Direction of current in external r is from circumference to center

57. Which of the following field line patterns could represent a magnetic field?

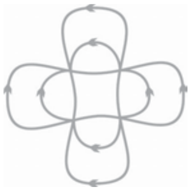
(a)



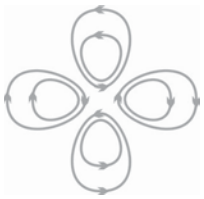
(b)



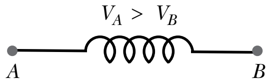
(c)



(d)



58. The potential at point A is higher than the potential at point B (shown in the following figure). Which of the following statements about the inductor current i could be true?



(a) i is from A to B and increasing

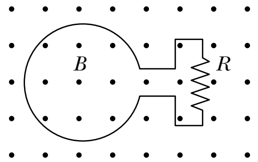
(b) i is from B to A and decreasing

(c) i is from B to A and increasing

(d) i is from A to B and decreasing

Linked Comprehension

***Paragraph for Questions 59–62*:** The figure shows a uniform, 3.0 T magnetic field that is normal to the plane of a conducting, circular loop with a resistance of 1.5 Ω and a radius of 0.024 m. The magnetic field is directed out of the paper as shown. **Note**: The area of the non-circular portion of the wire is considered negligible compared to that of the circular loop.



59. What is the magnitude of the average induced emf in the loop if the magnitude of the magnetic field is doubled in 0.4 s?

(a) 0.43 V **(b)** 0.014 V

(c) 0.65 V **(d)** 0.027 V

60. What is the average current around the loop if the magnitude of the magnetic field is doubled in 0.4 s?

(a) 2.8×10^{-3} A, clockwise

(b) 9.0×10^{-3} A, clockwise

(c) 4.5×10^{-3} A, clockwise

(d) 9.0×10^{-3} A, counterclockwise

61. If the magnetic field is held constant at 3.0 T and the loop is pulled out of the region that contains the field in 0.2 s, what is the magnitude of the average induced emf in the loop?

(a) 2.7×10^{-2} V **(b)** 6.4×10^{-2} V

(c) 9.8×10^{-2} V **(d)** 5.4×10^{-2} V

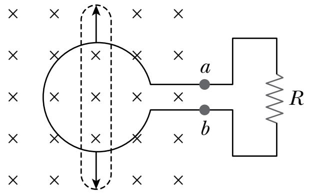
62. If the magnetic field is held constant at 3.0 T and the loop is pulled out of the region that contains the field in 0.2 s, at what rate is energy dissipated in R?

(a) 1.8×10^{-2} W **(b)** 3.8×10^{-3} W

(c) 4.9×10^{-4} W **(d)** 2.7×10^{-4} W

***Paragraph for Questions 63 and 64*:** A flexible circular loop 20 cm in diameter lies in a magnetic field with magnitude 1.0 T, directed into the plane of the page as shown in the following figure. The loop is pulled at the points indicated by the arrows, forming a loop of zero area in 0.314 s.

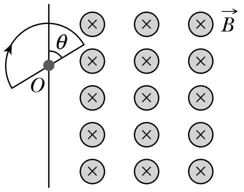
63. The average induced emf in the circuit is



(a) 0.2 V **(b)** 0.1 V
(c) 1 V **(d)** 10 V

64. If $R = 0.01\ \Omega$; the magnitude and direction of current flowing in the loop is
(a) 1 A, clockwise **(b)** 1 A, anticlockwise
(c) 10 A, clockwise **(d)** 10 A, anticlockwise

***Paragraph for Questions 65 and 66*:** A wire loop enclosing a semicircle of radius R is located on the boundary of a uniform magnetic field B as shown in the following figure. At the moment $t = 0$, the loop is set into rotation with constant angular acceleration a about an axis O conducting with the line of vector on the boundary. The clockwise emf direction is taken to be positive.



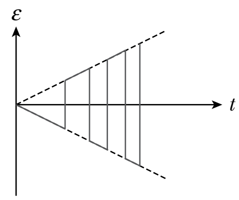
65. The variation of emf in function of time is

(a) $\frac{1}{2}BR^2\alpha t$ **(b)** $\frac{3}{2}BR^2\alpha t$

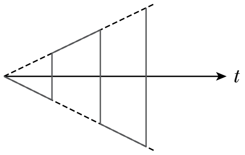
(c) $\sqrt{3}BR^2\alpha t$ **(d)** $\frac{BR^2\alpha t}{\sqrt{2}}$

66. Which among the four graphs shown in the below figure, the variation of emf as a function of time is correct?

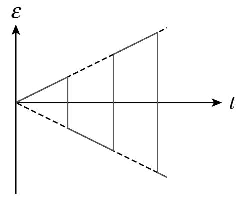
(a)



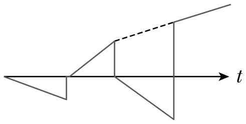
(b)



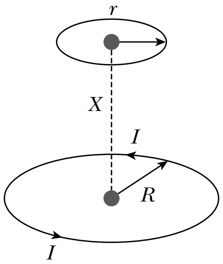
(c)



(d)



***Paragraph for Questions 67 and 68*:** The following figure shows two parallel and coaxial loops. The smaller loop (radius R) is above the large loop (radius R), by distance $x \gg R$. The magnetic field due to current I in the larger loop is nearly constant throughout the smaller loop. Suppose that x is increasing at constant rate $dx/dt = v$.



67. Determine the magnetic flux through the smaller loop as a function of x.

(a) $\frac{\mu_0 IR^2 \times \pi r^2}{(R^2 + x^2)^{3/2}}$ **(b)** $\frac{\mu_0 IR^2 \times \pi r^2}{2(R^2 + x^2)^{3/2}}$

(c) $\frac{2\mu_0 IR^2 \times \pi r^2}{(R^2 + x^2)^{3/2}}$ **(d)** $\frac{\sqrt{2}\mu_0 IR^2 \times \pi r^2}{(R^2 + x^2)^{3/2}}$

68. Find the induced emf and the direction of induced current in the smaller loop.

(a) $\frac{\mu_0 \pi IR^2 r^2}{x^4}v$ **(b)** $\frac{\mu_0 \pi IR^2 r^2}{2x^4}v$

(c) $\frac{3}{2}\frac{\mu_0 \pi IR^2 r^2}{x^4}v$ **(d)** $\frac{\mu_0 \pi IR^2 r^2}{3x^4}v$

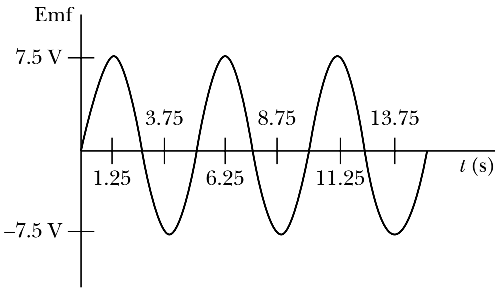
***Paragraph for Questions 69–71*:** In a conventional vehicle, power is dissipated due to friction. To reduce the loss of power due to friction MAGLEV train using an EDS (electrodynamic suspension) technology was developed. It works on the principle of Lenz's law. In this system, a train moves without contact with the track due to which there is no loss of power due to friction.

69. The advantage of EDS is that
(a) No power is required to propel the train
(b) No energy is lost due to conservative forces
(c) No power is dissipated against frictional force
(d) No force is required to overcome gravity

70. Which of the following statements about MAGLEV train is correct?
(a) It moves due to electrostatic attraction
(b) It is lifted due to magnetic repulsion
(c) It moves due to magnetic attraction
(d) It is lifted due to electrostatic repulsion

71. The disadvantage of EDS is that
(a) It cannot overcome the force of gravity
(b) Repulsive force is conservative so it lifts the vehicle and brings down
(c) Magnetic force tends to bring down the train
(d) Lot of energy is lost to start the motion as there is more opposition to the motion at lower speeds

***Paragraph for Questions 72–74*:** A single conducting loop with an area of 2.0 m^2 rotates in a uniform magnetic field so that the induced emf has a sinusoidal time dependence as shown.

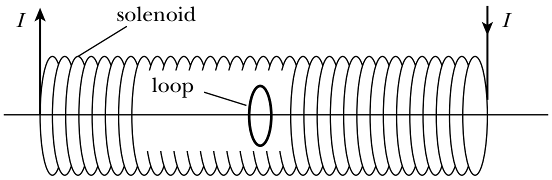


72. With what angular frequency does the loop rotate?
(a) 0.16 rad/s **(b)** 0.52 rad/s
(c) 1.26 rad/s **(d)** 0.30 rad/s

73. What is the period of the induced current?
(a) 1.25 s **(b)** 3.75 s
(c) 2.50 s **(d)** 5.00 s

74. Determine the strength of the magnetic field in which the loop rotates.
(a) 0.5 T **(b)** 3.0 T
(c) 2.4 T **(d)** 7.5 T

***Paragraph for Questions 75–77*:** A 0.100 m long solenoid has a radius of 0.050 m and 1.50 × 10^4 turns. The current in the solenoid changes at a rate of 6.0 A/s. A conducting loop of radius 0.0200 m is placed at the center of the solenoid with its axis the same as that of the solenoid as shown.



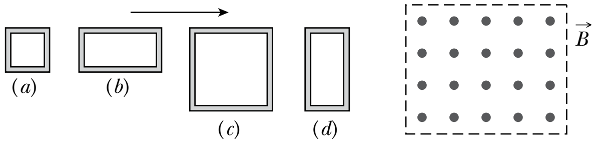
75. What is the magnetic flux through the small loop when the current through the solenoid is 2.50 A?
(a) 2.95×10^{-2} Wb
(b) 5.92×10^{-4} Wb
(c) 7.28×10^{-2} Wb
(d) 4.38×10^{-4} Wb

76. Determine the mutual inductance of this combination.
(a) 1.8×10^{-4} H **(b)** 3.6×10^{-4} H
(c) 2.4×10^{-4} H **(d)** 4.4×10^{-4} H

77. Determine the induced emf in the loop.
(a) 0.7×10^{-3} V **(b)** 2.8×10^{-3} V
(c) 1.4×10^{-3} V **(d)** 5.6×10^{-3} V

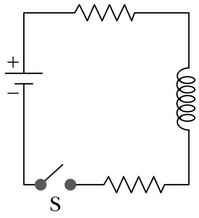
Matrix-Match

78. The following figure shows four wire loops, with edge lengths of either L or $2L$. All four loops will move through a region of uniform magnetic field $\vec{B}$ (directed out of the page) at the same constant velocity. (Corresponding to loop mentioned in Column II, information on induced emf is provided in Column I; match them correctly).

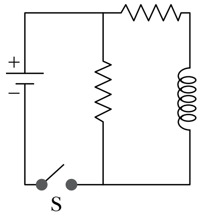


Column I	Column II
(a) Maximum emf	**(p)** a
(b) Minimum emf	**(q)** b
(c) Greater equal emf	**(r)** c
(d) Lower equal emf	**(s)** d

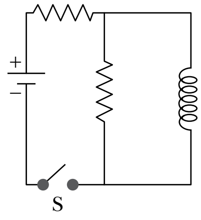
79. The following figure shows three circuits with identical batteries, inductors, and resistors.



(1)



(2)

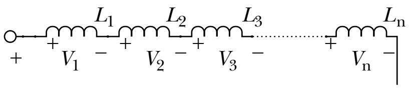
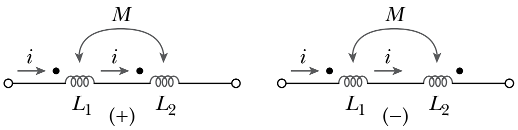
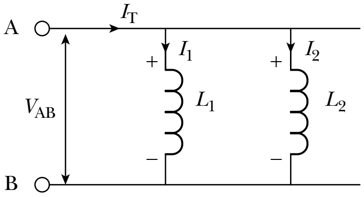
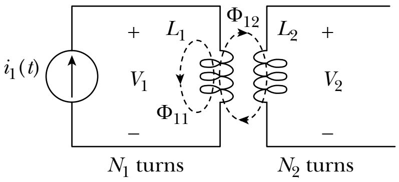


(3)

Column I	Column II
(a) Greatest current circuit when just switch is closed	**(p)** (1)
(b) Minimum current circuit when just switch is closed	**(q)** (2)
(c) Greater current circuit after long time switch is closed	**(r)** (3)
(d) Minimum current circuit after long time switch is closed	**(s)** Impossible to determine

Directions for Questions 80 and 81: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

80. In the given table Column I lists the elements in the circuit, Column II shows their combination and Column III shows the circuit diagram.

Column I	Column II	Column III
(I) If two inductors	**(i)** L_1 and L_2 are over each other	**(J)** 
(II) If two conductors	**(ii)** connected in series and they have mutual inductance M	**(K)** 
(III) If two coils of self-inductances	**(iii)** kept far apart and joined in series	**(L)** 
(IV) If semi-conductor	**(iv)** connected in parallel and are kept far apart	**(M)** 

(1) What are the conditions for $L = L_1 + L_2$?
(a) (I) (ii) (J)
(b) (IV) (ii) (M)
(c) (II) (i) (M)
(d) (I) (iii) (J)

(2) What are the conditions for $1/L = 1/L_1 + 1/L_2$?
(a) (I) (iv) (L)
(b) (IV) (iii) (L)
(c) (II) (iii) (L)
(d) (I) (i) (M)

(3) What are the conditions for $M = K\sqrt{L_1 L_2}$?
(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (III (i) (M) **(d)** (II) (iii) (M)

81. Let magnetic flux be ϕ_B, induced emf be ξ and power P. All the three quantities related with each other, that is, if we vary the magnetic flux, induced emf and power change according the variation of magnetic flux. In the given table, Column I shows the variation of magnetic flux, Column II shows the effect on induced emf and Column III shows the effect on power.

Column I	Column II	Column III
(I) ϕ_B is equal to 0	**(i)** E is equal to L	**(J)** P is equal to E
(II) ϕ_B decreases gradually	**(ii)** E is equal to 0	**(K)** P is a positive constant
(III) ϕ_B remains constant	**(iii)** E is a positive constant	**(L)** P is equal to 0
(IV) ϕ_B increases gradually	**(iv)** E is a negative constant	**(M)** P is a negative constant

(1) What are the conditions for a coil entering the magnetic field?
(a) (I) (iii) (L)
(b) (IV) (iv) (M)
(c) (IV) (iv) (K)
(d) (I) (iii) (J)

(2) What are the conditions for a coil moving in the magnetic field?
(a) (III) (ii) (L)
(b) (IV) (iii) (L)
(c) (II) (iii) (K)
(d) (I) (i) (M)

(3) What are the conditions for a coil out of the field?
(a) (III) (i) (L) **(b)** (I) (i) (J)
(c) (III) (iii) (J) **(d)** (I) (ii) (L)

Integer Type

82. A straight conductor 1 m long moves at right angle to both its length and a uniform magnetic field. If the speed of conductor is 2.0 m/s and strength of magnetic field is 104 G, find the value of induced emf in volts.

83. A closely wound rectangular coil of 200 turns and size 0.3 m × 0.05 m is placed perpendicular to a magnetic field of induction 0.20 Wb/m^2. Calculate the induced emf in the coil when the magnetic induction drops to 0.15 Wb/m^2 in 0.02 s.

84. A coil of 100 turns is pulled from the magnetic field where its area includes 21×10^{-4} Wb to a place where its area includes 1×10^{-4} Wb. If the time for pulling is 0.05 s, find the emf induced in the coil.

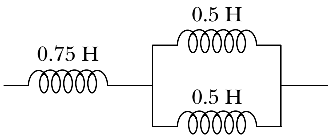
85. A metal disk of radius 200 cm is rotated at a constant angular speed of 60 rad/s in a plane at right angles to an external field of magnetic induction 0.05 Wb/m. Find the emf induced between the center and a point on the rim.

86. A uniform magnetic flux density of 0.1 Wb/m extends over a plane circuit of area 2 m^2 and is normal to it. How quickly must the field be reduced to zero, if an emf of 100 V is to be induced in the circuit?

87. The two rails of a railway track insulated from each other and the ground are connected to a millivoltmeter. What will be the reading of the millivoltmeter, when a train travels at a speed of 180 km/h along the track? Given that the vertical component of Earth's field is 2×10^{-5} Wb/m^2 and the rails are separated by 1 m.

88. A circular disc of radius 20 cm is rotating with a constant angular speed of 2.0 rad/s in a uniform magnetic field of 0.2 T. Find induced emf between the center and the rim of the disc. Assume that the magnetic field is along the axis of rotation of the disk.

89. Calculate the resultant inductance of the three inductances that are connected as shown in the following figure.



90. Magnetic flux of 20 μWb is linked with a coil when current of 5 mA is flown through it. What is the self-inductance of the coil (in $\times 10^{-3}$ H) ?

91. A bicycle generates 3 V when it is traveling at a speed of 9 km/h. How much emf is generated when the bicycle is traveling at 15 km/h?

92. Magnetic flux of 5 μWb is linked with a coil, when a current of 1 mA flows through it. What is the self-inductance of the coil (in mH)?

93. A magnetic field of flux density 1.0 Wb/m^2 acts normal to an 80 turn coil of 0.01 m^2 area. Find the emf induced in it, if this coil is removed from the field in 0.1 s.

94. A coil has an inductance of 1.5×10^{-2} H. Calculate the emf induced, when current in the coil changes at the rate 200 A/s.

95. An ac generator consists of a coil if 50 turns and area 2.5 m^2 rotating at an angular speed of 60 rad/s in a uniform magnetic field $B = 0.2$ T between the two fixed pole pieces. The resistance of the circuit including that of the coil is 500 Ω. Calculate the maximum current drawn from the

ANSWER KEY

Checkpoints

1. *b*, then *d* and *e* tie, and then *a* and *c* tie (zero)

2. *a* and *b* tie, then *c* (zero)

3. (a) We can break the motion in two parts. (a) Pure rotation about any point on the rod and (b) pure translation with the velocity of that point. The motional emf due to both kinds of motion can be added to give the total induced emf; (b) $\dfrac{B\omega l^2}{2} \pm Blv$

4. *c* and *d* tie, then *a* and *b* tie

5. *b*, out; *c*, out; *d*, into; *e* into

6. The voltmeter will read zero because although there is induced electric field in space, but the loop forming the voltmeter and the metal ring has no induced emf across it.

7. *d* and *e*

8. (a) 2, 3, 1 (zero); (b) 2, 3, 1

9. *a* and *b* tie, then *c*

Problems

1. (a) 0.0119 V; (b) 6.60×10^{-4} A; (c) 7.83×10^{-6} W; (d) 2.06×10^{-5} N

2. (a) 1.5 μWb; (b)10×10^2 mV; (c) 90 Wb; (d) 12 mV

3. (a) 0.60; (b) up; (c) 1.5 A; (d) clockwise; (e) 0.90 W; (f) 0.18 N; (g) 0.90 W

5. 3.0×10^{-4} W

6. (a) $\mu_0 i R^2 \pi r^2/2x^3$; (b) $3\mu_0 i \pi R^2 r^2 v/2x^4$; (c) counterclockwise

7. $\varepsilon = 42 + 20t$

8. 0.60 V/m

9. 1.23

10. 5.0 A/s

11. 2.1×10^5 H

12. (a) 3.33 A; (b) 3.33 A; (c) 4.55 A; (d) 2.73 A; (e) 0; (f) –182 A; (g) 0; (h) 0

13. (a) 97.9 H; (b) 0.196 mJ

14. (a) 1.19 mH; (b) 1.61 mWb

15. (a) 1.00; (b) 0.135; (0.693)

16. 2.37 μH/m

17. no induced current

18. 2.6 Ω
19. (a) 2.4×10^2 W; (b) 1.5×10^2 W; (c) 3.9×10^2 W
20. (a) 0.12 J/m^3; (b) 5.9×10^{-16} J/m^3
21. (a) 85.2 Wb; (b) 56.8 V; (c) 1
22. (a) 4.13 ns; (b) 4.59 mA
23. (a) 55.5 J/m^3; (b) 8.01×10^{-2} J
24. 0.018 V
25. (a) 1.74 J; (b) 2.19×10^{-2} J ; (c) 1.72 J
26. $i[1 - \exp(-Rt/L)]$; (b) $(L/R) \ln 3$
27. 5.58 A
28. 41.6 ms
29. 13 H
30. (a) 31.4 μT; (b) 0; (c) 31.4 μT; (d) yes; (e) 12.6 nV
31. (a) 1.4×10^{-8} Wb; (b) 1.0×10^{-5} A
32. (a) 30 Hz; (b) 1.2 mV
33. 12.3 s
34. (a) 0.34 mV; (b) clockwise
35. 6.00×10^6 V/m
36. (a) 137 μV/m; (b) 114 μV/m
37. (b) $L_{eq} = \Sigma L_j\, L_{eq}\_L_j$, sum from $j = 1$ to $j = N$
38. (b) have the turns of the two solenoids wrapped in opposite directions, $L_{eq} = L_1 + L_2 - 2M$
39. (b) $\dfrac{1}{L_{eq}} = \sum_{n=1}^{N} \dfrac{1}{L_n}$
40. 71.0 μV
41. 1.55×10^{-5} C
42. (a) 14.5; (b) current is in the sense of the total emf (counterclockwise)
43. (a) 322 μV; (b) 0.805 mA; (c) 0.259 μW; (d) 51.8 nN; (e) 0.259 μW
44. (a) 2.9×10^{-4} H; (b) 2.9×10^{-4}
45. (a) decreasing; (b) 1.3 mH
46. 4.61
47. 0.031 T/s
48. (a) 5.51 mWb; (b) 1.97 mH
49. 0.452 V
50. 81.0 mH
51. (a) 7.8 μWb/m; (b) 22%; (c) 0
52. 0.32 mA
53. 7.50×10^{-10} J
54. (a) -1.07×10^{-3} V; (b) -2.40×10^{-3} V; (c) 1.33×10^{-3} V
55. Zero
56. 1.0×10^{-7} Wb
57. 64.0 mC
58. (a) 2.4×10^{-5} V; (b) the current in segment *bc* flows from *c* to *b* (following Lenz's law)
59. (a) 4.5 s
60. $v_t = mgR/B^2L^2$
61. (a) 16 mV; (b) left
62. (a) $(4.4 \times 10^7\ \text{m/s}^2)\hat{\text{i}}$; (b) 0; (c) $-(4.4\times 10^7\ \text{m/s}^2)\hat{\text{i}}$
63. 1.25×10^{-3} Tm^2
64. +3 Wb
65. 0.2 V
66. (a) zero; (b) $11\sqrt{2} \times 10^{-7}$ Tm; (c) 22×10^{-7} Tm^2; (d) zero
67. (a) 0.315 V; (b) no
68. 80π mT

Practice Questions

Single Correct Choice Type

1. (b)
2. (a)
3. (a)
4. (a)
5. (b)
6. (a)
7. (b)
8. (c)
9. (a)
10. (c)
11. (c)
12. (d)
13. (d)
14. (c)
15. (a)
16. (c)
17. (c)
18. (a)
19. (c)
20. (d)
21. (d)
22. (b)
23. (d)
24. (d)
25. (d)
26. (a)
27. (d)
28. (c)
29. (a)
30. (d)
31. (d)
32. (b)
33. (c)
34. (b)
35. (d)
36. (d)
37. (d)
38. (c)
39. (b)
40. (a)
41. (c)
42. (a)
43. (a)
44. (b)
45. (d)

More than One Correct Choice Type

46. (a), (d)
47. (a), (b), (d)
48. (a), (b), (c)
49. (a), (c), (d)
50. (a), (c)
51. (b), (c), (d)
52. (a), (b), (c)
53. (a), (c), (d)
54. (a), (c)
55. (a), (b), (d)
56. (b), (d)
57. (b), (d)
58. (a), (b)

Linked Comprehension

59. (b)
60. (b)
61. (a)
62. (c)
63. (b)
64. (c)
65. (a)
66. (a)
67. (b)
68. (c)
69. (c)
70. (b)
71. (d)
72. (c)
73. (d)
74. (b)
75. (b)
76. (c)
77. (c)

Matrix-Match

78. (a) → (r), (s); (b) → (p), (q); (c) → (r), (s); (d) → (p), (q)
79. (a) → (q); (b) → (p); (c) → (p); (d) → (r)
80. (1) → (d); (2) → (a); (3) → (c)
81. (1) → (c); (2) → (a); (3) → (d)

Integer Type

82. 2
83. 7
84. 4
85. 6
86. 2
87. 1
88. 8
89. 1
90. 4
91. 5
92. 5
93. 8
94. 3
95. 3

CHAPTER

31

Electromagnetic Oscillations and Alternating Current

Contents

31.1 What is Physics?
31.2 *LC* Oscillations, Qualitatively
31.3 The Electrical–Mechanical Analogy
31.4 *LC* Oscillations, Quantitatively
31.5 Alternating Current
31.6 Three Simple Circuits
31.7 The Series *RLC* Circuit
31.8 Power in Alternating-Current Circuits
31.9 Transformers

31.1 | WHAT IS PHYSICS?

We have explored the basic physics of electric and magnetic fields and how energy can be stored in capacitors and inductors. We next turn to the associated applied physics, in which the energy stored in one location can be transferred to another location so that it can be put to use. For example, energy produced at a power plant can show up at your home to run a computer. The total worth of this applied physics is now so high that its estimation is almost impossible. Indeed, modern civilization would be impossible without this applied physics.

In most parts of the world, electrical energy is transferred not as a direct current but as a sinusoidally oscillating current (alternating current, or ac). The challenge to both physicists and engineers is to design ac systems that transfer energy efficiently and to build appliances that make use of that energy. Our first step here is to study the oscillations in a circuit with inductance L and capacitance C.

31.2 | *LC* OSCILLATIONS, QUALITATIVELY

Key Concepts

- In an oscillating LC circuit, energy is shuttled periodically between the electric field of the capacitor and the magnetic field of the inductor; instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2},$$

  where q is the instantaneous charge on the capacitor and i is the instantaneous current through the inductor.
- The total energy $U\ (= U_E + U_B)$ remains constant.

Of the three circuit elements, resistance R, capacitance C, and inductance L, we have so far discussed the series combinations RC (in Section 27.7)

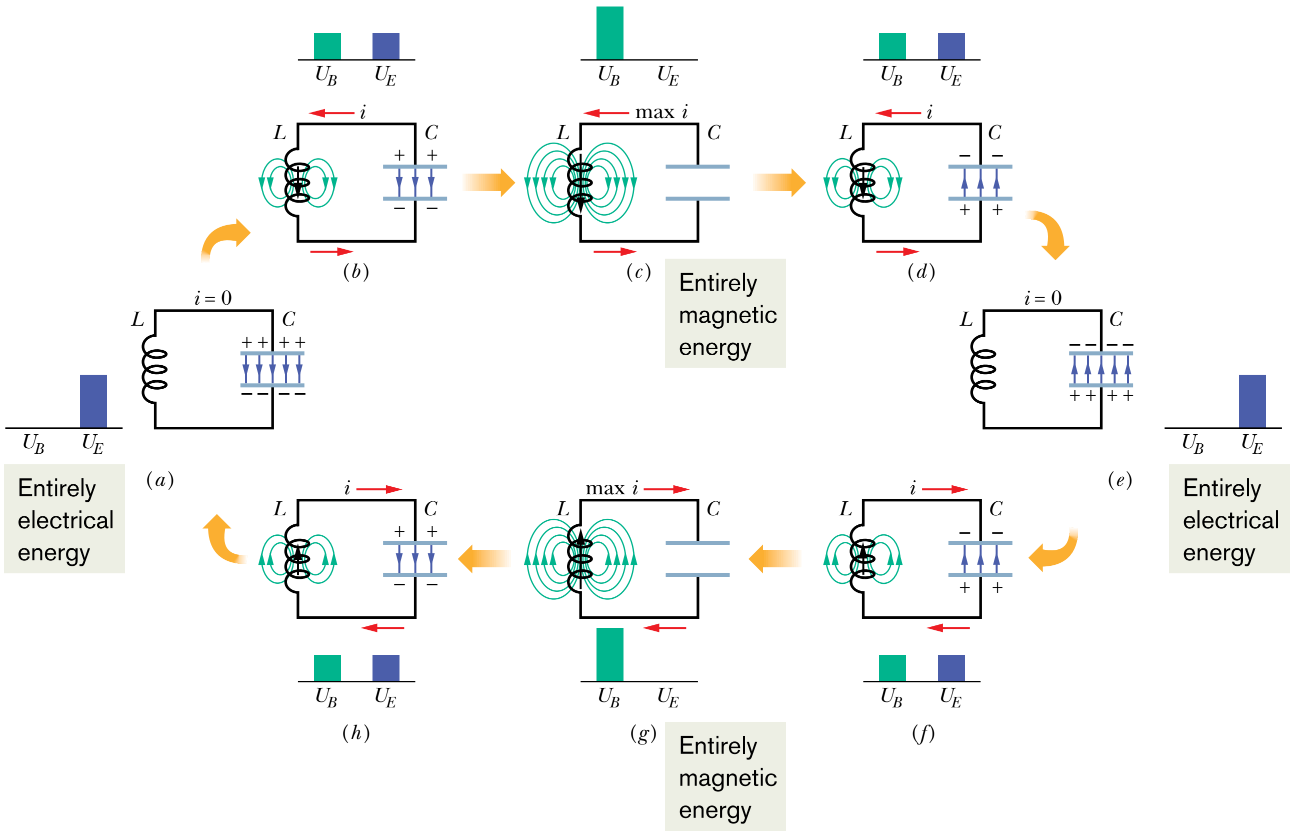


Figure 31-1 Eight stages in a single cycle of oscillation of a resistanceless LC circuit. The bar graphs by each figure show the stored magnetic and electrical energies. The magnetic field lines of the inductor and the electric field lines of the capacitor are shown. (*a*) Capacitor with maximum charge, no current. (*b*) Capacitor discharging, current increasing. (*c*) Capacitor fully discharged, current maximum. (*d*) Capacitor charging but with polarity opposite that in (*a*), current decreasing. (*e*) Capacitor with maximum charge having polarity opposite that in (*a*), no current. (*f*) Capacitor discharging, current increasing with direction opposite that in (*b*). (*g*) Capacitor fully discharged, current maximum. (*h*) Capacitor charging, current decreasing.

and RL (in Section 30.11). In these two kinds of circuit we found that the charge, current, and potential difference grow and decay exponentially. The time scale of the growth or decay is given by a *time constant* τ, which is either capacitive or inductive.

We now examine the remaining two-element circuit combination LC. You will see that in this case the charge, current, and potential difference do not decay exponentially with time but vary sinusoidally (with period T and angular frequency ω). The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations**. Such a circuit is said to oscillate.

Parts *a* through *h* of Fig. 31-1 show succeeding stages of the oscillations in a simple LC circuit. From Eq. 25-21, the energy stored in the electric field of the capacitor at any time is

$$U_E = \frac{q^2}{2C}, \tag{31-1}$$

where q is the charge on the capacitor at that time. From Eq. 30-51, the energy stored in the magnetic field of the inductor at any time is

$$U_B = \frac{Li^2}{2}, \tag{31-2}$$

where i is the current through the inductor at that time.

We now adopt the convention of representing *instantaneous values* of the electrical quantities of a sinusoidally oscillating circuit with small letters, such as q, and the *amplitudes* of those quantities with capital letters, such as Q. With this convention in mind, let us assume that initially the charge q on the capacitor in Fig. 31-1 is at its maximum value Q and that the current i through the inductor is zero. This initial state of the circuit is shown in Fig. 31-1*a*. The bar graphs for energy included there indicate that at this instant, with zero current through the inductor and maximum charge on the capacitor, the energy U_B of the magnetic field is zero and the energy U_E of the electric field is a maximum. As the circuit oscillates, energy shifts back and forth from one type of stored energy to the other, but the total amount is conserved.

The capacitor now starts to discharge through the inductor, positive charge carriers moving counterclockwise, as shown in Fig. 31-1*b*. This means that a current i, given by dq/dt and pointing down in the inductor, is established. As the capacitor's charge decreases, the energy stored in the electric field within the capacitor also decreases. This energy is transferred to the magnetic field that appears around the inductor because of the current i that is building up there. Thus, the electric field decreases and the magnetic field builds up as energy is transferred from the electric field to the magnetic field.

The capacitor eventually loses all its charge (Fig. 31-1*c*) and thus also loses its electric field and the energy stored in that field. The energy has then been fully transferred to the magnetic field of the inductor. The magnetic field is then at its maximum magnitude, and the current through the inductor is then at its maximum value I.

Although the charge on the capacitor is now zero, the counterclockwise current must continue because the inductor does not allow it to change suddenly to zero. The current continues to transfer positive charge from the top plate to the bottom plate through the circuit (Fig. 31-1*d*). Energy now flows from the inductor back to the capacitor as the electric field within the capacitor builds up again. The current gradually decreases during this energy transfer. When, eventually, the energy has been transferred completely back to the capacitor (Fig. 31-1*e*), the current has decreased to zero (momentarily). The situation of Fig. 31-1*e* is like the initial situation, except that the capacitor is now charged oppositely.

The capacitor then starts to discharge again but now with a clockwise current (Fig. 31-1*f*). Reasoning as before, we see that the clockwise current builds to a maximum (Fig. 31-1*g*) and then decreases (Fig. 31-1*h*), until the circuit eventually returns to its initial situation (Fig. 31-1*a*). The process then repeats at some frequency f and thus at an angular frequency $\omega = 2\pi f$. In the ideal *LC* circuit with no resistance, there are no energy transfers other than that between the electric field of the capacitor and the magnetic field of the inductor. Because of the conservation of energy, the oscillations continue indefinitely. The oscillations need not begin with the energy all in the electric field; the initial situation could be any other stage of the oscillation.

To determine the charge q on the capacitor as a function of time, we can put in a voltmeter to measure the time-varying potential difference (or *voltage*) v_C that exists across the capacitor C. From Eq. 25-1, we can write

$$v_C = \left(\frac{1}{C}\right)q,$$

which allows us to find q. To measure the current, we can connect a small resistance R in series with the capacitor and inductor and measure the time-varying potential difference v_R across it; v_R is proportional to i through the relation

$$v_R = iR.$$

We assume here that R is so small that its effect on the behavior of the circuit is negligible. The variations in time of v_C and v_R, and thus of q and i, are shown in Fig. 31-2. All four quantities vary sinusoidally.

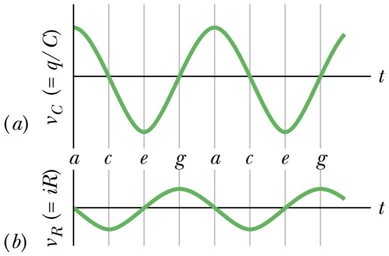
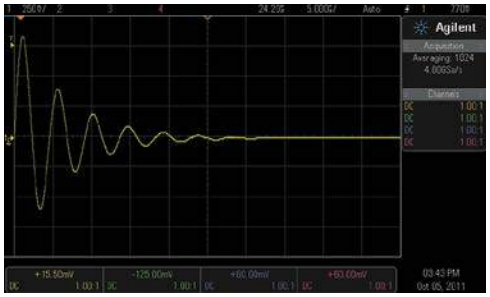


Figure 31-2 (*a*) The potential difference across the capacitor in the circuit of Fig. 31-1 as a function of time. This quantity is proportional to the charge on the capacitor. (*b*) A potential proportional to the current in the circuit of Fig. 31-1. The letters refer to the correspondingly labelled oscillation stages in Fig. 31-1.

In an actual LC circuit, the oscillations will not continue indefinitely because there is always some resistance present that will drain energy from the electric and magnetic fields and dissipate it as thermal energy (the circuit may become warmer). The oscillations, once started, will die away as Fig. 31-3 suggests. Compare this figure with Fig. 15-21, which shows the decay of mechanical oscillations caused by frictional damping in a block–spring system.



Courtesy Agilent Technologies

Figure 31-3 An oscilloscope trace showing how the oscillations in an RLC circuit actually die away because energy is dissipated in the resistor as thermal energy.

CHECKPOINT 1

A charged capacitor and an inductor are connected in series at time $t = 0$. In terms of the period T of the resulting oscillations, determine how much later the following reach their maximum value: (a) the charge on the capacitor; (b) the voltage across the capacitor, with its original polarity; (c) the energy stored in the electric field; and (d) the current.

SAMPLE PROBLEM 31.01

LC oscillator: current in the coil

A 1.5 μF capacitor is charged to 57 V. The charging battery is then disconnected, and a 12 mH coil is connected in series with the capacitor so that LC oscillations occur. What is the maximum current in the coil? Assume that the circuit contains no resistance, which would cause the energy to be dissipated.

KEY IDEA

(1) Because the circuit contains no resistance, the electromagnetic energy of the circuit is conserved as the energy is transferred back and forth between the electric field of the capacitor and the magnetic field of the coil (inductor).

(2) At any time t, the energy $U_B(t)$ of the magnetic field is related to the current $i(t)$ through the coil by Eq. 31-2 ($U_B = Li^2/2$). When all the energy is stored as magnetic energy, the current is at its maximum value I and that energy is $U_{B,\max} = LI^2/2$. At any time t, the energy $U_E(t)$ of the electric field is related to the charge $q(t)$ on the capacitor by Eq. 31-1 ($U_E = q^2/2C$). When all the energy is stored as electrical energy, the charge is at its maximum value Q and that energy is $U_{E,\max} = Q^2/2C$.

Calculation: With these ideas, we can now write the conservation of energy as

$$U_{B,\max} = U_{E,\max}$$

or,

$$\frac{LI^2}{2} = \frac{Q^2}{2C}.$$

Solving for I gives us

$$I = \sqrt{\frac{Q^2}{LC}}.$$

We know L and C, but not Q. However, with Eq. 25-1 ($q = CV$) we can relate Q to the maximum potential difference V across the capacitor, which is the initial potential difference of 57 V. Thus, substituting $Q = CV$ leads to

$$I = V\sqrt{\frac{C}{L}} = (57\text{ V})\sqrt{\frac{1.5\times10^{-6}\text{ F}}{12\times10^{-3}\text{ H}}} = 0.637\text{ A} \cong 640\text{ mA}$$
(Answer)

Note: As will be shown later in the chapter, the charge on the plates follows the equation of SHM.

31.3 | THE ELECTRICAL–MECHANICAL ANALOGY

Key Concepts

- The principle of conservation of energy leads to

$$L\frac{d^2q}{dt^2}+\frac{1}{C}q=0 \quad (LC \text{ oscillations})$$

as the differential equation of LC oscillations (with no resistance).

- The solution of this differential equation is

$$q = Q\cos(\omega t + \phi) \quad \text{(charge)},$$

in which Q is the charge amplitude (maximum charge on the capacitor) and the angular frequency ω of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}.$$

Let us look a little closer at the analogy between the oscillating LC system of Fig. 31-1 and an oscillating block–spring system. Two kinds of energy are involved in the block–spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block. These two energies are given by the formulas in the first energy column in Table 31-1.

Table 31-1 Comparison of the Energy in Two Oscillating Systems

Block–Spring System		*LC* Oscillator	
Element	**Energy**	**Element**	**Energy**
Spring	Potential, $1/2kx^2$	Capacitor	Electrical, $1/2(1/C)q^2$
Block	Kinetic, $1/2mv^2$	Inductor	Magnetic, $1/2Li^2$
	$v = dx/dt$		$i = dq/dt$

The table also shows, in the second energy column, the two kinds of energy involved in LC oscillations. By looking across the table, we can see an analogy between the forms of the two pairs of energies—the mechanical energies of the block–spring system and the electromagnetic energies of the LC oscillator. The equations for v and i at the bottom of the table help us see the details of the analogy. They tell us that q corresponds to x and i corresponds to v (in both equations, the former is differentiated to obtain the latter). These correspondences then suggest that, in the energy expressions, $1/C$ corresponds to k and L corresponds to m. Thus,

$$q \text{ corresponds to } x, \qquad 1/C \text{ corresponds to } k,$$
$$i \text{ corresponds to } v, \quad \text{and} \quad L \text{ corresponds to } m.$$

These correspondences suggest that in an LC oscillator, the capacitor is mathematically like the spring in a block–spring system and the inductor is like the block.

In Section 15.3 we saw that the angular frequency of oscillation of a (frictionless) block–spring system is

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(block-spring system)}. \tag{31-3}$$

The correspondences listed above suggest that to find the angular frequency of oscillation for an ideal (resistanceless) LC circuit, k should be replaced by $1/C$ and m by L, yielding

$$\omega = \frac{1}{\sqrt{LC}} \quad (LC \text{ circuit}). \tag{31-4}$$

31.4 | *LC* OSCILLATIONS, QUANTITATIVELY

Key Concepts

- The phase constant ϕ is determined by the initial conditions (at $t = 0$) of the system.
- The current i in the system at any time t is
  $$i = -\omega Q \sin(\omega t + \phi) \quad \text{(current)},$$
  in which ωQ is the current amplitude I.

Here we want to show explicitly that Eq. 31-4 for the angular frequency of *LC* oscillations is correct. At the same time, we want to examine even more closely the analogy between *LC* oscillations and block–spring oscillations. We start by extending somewhat our earlier treatment of the mechanical block–spring oscillator.

The Block–Spring Oscillator

We analyzed block–spring oscillations in Chapter 15 in terms of energy transfers and did not—at that early stage—derive the fundamental differential equation that governs those oscillations. We do so now.

We can write, for the total energy U of a block–spring oscillator at any instant,

$$U = U_b + U_s = \frac{1}{2}mv^2 + \frac{1}{2}kx^2, \tag{31-5}$$

where U_b and U_s are, respectively, the kinetic energy of the moving block and the potential energy of the stretched or compressed spring. If there is no friction—which we assume—the total energy U remains constant with time, even though v and x vary. In more formal language, $dU/dt = 0$. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0. \tag{31-6}$$

Substituting $v = dx/dt$ and $dv/dt = d^2x/dt^2$, we find

$$m\frac{d^2x}{dt^2} + kx = 0 \quad \text{(block-spring oscillations)}. \tag{31-7}$$

Equation 31-7 is the fundamental *differential equation* that governs the frictionless block–spring oscillations.

The general solution to Eq. 31-7 is (as we saw in Eq. 15-3)

$$x = X\cos(\omega t + \phi) \quad \text{(displacement)}, \tag{31-8}$$

in which X is the amplitude of the mechanical oscillations (x_m in Chapter 15), ω is the angular frequency of the oscillations, and ϕ is a phase constant.

The *LC* Oscillator

Now let us analyze the oscillations of a resistanceless *LC* circuit, proceeding exactly as we just did for the block–spring oscillator. The total energy U present at any instant in an oscillating *LC* circuit is given by

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}, \tag{31-9}$$

in which U_B is the energy stored in the magnetic field of the inductor and U_E is the energy stored in the electric field of the capacitor. Since we have assumed the circuit resistance to be zero, no energy is transferred to thermal energy and U remains constant with time. In more formal language, dU/dt must be zero. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0. \tag{31-10}$$

However, $i = dq/dt$ and $di/dt = d^2q/dt^2$. With these substitutions, Eq. 31-10 becomes

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad (LC \text{ oscillations}). \tag{31-11}$$

This is the *differential equation* that describes the oscillations of a resistanceless LC circuit. Equations 31-11 and 31-7 are exactly of the same mathematical form.

Charge and Current Oscillations

Since the differential equations are mathematically identical, their solutions must also be mathematically identical. Because q corresponds to x, we can write the general solution of Eq. 31-11, by analogy to Eq. 31-8, as

$$q = Q\cos(\omega t + \phi) \quad (\text{charge}), \tag{31-12}$$

where Q is the amplitude of the charge variations, ω is the angular frequency of the electromagnetic oscillations, and ϕ is the phase constant. Taking the first derivative of Eq. 31-12 with respect to time gives us the current:

$$i = \frac{dq}{dt} = -\omega Q\sin(\omega t + \phi) \quad (\text{current}). \tag{31-13}$$

The amplitude I of this sinusoidally varying current is

$$I = \omega Q, \tag{31-14}$$

and so we can rewrite Eq. 31-13 as

$$i = -I\sin(\omega t + \phi). \tag{31-15}$$

Angular Frequencies

We can test whether Eq. 31-12 is a solution of Eq. 31-11 by substituting Eq. 31-12 and its second derivative with respect to time into Eq. 31-11. The first derivative of Eq. 31-12 is Eq. 31-13. The second derivative is then

$$\frac{d^2q}{dt^2} = -\omega^2 Q\cos(\omega t + \phi).$$

Substituting for q and d^2q/dt^2 in Eq. 31-11, we obtain

$$-L\omega^2 Q\cos(\omega t + \phi) + \frac{1}{C}Q\cos(\omega t + \phi) = 0.$$

Canceling $Q\cos(\omega t + \phi)$ and rearranging lead to

$$\omega = \frac{1}{\sqrt{LC}}.$$

Thus, Eq. 31-12 is indeed a solution of Eq. 31-11 if ω has the constant value $1/\sqrt{LC}$. Note that this expression for ω is exactly that given by Eq. 31-4.

The phase constant ϕ in Eq. 31-12 is determined by the conditions that exist at any certain time—say, $t = 0$. If the conditions yield $\phi = 0$ at $t = 0$, Eq. 31-12 requires that $q = Q$ and Eq. 31-13 requires that $i = 0$; these are the initial conditions represented by Fig. 31-1*a*.

Electrical and Magnetic Energy Oscillations

The electrical energy stored in the LC circuit at time t is, from Eqs. 31-1 and 31-12,

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi). \tag{31-16}$$

The magnetic energy is, from Eqs. 31-2 and 31-13,

$$U_B = \frac{1}{2}Li^2 = \frac{1}{2}L\omega^2Q^2\sin^2(\omega t + \phi).$$

Substituting for ω from Eq. 31-4 then gives us

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi). \tag{31-17}$$

Figure 31-4 shows plots of $U_E(t)$ and $U_B(t)$ for the case of $\phi = 0$. Note that

1. The maximum values of U_E and U_B are both $Q^2/2C$.
2. At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
3. When U_E is maximum, U_B is zero, and conversely.

The electrical and magnetic energies vary but the total is constant.

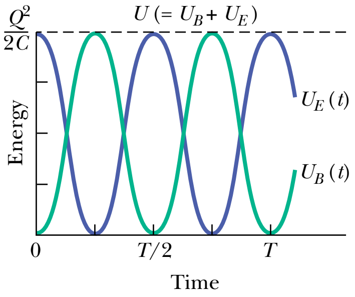


Figure 31-4 The stored magnetic energy and electrical energy in the circuit of Fig. 31-1 as a function of time. Note that their sum remains constant. T is the period of oscillation.

CHECKPOINT 2

A capacitor in an LC oscillator has a maximum potential difference of 17 V and a maximum energy of 160 μJ. When the capacitor has a potential difference of 5 V and an energy of 10 μJ, what are (a) the emf across the inductor and (b) the energy stored in the magnetic field?

SAMPLE PROBLEM 31.02

LC oscillator: potential change, rate of current change

For the situation described in Sample Problem 31.01, at time $t = 0$, a 12 mH coil is connected in series with the capacitor to form an LC oscillator (Fig. 31-1).

(a) What is the potential difference $v_L(t)$ across the inductor as a function of time?

KEY IDEAS

(1) The current and potential differences of the circuit (both the potential difference of the capacitor and the potential difference of the coil) undergo sinusoidal oscillations.

(2) We can still apply the loop rule to these oscillating potential differences, just as we did for the nonoscillating circuits of Chapter 27.

Calculations: At any time t during the oscillations, the loop rule and Fig. 31-1 give us

$$v_L(t) = v_C(t); \tag{31-18}$$

that is, the potential difference v_L across the inductor must always be equal to the potential difference v_C across the capacitor, so that the net potential difference around the circuit is zero. Thus, we will find $v_L(t)$ if we can find $v_C(t)$, and we can find $v_C(t)$ from $q(t)$ with Eq. 25-1 ($q = CV$).

Because the potential difference $v_C(t)$ is maximum when the oscillations begin at time $t = 0$, the charge q on the capacitor must also be maximum then. Thus, phase constant ϕ must be zero; so Eq. 31-12 gives us

$$q = Q \cos \omega t. \tag{31-19}$$

(Note that this cosine function does indeed yield maximum $q(= Q)$ when $t = 0$.) To get the potential difference $v_C(t)$, we divide both sides of Eq. 31-19 by C to write

$$\frac{q}{C} = \frac{Q}{C} \cos \omega t,$$

and then use Eq. 25-1 to write

$$v_C = V_C \cos \omega t. \tag{31-20}$$

Here, V_C is the amplitude of the oscillations in the potential difference v_C across the capacitor.

Next, substituting $v_C = v_L$ from Eq. 31-18, we find

$$v_L = V_C \cos \omega t. \tag{31-21}$$

We can evaluate the right side of this equation by first noting that the amplitude V_C is equal to the initial (maximum) potential difference of 57 V across the capacitor. Then we find ω with Eq. 31-4:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{[(0.012\text{ H})(1.5\times10^{-6}\text{ F})]^{0.5}}$$
$$= 7454\text{ rad/s} \approx 7500\text{ rad/s}.$$

Thus, Eq. 31-21 becomes

$$v_L = (57\text{ V}) \cos(7500\text{ rad/s})t. \quad \text{(Answer)}$$

(b) What is the maximum rate $(di/dt)_{\max}$ at which the current i changes in the circuit?

KEY IDEA

With the charge on the capacitor oscillating as in Eq. 31-12, the current is in the form of Eq. 31-13. Because $\phi = 0$, that equation gives us

$$i = -\omega Q \sin \omega t.$$

Calculations: Taking the derivative, we have

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q \sin \omega t) = -\omega^2 Q \cos \omega t.$$

We can simplify this equation by substituting CV_C for Q (because we know C and V_C but not Q) and $1/\sqrt{LC}$ for ω according to Eq. 31-4. We get

$$\frac{di}{dt} = -\frac{1}{LC} CV_C \cos \omega t = -\frac{V_C}{L} \cos \omega t.$$

This tells us that the current changes at a varying (sinusoidal) rate, with its maximum rate of change being

$$\frac{V_C}{L} = \frac{57\text{ V}}{0.012\text{ H}} = 4750\text{ A/s} \approx 4800\text{ A/s}. \quad \text{(Answer)}$$

31.5 | ALTERNATING CURRENT

Key Concepts

- A series RLC circuit may be set into forced oscillation at a driving angular frequency ω_d by an external alternating emf

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t.$$

- The current driven in the circuit is

$$i = I \sin(\omega_d t - \phi),$$

where ϕ is the phase constant of the current.

The oscillations in an RLC circuit will not damp out if an external emf device supplies enough energy to make up for the energy dissipated as thermal energy in the resistance R. Circuits in homes, offices, and factories, including countless RLC circuits, receive such energy from local power companies. In most countries the energy is supplied via oscillating emfs and currents—the current is said to be an **alternating current**, or **ac** for short. (The nonoscillating current from a battery is said to be a **direct current**, or **dc**.) These oscillating emfs and currents vary sinusoidally with time, reversing direction (in India) 100 times per second and thus having frequency $f = 50$ Hz.

Electron Oscillations. At first sight this may seem to be a strange arrangement. We have seen that the drift speed of the conduction electrons in household wiring may typically be 4×10^{-5} m/s. If we now reverse their direction every

1/120 s, such electrons can move only about 3×10^{-7} m in a half-cycle. At this rate, a typical electron can drift past no more than about 10 atoms in the wiring before it is required to reverse its direction. How, you may wonder, can the electron ever get anywhere?

Although this question may be worrisome, it is a needless concern. The conduction electrons do not have to "get anywhere." When we say that the current in a wire is one ampere, we mean that charge passes through any plane cutting across that wire at the rate of one coulomb per second. The speed at which the charge carriers cross that plane does not matter directly; one ampere may correspond to many charge carriers moving very slowly or to a few moving very rapidly. Furthermore, the signal to the electrons to reverse directions—which originates in the alternating emf provided by the power company's generator—is propagated along the conductor at a speed close to that of light. All electrons, no matter where they are located, get their reversal instructions at about the same instant. Finally, we note that for many devices, such as lightbulbs and toasters, the direction of motion is unimportant as long as the electrons do move so as to transfer energy to the device via collisions with atoms in the device.

Why ac? The basic advantage of alternating current is this: *As the current alternates, so does the magnetic field that surrounds the conductor*. This makes possible the use of Faraday's law of induction, which, among other things, means that we can step up (increase) or step down (decrease) the magnitude of an alternating potential difference at will, using a device called a transformer, as we shall discuss later. Moreover, alternating current is more readily adaptable to rotating machinery such as generators and motors than is (nonalternating) direct current.

Emf and Current. Figure 31-5 shows a simple model of an ac generator. As the conducting loop is forced to rotate through the external magnetic field $\vec{B}$, a sinusoidally oscillating emf $\mathscr{E}$ is induced in the loop:

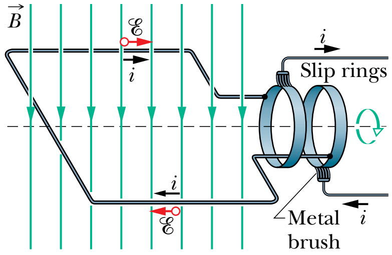


Figure 31-5 The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field. In practice, the alternating emf induced in a coil of many turns of wire is made accessible by means of slip rings attached to the rotating loop. Each ring is connected to one end of the loop wire and is electrically connected to the rest of the generator circuit by a conducting brush against which the ring slips as the loop (and it) rotates.

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t. \tag{31-22}$$

The *angular frequency* ω_d of the emf is equal to the angular speed with which the loop rotates in the magnetic field, the *phase* of the emf is $\omega_d t$, and the *amplitude* of the emf is $\mathscr{E}_m$ (where the subscript stands for maximum). When the rotating loop is part of a closed conducting path, this emf produces (*drives*) a sinusoidal (alternating) current along the path with the same angular frequency ω_d, which then is called the **driving angular frequency**. We can write the current as

$$i = I \sin(\omega_d t - \phi), \tag{31-23}$$

in which I is the amplitude of the driven current. (The phase $\omega_d t - \phi$ of the current is traditionally written with a minus sign instead of as $\omega_d t + \phi$.) We include a phase constant ϕ in Eq. 31-23 because the current i may not be in phase with the emf $\mathscr{E}$. (As you will see, the phase constant depends on the circuit to which the generator is connected.) We can also write the current i in terms of the **driving frequency** f_d of the emf, by substituting $2\pi f_d$ for ω_d in Eq. 31-23.

Forced Oscillations

We have seen that once started, the charge, potential difference, and current in both undamped LC circuits and damped RLC circuits (with small enough R) oscillate at angular frequency $\omega = 1/\sqrt{LC}$. Such oscillations are said to be *free oscillations* (free of any external emf), and the angular frequency ω is said to be the circuit's **natural angular frequency**.

When the external alternating emf of Eq. 31-22 is connected to an RLC circuit, the oscillations of charge, potential difference, and current are said to be *driven oscillations* or *forced oscillations*. These oscillations always occur at the driving angular frequency ω_d:

Whatever the natural angular frequency ω of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency ω_d.

However, as you will see in Section 31.7, the amplitudes of the oscillations very much depend on how close ω_d is to ω. When the two angular frequencies match—a condition known as **resonance**—the amplitude I of the current in the circuit is maximum.

31.6 | THREE SIMPLE CIRCUITS

Key Concepts

- The alternating potential difference across a resistor has amplitude $V_R = IR$; the current is in phase with the potential difference.
- For a capacitor, $V_C = IX_C$, in which $X_C = 1/\omega_d C$ is the capacitive reactance; the current here leads the potential difference by 90° ($\phi = -90° = -\pi/2$ rad).
- For an inductor, $V_L = IX_L$, in which $X_L = \omega_d L$ is the inductive reactance; the current here lags the potential difference by 90° ($\phi = +90° = +\pi/2$ rad).

Later in this chapter, we shall connect an external alternating emf device to a series RLC circuit as in Fig. 31-6. We shall then find expressions for the amplitude I and phase constant ϕ of the sinusoidally oscillating current in terms of the amplitude $\mathscr{E}_m$ and angular frequency ω_d of the external emf. First, let's consider three simpler circuits, each having an external emf and only one other circuit element: R, C, or L. We start with a resistive element (a purely *resistive load*).

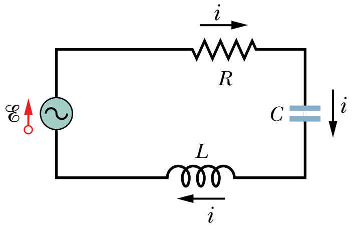


Figure 31-6 A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the directions of the emf and current are indicated here at only one instant.

A Resistive Load

Figure 31-7 shows a circuit containing a resistance element of value R and an ac generator with the alternating emf of Eq. 31-22. By the loop rule, we have

$$\mathscr{E} - v_R = 0.$$

With Eq. 31-22, this gives us

$$v_R = \mathscr{E}_m \sin \omega_d t.$$

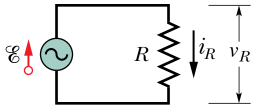


Figure 31-7 A resistor is connected across an alternating-current generator.

Because the amplitude V_R of the alternating potential difference (or voltage) across the resistance is equal to the amplitude $\mathscr{E}_m$ of the alternating emf, we can write this as

$$v_R = V_R \sin \omega_d t. \tag{31-24}$$

From the definition of resistance ($R = V/i$), we can now write the current i_R in the resistance as

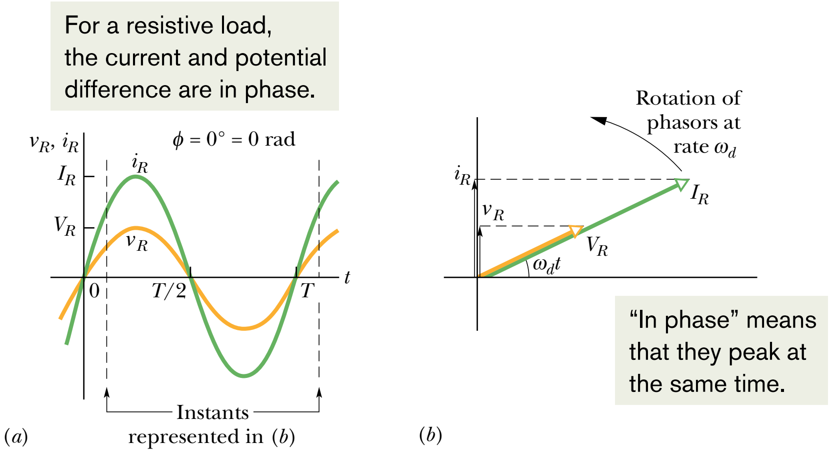
$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t. \tag{31-25}$$

From Eq. 31-23, we can also write this current as

$$i_R = I_R \sin(\omega_d t - \phi), \tag{31-26}$$

where I_R is the amplitude of the current i_R in the resistance. Comparing Eqs. 31-25 and 31-26, we see that for a purely resistive load the phase constant $\phi = 0°$. We also see that the voltage amplitude and current amplitude are related by

$$V_R = I_R R \quad \text{(resistor)}. \tag{31-27}$$



For a resistive load, the current and potential difference are in phase.

"In phase" means that they peak at the same time.

Figure 31-8 (*a*) The current i_R and the potential difference v_R across the resistor are plotted on the same graph, both versus time t. They are in phase and complete one cycle in one period T. (*b*) A phasor diagram shows the same thing as (*a*).

Although we found this relation for the circuit of Fig. 31-7, it applies to any resistance in any ac circuit.

By comparing Eqs. 31-24 and 31-25, we see that the time-varying quantities v_R and i_R are both functions of sin $\omega_d t$ with $\phi = 0°$. Thus, these two quantities are *in phase*, which means that their corresponding maxima (and minima) occur at the same times. Figure 31-8*a*, which is a plot of $v_R(t)$ and $i_R(t)$, illustrates this fact. Note that v_R and i_R do not decay here because the generator supplies energy to the circuit to make up for the energy dissipated in R.

The time-varying quantities v_R and i_R can also be represented geometrically by *phasors*. Recall that phasors are vectors that rotate around an origin. Those that represent the voltage across and current in the resistor of Fig. 31-7 are shown in Fig. 31-8*b* at an arbitrary time t. Such phasors have the following properties:

Angular speed: Both phasors rotate counterclockwise about the origin with an angular speed equal to the angular frequency ω_d of v_R and i_R.

Length: The length of each phasor represents the amplitude of the alternating quantity: V_R for the voltage and I_R for the current.

Projection: The projection of each phasor on the *vertical* axis represents the value of the alternating quantity at time t: v_R for the voltage and i_R for the current.

Rotation angle: The rotation angle of each phasor is equal to the phase of the alternating quantity at time t. In Fig. 31-8*b*, the voltage and current are in phase; so their phasors always have the same phase $\omega_d t$ and the same rotation angle, and thus they rotate together.

Mentally follow the rotation. Can you see that when the phasors have rotated so that $\omega_d t = 90°$ (they point vertically upward), they indicate that just then $v_R = V_R$ and $i_R = I_R$? Equations 31-24 and 31-26 give the same results.

CHECKPOINT 3

If we increase the driving frequency in a circuit with a purely resistive load, do (a) amplitude V_R and (b) amplitude I_R increase, decrease, or remain the same?

SAMPLE PROBLEM 31.03

Purely resistive load: potential difference and current

In Fig. 31-7, resistance R is 200 Ω and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What is the potential difference $v_R(t)$ across the resistance as a function of time t, and what is the amplitude V_R of $v_R(t)$?

KEY IDEA

In a circuit with a purely resistive load, the potential difference $v_R(t)$ across the resistance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: For our situation, $v_R(t) = \mathscr{E}(t)$ and $V_R = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we can write

$$V_R = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_R(t)$, we use Eq. 31-22 to write

$$v_R(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t \qquad (31\text{-}28)$$

and then substitute $\mathscr{E}_m = 36.0$ V and

$$\omega_d = 2\pi f_d = 2\pi(60 \text{ Hz}) = 120\pi$$

to obtain

$$v_R = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

We can leave the argument of the sine in this form for convenience, or we can write it as (377 rad/s)t or as (377 s^{-1})t.

(b) What are the current $i_R(t)$ in the resistance and the amplitude I_R of $i_R(t)$?

KEY IDEA

In an ac circuit with a purely resistive load, the alternating current $i_R(t)$ in the resistance is *in phase* with the alternating potential difference $v_R(t)$ across the resistance; that is, the phase constant ϕ for the current is zero.

Calculations: Here we can write Eq. 31-23 as

$$i_R = I_R \sin(\omega_d t - \phi) = I_R \sin \omega_d t. \qquad (31\text{-}29)$$

From Eq. 31-27, the amplitude I_R is

$$I_R = \frac{V_R}{R} = \frac{36.0 \text{ V}}{200\ \Omega} = 0.180 \text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-29, we have

$$i_R = (0.180 \text{ A}) \sin(120\pi t). \qquad \text{(Answer)}$$

A Capacitive Load

Figure 31-9 shows a circuit containing a capacitance and a generator with the alternating emf of Eq. 31-22. Using the loop rule and proceeding as we did when we obtained Eq. 31-24, we find that the potential difference across the capacitor is

$$v_C = V_C \sin \omega_d t, \qquad (31\text{-}30)$$

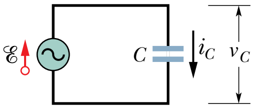


Figure 31-9 A capacitor is connected across an alternating-current generator.

where V_C is the amplitude of the alternating voltage across the capacitor. From the definition of capacitance we can also write

$$q_C = Cv_C = CV_C \sin \omega_d t. \qquad (31\text{-}31)$$

Our concern, however, is with the current rather than the charge. Thus, we differentiate Eq. 31-31 to find

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t. \qquad (31\text{-}32)$$

We now modify Eq. 31-32 in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_C, called the **capacitive reactance** of a capacitor, defined as

$$X_C = \frac{1}{\omega_d C} \quad \text{(capacitive reactance).} \qquad (31\text{-}33)$$

Its value depends not only on the capacitance but also on the driving angular frequency ω_d. We know from the definition of the capacitive time constant ($\tau = RC$) that the SI unit for C can be expressed as seconds per ohm. Applying this to Eq. 31-33 shows that the SI unit of X_C is the *ohm*, just as for resistance R.

Second, we replace $\cos \omega_d t$ in Eq. 31-32 with a phase-shifted sine

$$\cos \omega_d t = \sin(\omega_d t + 90°).$$

You can verify this identity by shifting a sine curve 90° in the negative direction.

With these two modifications, Eq. 31-32 becomes

$$i_C = \left(\frac{V_C}{X_C}\right)\sin(\omega_d t + 90°). \tag{31-34}$$

From Eq. 31-23, we can also write the current i_C in the capacitor of Fig. 31-9 as

$$i_C = I_C \sin(\omega_d t - \phi), \tag{31-35}$$

where I_C is the amplitude of i_C. Comparing Eqs. 31-34 and 31-35, we see that for a purely capacitive load the phase constant ϕ for the current is $-90°$. We also see that the voltage amplitude and current amplitude are related by

$$V_C = I_C X_C \quad \text{(capacitor)}. \tag{31-36}$$

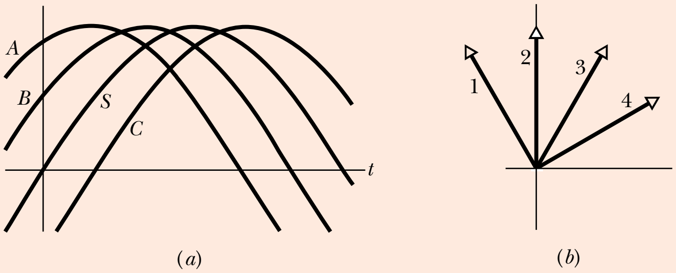
Although we found this relation for the circuit of Fig. 31-9, it applies to any capacitance in any ac circuit.

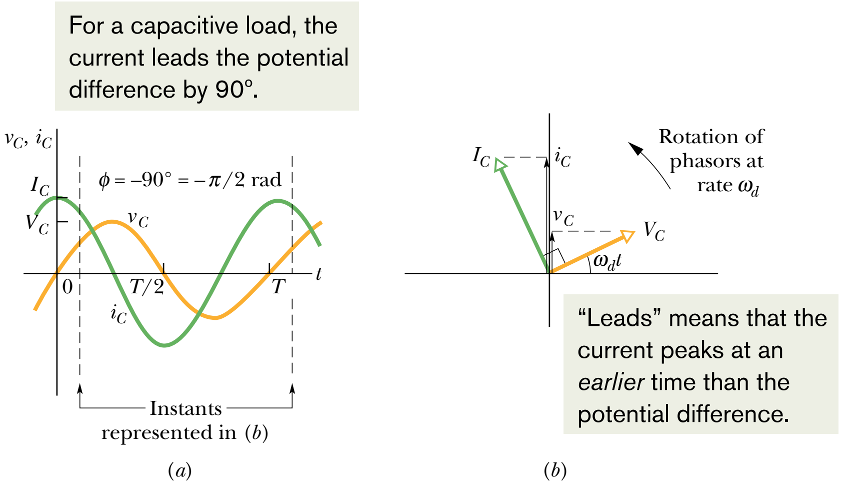
Comparison of Eqs. 31-30 and 31-34, or inspection of Fig. 31-10*a*, shows that the quantities v_C and i_C are 90°, $\pi/2$ rad, or one-quarter cycle, out of phase. Furthermore, we see that i_C *leads* v_C, which means that, if you monitored the current i_C and the potential difference v_C in the circuit of Fig. 31-9, you would find that i_C reaches its maximum *before* v_C does, by one-quarter cycle.

This relation between i_C and v_C is illustrated by the phasor diagram of Fig. 31-10*b*. As the phasors representing these two quantities rotate counterclockwise together, the phasor labeled I_C does indeed lead that labeled V_C, and by an angle of 90°; that is, the phasor I_C coincides with the vertical axis one-quarter cycle before the phasor V_C does. Be sure to convince yourself that the phasor diagram of Fig. 31-10*b* is consistent with Eqs. 31-30 and 31-34.

CHECKPOINT 4

The figure shows, in (*a*), a sine curve $S(t) = \sin(\omega_d t)$ and three other sinusoidal curves $A(t)$, $B(t)$, and $C(t)$, each of the form $\sin(\omega_d t - \phi)$. (a) Rank the three other curves according to the value of ϕ, most positive first and most negative last. (b) Which curve corresponds to which phasor in (*b*) of the figure? (c) Which curve leads the others?




For a capacitive load, the current leads the potential difference by 90°.

"Leads" means that the current peaks at an *earlier* time than the potential difference.
v_C, i_C
I_C
V_C
$\phi = -90° = -\pi/2$ rad
v_C
0
$T/2$
T
t
i_C
Instants represented in (b)
(a)
I_C
i_C
Rotation of phasors at rate ω_d
v_C
V_C
$\omega_d t$
(b)

Figure 31-10 (*a*) The current in the capacitor leads the voltage by 90° (= $\pi/2$ rad). (*b*) A phasor diagram shows the same thing.

SAMPLE PROBLEM 31.04

Purely capacitive load: potential difference and current

In Fig. 31-9, capacitance C is 15.0 μF and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What are the potential difference $v_C(t)$ across the capacitance and the amplitude V_C of $v_C(t)$?

KEY IDEA

In a circuit with a purely capacitive load, the potential difference $v_C(t)$ across the capacitance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: Here we have $v_C(t) = \mathscr{E}(t)$ and $V_C = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we have

$$V_C = \mathscr{E}_m = 36.0\text{ V}. \qquad \text{(Answer)}$$

To find $v_C(t)$, we use Eq. 31-22 to write

$$v_C(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (31\text{-}37)$$

Then, substituting $\mathscr{E}_m = 36.0$ V and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-37, we have

$$v_C = (36.0\text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_C(t)$ in the circuit as a function of time and the amplitude I_C of $i_C(t)$?

KEY IDEA

In an ac circuit with a purely capacitive load, the alternating current $i_C(t)$ in the capacitance leads the alternating potential difference $v_C(t)$ by 90°; that is, the phase constant ϕ for the current is $-90°$, or $-\pi/2$ rad.

Calculations: Thus, we can write Eq. 31-23 as

$$i_C = I_C \sin(\omega_d t - \phi) = I_C \sin(\omega_d t + \pi/2). \qquad (31\text{-}38)$$

We can find the amplitude I_C from Eq. 31-36 ($V_C = I_C X_C$) if we first find the capacitive reactance X_C. From Eq. 31-33 ($X_C = 1/\omega_d C$), with $\omega_d = 2\pi f_d$, we can write

$$X_C = \frac{1}{2\pi f_d C} = \frac{1}{(2\pi)(60.0\text{ Hz})(15.0 \times 10^{-6}\text{ F})} = 177\ \Omega.$$

Then Eq. 31-36 tells us that the current amplitude is

$$I_C = \frac{V_C}{X_C} = \frac{36.0\text{ V}}{177\ \Omega} = 0.203\text{ A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-38, we have

$$i_C = (0.203\text{ A}) \sin(120\pi t + \pi/2). \qquad \text{(Answer)}$$

An Inductive Load

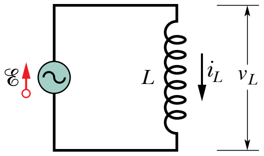


Figure 31-11 An inductor is connected across an alternating-current generator.

Figure 31-11 shows a circuit containing an inductance and a generator with the alternating emf of Eq. 31-22. Using the loop rule and proceeding as we did to obtain Eq. 31-24, we find that the potential difference across the inductance is

$$v_L = V_L \sin \omega_d t, \qquad (31\text{-}39)$$

where V_L is the amplitude of v_L. From Eq. 30-37 ($\mathscr{E}_L = -L\,di/dt$), we can write the potential difference across an inductance L in which the current is changing at the rate di_L/dt as

$$v_L = L\frac{di_L}{dt}. \qquad (31\text{-}40)$$

If we combine Eqs. 31-39 and 31-40, we have

$$\frac{di_L}{dt} = \frac{V_L}{L} \sin \omega_d t. \qquad (31\text{-}41)$$

Our concern, however, is with the current, so we integrate

$$i_L = \int di_L = \frac{V_L}{L}\int \sin \omega_d t\, dt = -\left(\frac{V_L}{\omega_d L}\right)\cos \omega_d t. \qquad (31\text{-}42)$$

We now modify this equation in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_L, called the **inductive reactance** of an inductor, which is defined as

$$X_L = \omega_d L \quad \text{(inductive reactance).} \tag{31-43}$$

The value of X_L depends on the driving angular frequency ω_d. The unit of the inductive time constant τ_L indicates that the SI unit of X_L is the *ohm*, just as it is for X_C and for R.

Second, we replace $-\cos \omega_d t$ in Eq. 31-42 with a phase-shifted sine:

$$-\cos \omega_d t = \sin(\omega_d t - 90°).$$

You can verify this identity by shifting a sine curve 90° in the positive direction.

With these two changes, Eq. 31-42 becomes

$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90°). \tag{31-44}$$

From Eq. 31-23, we can also write this current in the inductance as

$$i_L = I_L \sin(\omega_d t - \phi), \tag{31-45}$$

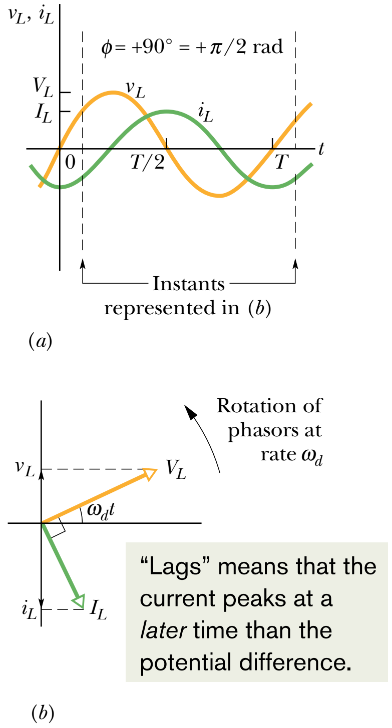
where I_L is the amplitude of the current i_L. Comparing Eqs. 31-44 and 31-45, we see that for a purely inductive load the phase constant ϕ for the current is +90°. We also see that the voltage amplitude and current amplitude are related by

$$V_L = I_L X_L \quad \text{(inductor).} \tag{31-46}$$

Although we found this relation for the circuit of Fig. 31-11, it applies to any inductance in any ac circuit.

Comparison of Eqs. 31-39 and 31-44, or inspection of Fig. 31-12*a*, shows that the quantities i_L and v_L are 90° out of phase. In this case, however, i_L *lags* v_L; that is, monitoring the current i_L and the potential difference v_L in the circuit of Fig. 31-11 shows that i_L reaches its maximum value *after* v_L does, by one-quarter cycle. The phasor diagram of Fig. 31-12*b* also contains this information. As the phasors rotate counterclockwise in the figure, the phasor labeled I_L does indeed lag that labeled V_L, and by an angle of 90°. Be sure to convince yourself that Fig. 31-12*b* represents Eqs. 31-39 and 31-44.

For an inductive load, the current lags the potential difference by 90°.



"Lags" means that the current peaks at a *later* time than the potential difference.

Figure 31-12 (*a*) The current in the inductor lags the voltage by 90° (= π/2 rad). (*b*) A phasor diagram shows the same thing.

CHECKPOINT 5

If we increase the driving frequency in a circuit with a purely capacitive load, do (a) amplitude V_C and (b) amplitude I_C increase, decrease, or remain the same? If, instead, the circuit has a purely inductive load, do (c) amplitude V_L and (d) amplitude I_L increase, decrease, or remain the same?

Application. One familiar example of an ac circuit element is the tube light we use at our homes. It consists of a choke coil and a tube as shown in Fig. 31-13. (Note choke coil is an inductor.)

Since the inductor and the resistor are in series, the currents through them are the same. However, at any instant, the voltage across them will not be necessarily the same. The instantaneous value of the voltage across the combination can be found by adding the voltage across

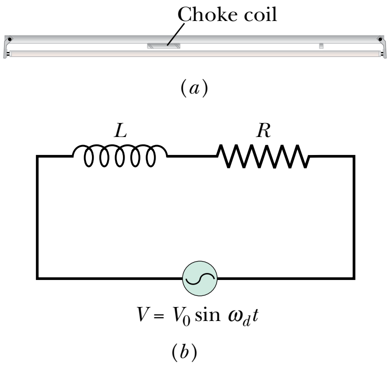


Figure 31-13 (*a*) A tubelight. (*b*) The equivalent circuit of a tubelight. The tube effectively acts as a resistance R and the choke coil has inductance L.

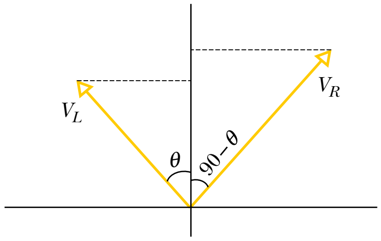


Figure 31-14 The phasor of the voltages across the inductance and the resistance.

the two, as in Kirchhoff's law. As we can see in the phase diagram (Fig. 31-14), the instantaneous voltages will have a phase difference of 90° and so the total voltage across the combination will be

$$V = V_L \cos\theta + V_R \sin\theta.$$

As we know from our mathematics classes, the maximum value of V is

$$V_{\max} = \sqrt{V_L^2 + V_R^2} = I_{\max}\sqrt{(\omega_d L)^2 + R^2}.$$

Interestingly, as the equation suggests, the value of V is maximum when neither V_L nor V_R are maximum. If you remember, we encountered a similar situation before, while studying Simple Harmonic Motion in Chapter 15, Vol. I. There, in order to find out the resultant amplitude when two or more SHM are added, we added the amplitudes vectorially, taking into account the phase difference between them.

Mathematically, the situation that we have here is the same. So, in the same spirit, we can add the amplitude of the voltage across the inductor and the resistor vectorially to obtain the amplitude of the applied voltage. Actually, we are already aware of the amplitude of the applied voltage. We are more interested in the amplitude of current. The current amplitude is

$$I_{\max} = \frac{V_{\max}}{\sqrt{(\omega_d L)^2 + R^2}}$$

This gives that the impedance of the circuit as

$$Z = \frac{V_{\max}}{I_{\max}} = \sqrt{(\omega_d L)^2 + R^2}.$$

We can also see from the phase diagram that the current and the voltage across the resistor (tube in this case) are in the same phase. But here V_R will not be in the same phase as that of the applied voltage. Since the equation for the applied voltage is given by

$$V = V_0 \sin \omega_d t;$$

Thus, the equation for current at any instant is

$$I = i_0 \sin(\omega_d t - \phi),$$

where ϕ is given by

$$\tan\phi = \frac{V_L}{V_R} = \frac{\omega_d L}{R}.$$

CHECKPOINT 6

A long wire of finite resistance is connected to an ac generator. This wire is then wound into a coil of many loops and reconnected to the generator. Is the current in the circuit with the coil greater than, less than, or the same as the current in the circuit with the uncoiled wire?

PROBLEM-SOLVING TACTICS

Tactic 1: Leading and Lagging in ac Circuits Table 31-2 summarizes the relations between the current i and the voltage v for each of the three kinds of circuit elements we have considered. When an applied alternating voltage produces an alternating current in these elements, the current is always in phase with the voltage across a resistor, always leads the voltage across a capacitor, and always lags the voltage across an inductor.

Many students remember these results with the mnemonic "*ELI* the *ICE* man." *ELI* contains the letter L (for inductor), and in it the letter I (for current) comes *after* the letter E (for emf or voltage). Thus, for an inductor, the current *lags* (comes after) the voltage. Similarly, *ICE* (which contains a C for capacitor) means that the current *leads* (comes before) the voltage. You might also use the modified mnemonic "*ELI positively* is the *ICE* man" to remember that the phase constant ϕ is positive for an inductor.

If you have difficulty in remembering whether X_C is equal to $\omega_d C$ (wrong) or $1/\omega_d C$ (right), try remembering that C is in the "cellar"—that is, in the denominator.

Table 31-2 Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	0° (= 0 rad)	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by 90° (= $\pi/2$ rad)	−90° (= $-\pi/2$ rad)	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Leads v_L by 90° (= $\pi/2$ rad)	+90° (= $+\pi/2$ rad)	$V_L = I_L X_L$

SAMPLE PROBLEM 31.05

Purely inductive load: potential difference and current

In Fig. 31-11, inductance L is 230 mH and the sinusoidal alternating emf device operates at amplitude $\mathscr{E}_m = 36.0$ V and frequency $f_d = 60.0$ Hz.

(a) What are the potential difference $v_L(t)$ across the inductance and the amplitude V_L of $v_L(t)$?

KEY IDEA

In a circuit with a purely inductive load, the potential difference $v_L(t)$ across the inductance is always equal to the potential difference $\mathscr{E}(t)$ across the emf device.

Calculations: Here, we have $v_L(t) = \mathscr{E}(t)$ and $V_L = \mathscr{E}_m$. Since $\mathscr{E}_m$ is given, we know that

$$V_L = \mathscr{E}_m = 36.0 \text{ V}. \qquad \text{(Answer)}$$

To find $v_L(t)$, we use Eq. 31-22 to write

$$V_L = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t. \qquad (31\text{-}47)$$

Then, substituting $\mathscr{E}_m = 36.0$ V and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-47, we have

$$V_L = (36.0 \text{ V}) \sin(120\pi t). \qquad \text{(Answer)}$$

(b) What are the current $i_L(t)$ in the circuit as a function of time and the amplitude I_L of $i_L(t)$?

KEY IDEA

In an ac circuit with a purely inductive load, the alternating current $i_L(t)$ in the inductance lags the alternating potential difference $v_L(t)$ by 90°. (In the mnemonic of the problem-solving tactic, this circuit is "positively an *ELI* circuit," which tells us that the emf *E* leads the current *I* and that ϕ is *positive*.)

Calculations: Because the phase constant ϕ for the current is +90°, or $+\pi/2$ rad, we can write Eq. 31-23 as

$$i_L = I_L \sin(\omega_d t - \phi) = I_L \sin(\omega_d t - \pi/2). \qquad (31\text{-}48)$$

We can find the amplitude I_L from Eq. 31-46 ($V_L = I_L X_L$) if we first find the inductive reactance X_L. From Eq. 31-43 ($X_L = \omega_d L$), with $\omega_d = 2\pi f_d$, we can write

$$X_L = 2\pi f_d L = (2\pi)(60.0\ \text{Hz})(230 \times 10^{-3}\ \text{H})$$
$$= 86.7\ \Omega.$$

Then Eq. 31-46 tells us that the current amplitude is

$$I_L = \frac{V_L}{X_L} = \frac{36.0\ \text{V}}{86.7\ \Omega} = 0.415\ \text{A}. \qquad \text{(Answer)}$$

Substituting this and $\omega_d = 2\pi f_d = 120\pi$ into Eq. 31-48, we have

$$i_L = (0.415\ \text{A}) \sin(120\pi t - \pi/2). \qquad \text{(Answer)}$$

31.7 | THE SERIES *RLC* CIRCUIT

Key Concepts

- For a series *RLC* circuit with an external emf given by

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t,$$

and current given by

$$i = I \sin(\omega_d t - \phi),$$

the current amplitude is given by

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \qquad \text{(current amplitude)}.$$

- The phase constant is given by

$$\tan\phi = \frac{X_L - X_C}{R} \qquad \text{(phase constant)}.$$

- The impedance *Z* of the circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad \text{(impedance)}.$$

- We relate the current amplitude and the impedance with $I = \mathscr{E}_m/Z.$
- The current amplitude *I* is maximum ($I = \mathscr{E}_m/R$) when the driving angular frequency ω_d equals the natural angular frequency ω of the circuit, a condition known as resonance. Then $X_C = X_L$, $\phi = 0$, and the current is in phase with the emf.

We are now ready to apply the alternating emf of Eq. 31-22,

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t, \quad \text{(applied emf)}, \qquad (31\text{-}49)$$

to the full *RLC* circuit of Fig. 31-6. Because *R*, *L*, and *C* are in series, the same current

$$i = I \sin(\omega_d t - \phi) \qquad (31\text{-}50)$$

is driven in all three of them. We wish to find the current amplitude *I* and the phase constant ϕ and to investigate how these quantities depend on the driving angular frequency ω_d. The solution is simplified by the use of phasor diagrams as introduced for the three basic circuits of Section 31.6: capacitive load, inductive load, and resistive load. In particular we shall make use of how the voltage phasor is related to the current phasor for each of those basic circuits. We shall find that series *RLC* circuits can be separated into three types: mainly capacitive circuits, mainly inductive circuits, and circuits that are in resonance.

The Current Amplitude

We start with Fig. 31-15*a*, which shows the phasor representing the current of Eq. 31-50 at an arbitrary time *t*. The length of the phasor is the current amplitude *I*, the projection of the phasor on the vertical axis is the current *i* at time *t*, and the angle of rotation of the phasor is the phase $\omega_d t - \phi$ of the current at time *t*.

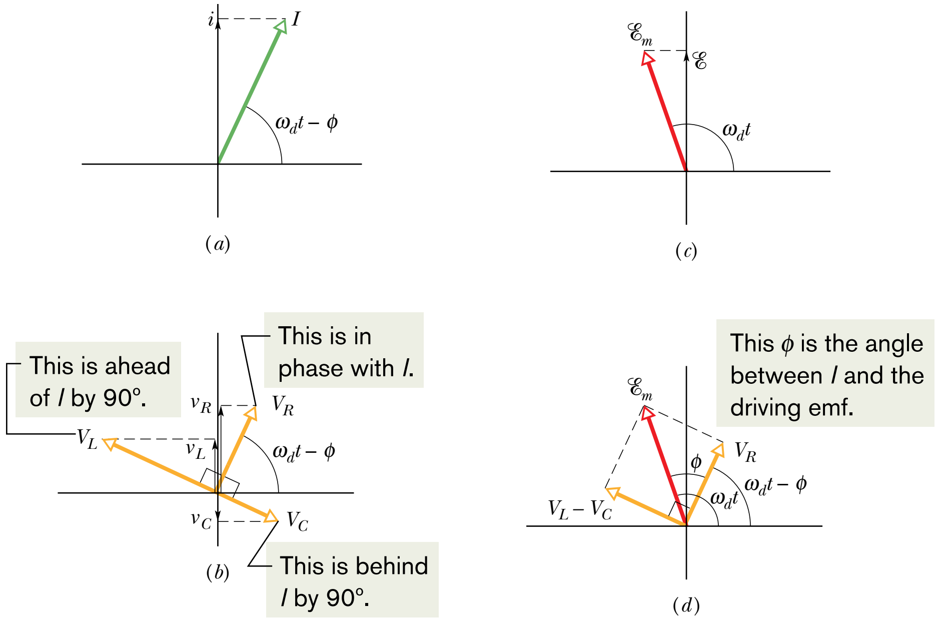


Figure 31-15 (*a*) A phasor representing the alternating current in the driven *RLC* circuit of Fig. 31-6 at time t. The amplitude I, the instantaneous value i, and the phase $(\omega_d t - \phi)$ are shown. (*b*) Phasors representing the voltages across the inductor, resistor, and capacitor, oriented with respect to the current phasor in (*a*). (*c*) A phasor representing the alternating emf that drives the current of (*a*). (*d*) The emf phasor is equal to the vector sum of the three voltage phasors of (*b*). Here, voltage phasors V_L and V_C have been added vectorially to yield their net phasor $(V_L - V_C)$.

Figure 31-15*b* shows the phasors representing the voltages across R, L, and C at the same time t. Each phasor is oriented relative to the angle of rotation of current phasor I in Fig. 31-15*a*, based on the information in Table 31-2:

Resistor: Here current and voltage are in phase; so the angle of rotation of voltage phasor V_R is the same as that of phasor I.

Capacitor: Here current leads voltage by 90°; so the angle of rotation of voltage phasor V_C is 90° less than that of phasor I.

Inductor: Here current lags voltage by 90°; so the angle of rotation of voltage phasor v_L is 90° greater than that of phasor I.

Figure 31-15*b* also shows the instantaneous voltages v_R, v_C, and v_L across R, C, and L at time t; those voltages are the projections of the corresponding phasors on the vertical axis of the figure.

Figure 31-15*c* shows the phasor representing the applied emf of Eq. 31-49. The length of the phasor is the emf amplitude $\mathscr{E}_m$, the projection of the phasor on the vertical axis is the emf $\mathscr{E}$ at time t, and the angle of rotation of the phasor is the phase $\omega_d t$ of the emf at time t.

From the loop rule we know that at any instant the sum of the voltages v_R, v_C, and v_L is equal to the applied emf $\mathscr{E}$:

$$\mathscr{E} = v_R + v_C + v_L. \tag{31-51}$$

Thus, at time t the projection $\mathscr{E}$ in Fig. 31-15*c* is equal to the algebraic sum of the projections v_R, v_C, and v_L in Fig. 31-15*b*. In fact, as the phasors rotate together, this equality always holds. This means that phasor $\mathscr{E}_m$ in Fig. 31-15*c* must be equal to the vector sum of the three voltage phasors V_R, V_C, and V_L in Fig. 31-15*b*.

That requirement is indicated in Fig. 31-15*d*, where phasor $\mathscr{E}_m$ is drawn as the sum of phasors V_R, V_L, and V_C. Because phasors V_L and V_C have opposite directions in the figure, we simplify the vector sum by first combining V_L and V_C to form the single phasor $V_L - V_C$. Then we combine that single phasor with V_R to find the net phasor. Again, the net phasor must coincide with phasor $\mathscr{E}_m$, as shown.

Both triangles in Fig. 31-15*d* are right triangles. Applying the Pythagorean theorem to either one yields

$$\mathscr{E}_m^2 = V_R^2 + (V_L - V_C)^2. \tag{31-52}$$

From the voltage amplitude information displayed in the rightmost column of Table 31-2, we can rewrite this as

$$\mathscr{E}_m^2 = (IR)^2 + (IX_L - IX_C)^2, \tag{31-53}$$

and then rearrange it to the form

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}. \tag{31-54}$$

The denominator in Eq. 31-54 is called the **impedance** Z of the circuit for the driving angular frequency ω_d:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(impedance defined).} \tag{31-55}$$

We can then write Eq. 31-54 as

$$I = \frac{\mathscr{E}_m}{Z}. \tag{31-56}$$

If we substitute for X_C and X_L from Eqs. 31-33 and 31-43, we can write Eq. 31-54 more explicitly as

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude).} \tag{31-57}$$

We have now accomplished half our goal: We have obtained an expression for the current amplitude I in terms of the sinusoidal driving emf and the circuit elements in a series *RLC* circuit.

The value of I depends on the difference between $\omega_d L$ and $1/\omega_d C$ in Eq. 31-57 or, equivalently, the difference between X_L and X_C in Eq. 31-54. In either equation, it does not matter which of the two quantities is greater because the difference is always squared.

The current that we have been describing in this module is the *steady-state current* that occurs after the alternating emf has been applied for some time. When the emf is first applied to a circuit, a brief *transient current* occurs. Its duration (before settling down into the steady-state current) is determined by the time constants $\tau_L = L/R$ and $\tau_C = RC$ as the inductive and capacitive elements "turn on." This transient current can, for example, destroy a motor on start-up if it is not properly taken into account in the motor's circuit design.

The Phase Constant

From the right-hand phasor triangle in Fig. 31-15*d* and from Table 31-2 we can write

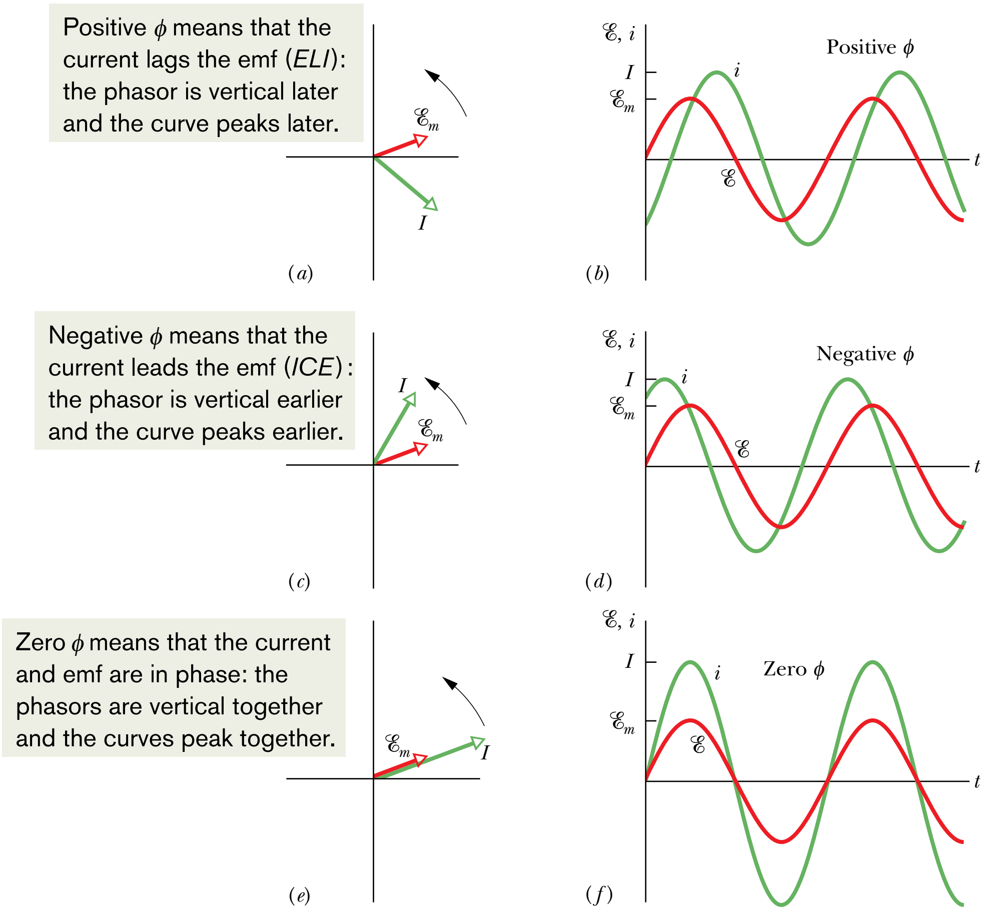
$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR}, \tag{31-58}$$

which gives us

$$\tan\phi = \frac{X_L - X_C}{R} \quad \text{(phase constant).} \tag{31-59}$$

This is the other half of our goal: an equation for the phase constant ϕ in the sinusoidally driven series *RLC* circuit of Fig. 31-6. In essence, it gives us three different results for the phase constant, depending on the relative values of the reactances X_L and X_C:

$X_L > X_C$: The circuit is said to be *more inductive than capacitive*. Equation 31-59 tells us that ϕ is positive for such a circuit, which means that phasor I rotates behind phasor $\mathscr{E}_m$ (Fig. 31-16*a*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 31-16*b*. (Figures 31-15*c* and *d* were drawn assuming $X_L > X_C$.)



Positive ϕ means that the current lags the emf (*ELI*): the phasor is vertical later and the curve peaks later.

Negative ϕ means that the current leads the emf (*ICE*): the phasor is vertical earlier and the curve peaks earlier.

Zero ϕ means that the current and emf are in phase: the phasors are vertical together and the curves peak together.

Figure 31-16 Phasor diagrams and graphs of the alternating emf $\mathscr{E}$ and current i for the driven *RLC* circuit of Fig. 31-6. In the phasor diagram of (*a*) and the graph of (*b*), the current i lags the driving emf $\mathscr{E}$ and the current's phase constant ϕ is positive. In (*c*) and (*d*), the current i leads the driving emf $\mathscr{E}$ and its phase constant ϕ is negative. In (*e*) and (*f*), the current i is in phase with the driving emf $\mathscr{E}$ and its phase constant ϕ is zero.

$X_C > X_L$: The circuit is said to be *more capacitive than inductive*. Equation 31-59 tells us that ϕ is negative for such a circuit, which means that phasor I rotates ahead of phasor $\mathscr{E}_m$ (Fig. 31-16*c*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 31-16*d*.

$X_C = X_L$: The circuit is said to be in *resonance*, a state that is discussed next. Equation 31-59 tells us that $\phi = 0°$ for such a circuit, which means that phasors $\mathscr{E}_m$ and I rotate together (Fig. 31-16*e*). A plot of $\mathscr{E}$ and i versus time is like that in Fig. 31-16*f*.

As illustration, let us reconsider two extreme circuits: In the *purely inductive circuit* of Fig. 31-11, where X_L is nonzero and $X_C = R = 0$, Eq. 31-59 tells us that the circuit's phase constant is $\phi = +90°$ (the greatest value of ϕ), consistent with Fig. 31-12*b*. In the *purely capacitive circuit* of Fig. 31-9, where X_C is nonzero and $X_L = R = 0$, Eq. 31-59 tells us that the circuit's phase constant is $\phi = -90°$ (the least value of ϕ), consistent with Fig. 31-10*b*.

Resonance

Equation 31-57 gives the current amplitude I in an *RLC* circuit as a function of the driving angular frequency ω_d of the external alternating emf. For a given resistance R, that amplitude is a maximum when the quantity $\omega_d L - 1/\omega_d C$ in the denominator is zero—that is, when

$$\omega_d L = \frac{1}{\omega_d C}$$

or

$$\omega_d = \frac{1}{\sqrt{LC}} \quad \text{(maximum } I\text{)}. \tag{31-60}$$

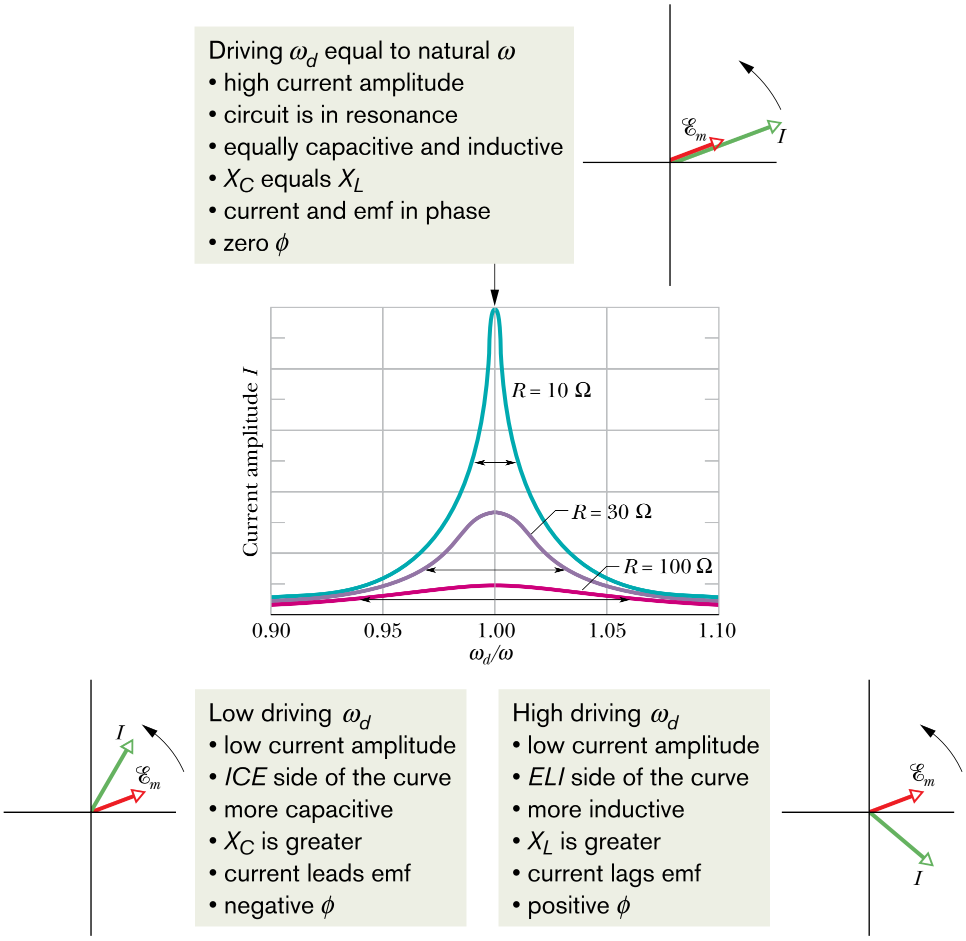
Because the natural angular frequency ω of the RLC circuit is also equal to $1/\sqrt{LC}$, the maximum value of I occurs when the driving angular frequency matches the natural angular frequency—that is, at resonance. Thus, in an RLC circuit, resonance and maximum current amplitude I occur when

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \quad \text{(resonance)}. \tag{31-61}$$

Resonance Curves. Figure 31-17 shows three *resonance curves* for sinusoidally driven oscillations in three series RLC circuits differing only in R. Each curve peaks at its maximum current amplitude I when the ratio ω_d/ω is 1.00, but the maximum value of I decreases with increasing R. (The maximum I is always $\mathscr{E}_m/R$; to see why, combine Eqs. 31-55 and 31-56.) In addition, the curves increase in width (measured in Fig. 31-17 at half the maximum value of I) with increasing R.

To make physical sense of Fig. 31-17, consider how the reactances X_L and X_C change as we increase the driving angular frequency ω_d, starting with a value much less than the natural frequency ω. For small ω_d, reactance X_L $(= \omega_d L)$ is small and reactance X_C $(= 1/\omega_d C)$ is large. Thus, the circuit is mainly capacitive and the impedance is dominated by the large X_C, which keeps the current low.

As we increase ω_d, reactance X_C remains dominant but decreases while reactance X_L increases. The decrease in X_C decreases the impedance, allowing the current to increase, as we see on the left side of any resonance curve in Fig. 31-17. When the increasing X_L and the decreasing X_C reach equal values, the current is greatest and the circuit is in resonance, with $\omega_d = \omega$.



Driving ω_d equal to natural ω
- high current amplitude
- circuit is in resonance
- equally capacitive and inductive
- X_C equals X_L
- current and emf in phase
- zero ϕ

Low driving ω_d
- low current amplitude
- *ICE* side of the curve
- more capacitive
- X_C is greater
- current leads emf
- negative ϕ

High driving ω_d
- low current amplitude
- *ELI* side of the curve
- more inductive
- X_L is greater
- current lags emf
- positive ϕ

Figure 31-17 *Resonance curves* for the driven RLC circuit of Fig. 31-6 with $L = 100\ \mu\text{H}$, $C = 100$ pF, and three values of R. The current amplitude I of the alternating current depends on how close the driving angular frequency ω_d is to the natural angular frequency ω. The horizontal arrow on each curve measures the curve's *half-width*, which is the width at the half-maximum level and is a measure of the sharpness of the resonance. To the left of $\omega_d/\omega = 1.00$, the circuit is mainly capacitive, with $X_C > X_L$; to the right, it is mainly inductive, with $X_L > X_C$.

As we continue to increase ω_d, the increasing reactance X_L becomes progressively more dominant over the decreasing reactance X_C. The impedance increases because of X_L and the current decreases, as on the right side of any resonance curve in Fig. 31-17. In summary, then: The low-angular-frequency side of a resonance curve is dominated by the capacitor's reactance, the high-angular-frequency side is dominated by the inductor's reactance, and resonance occurs in the middle.

☑ CHECKPOINT 7

Here are the capacitive reactance and inductive reactance, respectively, for three sinusoidally driven series *RLC* circuits: (1) 50 Ω, 100 Ω; (2) 100 Ω, 50 Ω; (3) 50 Ω, 50 Ω. (a) For each, does the current lead or lag the applied emf, or are the two in phase? (b)Which circuit is in resonance?

SAMPLE PROBLEM 31.06

Current amplitude, impedance, and phase constant

In Fig. 31-6, let $R = 200\ \Omega$, $C = 15.0\ \mu\text{F}$, $L = 230$ mH, $f_d = 60.0$ Hz, and $\mathscr{E}_m = 36.0$ V. (These parameters are those used in the earlier sample problems.)

(a) What is the current amplitude I?

KEY IDEA

The current amplitude I depends on the amplitude $\mathscr{E}_m$ of the driving emf and on the impedance Z of the circuit, according to Eq. 31-56 ($I = \mathscr{E}_m/Z$).

Calculations: So, we need to find Z, which depends on resistance R, capacitive reactance X_C, and inductive reactance X_L. The circuit's resistance is the given resistance R. Its capacitive reactance is due to the given capacitance and, from an earlier sample problem, $X_C = 177\ \Omega$. Its inductive reactance is due to the given inductance and, from another sample problem, $X_L = 86.7\ \Omega$. Thus, the circuit's impedance is

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(200\ \Omega)^2 + (86.7\ \Omega - 177\ \Omega)^2} \\ &= 219\ \Omega. \end{aligned}$$

We then find

$$I = \frac{\mathscr{E}_m}{Z} = \frac{36.0\ \text{V}}{219\ \Omega} = 0.164\ \text{A}. \qquad \text{(Answer)}$$

(b) What is the phase constant ϕ of the current in the circuit relative to the driving emf?

KEY IDEA

The phase constant depends on the inductive reactance, the capacitive reactance, and the resistance of the circuit, according to Eq. 31-59.

Calculation: Solving Eq. 31-59 for ϕ leads to

$$\begin{aligned} \phi &= \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{86.7\ \Omega - 177\ \Omega}{200\ \Omega} \\ &= -24.3° = -0.424\ \text{rad}. \qquad \text{(Answer)} \end{aligned}$$

The negative phase constant is consistent with the fact that the load is mainly capacitive; that is, $X_C > X_L$. In the common mnemonic for driven series *RLC* circuits, this circuit is an *ICE* circuit—the current *leads* the driving emf.

31.8 | POWER IN ALTERNATING-CURRENT CIRCUITS

Key Concepts

- In a series *RLC* circuit, the average power P_{avg} of the generator is equal to the production rate of thermal energy in the resistor:

$$P_{\text{avg}} = I_{\text{rms}}^2 R = \mathscr{E}_{\text{rms}} I_{\text{rms}} \cos\phi.$$

- The abbreviation rms stands for root-mean-square; the rms quantities are related to the maximum quantities by $I_{\text{rms}} = I/\sqrt{2}$, $V_{\text{rms}} = V/\sqrt{2}$, and $\mathscr{E}_{\text{rms}} = \mathscr{E}_m/\sqrt{2}$. The term $\cos\phi$ is called the power factor of the circuit.

In the *RLC* circuit of Fig. 31-6, the source of energy is the alternating-current generator. Some of the energy that it provides is stored in the electric field in the capacitor, some is stored in the magnetic field in the inductor, and some is dissipated as thermal energy in the resistor. In steady-state operation, the average stored energy remains constant. The net transfer of energy is thus from the generator to the resistor, where energy is dissipated.

The instantaneous rate at which energy is dissipated in the resistor can be written, with the help of Eqs. 26-27 and 31-23, as

$$P = i^2R = [I\sin(\omega_d t - \phi)]^2R = I^2R\sin^2(\omega_d t - \phi). \tag{31-62}$$

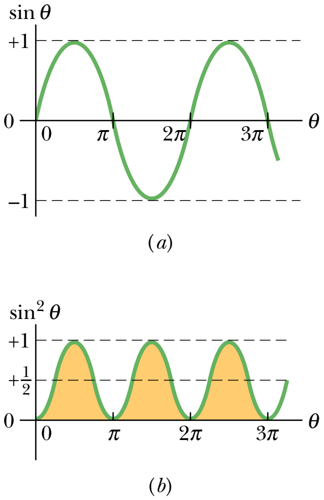


Figure 31-18 (*a*) A plot of sin θ versus θ. The average value over one cycle is zero. (*b*) A plot of sin² θ versus θ. The average value over one cycle is 1/2.

The *average* rate at which energy is dissipated in the resistor, however, is the average of Eq. 31-62 over time. Over one complete cycle, the average value of sin θ, where θ is any variable, is zero (Fig. 31-18*a*) but the average value of $\sin^2\theta$ is 1/2 (Fig. 31-18*b*). (Note in Fig. 31-18*b* how the shaded areas under the curve but above the horizontal line marked +1/2 exactly fill in the unshaded spaces below that line.) Thus, we can write, from Eq. 31-62,

$$P_{\text{avg}} = \frac{I^2R}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R. \tag{31-63}$$

The quantity $I/\sqrt{2}$ is called the **root-mean-square**, or **rms**, value of the current *i*

$$I_{\text{rms}} = \frac{I}{\sqrt{2}} \quad \text{(rms current).} \tag{31-64}$$

We can now rewrite Eq. 31-63 as

$$P_{\text{avg}} = I_{\text{rms}}^2 R \quad \text{(average power).} \tag{31-65}$$

Equation 31-65 has the same mathematical form as Eq. 26-27 ($P = i^2R$); the message here is that if we switch to the rms current, we can compute the average rate of energy dissipation for alternating-current circuits just as for direct-current circuits.

We can also define rms values of voltages and emfs for alternating-current circuits:

$$V_{\text{rms}} = \frac{V}{\sqrt{2}} \quad \text{and} \quad \mathscr{E}_{\text{rms}} = \frac{\mathscr{E}_m}{\sqrt{2}} \quad \text{(rms voltage; rms emf).} \tag{31-66}$$

Alternating-current instruments, such as ammeters and voltmeters, are usually calibrated to read I_{rms}, V_{rms}, and $\mathscr{E}_{\text{rms}}$. Thus, if you plug an alternating-current voltmeter into a household electrical outlet and it reads 120 V, that is an rms voltage. The *maximum* value of the potential difference at the outlet is $\sqrt{2} \times (120\text{ V})$, or 170 V. Generally scientists and engineers report rms values instead of maximum values.

Because the proportionality factor $1/\sqrt{2}$ in Eqs. 31-64 and 31-66 is the same for all three variables, we can write Eqs. 31-56 and 31-54 as

$$I_{\text{rms}} = \frac{\mathscr{E}_{\text{rms}}}{Z} = \frac{\mathscr{E}_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}}, \tag{31-67}$$

and, indeed, this is the form that we almost always use.

We can use the relationship $I_{\text{rms}} = \mathscr{E}_{\text{rms}}/Z$ to recast Eq. 31-65 in a useful equivalent way. We write

$$P_{\text{avg}} = \frac{\mathscr{E}_{\text{rms}}}{Z} I_{\text{rms}} R = \mathscr{E}_{\text{rms}} I_{\text{rms}} \frac{R}{Z}. \tag{31-68}$$

From Fig. 31-15*d*, Table 31-2, and Eq. 31-56, however, we see that R/Z is just the cosine of the phase constant ϕ:

$$\cos\phi = \frac{V_R}{\mathscr{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}. \tag{31-69}$$

Equation 31-68 then becomes

$$P_{\text{avg}} = \mathscr{E}_{\text{rms}} I_{\text{rms}} \cos\phi \quad \text{(average power)}, \tag{31-70}$$

in which the term $\cos\phi$ is called the **power factor**. Because $\cos\phi = \cos(-\phi)$, Eq. 31-70 is independent of the sign of the phase constant ϕ.

To maximize the rate at which energy is supplied to a resistive load in an *RLC* circuit, we should keep the power factor $\cos\phi$ as close to unity as possible. This is equivalent to keeping the phase constant ϕ in Eq. 31-23 as close to zero as possible. If, for example, the circuit is highly inductive, it can be made less so by putting more capacitance in the circuit, connected in series. (Recall that putting an additional capacitance into a series of capacitances decreases the equivalent capacitance C_{eq} of the series.) Thus, the resulting decrease in C_{eq} in the circuit reduces the phase constant and increases the power factor in Eq. 31-70. Power companies place series-connected capacitors throughout their transmission systems to get these results.

CHECKPOINT 8

(a) If the current in a sinusoidally driven series *RLC* circuit leads the emf, would we increase or decrease the capacitance to increase the rate at which energy is supplied to the resistance? (b) Would this change bring the resonant angular frequency of the circuit closer to the angular frequency of the emf or put it farther away?

SAMPLE PROBLEM 31.07

Driven *RLC* circuit: power factor and average power

A series *RLC* circuit, driven with $\mathscr{E}_{\text{rms}} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with inductive reactance $X_L = 80.0\ \Omega$, and a capacitance with capacitive reactance $X_C = 150\ \Omega$.

(a) What are the power factor $\cos\phi$ and phase constant ϕ of the circuit?

KEY IDEA

The power factor $\cos\phi$ can be found from the resistance R and impedance Z via Eq. 31-69 ($\cos\phi = R/Z$).

Calculations: To calculate Z, we use Eq. 31-55

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(200\ \Omega)^2 + (80.0\ \Omega - 150\ \Omega)^2} \\ &= 211.90\ \Omega. \end{aligned}$$

Equation 31-69 then gives us

$$\cos\phi = \frac{R}{Z} = \frac{200\ \Omega}{211.90\ \Omega} = 0.9438 \approx 0.944. \qquad \text{(Answer)}$$

Taking the inverse cosine then yields

$$\phi = \cos^{-1} 0.944 = \pm 19.3°.$$

The inverse cosine on a calculator gives only the positive answer here, but both +19.3° and −19.3° have a cosine of 0.944. To determine which sign is correct, we must consider whether the current leads or lags the driving emf. Because $X_C > X_L$, this circuit is mainly capacitive, with the current leading the emf. Thus, ϕ must be negative:

$$\phi = -19.3°. \qquad \text{(Answer)}$$

We could, instead, have found ϕ with Eq. 31-59. A calculator would then have given us the answer with the minus sign.

(b) What is the average rate P_{avg} at which energy is dissipated in the resistance?

KEY IDEAS

There are two ways and two ideas to use: (1) Because the circuit is assumed to be in steady-state operation, the rate at which energy is dissipated in the resistance is equal to the rate at which energy is supplied to the circuit, as given by Eq. 31-70 ($P_{\text{avg}} = \mathscr{E}_{\text{rms}} I_{\text{rms}} \cos\phi$). (2) The rate at which energy is dissipated in a resistance R depends on the square of the rms current I_{rms} through it, according to Eq. 31-65 ($P_{\text{avg}} = I^2_{\text{rms}} R$).

First way: We are given the rms driving emf $\mathscr{E}_{\text{rms}}$ and we already know $\cos\phi$ from part (a). The rms current I_{rms} is determined by the rms value of the driving emf and the circuit's impedance Z (which we know), according to Eq. 31-67

$$I_{\text{rms}} = \frac{\mathscr{E}_{\text{rms}}}{Z}.$$

Substituting this into Eq. 31-70 then leads to

$$\begin{aligned} P_{\text{avg}} &= \mathscr{E}_{\text{rms}} I_{\text{rms}} \cos\phi = \frac{\mathscr{E}^2_{\text{rms}}}{Z} \cos\phi \\ &= \frac{(120\ \text{V})^2}{211.90\ \Omega}(0.9438) = 64.1\ \text{W}. \qquad \text{(Answer)} \end{aligned}$$

Second way: Instead, we can write

$$\begin{aligned} P_{\text{avg}} &= I^2_{\text{rms}} R = \frac{\mathscr{E}^2_{\text{rms}}}{Z^2} R \\ &= \frac{(120\ \text{V})^2}{(211.90\ \Omega)^2}(200\ \Omega) = 64.1\ \text{W}. \qquad \text{(Answer)} \end{aligned}$$

(c) What new capacitance C_{new} is needed to maximize P_{avg} if the other parameters of the circuit are not changed?

KEY IDEAS

(1) The average rate P_{avg} at which energy is supplied and dissipated is maximized if the circuit is brought into resonance with the driving emf. (2) Resonance occurs when $X_C = X_L$.

Calculations: From the given data, we have $X_C > X_L$. Thus, we must decrease X_C to reach resonance. From Eq. 31-33 ($X_C = 1/\omega_d C$), we see that this means we must increase C to the new value C_{new}.

Using Eq. 31-33, we can write the resonance condition $X_C = X_L$ as

$$\frac{1}{\omega_d C_{\text{new}}} = X_L.$$

Substituting $2\pi f_d$ for ω_d (because we are given f_d and not ω_d) and then solving for C_{new}, we find

$$\begin{aligned} C_{\text{new}} &= \frac{1}{2\pi f_d X_L} = \frac{1}{(2\pi)(60\ \text{Hz})(80.0\ \Omega)} \\ &= 3.32 \times 10^{-5}\ \text{F} = 33.2\ \mu\text{F}. \qquad \text{(Answer)} \end{aligned}$$

Following the procedure of part (b), you can show that with C_{new}, the average power of energy dissipation P_{avg} would then be at its maximum value of

$$P_{\text{avg, max}} = 72.0\ \text{W}.$$

SAMPLE PROBLEM 31.08

RLC circuit with elements in parallel: current, phase angle and impedance

For the circuit of Fig. 31-19, find the instantaneous current through each element. Also find (a) the total instantaneous current through the source and (b) the expression for the phase angle of this current and the impedance of the circuit. Draw a phasor diagram to illustrate the relationship of the instantaneous current to the instantaneous emf.

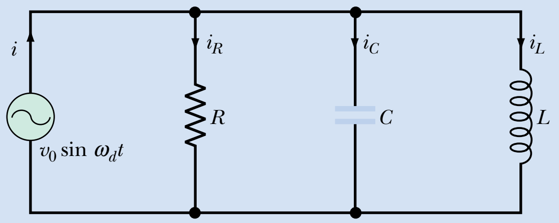


Figure 31-19 Three circuit elements are in parallel. The voltages across them will be the same, but not the current.

KEY IDEA

Till now we have considered only the series circuits. In them the voltage across each element was added to the voltage across the other elements. The current was taken as reference phasor because the current was common through all of them. However, here we should take the voltage as the reference phasor as the voltage across each of the element is the same.

Here we can see that the circuit elements are in parallel. So, the voltage across them will be the same but the current will be different.

Calculation: The three current equations are

$$Ri_R = V_e; \quad L\frac{di_L}{dt} = V_e; \quad \frac{1}{C}i_C = \frac{dV_e}{dt}.$$

The steady-state solutions of these equations are

$$i_R = \frac{V_e}{R}\sin\omega_d t = I_R \sin\omega_d t;$$

$$i_L = -\frac{V_e}{\omega_e L}\cos\omega_d t = -\frac{V_e}{X_L}\cos\omega_d t = -I_L \cos\omega_d t;$$

$$i_C = -V_e\omega_d \cos\omega_d t = -\frac{V_e}{X_C}\cos\omega_d t = I_C \cos\omega_d t.$$

For the total current, we have

$$i = i_R + i_L + i_C = V_e\left[\frac{1}{R}\sin\omega_d t + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)\cos\omega_d t\right]$$

$$\equiv I\sin(\omega_d t + \phi),$$

where $I = V_e/Z$. The impedance and phase angle are then given by

$$\frac{1}{Z} \equiv \left[\left(\frac{1}{R}\right)^2 + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2\right]^{1/2}$$

$$\tan\phi \equiv \frac{(1/X_C) - (1/X_L)}{1/R}$$

Figure 31-20 shows five phasors v_0, I_R, I_L, I_C, I, where I_R is in-phase with V_e, I_C leads V_e by 90°, I_L lags V_e by 90°, and I leads or lags V_e by $|\phi|$. The phasor equality

$$I = I_R + I_L + I_C$$

reflects conservation of charge in the circuit.

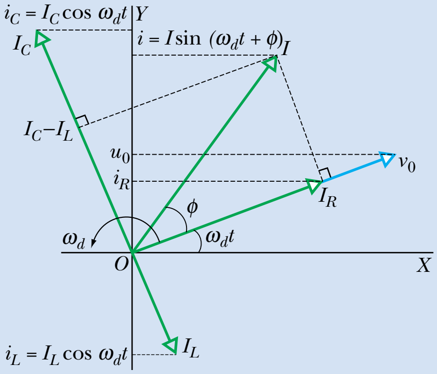

$i_C = I_C \cos\omega_d t$
Y
I_C
$i = I\sin(\omega_d t + \phi)$
I
$I_C - I_L$
u_0
v_0
i_R
I_R
ϕ
ω_d
$\omega_d t$
O
X
$i_L = I_L \cos\omega_d t$
I_L

Figure 31-20 Phasors representing the current through each of the circuit element.

Learn: In addition to the series *RCL* and the parallel circuit, there are many different ways to connect resistors, capacitors, and inductors. The analysis of these circuits may be complex, but it helps to keep in mind the behavior of capacitors and inductors at the extreme limits of the frequency. When the frequency approaches zero (i.e., dc conditions), the reactance of a capacitor becomes so large that no charge can flow through the capacitor. It is as if the capacitor were cut out of circuit, leaving an open gap in the connecting wire. In the limit of zero frequency the reactance of an inductor is vanishingly small. The inductor offers no opposition to a dc current. It is as if the inductor were replaced with a wire of zero resistance. In the limit of very large frequency, the behavior of a capacitor and an inductor are reversed. The capacitor has a very small reactance and offers little opposition to the current, as if it were replaced by a wire with zero resistance. In contrast, the inductor has a very large reactance at a high frequency. The inductor offers so much opposition to the current that it might as well be cut out of the circuit, leaving an open gap in the connecting wire.

SAMPLE PROBLEM 31.09

RLC circuit: rms current through the circuit

Figure 31-21 shows a circuit. The rms voltage of the generator is V_0. The frequency of the ac generator is very near zero. What is the rms current through the circuit?

KEY IDEA

As we just now discussed, at small frequencies, the circuit is like a dc circuit. So, the capacitor can be replaced by a open circuit and inductor by a short circuit. Figure 31-22 shows the circuits as they would appear according to these changes.

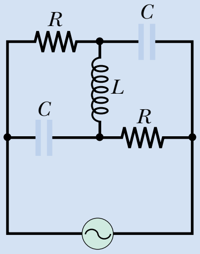


Figure 31-21 Circuit covered in Sample Problem 31.09.

It is obvious from the circuit that it is not a circuit of series elements, nor elements in parallel.

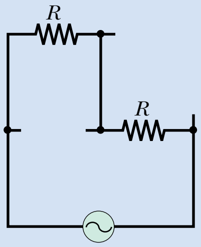


Figure 31-22 The equivalent circuit of Fig. 31-21 at low frequency.

Calculation: According to Eq. 31-67, the rms current is given by

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z}.$$

However, the impedance Z is not given by Eq. 31-55 because the circuit in Fig. 31-21 is not series RCL circuit. The circuit behaves as if it contained only two identical resistors in series, with a total impedance of $Z = 2R$. Thus circuit has

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{2R}.$$

Learn: If the source had a very high frequency (much higher than the resonance frequency), the capacitor would be treated as short-circuited and inductor as open-circuited. Figure 31-23 shows the circuits as they would appear according to these changes. So, at this frequency, the impedance of the circuit would be $R/2$. Hence

$$I_{\text{rms}} = \frac{2V_{\text{rms}}}{R}.$$

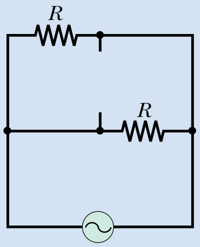


Figure 31-23 Equivalent circuit at high frequency.

31.9 | TRANSFORMERS

Key Concepts

- A transformer (assumed to be ideal) is an iron core on which are wound a primary coil of N_p turns and a secondary coil of N_s turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_S = V_p \frac{N_S}{N_p} \quad \text{(transformation of voltage).}$$

- The currents through the coils are related by

$$I_S = I_p \frac{N_p}{N_S} \quad \text{(transformation of currents).}$$

- The equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{\text{eq}} = \left(\frac{N_p}{N_S}\right)^2 R,$$

where R is the resistive load in the secondary circuit. The ratio N_p/N_s is called the transformer's turns ratio.

Energy Transmission Requirements

When an ac circuit has only a resistive load, the power factor in Eq. 31-70 is cos 0° = 1 and the applied rms emf $\mathscr{E}_{\text{rms}}$ is equal to the rms voltage V_{rms} across the load. Thus, with an rms current I_{rms} in the load, energy is supplied and dissipated at the average rate of

$$P_{\text{avg}} = \mathscr{E}I = IV. \tag{31-71}$$

(In Eq. 31-71 and the rest of this section, we follow conventional practice and drop the subscripts identifying rms quantities. Engineers and scientists assume that all time-varying currents and voltages are reported as rms values; that is what the meters read.) Equation 31-71 tells us that, to satisfy a given power requirement, we have a range of choices for I and V, provided only that the product IV is as required.

In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory). Nobody wants an electric toaster to operate at, say, 10 kV. However, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize I^2R losses (often called *ohmic losses*) in the transmission line.

As an example, consider the 735 kV line used to transmit electrical energy from the La Grande 2 hydroelectric plant in Quebec to Montreal, 1000 km away. Suppose that the current is 500 A and the power factor is close to unity. Then from Eq. 31-71, energy is supplied at the average rate

$$P_{\text{avg}} = \mathscr{E}I = (7.35 \times 10^5\ \text{V})(500\ \text{A}) = 368\ \text{MW}.$$

The resistance of the transmission line is about 0.220 Ω/km; thus, there is a total resistance of about 220 Ω for the 1000 km stretch. Energy is dissipated due to that resistance at a rate of about

$$P_{\text{avg}} = I^2R = (500\ \text{A})^2(220\ \Omega) = 55.0\ \text{MW},$$

which is nearly 15% of the supply rate.

Imagine what would happen if we doubled the current and halved the voltage. Energy would be supplied by the plant at the same average rate of 368 MW as previously, but now energy would be dissipated at the rate of about

$$P_{\text{avg}} = I^2R = (1000\ \text{A})^2(220\ \Omega) = 220\ \text{MW},$$

which is *almost 60% of the supply rate*. Hence the general energy transmission rule: Transmit at the highest possible voltage and the lowest possible current.

The Ideal Transformer

The transmission rule leads to a fundamental mismatch between the requirement for efficient high-voltage transmission and the need for safe low-voltage generation and consumption. We need a device with which we can raise (for transmission) and lower (for use) the ac voltage in a circuit, keeping the product current × voltage essentially constant. The **transformer** is such a device. It has no moving parts, operates by Faraday's law of induction, and has no simple direct-current counterpart.

The *ideal transformer* in Fig. 31-24 consists of two coils, with different numbers of turns, wound around an iron core. (The coils are insulated from the core.) In use, the primary winding, of N_p turns, is connected to an alternating-current generator whose emf $\mathscr{E}$ at any time t is given by

$$\mathscr{E} = \mathscr{E}_m \sin \omega t. \tag{31-72}$$

The secondary winding, of N_s turns, is connected to load resistance R, but its circuit is an open circuit as long as switch S is open (which we assume for the present). Thus, there can be no current through the secondary coil. We assume further for this ideal transformer that the resistances of the primary and secondary windings are negligible. Well-designed, high-capacity transformers can have energy losses as low as 1%; so our assumptions are reasonable.

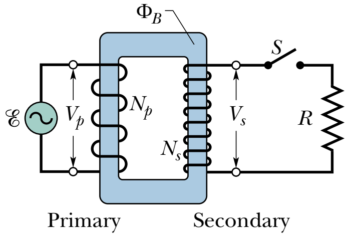


Figure 31-24 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load R when switch S is closed.

For the assumed conditions, the primary winding (or *primary*) is a pure inductance and the primary circuit is like that in Fig. 31-11. Thus, the (very small) primary current, also called the *magnetizing current* I_{mag}, lags the primary voltage V_p by 90°; the primary's power factor (= cos ϕ in Eq. 31-70) is zero; so no power is delivered from the generator to the transformer.

However, the small sinusoidally changing primary current I_{mag} produces a sinusoidally changing magnetic flux Φ_B in the iron core. The core acts to strengthen the flux and to bring it through the secondary winding (or *secondary*). Because Φ_B varies, it induces an emf $\mathscr{E}_{\text{turn}}$ (= $d\Phi_B/dt$) in each turn of the secondary. In fact, this emf per turn $\mathscr{E}_{\text{turn}}$ is the same in the primary and the secondary. Across the primary, the voltage V_p is the product of $\mathscr{E}_{\text{turn}}$ and the number of turns N_p; that is, $V_p = \mathscr{E}_{\text{turn}}N_p$. Similarly, across the secondary the voltage is $V_s = \mathscr{E}_{\text{turn}}N_s$. Thus, we can write

$$\mathscr{E}_{\text{turn}} = \frac{V_p}{N_p} = \frac{V_S}{N_S},$$

or $$V_S = V_p \frac{N_S}{N_p} \quad \text{(transformation of voltage).} \tag{31-73}$$

If $N_s > N_p$, the device is a *step-up transformer* because it steps the primary's voltage V_p *up* to a higher voltage V_s. Similarly, if $N_s < N_p$, it is a *step-down transformer*.

With switch S open, no energy is transferred from the generator to the rest of the circuit, but when we close S to connect the secondary to the resistive load R, energy *is* transferred. (In general, the load would also contain inductive and capacitive elements, but here we consider just resistance R.) Here is the process:

1. An alternating current I_s appears in the secondary circuit, with corresponding energy dissipation rate $I_S^2R\ (= V_S^2/R)$ in the resistive load.
2. This current produces its own alternating magnetic flux in the iron core, and this flux induces an opposing emf in the primary windings.
3. The voltage V_p of the primary, however, cannot change in response to this opposing emf because it must always be equal to the emf $\mathscr{E}$ that is provided by the generator; closing switch S cannot change this fact.
4. To maintain V_p, the generator now produces (in addition to I_{mag}) an alternating current I_p in the primary circuit; the magnitude and phase constant of I_p are just those required for the emf induced by I_p in the primary to exactly cancel the emf induced there by I_s. Because the phase constant of I_p is not 90° like that of I_{mag}, this current I_p can transfer energy to the primary.

Energy Transfers. We want to relate I_s to I_p. However, rather than analyze the foregoing complex process in detail, let us just apply the principle of conservation of energy. The rate at which the generator transfers energy to the primary is equal to I_pV_p. The rate at which the primary then transfers energy to the secondary (via the alternating magnetic field linking the two coils) is I_sV_s. Because we assume that no energy is lost along the way, conservation of energy requires that

$$I_pV_p = I_sV_s.$$

Substituting for V_s from Eq. 31-73, we find that

$$I_s = I_p \frac{N_p}{N_s} \quad \text{(transformation of currents).} \tag{31-74}$$

This equation tells us that the current I_s in the secondary can differ from the current I_p in the primary, depending on the *turns ratio* N_p/N_s.

Current I_p appears in the primary circuit because of the resistive load R in the secondary circuit. To find I_p, we substitute $I_s = V_s/R$ into Eq. 31-74 and then we substitute for V_s from Eq. 31-73. We find

$$I_p = \frac{1}{R}\left(\frac{N_S}{N_p}\right)^2 V_p. \tag{31-75}$$

This equation has the form $I_p = V_p/R_{\text{eq}}$, where equivalent resistance R_{eq} is

$$R_{\text{eq}} = \left(\frac{N_p}{N_S}\right)^2 R, \tag{31-76}$$

This R_{eq} is the value of the load resistance as "seen" by the generator; the generator produces the current I_p and voltage V_p as if the generator were connected to a resistance R_{eq}.

Impedance Matching

Equation 31-76 suggests still another function for the transformer. For maximum transfer of energy from an emf device to a resistive load, the resistance of the emf device must equal the resistance of the load. The same relation holds for ac circuits except that the *impedance* (rather than just the resistance) of the generator must equal that of the load. Often this condition is not met. For example, in a music-playing system, the amplifier has high impedance and the speaker set has low impedance. We can match the impedances of the two devices by coupling them through a transformer that has a suitable turns ratio N_p/N_s.

Solar Activity and Power-Grid Systems

In a *solar flare,* a huge loop of electrons and protons extends outward from the surface of the Sun, as shown in the photograph that opens this chapter. Some solar flares explode, shooting those charged particles into space. On March 10, 1989, a gigantic solar flare exploded toward Earth. When the particles arrived three days later, they produced a 106 A current, called an *electrojet,* in the high-altitude atmosphere above the Northern Hemisphere.

Because it is a current, an electrojet sets up a magnetic field $\vec{B}$ around itself, including along Earth's surface. Using the right-hand rule of Fig. 29-5, we see that the electrojet in Fig. 31-25 sets up a magnetic field component B_x along Earth's surface, directed perpendicular to the long power transmission line shown there. Grounded step-up or step-down transformers are attached at each end of the transmission line. Note that the transmission line, the ground, and the wires grounding the transformers form a conducting loop. A magnetic flux F due to B_x penetrates that loop.

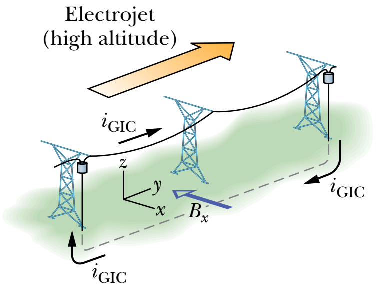


Figure 31-25 An electrojet (current) in the ionosphere produces a magnetic field B_x through a vertical loop formed by a transmission line, the ground, and the wires grounding the transformers (located inside the cylinders at the ends of the transmission lines). Variations in B_x induce current i_{GIC} around the loop.

An electrojet varies in both size and location, and the resulting variations in F induce emf and current in the loop. The current i_{GIC}, called the geomagnetically induced current (GIC), is directed along the transmission line and (more important) through the transformers.

Transmission of power by a power-grid system depends on the proper sinusoidal variations in current and voltage throughout the system. The presence of i_{GIC} through a transformer ruins the ability of the transformer's core to transfer the sinusoidal variations in the primary to the secondary. The reason is that the added flux in the core due to the i_{GIC} *saturates* the core, making it unable to respond properly to sinusoidal variations in the primary. The result is that the current and voltage in the secondary are highly distorted and no longer sinusoidal, and this distortion disrupts the power transmission.

Today, whenever a solar flare explodes toward Earth, astronomers immediately warn power-grid engineers so that the engineers can brace for grid disruptions.

CHECKPOINT 9

An alternating-current emf device in a certain circuit has a smaller resistance than that of the resistive load in the circuit; to increase the transfer of energy from the device to the load, a transformer will be connected between the two. (a) Should N_s be greater than or less than N_p? (b) Will that make it a step-up or step-down transformer?

SAMPLE PROBLEM 31.10

Transformer: turns ratio, average power, rms currents

A transformer on a utility pole operates at $V_p = 8.5$ kV on the primary side and supplies electrical energy to a number of nearby houses at $V_s = 120$ V, both quantities being rms values. Assume an ideal step-down transformer, a purely resistive load, and a power factor of unity.

(a) What is the turns ratio N_p/N_s of the transformer?

KEY IDEA

The turns ratio N_p/N_s is related to the (given) rms primary and secondary voltages via Eq. 31-73 ($V_s = V_pN_s/N_p$).

Calculation: We can write Eq. 31-73 as

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}. \tag{31-77}$$

(Note that the right side of this equation is the *inverse* of the turns ratio.) Inverting both sides of Eq. 31-77 gives us

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5 \times 10^3\ \text{V}}{120\ \text{V}} = 70.83 \approx 71. \qquad \text{(Answer)}$$

(b) The average rate of energy consumption (or dissipation) in the houses served by the transformer is 78 kW. What are the rms currents in the primary and secondary of the transformer?

KEY IDEA

For a purely resistive load, the power factor cos ϕ is unity; thus, the average rate at which energy is supplied and dissipated is given by Eq. 31-71 ($P_{\text{avg}} = \mathscr{E}I = IV$).

Calculations: In the primary circuit, with $V_p = 8.5$ kV, Eq. 31-71 yields

$$I_p = \frac{P_{\text{avg}}}{V_p} = \frac{78 \times 10^3\ \text{W}}{8.5 \times 10^3\ \text{V}} = 9.176\ \text{A} \approx 9.2\ \text{A}. \qquad \text{(Answer)}$$

Similarly, in the secondary circuit,

$$I_s = \frac{P_{\text{avg}}}{V_s} = \frac{78 \times 10^3\ \text{W}}{120\ \text{V}} = 650\ \text{A}. \qquad \text{(Answer)}$$

You can check that $I_s = I_p(N_p/N_s)$ as required by Eq. 31-74.

(c) What is the resistive load R_s in the secondary circuit? What is the corresponding resistive load R_p in the primary circuit?

One way: We can use $V = IR$ to relate the resistive load to the rms voltage and current. For the secondary circuit, we find

$$R_s = \frac{V_s}{I_s} = \frac{120\ \text{V}}{650\ \text{A}} = 0.1846\ \Omega \approx 0.18\ \Omega. \qquad \text{(Answer)}$$

Similarly, for the primary circuit we find

$$R_p = \frac{V_p}{I_p} = \frac{8.5 \times 10^3\ \text{V}}{9.176\ \text{A}} = 926\ \Omega \approx 930\ \Omega. \qquad \text{(Answer)}$$

Second way: We use the fact that R_p equals the equivalent resistive load "seen" from the primary side of the transformer, which is a resistance modified by the turns ratio and given by Eq. 31-76 ($R_{\text{eq}} = (N_p/N_s)^2R$). If we substitute R_p for R_{eq} and R_s for R, that equation yields

$$R_p = \left(\frac{N_p}{N_s}\right)^2 R_s = (70.83)^2(0.1846\ \Omega)$$
$$= 926\ \Omega \approx 930\ \Omega. \qquad \text{(Answer)}$$

REVIEW AND SUMMARY

***LC* Energy Transfers** In an oscillating *LC* circuit, energy is shuttled periodically between the electric field of the capacitor and the magnetic field of the inductor; instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2}, \qquad (31\text{-}1, 31\text{-}2)$$

where q is the instantaneous charge on the capacitor and i is the instantaneous current through the inductor. The total energy $U(= U_E + U_B)$ remains constant.

***LC* Charge and Current Oscillations** The principle of conservation of energy leads to

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad (LC\ \text{oscillations}) \qquad (31\text{-}11)$$

as the differential equation of *LC* oscillations (with no resistance). The solution of Eq. 31-11 is

$$q = Q\cos(\omega t + \phi)\ \text{(charge)}, \qquad (31\text{-}12)$$

in which Q is the *charge amplitude* (maximum charge on the capacitor) and the angular frequency ω of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}. \qquad (31\text{-}4)$$

The phase constant ϕ in Eq. 31-12 is determined by the initial conditions (at $t = 0$) of the system.

The current i in the system at any time t is

$$i = -\omega Q\sin(\omega t - \phi) \quad \text{(current)}, \qquad (31\text{-}13)$$

in which ωQ is the *current amplitude I*.

Alternating Currents; Forced Oscillations A series *RLC* circuit may be set into *forced oscillation* at a *driving angular frequency* ω_d by an external alternating emf

$$\mathscr{E} = \mathscr{E}_m \sin\omega_d t. \qquad (31\text{-}22)$$

The current driven in the circuit is

$$i = I\sin(\omega_d t - \phi), \qquad (31\text{-}23)$$

where ϕ is the phase constant of the current.

Resonance The current amplitude I in a series RLC circuit driven by a sinusoidal external emf is a maximum ($I = \mathscr{E}_m/R$) when the driving angular frequency ω_d equals the natural angular frequency ω of the circuit (that is, at *resonance*). Then $X_C = X_L$, $\phi = 0$, and the current is in phase with the emf.

Single Circuit Elements The alternating potential difference across a resistor has amplitude $V_R = IR$; the current is in phase with the potential difference.

For a *capacitor*, $V_C = IX_C$, in which $X_C = 1/\omega_d C$ is the **capacitive reactance**; the current here leads the potential difference by 90° ($\phi = -90° = -\pi/2$ rad).

For an *inductor*, $V_L = IX_L$, in which $X_L = \omega_d L$ is the **inductive reactance**; the current here lags the potential difference by 90° ($\phi = +90° = +\pi/2$ rad).

Series *RLC* Circuits For a series RLC circuit with an alternating external emf given by Eq. 31-22 and a resulting alternating current given by Eq. 31-23,

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude)} \qquad \text{(31-54, 31-57)}$$

and

$$\tan\phi = \frac{X_L - X_C}{R} \quad \text{(phase constant).} \qquad \text{(31-59)}$$

Defining the impedance Z of the circuit as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(impedance)} \qquad \text{(31-55)}$$

allows us to write Eq. 31-54 as $I = \mathscr{E}_m/Z$.

Power In a series RLC circuit, the **average power** P_{avg} of the generator is equal to the production rate of thermal energy in the resistor:

$$P_{\text{avg}} = I^2_{\text{rms}}R = \mathscr{E}_{\text{rms}} I_{\text{rms}} \cos\phi. \qquad \text{(31-65, 31-70)}$$

Here rms stands for **root-mean-square**; the rms quantities are related to the maximum quantities by $I_{\text{rms}} = I/\sqrt{2}$, $V_{\text{rms}} = V/\sqrt{2}$, and $\mathscr{E}_{\text{rms}} = \mathscr{E}_m/\sqrt{2}$. The term $\cos\phi$ is called the **power factor** of the circuit.

Transformers A *transformer* (assumed to be ideal) is an iron core on which are wound a primary coil of N_p turns and a secondary coil of N_s turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_S = V_p \frac{N_S}{N_p} \quad \text{(transformation of voltage).} \qquad \text{(31-73)}$$

The currents through the coils are related by

$$I_S = I_p \frac{N_p}{N_S} \quad \text{(transformation of currents),} \qquad \text{(31-74)}$$

and the equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{\text{eq}} = \left(\frac{N_p}{N_S}\right)^2 R, \qquad \text{(31-76)}$$

where R is the resistive load in the secondary circuit. The ratio N_p/N_s is called the transformer's *turns ratio*.

PROBLEMS

1. In an oscillating LC circuit with $L = 79$ mH and $C = 4.0\ \mu$F, the current is initially a maximum. How long will it take before the capacitor is fully charged for (a) the first time and (b) the second time?

2. An ac generator with emf $\mathscr{E} = \mathscr{E}_m \sin \omega_d t$, where $\mathscr{E}_m = 25.0$ V and $\omega_d = 377$ rad/s, is connected to a 4.15 μF capacitor. (a) What is the maximum value of the current? (b) When the current is a maximum, what is the emf of the generator? (c) When the emf of the generator is −12.5 V and increasing in magnitude, what is the current?

3. In Fig. 31-26, a generator with an adjustable frequency of oscillation is connected to resistance $R = 100\ \Omega$, inductances $L_1 = 9.70$ mH and $L_2 = 2.30$ mH, and capacitances $C_1 = 8.40\ \mu$F, $C_2 = 2.50\ \mu$F, and $C_3 = 3.50\ \mu$F. (a) What is the resonant frequency of the circuit? What happens to the resonant frequency if (b) R is increased, (c) L_1 is increased, (d) C_3 is removed from the circuit, and (e) L_2 is removed?

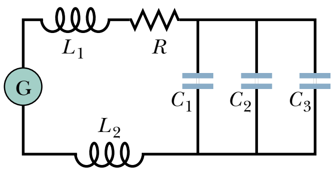


Figure 31-26 Problem 3.

4. In Fig. 31-6, set $R = 400\ \Omega$, $C = 70.0\ \mu$F, $L = 920$ mH, $f_d = 30.0$ Hz, and $\mathscr{E}_m = 72.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

5. (a) In an RLC circuit, can the amplitude of the voltage across an inductor be greater than the amplitude of the generator emf? (b) Consider an RLC circuit with emf amplitude $\mathscr{E}_m = 10$ V, resistance $R = 5.0\ \Omega$, inductance $L = 1.0$ H, and capacitance $C = 1.0\ \mu$F. Find the amplitude of the voltage across the inductor at resonance.

6. An air conditioner connected to a 125 V rms ac line is equivalent to a 9.20 Ω resistance and a 4.70 Ω inductive reactance in series. Calculate (a) the impedance of the air conditioner and (b) the average rate at which energy is supplied to the appliance.

7. Remove the capacitor from the circuit in Fig. 31-6 and set $R = 400\ \Omega$, $L = 230$ mH, $f_d = 120$ Hz, and $\mathscr{E}_m = 72.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

8. An alternating source drives a series *RLC* circuit with an emf amplitude of 6.00 V, at a phase angle of +30.0°. When the potential difference across the capacitor reaches its maximum positive value of +5.00 V, what is the potential difference across the inductor (sign included)?

9. A coil of inductance 62 mH and unknown resistance and a 0.94 μF capacitor are connected in series with an alternating emf of frequency 930 Hz. If the phase constant between the applied voltage and the current is 82°, what is the resistance of the coil?

10. A generator supplies 100 V to a transformer's primary coil, which has 100 turns. If the secondary coil has 500 turns, what is the secondary voltage?

11. What direct current will produce the same amount of thermal energy, in a particular resistor, as an alternating current that has a maximum value of 7.82 A?

12. A single loop consists of inductors (L_1, L_2, …), capacitors (C_1, C_2, …), and resistors (R_1, R_2, …) connected in series as shown, for example, in Fig. 31-27*a*. Show that regardless of the sequence of these circuit elements in the loop, the behavior of this circuit is identical to that of the simple *LC* circuit shown in Fig. 31-27*b*.

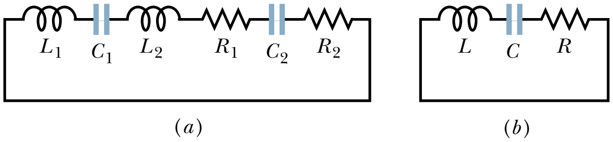


Figure 31-27 Problem 12.

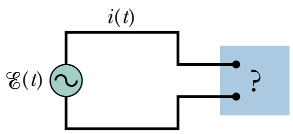


Figure 31-28 Problem 13.

13. Figure 31-28 shows an ac generator connected to a "black box" through a pair of terminals. The box contains an *RLC* circuit, possibly even a multiloop circuit, whose elements and connections we do not know. Measurements outside the box reveal that

$$\mathscr{E}(t) = (61.4\ \text{V})\sin \omega_d t$$

and $$i(t) = (0.930\ \text{A})\sin(\omega_d t = 42.0°).$$

(a) What is the power factor? (b) Does the current lead or lag the emf? (c) Is the circuit in the box largely inductive or largely capacitive? (d) Is the circuit in the box in resonance? (e) Must there be a capacitor in the box? (f) An inductor? (g) A resistor? (h) At what average rate is energy delivered to the box by the generator? (i) Why do not you need to know ω_d to answer all these questions?

14. What is the capacitance of an oscillating *LC* circuit if the maximum charge on the capacitor is 1.60 μC and the total energy is 140 μJ?

15. Remove the inductor from the circuit in Fig. 31-6 and set $R = 400\ \Omega$, $C = 15.0\ \mu$F, $f_d = 30.0$ Hz, and $\mathscr{E}_m = 72.0$ V. What are (a) Z, (b) ϕ, and (c) I? (d) Draw a phasor diagram.

16. Figure 31-29 shows a driven *RLC* circuit that contains two identical capacitors and two switches. The emf amplitude is set at 12.0 V, and the driving frequency is set at 60.0 Hz. With both switches open, the current leads the emf by 30°. With switch S_1 closed and switch S_2 still open, the emf leads the current by 30°. With both switches closed, the current amplitude is 447 mA. What are (a) R, (b) C, and (c) L?

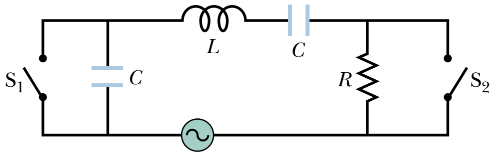


Figure 31-29 Problem 16.

17. In an oscillating *LC* circuit, $L = 5.97$ mH and $C = 4.00\ \mu$F. The maximum charge on the capacitor is 3.00 μC. Find (a) the maximum current and (b) the oscillation period.

18. In an oscillating *LC* circuit in which $C = 4.00\ \mu$F, the maximum potential difference across the capacitor during the oscillations is 1.50 V and the maximum current through the inductor is 50.0 mA. What are (a) the inductance L and (b) the frequency of the oscillations? (c) How much time is required for the charge on the capacitor to rise from zero to its maximum value?

19. A 85.0 mH inductor is connected as in Fig. 31-11 to an ac generator with $\mathscr{E}_m = 30.0$ V. What is the amplitude of the resulting alternating current if the frequency of the emf is (a) 1.00 kHz and (b) 5.00 kHz?

20. An inductor is connected across a capacitor whose capacitance can be varied by turning a knob. We wish to make the frequency of oscillation of this *LC* circuit vary linearly with the angle of rotation of the knob, going from 2×10^5 to 4×10^5 Hz as the knob turns through 180°. If $L = 1.0$ mH, plot the required capacitance C as a function of the angle of rotation of the knob.

21. A transformer has 400 primary turns and 10 secondary turns. (a) If V_p is 120 V (rms), what is V_s with an open circuit? If the secondary now has a resistive load of 27 Ω, what is the current in the (b) primary and (c) secondary?

22. A typical light dimmer used to dim the stage lights in a theater consists of a variable inductor L (whose inductance is adjustable between zero and L_{max}) connected in series with a lightbulb B, as shown in Fig. 31-30. The electrical supply is 120 V (rms) at 60.0 Hz; the lightbulb is rated at 120 V, 1000 W. (a) What L_{max} is required if the rate of energy dissipation in the lightbulb is to be varied by a factor of 5 from its upper limit of 1000 W? Assume that the resistance of the lightbulb is independent of its temperature. (b) Could one use a variable resistor (adjustable between zero and R_{max}) instead of an inductor? (c) If so, what R_{max} is required? (d) Why isn't this done?

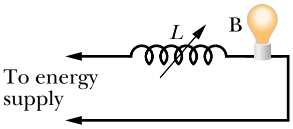


Figure 31-30 Problem 22.

23. In a certain oscillating LC circuit, the total energy is converted from electrical energy in the capacitor to magnetic energy in the inductor in 2.50 μs. What are (a) the period of oscillation and (b) the frequency of oscillation? (c) How long after the magnetic energy is a maximum will it be a maximum again? (d) In one full cycle, how many times will the electrical energy be maximum?

24. An ac voltmeter with large impedance is connected in turn across the inductor, the capacitor, and the resistor in a series circuit having an alternating emf of 125 V (rms); the meter gives the same reading in volts in each case. What is this reading?

25. In Fig. 31-6, $R = 25.0\ \Omega$, $C = 4.70\ \mu$F, and $L = 25.0$ mH. The generator provides an emf with rms voltage 75.0 V and frequency 550 Hz. (a) What is the rms current? What is the rms voltage across (b) R, (c) C, (d) L, (e) C and L together, and (f) R, C, and L together? At what average rate is energy dissipated by (g) R, (h) C, and (i) L?

26. A series circuit containing inductance L_1 and capacitance C_1 oscillates at angular frequency ω. A second series circuit, containing inductance L_2 and capacitance C_2, oscillates at the same angular frequency. In terms of ω, what is the angular frequency of oscillation of a series circuit containing all four of these elements? Neglect resistance.

27. What is the maximum value of an ac voltage whose rms value is 220 V?

28. In an oscillating LC circuit, $L = 25.0$ mH and $C = 2.89\ \mu$F. At time $t = 0$ the current is 9.20 mA, the charge on the capacitor is 3.80 μC, and the capacitor is charging. What are (a) the total energy in the circuit, (b) the maximum charge on the capacitor, and (c) the maximum current? (d) If the charge on the capacitor is given by $q = Q\cos(\omega t + \phi)$, what is the phase angle ϕ? (e) Suppose the data are the same, except that the capacitor is discharging at $t = 0$. What then is ϕ?

29. An electric motor has an effective resistance of 61.0 Ω and an inductive reactance of 52.0 Ω when working under load. The rms voltage across the alternating source is 420 V. Calculate the rms current.

30. A 0.50 kg body oscillates in SHM on a spring that, when extended 2.0 mm from its equilibrium position, has an 8.0 N restoring force. What are (a) the angular frequency of oscillation, (b) the period of oscillation, and (c) the capacitance of an LC circuit with the same period if L is 5.0 H?

31. LC oscillators have been used in circuits connected to loudspeakers to create some of the sounds of electronic music. What inductance must be used with a 6.7 μF capacitor to produce a frequency of 10 kHz, which is near the middle of the audible range of frequencies?

32. In Fig. 31-31, $R = 14.0\ \Omega$, $C = 31.2\ \mu$F, and $L = 54.0$ mH, and the ideal battery has emf $\mathscr{E} = 34.0$ V. The switch is kept at a for a long time and then thrown to position b. What are the (a) frequency and (b) current amplitude of the resulting oscillations?

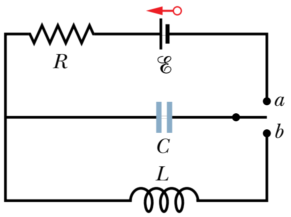


Figure 31-31 Problem 32.

33. Figure 31-32 shows an "autotransformer." It consists of a single coil (with an iron core). Three taps T_i are provided. Between taps T_1 and T_2 there are 200 turns, and between taps T_2 and T_3 there are 800 turns. Any two taps can be chosen as the primary terminals, and any two taps can be chosen as the secondary terminals. For choices producing a step-up transformer, what are the (a) smallest, (b) second smallest, and (c) largest values of the ratio V_s/V_p? For a step-down transformer, what are the (d) smallest, (e) second smallest, and (f) largest values of V_s/V_p?

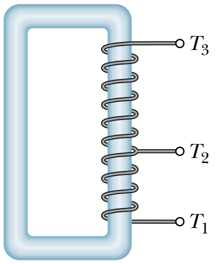


Figure 31-32 Problem 33.

34. In an oscillating LC circuit, $L = 3.00$ mH and $C = 3.90\ \mu$F. At $t = 0$ the charge on the capacitor is zero and the current is 1.75 A. (a) What is the maximum charge that will appear on the capacitor? (b) At what earliest time $t > 0$ is the rate at which energy is stored in the capacitor greatest, and (c) what is that greatest rate?

35. An oscillating LC circuit consists of a 75.0 mH inductor and a 3.60 μF capacitor. If the maximum charge on the capacitor is 5.00 μC, what are (a) the total energy in the circuit, (b) the maximum current, and (c) the period of the oscillations?

36. In an oscillating LC circuit with $C = 64.0\ \mu$F, the current is given by $i = (1.60)\sin(4100t + 0.680)$, where t is in seconds, i in amperes, and the phase constant in radians. (a) How soon after $t = 0$ will the current reach its maximum value? What are (b) the inductance L and (c) the total energy?

37. An oscillating LC circuit has a current amplitude of 7.50 mA, a potential amplitude of 250 mV, and a capacitance of 220 μF. What are (a) the period of oscillation, (b) the maximum energy stored in the capacitor, (c) the maximum energy stored in the inductor, (d) the maximum rate at which the current changes, and (e) the maximum rate at which the inductor gains energy?

38. An oscillating LC circuit consisting of a 1.0 μF capacitor and a 9.0 mH coil has a maximum voltage of 3.0 V. What are (a) the maximum charge on the capacitor, (b) the maximum current through the circuit, and (c) the maximum energy stored in the magnetic field of the coil?

39. In an oscillating LC circuit, when 75.0% of the total energy is stored in the inductor's magnetic field, (a) what multiple of the maximum charge is on the capacitor and (b) what multiple of the maximum current is in the inductor?

40. Using the loop rule, derive the differential equation for an LC circuit (Eq. 31-11).

41. The frequency of oscillation of a certain LC circuit is 200 kHz. At time $t = 0$, plate A of the capacitor has maximum positive charge. At what earliest time $t > 0$ will (a) plate A again have maximum positive charge, (b) the other plate of the capacitor have maximum positive charge, and (c) the inductor have maximum magnetic field?

42. (a) At what frequency would a 12 mH inductor and a 10 μF capacitor have the same reactance? (b) What would the reactance be? (c) Show that this frequency would be the natural frequency of an oscillating circuit with the same L and C.

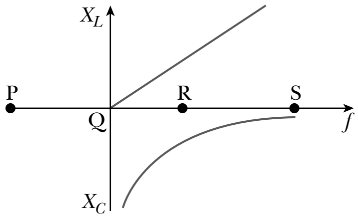
PRACTICE QUESTIONS

Single Correct Choice Type

1. The voltage of an ac source varies with time according to the equation $V = 100 \sin 100\pi t \cos 100\pi t$ where t is in seconds and V is in volts. Then,

(a) The peak voltage of the source is 100 V
(b) The peak voltage of the source is 50 V
(c) The peak voltage of the source is $100/\sqrt{2}$
(d) The frequency of the source is 50 Hz

2. The resonance point in X_L–f and X_C–f curves is as shown in the given figure.



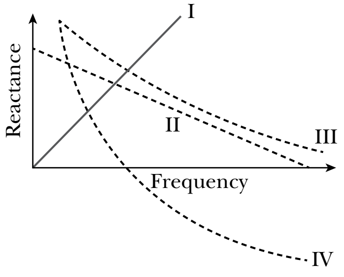
(a) P **(b)** Q
(c) R **(d)** S

3. An ac source of angular frequency ω is fed across a resistor r and a capacitor C in series. The current registered is I. If the frequency of source is changed to $\omega/3$ (maintaining the same voltage), the current in the circuit is found to be halved. Calculate the ratio of reactance to resistance at the original frequency ω.

(a) $\sqrt{\dfrac{3}{5}}$ **(b)** $\sqrt{\dfrac{2}{5}}$

(c) $\sqrt{\dfrac{1}{5}}$ **(d)** $\sqrt{\dfrac{4}{5}}$

4. Which of the plots shown in the following figure may represent the reactance of series LC combination?



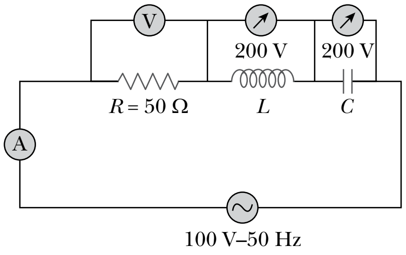
(a) I **(b)** II
(c) III **(d)** IV

5. A coil having n turns and resistance R Ω is connected with a galvanometer of resistance $4R$ Ω. This combination is moved in time t seconds from a magnetic field W_1 Weber to W_2 Weber. The induced current in the circuit is

(a) $-\dfrac{W_2 - W_1}{5Rnt}$ **(b)** $-\dfrac{n(W_2 - W_1)}{5Rnt}$

(c) $-\dfrac{(W_2 - W_1)}{Rnt}$ **(d)** $-\dfrac{n(W_2 - W_1)}{Rnt}$

6. In the series LCR circuit shown in the following figure, the readings of voltmeter and ammeter are



(a) $V = 100$ V, $I = 2$ A **(b)** $V = 100$ V, $I = 5$ A
(c) $V = 1000$ V, $I = 2$ A **(d)** $V = 300$ V, $I = 1$ A

7. If the total charge stored in an LC circuit is Q_0, then for $t \geq 0$,

(a) the charge on the capacitor is $Q = Q_0 \cos\left(\dfrac{\pi}{2} + \dfrac{t}{\sqrt{LC}}\right)$

(b) the charge on the capacitor is $Q = Q_0 \cos\left(\dfrac{\pi}{2} - \dfrac{t}{\sqrt{LC}}\right)$

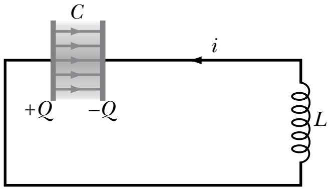
(c) the charge on the capacitor is $Q = -LC\dfrac{d^2Q}{dt^2}$

(d) the charge on the capacitor is $Q = \dfrac{1}{\sqrt{LC}}\dfrac{d^2Q}{dt^2}$

8. A charged capacitor and an inductor are connected in series. At time $t = 0$, the current is zero, but the capacitor is charged. If T is the period of the resulting oscillations, the next time after $t = 0$, the maximum voltage across the inductor is

(a) T **(b)** $T/4$
(c) $T/2$ **(d)** T

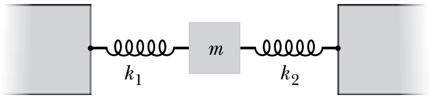
9. In the given LC circuit the direction of current and the sign of the charges on the capacitor plates are as shown in the figure. At this time



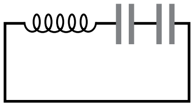
(a) i is increasing and Q is increasing
(b) i is increasing and Q is decreasing
(c) i is decreasing and Q is increasing
(d) i is decreasing and Q is decreasing

10. The electrical analog of a spring constant k is
(a) L
(b) $1/L$
(c) C
(d) $1/C$

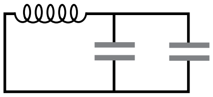
11. Consider the mechanical system consisting of two springs and a block, as shown in the following figure. Which one of the five electrical circuits is the analog of the mechanical system?



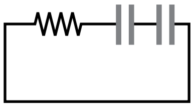
(a)



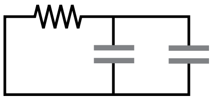
(b)



(c)



(d)



12. An LC series circuit with an inductance L and a capacitance C has an oscillation frequency f. Two isolated inductors, each with inductance L, and two capacitors, each with capacitance C, are all wired in series and the circuit is completed. The oscillation frequency is
(a) $f/4$
(b) $f/2$
(c) f
(d) $2f$

13. In an oscillating LC circuit, the total stored energy is U and the maximum charge on the capacitor is Q. When the charge on the capacitor is $Q/2$, the energy stored in the inductor is
(a) $U/2$
(b) $U/4$
(c) $(4/3)U$
(d) $3U/4$

14. The resonance frequency of a certain RLC series circuit is ω_0. A source of sinusoidal emf, with angular frequency $2\omega_0$, is inserted into the circuit. After transients die out the angular frequency of the current oscillations is
(a) $\omega_0/2$
(b) ω_0
(c) $2\omega_0$
(d) $1.5\omega_0$

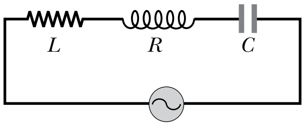
15. A variable inductor is connected to an ac source. What effect does increasing the inductance have on the reactance and current in this circuit?

	Reactance	**Current**
(a)	No change	No change
(b)	Decreases	No change
(c)	Increases	Increases
(d)	Increases	Decreases

16. A variable capacitor is connected to an ac source. What effect does *decreasing* the capacitance have on the reactance and current in this circuit?

	Reactance	**Current**
(a)	No change	No change
(b)	Decreases	No change
(c)	Increases	Increases
(d)	Increases	Decreases

17. For the circuit shown in the following figure, the voltage of the source at any instant is equal to



(a) The sum of the maximum voltages across the elements
(b) The voltage drop across the resistor
(c) The sum of the instantaneous voltages across the elements
(d) The sum of rms voltages across the elements

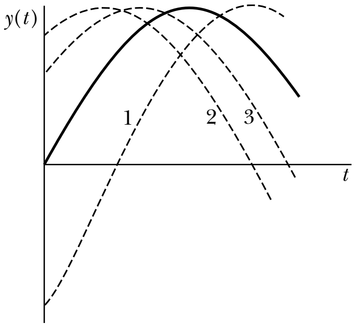
18. In an RLC series circuit, if the ac frequency is increased to a very large value, what value does the phase angle between the current and voltage approach?
(a) 90°
(b) 0°
(c) 30°
(d) 45°

19. A series LCR circuit powered by a frequency generator has equal resistance, capacitive reactance, and inductive reactance. What is the phase angle between the voltages across the *generator* and the *resistor*?
(a) 0°
(b) 45°
(c) 60°
(d) 90°

20. In an RL series circuit driven by an ac voltage source with fixed amplitude,
(a) The source current leads the source voltage in phase
(b) The source current lags the source voltage in phase
(c) The source current amplitude increases as source frequency increase
(d) The inductor voltage lags the inductor current in phase

21. An inductor, a capacitor, and a resistor are connected in series with a frequency generator to complete a circuit. The frequency generator can supply alternating current at frequencies ranging from about 0 Hz to 100 MHz. As the frequency is increased from the lowest to the highest value of the generator there is a particular value for which the voltage across the resistor is maximum. At that frequency
(a) The voltage across the inductor is zero at all times
(b) The voltage across the capacitor is zero at all times
(c) The voltage across the inductor and the capacitor are zero at all times
(d) The voltage across the inductor and the capacitor are opposite in phase at all times

22. In the given figure, the function $y(t) = y_m \sin(\omega t)$ is plotted as a solid curve. The other three curves have the form $y(t) = y_m \sin(\omega t + f)$, where f is between $-\pi/2$ and $+\pi/2$. Rank the curves according to the value of f from the most negative to the most positive.

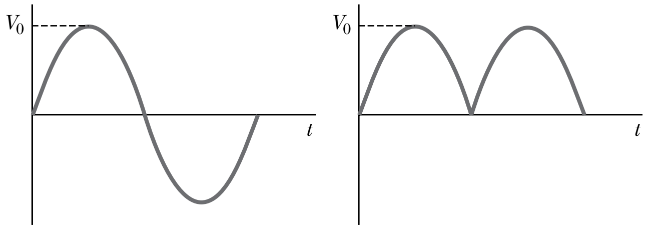


(a) 1, 2, 3
(b) 2, 3, 1
(c) 3, 2, 1
(d) 1, 3, 2

23. Consider an ac circuit where an inductor is in series with an incandescent light bulb. If the frequency of the generator is increased, what will happen to the brightness of the bulb?
(a) The bulb will have the same brightness
(b) The bulb will shine brighter
(c) The bulb will become dimmer
(d) Depends on the value of L and R

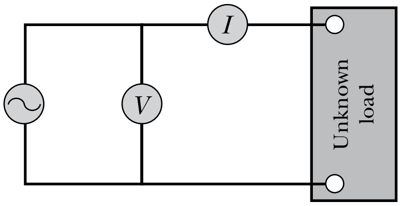
24. Which one of the following is true for an ideal capacitor connected to a sinusoidal voltage source?
(a) Neither the average power nor the average current is zero
(b) The average current is zero but the average power is not zero
(c) The average power is zero but the average current is not zero
(d) Both the average power and the average current are zero

25. The rms value of a sinusoidal voltage is $\frac{V_0}{\sqrt{2}}$, where V_0 is the amplitude (as shown in the following figure). What is the rms value of its fully rectified wave? Recall that $V_{\text{rect}}(t) = |V(t)|$.



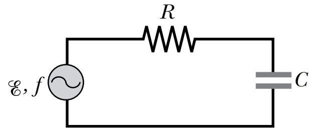
(a) $\frac{V_0}{\sqrt{2}}$
(b) $V_0\sqrt{2}$
(c) V_0
(d) $\frac{V_0}{2}$

26. The ideal meters shown in the given figure read rms current and voltage. The average power delivered to the load is



(a) Definitely equal to $V \times I$
(b) Definitely more than $V \times I$
(c) Possibly equal to $V \times I$ even if the load contains an inductor and a capacitor
(d) Definitely less than $V \times I$

27. The average power supplied to the circuit shown in the following figure passes through a maximum then which one of the following is increased continuously from a very low to a very high value?



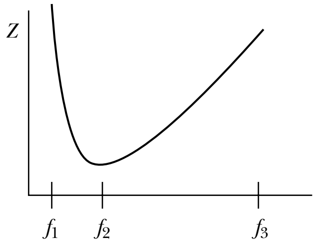
(a) Source emf ξ
(b) R
(c) C
(d) Source frequency f

28. In a sinusoidally driven series RLC circuit the current lags the applied emf. The rate at which energy is dissipated in the resistor can be increased by
(a) Decreasing the capacitance and making no other changes
(b) Increasing the capacitance and making no other changes
(c) Increasing the inductance and making no other changes
(d) Increasing the driving frequency and making no other changes

29. The core of a transformer is made in a laminated form to
(a) Facilitate easy assembly
(b) Reduce i^2R losses in the coils
(c) Increase the magnetic flux
(d) Prevent Eddy currents

30. The primary coil of an ideal transformer has 100 turns and the secondary coil has 600 turns. Then
(a) The power in the primary circuit is less than that in the secondary circuit
(b) The currents in the two circuits are the same
(c) The voltages in the two circuits are the same
(d) The primary current is six times the secondary current

31. The graph shows the impedance as a function of frequency for a series RCL circuit. At what frequency does the capacitor make the largest contribution?



(a) f_1
(b) f_1 or f_3
(c) f_2
(d) f_1 or f_2 or f_3

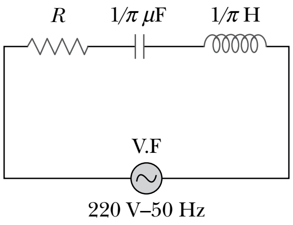
32. Complete the following statement: When the current in an oscillating LC circuit is zero,
(a) The charge on the capacitor is zero.
(b) The energy in the electric field is maximum.
(c) The energy in the magnetic field is a maximum.
(d) The charge is moving through the inductor.

33. A series RCL circuit operating at 60.0 Hz contains a 35 Ω resistor and an 8.2 μF capacitor. If the power factor of the circuit is +1.00, what is the inductance of the inductor in this circuit?
(a) 0.86 H **(b)** 2.3 H
(c) 1.1 H **(d)** 57 H

34. A series RCL circuit contains a 222 Ω resistor, a 1.40 μF capacitor, and a 0.125 H inductor. The 444 Hz ac generator in the circuit has an rms voltage of 208 V. What is the average electric power dissipated by the circuit?
(a) 135 W **(b)** 166 W
(c) 81 W **(d)** 191 W

More than One Correct Choice Type

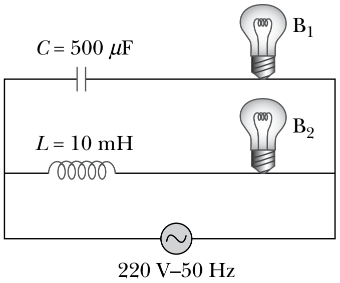
35. In an ac circuit shown in the following figure, the supply voltage has a constant rms value V but variable frequency. At resonance, the circuit



(a) Has a current I given by $I = \dfrac{V}{R}$
(b) Has a resonance frequency 50 Hz
(c) Has a voltage across the capacitor which is 180° out of phase with that across the inductor
(d) Has a current given by $I = \dfrac{V}{\sqrt{R^2 + \left(\dfrac{1}{\pi} + \dfrac{1}{\pi}\right)^2}}$

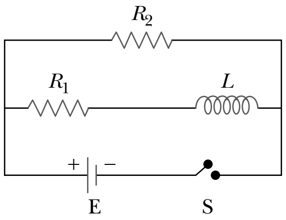
36. The symbols L, C, and R represent inductance, capacitance, and resistance, respectively. The dimensions of frequency are given by the combination
(a) I/RC
(b) R/L
(c) $1/\sqrt{LC}$
(d) C/L

37. In the circuit shown in the following figure, both the bulbs B_1 and B_2 are identical. Then,



(a) Their brightness will be the same.
(b) B_2 will be brighter than B_1.
(c) As the frequency of supply voltage is increased, the brightness of B_1 will increase and that of B_2 will decrease.
(d) Only B_2 will glow because the capacitor has infinite impedance.

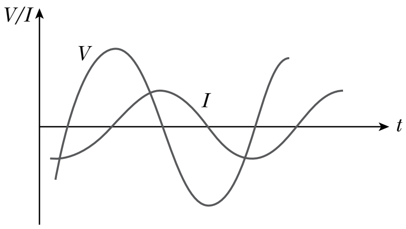
38. In the circuit shown in the following figure, resistance R = 100 Ω, inductance $L = (2/\pi)$ H and capacitance $C = (8/\pi)$ μF are connected in series with an ac source of 200 V and frequency f. If the readings of the hot wire voltmeters V_1 and V_2 are same, then



(a) $f = 250\pi$ Hz
(b) $f = 125$ Hz
(c) Current through R is 2 A
(d) $V_1 = V_2 = 1000$ V

39. An ac source producing $V = V_0 \sin \omega t + V_0 \sin 2\omega t$ is connected in series with a box containing either capacitor or inductor and resistance. The current found in the circuit is $I = I_1 \sin(\omega t + \phi_1) + I_2 \sin(2\omega t + \phi_2)$. Here, ϕ_1 and ϕ_2 may be positive or negative.
(a) If $I_1 > I_2$, box has inductor and resistor
(b) If $I_1 > I_2$, box has capacitor and resistor
(c) If $I_2 > I_1$, box has inductor and resistor
(d) If $I_2 > I_1$, box has capacitor and resistor

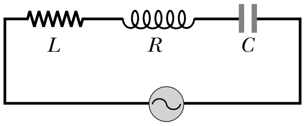
40. Graph shown in the following figure shows variation of source emf V and current I in a series RLC circuit with time. Then,



(a) To increase the rate at which energy is transferred to the resistive load, L should be decreased

(b) To increase the rate at which energy is transferred to the resistive load, C should be decreased
(c) The circuit is more inductive than capacitive
(d) The current leads the emf in the circuit

41. Resonance in the circuit shown in the following figure, will occur



(a) When the angular frequency of the source is equal to $1/\sqrt{LC}$
(b) When the capacitive and inductive reactances of the circuit are equal
(c) Independently of the value of R
(d) Independently of the maximum value of the voltage output of the source

42. A power outlet puts out 60 Hz ac. Which of the following statements is true?
(a) Voltage goes to zero only 120 times per second
(b) Current goes to zero only 120 times per second
(c) Power output goes to zero only 60 times per second
(d) Voltage, current, and power output stay constant

Linked Comprehension

Paragraph for Questions 43–46: The following table gives the reactance and rms voltage across the elements of a series RCL circuit:

Circuit element	Reactance	Voltage across element
Resistor	$2.00 \times 10^2\ \Omega$	86 V
Capacitor	$6.63 \times 10^2\ \Omega$	285 V
Inductor	$3.77 \times 10^2\ \Omega$	162 V

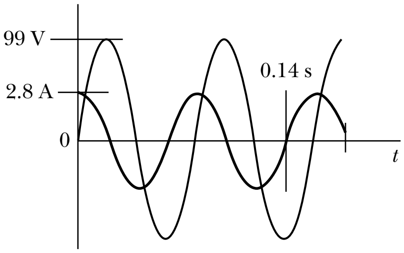
43. What is the rms current in the circuit?
(a) 0.25 A **(b)** 0.50 A
(c) 0.43 A **(d)** 0.86 A

44. What is the impedance of the circuit?
(a) 11 Ω **(c)** 349 Ω
(b) 22 Ω **(d)** 486 Ω

45. Determine the peak (not rms) voltage of the ac generator.
(a) 150 V **(b)** 300 V
(c) 212 V **(d)** 414 V

46. What is the power factor for this circuit?
(a) 0.40 **(b)** 0.81
(c) 0.57 **(d)** 1.4

Paragraph for Questions 47–49: The graph shows the voltage across and the current through a single circuit element connected to an ac generator.



47. Determine the frequency of the generator.
(a) 0.14 Hz **(b)** 25.0 Hz
(c) 7.14 Hz **(d)** 12.5 Hz

48. Determine the rms voltage across this element.
(a) 49.5 V **(b)** 112 V
(c) 70.0 V **(d)** 140 V

49. Determine the rms current through this element.
(a) 1.4 A **(b)** 3.4 A
(c) 2.0 A **(d)** 3.9 A

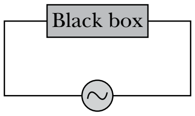
Paragraph for Questions 50–52: A 750 Hz–20 V source is connected to a resistance of 100 Ω, an inductance of 0.1803 H and a capacitor of 10 μF all in series. Resistance will get heated up by 10°C in time t. Thermal capacity is 2 J/°C.

50. Impedance of circuit is
(a) 834 Ω **(b)** 438 Ω
(c) 348 Ω **(d)** 893 Ω

51. Power loss in LCR circuit is
(a) 0.575 W **(b)** 0.999 W
(c) 0.0575 W **(d)** 0.0755 W

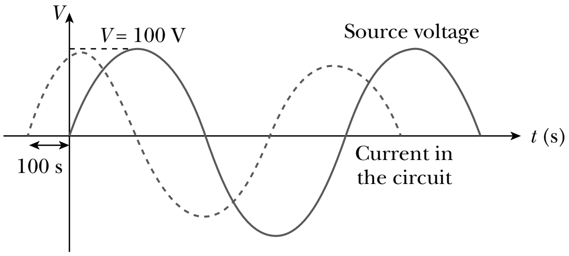
52. Time t in which resistance heats up is
(a) 348 s **(b)** 438 s
(c) 843 s **(d)** 834 s

Paragraph for Questions 53–55: A voltage source $V = V_0 \sin(100t)$ is connected to a black box as shown in the following figure, in which there can be either one element out of L, C, and R or any two of them connected in series.



$V = V_0 \sin(100t)$

At steady state, the variation of current in the circuit and the source voltage are plotted together with time, using an oscilloscope, as shown in the following figure.



53. The element(s) present in black box is/are
(a) only C
(b) LC
(c) L and R
(d) R and C

54. Values of the parameters of the elements present in the black box are
(a) $R = 50\ \Omega$, $C = 200\ \mu$F
(b) $R = 50\ \Omega$, $L = 2$ HF
(c) $R = 400\ \Omega$, $C = 50\ \mu$F
(d) None of these

55. If ac source is removed, the circuit is shorted and then at $t = 0$, a battery of constant emf is connected across the black box. The current in the circuit will
    (a) Increase exponentially with constant = 0.02 s
    (b) Decrease exponentially with time constant = 0.01 s
    (c) Oscillate with angular frequency 20 s^{-1}
    (d) First increase and then decrease

Paragraph for Questions 56–59: An *LCR* series circuit with 100 Ω resistance is connected to an ac source of 200 V and angular frequency 300 rad/s. When only the capacitance is removed, the current lags behind the voltage by 60°. When only the inductance is removed, the current leads the voltage by 60°.

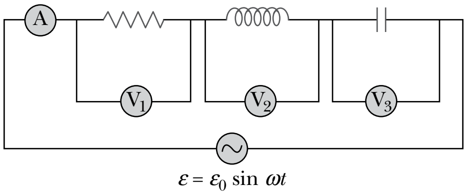
56. When capacitor is removed, inductive reactance of the circuit is
    (a) $100\sqrt{3}\ \Omega$ (b) $100\sqrt{2}\ \Omega$
    (c) $50\sqrt{2}\ \Omega$ (d) $50\sqrt{3}\ \Omega$

57. When inductor is removed, capacitive reactance of the circuit is
    (a) $\sqrt{3}\times 50\ \Omega$ (b) $100\sqrt{3}\ \Omega$
    (c) $50\sqrt{3}\ \Omega$ (d) $20\sqrt{2}\ \Omega$

58. Impedance of the *LCR* circuit is
    (a) 200 Ω   (b) 175 Ω
    (c) 150 Ω   (d) 100 Ω

59. Power dissipated in the circuit is
    (a) 250.5 W   (b) 500 W
    (c) 400 W   (d) 450 W

Paragraph for Questions 60–62: For the circuit shown in the following figure, a voltage of $\varepsilon = \varepsilon_0 \sin \omega t$ is applied. The voltmeter readings are $V_1 = 100$ V, $V_2 = 125$ V, and $V_3 = 150$ V and the ammeter reading is 5 A.



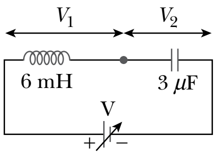
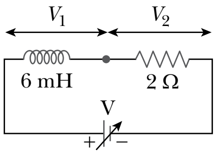
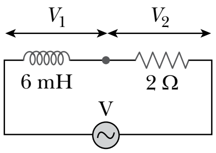
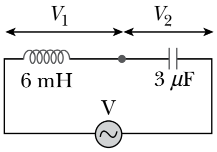
60. The net impedance of circuit is
    (a) 5 Ω   (b) $75\sqrt{5}\ \Omega$
    (c) $5\sqrt{75}\ \Omega$ (d) 75 Ω

61. The power factor of circuit is
    (a) 2.666   (b) 26.66
    (c) 0.2666   (d) 2.666

62. The value of ω_0 is
    (a) $\frac{\sqrt{5}}{30}\omega$ (b) $\frac{\sqrt{20}}{5}\omega$
    (c) $\frac{\sqrt{30}}{25}\omega$ (d) $\frac{\sqrt{30}}{5}\omega$

Matrix-Match

63. You are given many resistances, capacitors, and inductors. These are connected to a variable dc voltage source (the first two circuits) or an ac voltage source of 50 Hz frequency (the next three circuits) in different ways as shown in Column II. When a current *I* (steady state for dc or rms for ac) flows through the circuit, the corresponding voltages V_1 and V_2 (indicated in circuits) are related as shown in Column I.

Column I	Column II
(a) $I \neq 0$, V_1 is proportional to *I*.	**(p)** 
(b) $I \neq 0$, $V_2 > V_1$	**(q)** 
(c) $V_1 = 0$, $V_2 = V$	**(r)** 
(d) $I \neq 0$, V_2 is proportional to *I*.	**(s)** 

64. Column I gives certain situations in which a straight metallic wire of resistance *R* is used and Column II gives some resulting effects.

Column I	Column II
(a) A charged capacitor is connected to the ends of the wire	**(p)** A constant current flow through the wire
(b) The wire is moved perpendicular to its length with a constant velocity in a uniform magnetic field perpendicular to the plane of motion	**(q)** Thermal energy is generated in the wire
(c) The wire is placed in a constant electric field that has a direction along the length of the wire	**(r)** A constant potential difference develops between the ends of the wire
(d) A battery of constant emf is connected to the ends of the wire	**(s)** Charges of constant magnitude appear at the ends of the wire

Directions for Questions 65 and 66: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

65. There are three types of circuit elements in ac circuit, that is, resistance, inductance and capacitance. In the given table, Column I shows the different types of the ac circuit elements, Column II shows the amplitude or voltage across the different ac circuit elements and Column III shows the phase difference between the different types of elements.

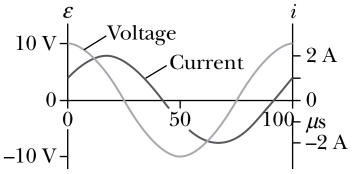
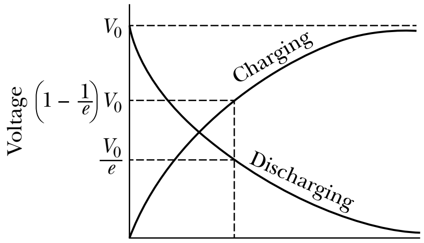
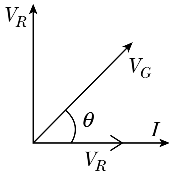
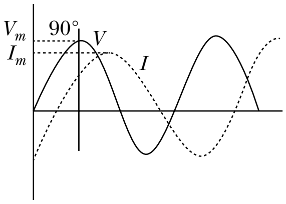
Column I	Column II	Column III
(I) Resistor	**(i)** $V_0 = i_0/R$	**(J)** In phase with i
(II) Capacitor	**(ii)** $V_0 = i_0X_L$	**(K)** Lags i by 90°
(III) Semi-conductor	**(iii)** $V_0 = i_0R$	**(L)** Leads i by 180°
(IV) Inductor	**(iv)** $V_0 = i_0X_C$	**(M)** Leads i by 90°

(1) Which circuit element has circuit quantity as $X = \omega L$?
- **(a)** (I) (ii) (J)
- **(b)** (IV) (ii) (M)
- **(c)** (II) (i) (M)
- **(d)** (I) (iii) (J)

(2) Which circuit element has circuit quantity as R?
- **(a)** (I) (iii) (J)
- **(b)** (IV) (iii) (L)
- **(c)** (II) (iii) (L)
- **(d)** (I) (i) (M)

(3) Which circuit element has circuit quantity as $X = 1/\omega C$
- **(a)** (III) (i) (K)
- **(b)** (I) (i) (L)
- **(c)** (II (iv) (K)
- **(d)** (II) (iii) (M)

66. There are four series combinations of different ac circuit elements, that is RL, RC, LC and LCR circuits. Each of these circuits has special properties and are used as filters. In the given table, Column I shows the types of filters made by using these ac circuits elements, Column II shows the use of the filters and Column III shows the graphs between current and voltage across the ac circuit elements.

Column I	Column II	Column III
(I) Can be used as analog infinite impulse response electronic filters	**(i)** helps in tuning radio transmitters and receivers	**(J)** 
(II) The two most common filters are the high-pass filters and low-pass filters	**(ii)** used for dc power supplies to RF amplifiers.	**(K)** 
(III) Can be used as band-pass filter	**(iii)** used in noise removers	**(L)**  (*b*)
(IV) Used as a band-pass filter, band-stop filter, low-pass filter or high-pass filter	**(iv)** as oscillator circuits	**(M)** 

(1) Which combination describes the characteristics of RLC circuit?

(a) (I) (iii) (L) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (IV) (iv) (M)

(2) Which combination describes the characteristics of LR circuit?

(a) (III) (ii) (L) **(b)** (I) (ii) (L)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) Which combination describes the characteristics of RC circuit?

(a) (II) (iii) (K) **(b)** (I) (i) (J)
(c) (III) (iii) (J) **(d)** (I) (ii) (L)

Integer Type

67. An ac source of frequency 1000 Hz is connected to a coil of $(200/\pi)$ mH and negligible resistance. If effective current through the coil is 7.5 mA, what is the voltage across the coil?

68. An ac source of voltage $V = 100 \sin 100\,\pi t$ is connected to a resistor of $(25/\sqrt{2})\ \Omega$. What is the rms value of current through the resistor?

69. An ac voltage is represented by $E = 220\sqrt{2}\cos(8\pi)t$. How many times will the current become zero in 1 s?

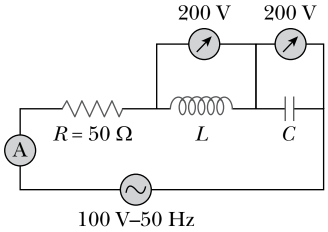
70. Two alternating currents are given by $I_1 = I_0 \sin(\omega t - \phi)$ and $I_2 = I_0 \cos(\omega t + \phi)$. What is the ratio of virtual values of the two currents?

71. An ac generator gives an output voltage of $E = 170 \sin 56.52t$. What is the frequency of alternating voltage produced?

72. A transformer working on 220 V ac line gives an output current of 4 A at 55 V. What is the primary current? Assume that there is no loss of energy.

73. Wavelength λ is transmitted by a transmitter using a condenser with capacitance 0.144 pF. If inductance of $1/\pi^2$ mH is used for resonance, what is the value of λ?

74. What would be the ammeter reading in the series LCR circuit as shown in the following figure?



ANSWER KEY

Checkpoints

1. (a) $T/2$; (b) T; (c) $T/2$; (d) $T/4$

2. (a) 5 V; (b) 150 μJ

3. (a) remains the same; (b) remains the same

4. (a) C, B, A; (b)1, A; 2, B; 3,S; 4, C; (c) A

5. (a) same; (b) increases; (c) same; (d) decreases

6. Less than

7. (a) 1, lags; 2, leads; 3, in phase; (b) 3 ($\omega_d = \omega$ when $X_L = X_C$)

8. (a) increase (circuit is mainly capacitive; increase C to decrease X_C to be closer to resonance for maximum P_{avg}); (b) closer

9. (a) greater; (b) step-up

Problems

1. (a) 8.8×10^{-4} s; (b) 2.6×10^{-3} s

2. (a) 39.1 mA; (b) zero; (c) 33.8 mA

3. (a) 383 Hz; (b) no change; (c) decreased; (d) increased; (e) increased

4. (a) 412 Ω; (b) 13.7°; (c) 175 mA

5. (a) yes; (b) 2.0 kV

6. (a) 10.3 Ω; (b) 1.35 kW

7. (a) 436 Ω; (b) 23.4°; (c) 165 mA

8. –8.00 V

9. 25 Ω

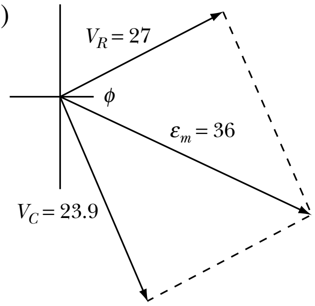
10. 500 V

11. 5.54 A

13. (a) 0.743; (b) lead; (c) capacitive; (d) no; (e) yes; (f) no; (g) yes; (h) 21.2 W

14. 9.14×10^{-9} F

15. (a) 534 Ω; (b) –41.5°; (c) 135 mA; (d)



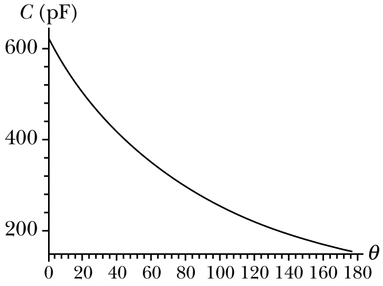
16. (a) $R = 100\ \Omega$; (b) $C = 30.6\ \mu$F; (c) $L = 301$ mH

17. (a) 19.4 mA; (b) 0.971 ms

18. (a) 3.60×10^{-3} H; (b) 1.33×10^{3} Hz; (c) 1.88×10^{-4} s

19. (a) 56.2 mA; (b) 11.2 mA

20.



21. (a) 3.0 V; (b) 2.7 mA; (c) 0.11 A

22. (a) 7.64×10^{-2} H; (b) yes; (c) 17.8 Ω; (d) this is not done because the resistors would consume power, rather than temporarily store electromagnetic energy

23. (a) 10.0 μs; (b) 100 kHz; (c) 5.00 μs; (d) twice

24. 125 V in all three cases

25. (a) 2.13 A; (b) 53.2 V; (c) 131 V; (d) 184 V; (e) 52.8 V; (f) 75.0 V; (g) 113 W; (h) 0; (i) 0

26. ω

27. 310.2 V

28. (a) 3.56 μJ; (b) 4.53 μC; (c) 16.9 mA; (d) −33.1°; (e) + 33.1°

29. 5.24 A

30. (a) 89 rad/s; (b) 7.0×10^{-2} s; (c) 2.5×10^{-5} F

31. 3.8×10^{-5} H

32. (a) 123 Hz; (b) 817 mA

33. (a) 1.25; (b) 4.00; (c) 5.00; (d) 0.200; (e) 0.250; (f) 0.800

34. (a) 0.189 mC; (b) 85.0 μs; (c) 42.5 W

35. (a) 3.47 μJ; (b) 9.62 mA; (c) 3.27 ms

36. (a) 217 μs; (b) 0.930 mH; (c) 1.19 mJ

37. (a) 46.1 μs; (b) 6.88 nJ; (c) 6.88 nJ; (d) 1.02×10^{3} A/s; (e) 0.938 mW

38. (a) 3.0 nC; (b) 1.0 mA; (c) 4.5 nJ

39. (a) 0.500; (b) 0.886

41. (a) 5.00 μs; (b) 2.50 μs; (c) 1.25 μs

42. (a) 0.46 kHz; (b) 35 Ω

Practice Questions

Single Correct Choice Type

1. (b) **2.** (c) **3.** (a) **4.** (d) **5.** (b)
6. (a) **7.** (c) **8.** (c) **9.** (b) **10.** (d)
11. (a) **12.** (c) **13.** (d) **14.** (c) **15.** (d)
16. (d) **17.** (c) **18.** (a) **19.** (a) **20.** (b)
21. (d) **22.** (d) **23.** (c) **24.** (d) **25.** (a)
26. (c) **27.** (b) **28.** (a) **29.** (d) **30.** (d)
31. (a) **32.** (b) **33.** (a) **34.** (b)

More than One Correct Choice Type

35. (a), (b), (c) **36.** (a), (b), (c) **37.** (b), (c) **38.** (b), (c), (d) **39.** (a), (d)
40. (a), (b), (c) **41.** (a), (b), (c), (d) **42.** (a), (b)

Linked Comprehension

43. (c) **44.** (b) **45.** (c) **46.** (c) **47.** (d)
48. (c) **49.** (c) **50.** (b) **51.** (c) **52.** (c)
53. (d) **54.** (c) **55.** (a) **56.** (a) **57.** (b)
58. (d) **59.** (c) **60.** (d) **61.** (c) **62.** (d)

Matrix-Match

63. (a) → (r), (s); (b) → (q), (r), (s); (c) → (p), (q); (d) → (q), (r), (s)

64. (a) → (p), (s); (b) → (q), (r); (c) → (p), (r); (d) → (s)

65. (1) → (b); (2) → (d); (3) → (c)

66. (1) → (d); (2) → (b); (3) → (a)

Integer Type

67. 3
68. 4
69. 8
70. 1
71. 9
72. 1
73. 7
74. 2

CHAPTER

32

Electromagnetic Waves

32.1 | WHAT IS PHYSICS?

The information age in which we live is based almost entirely on the physics of electromagnetic waves. Like it or not, we are now globally connected by television, telephones, and the web. And like it or not, we are constantly immersed in those signals because of television, radio, and telephone transmitters.

Much of this global interconnection of information processors was not imagined by even the most visionary engineers of 40 years ago. The challenge for today's engineers is trying to envision what the global interconnection will be like 40 years from now. The starting point in meeting that challenge is understanding the basic physics of electromagnetic waves, which come in so many different types that they are poetically said to form *Maxwell's rainbow*.

32.2 | MAXWELL'S RAINBOW

Key Concepts

- An electromagnetic wave consists of oscillating electric and magnetic fields.
- The various possible frequencies of electromagnetic waves form a spectrum, a small part of which is visible light.

The crowning achievement of James Clerk Maxwell was to show that a beam of light is a traveling wave of electric and magnetic fields—an **electromagnetic wave**—and thus that optics, the study of visible light, is a branch of electromagnetism. In this chapter we move from one to the other: we conclude our discussion of strictly electrical and magnetic phenomena, and we build a foundation for optics.

In Maxwell's time (the mid 1800s), the visible, infrared, and ultraviolet forms of light were the only electromagnetic waves known. Spurred on by Maxwell's work, however, Heinrich Hertz discovered what we now call radio waves and verified that they move through the laboratory at the same speed as visible light, indicating that they have the same basic nature as visible light.

Contents

32.1 What is Physics?
32.2 Maxwell's Rainbow
32.3 The Traveling Electromagnetic Wave, Qualitatively
32.4 The Traveling Electromagnetic Wave, Quantitatively
32.5 Energy Transport and the Poynting Vector
32.6 Radiation Pressure
32.7 Polarization
32.8 Reflection and Refraction
32.9 Polarization by Reflection
32.10 Polarization by Scattering

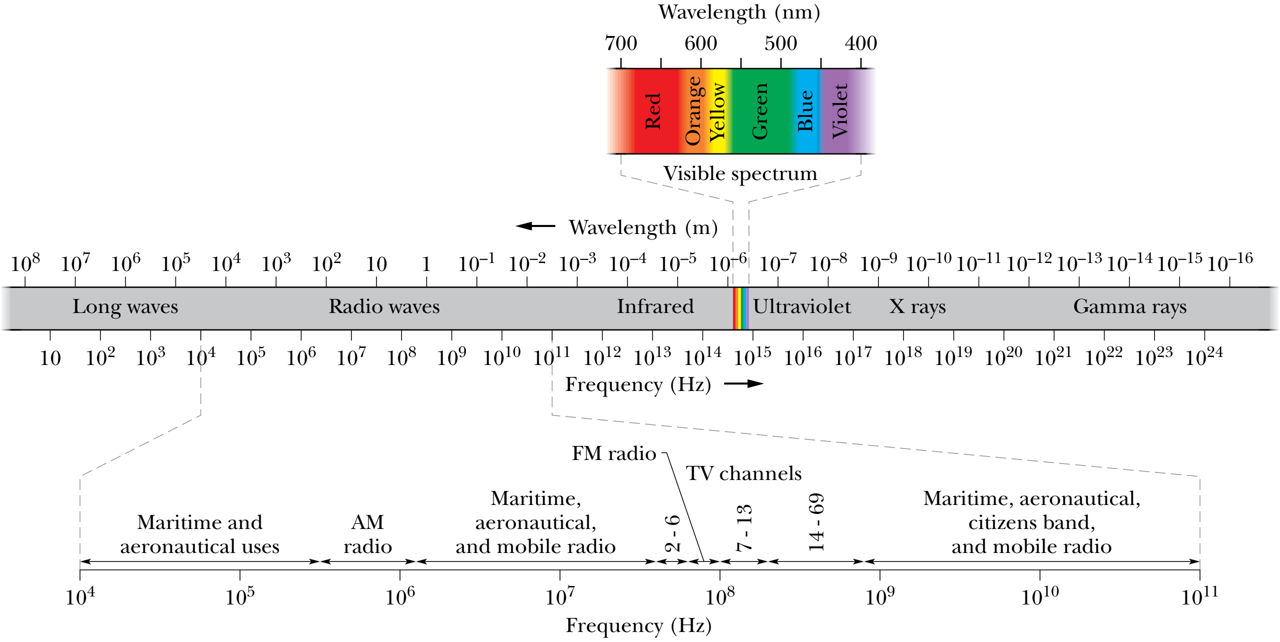


Figure 32-1 The electromagnetic spectrum.

As Fig. 32-1 shows, we now know a wide *spectrum* (or range) of electromagnetic waves: Maxwell's rainbow. Consider the extent to which we are immersed in electromagnetic waves throughout this spectrum. The Sun, whose radiations define the environment in which we as a species have evolved and adapted, is the dominant source. We are also crisscrossed by radio and television signals. Microwaves from radar systems and from telephone relay systems may reach us. There are electromagnetic waves from lightbulbs, from the heated engine blocks of automobiles, from x-ray machines, from lightning flashes, and from buried radioactive materials. Beyond this, radiation reaches us from stars and other objects in our galaxy and from other galaxies. Electromagnetic waves also travel in the other direction. Television signals, transmitted from Earth since about 1950, have now taken news about us (along with episodes of *I Love Lucy*, albeit *very* faintly) to whatever technically sophisticated inhabitants there may be on whatever planets may encircle the nearest 400 or so stars.

In the wavelength scale in Fig. 32-1 (and similarly the corresponding frequency scale), each scale marker represents a change in wavelength (and correspondingly in frequency) by a factor of 10. The scale is open-ended; the wavelengths of electromagnetic waves have no inherent upper or lower bound.

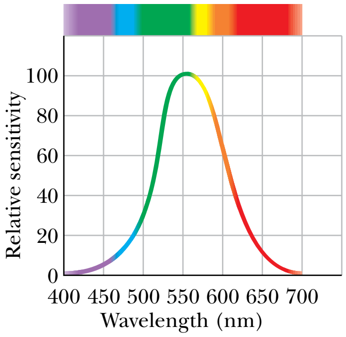


Figure 32-2 The relative sensitivity of the average human eye to electromagnetic waves at different wavelengths. This portion of the electromagnetic spectrum to which the eye is sensitive is called *visible light*.

Certain regions of the electromagnetic spectrum in Fig. 32-1 are identified by familiar labels, such as *x rays* and *radio waves*. These labels denote roughly defined wavelength ranges within which certain kinds of sources and detectors of electromagnetic waves are in common use. Other regions of Fig. 32-1, such as those labeled TV channels and AM radio, represent specific wavelength bands assigned by law for certain commercial or other purposes. There are no gaps in the electromagnetic spectrum—and all electromagnetic waves, no matter where they lie in the spectrum, travel through *free space* (vacuum) with the same speed c.

The visible region of the spectrum is of course of particular interest to us. Figure 32-2 shows the relative sensitivity of the human eye to light of various wavelengths. The center of the visible region is about 555 nm, which produces the sensation that we call yellow-green.

The limits of this visible spectrum are not well defined because the eye sensitivity curve approaches the zero-sensitivity line asymptotically at both long and short wavelengths. If we take the limits, arbitrarily, as the wavelengths at which eye sensitivity has dropped to 1% of its maximum value, these limits are about 430 and 690 nm; however, the eye can detect electromagnetic waves somewhat beyond these limits if they are intense enough.

32.3 | THE TRAVELING ELECTROMAGNETIC WAVE, QUALITATIVELY

Key Concept

- An electromagnetic wave traveling along an x axis has an electric field $\vec{E}$ and a magnetic field $\vec{B}$ with magnitudes that depend on x and t:

$$E = E_m \sin(kx - \omega t)$$

and

$$B = B_m \sin(kx - \omega t),$$

where E_m and B_m are the amplitudes of $\vec{E}$ and $\vec{B}$. The electric field induces the magnetic field and vice versa.

Some electromagnetic waves, including x rays, gamma rays, and visible light, are *radiated* (emitted) from sources that are of atomic or nuclear size, where quantum physics rules. Here we discuss how other electromagnetic waves are generated. To simplify matters, we restrict ourselves to that region of the spectrum (wavelength $\lambda \approx 1$ m) in which the source of the *radiation* (the emitted waves) is both macroscopic and of manageable dimensions.

Figure 32-3 shows, in broad outline, the generation of such waves. At its heart is an *LC oscillator*, which establishes an angular frequency $\omega\,(= 1/\sqrt{LC})$. Charges and currents in this circuit vary sinusoidally at this frequency, as depicted in Fig. 31-1. An external source—possibly an ac generator—must be included to supply energy to compensate both for thermal losses in the circuit and for energy carried away by the radiated electromagnetic wave.

The LC oscillator of Fig. 32-3 is coupled by a transformer and a transmission line to an *antenna*, which consists essentially of two thin, solid, conducting rods. Through this coupling, the sinusoidally varying current in the oscillator causes charge to oscillate sinusoidally along the rods of the antenna at the angular frequency ω of the LC oscillator. The current in the rods associated with this movement of charge also varies sinusoidally, in magnitude and direction, at angular frequency ω. The antenna has the effect of an electric dipole whose electric dipole moment varies sinusoidally in magnitude and direction along the antenna.

Because the dipole moment varies in magnitude and direction, the electric field produced by the dipole varies in magnitude and direction. Also, because the current varies, the magnetic field produced by the current varies in magnitude and direction. However, the changes in the electric and magnetic fields do not happen everywhere instantaneously; rather, the changes travel outward from the antenna at the speed of light c. Together the changing fields form an electromagnetic wave that travels away from the antenna at speed c. The angular frequency of this wave is ω, the same as that of the LC oscillator.

Electromagnetic Wave. Figure 32-4 shows how the electric field $\vec{E}$ and the magnetic field $\vec{B}$ change with time as one wavelength of the wave sweeps past the distant point P of Fig. 32-3; in each part of Fig. 32-4, the wave is traveling

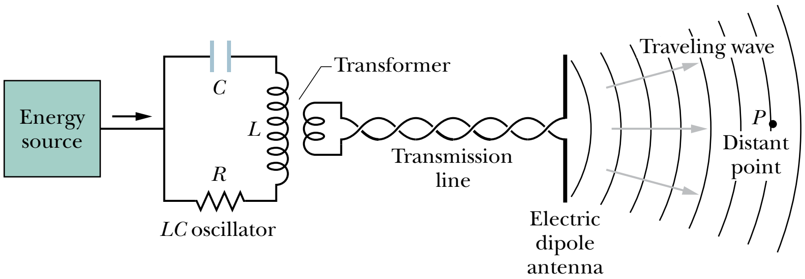


Figure 32-3 An arrangement for generating a traveling electromagnetic wave in the shortwave radio region of the spectrum: an LC oscillator produces a sinusoidal current in the antenna, which generates the wave. P is a distant point at which a detector can monitor the wave traveling past it.

directly out of the page. (We choose a distant point so that the curvature of the waves suggested in Fig. 32-3 is small enough to neglect. At such points, the wave is said to be a *plane wave*, and discussion of the wave is much simplified.) Note several key features in Fig. 32-4; they are present regardless of how the wave is created:

1. The electric and magnetic fields $\vec{E}$ and $\vec{B}$ are always perpendicular to the direction in which the wave is traveling. Thus, the wave is a *transverse wave*, as discussed in Chapter 16.
2. The electric field is always perpendicular to the magnetic field.
3. The cross product $\vec{E} \times \vec{B}$ always gives the direction in which the wave travels.
4. The fields always vary sinusoidally, just like the transverse waves discussed in Chapter 16. Moreover, the fields vary with the same frequency and *in phase* (in step) with each other.

In keeping with these features, we can assume that the electromagnetic wave is traveling toward P in the positive direction of an x axis, that the electric field in Fig. 32-4 is oscillating parallel to the y axis, and that the magnetic field is then oscillating parallel to the z axis (using a right-handed coordinate system, of course). Then we can write the electric and magnetic fields as sinusoidal functions of position x (along the path of the wave) and time t:

$$E = E_m \sin(kx - \omega t), \tag{32-1}$$

$$B = B_m \sin(kx - \omega t), \tag{32-2}$$

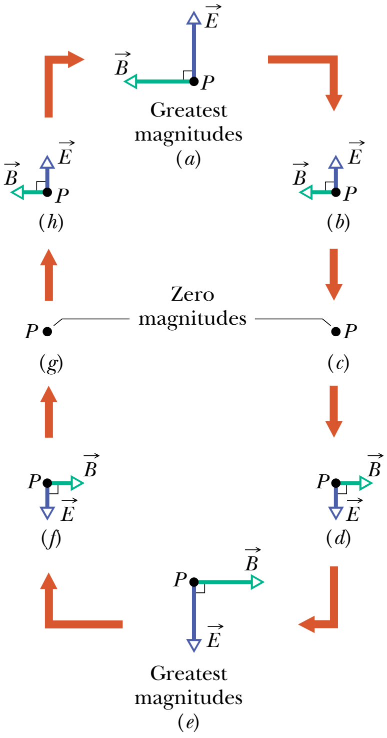


Figure 32-4 (*a*)–(*h*) The variation in the electric field $\vec{E}$ and the magnetic field $\vec{B}$ at the distant point P of Fig. 32-3 as one wavelength of the electromagnetic wave travels past it. In this perspective, the wave is traveling directly out of the page. The two fields vary sinusoidally in magnitude and direction. Note that they are always perpendicular to each other and to the wave's direction of travel.

in which E_m and B_m are the amplitudes of the fields and, as in Chapter 16, ω and k are the angular frequency and angular wave number of the wave, respectively. From these equations, we note that not only do the two fields form the electromagnetic wave but each also forms its own wave. Equation 32-1 gives the *electric wave component* of the electromagnetic wave, and Eq. 32-2 gives the *magnetic wave component*. As we shall discuss below, these two wave components cannot exist independently.

Wave Speed. From Eq. 16-13, we know that the speed of the wave is ω/k. However, because this is an electromagnetic wave, its speed (in vacuum) is given the symbol c rather than v. In the next section you will see that c has the value

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \text{(wave speed)}, \tag{32-3}$$

which is about 3.0×10^8 m/s. In other words,

All electromagnetic waves, including visible light, have the same speed c in vacuum.

You will also see that the wave speed c and the amplitudes of the electric and magnetic fields are related by

$$\frac{E_m}{B_m} = c \quad \text{(amplitude ratio)}. \tag{32-4}$$

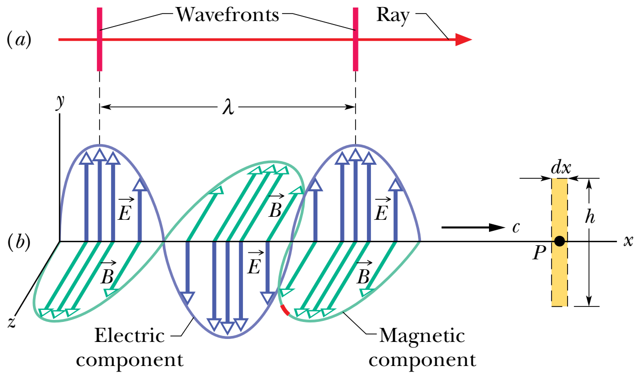


Figure 32-5 (*a*) An electromagnetic wave represented with a ray and two wavefronts; the wavefronts are separated by one wavelength λ. (*b*) The same wave represented in a "snapshot" of its electric field $\vec{E}$ and magnetic field $\vec{B}$ at points on the x axis, along which the wave travels at speed c. As it travels past point P, the fields vary as shown in Fig. 32-4. The electric component of the wave consists of only the electric fields; the magnetic component consists of only the magnetic fields. The dashed rectangle at P is used in Fig. 32-6.

If we divide Eq. 32-1 by Eq. 32-2 and then substitute with Eq. 32-4, we find that the magnitudes of the fields at every instant and at any point are related by

$$\frac{E}{B} = c \quad \text{(magnitude ratio).} \tag{32-5}$$

Rays and Wavefronts. We can represent the electromagnetic wave as in Fig. 32-5*a*, with a *ray* (a directed line showing the wave's direction of travel) or with *wavefronts* (imaginary surfaces over which the wave has the same magnitude of electric field), or both. The two wavefronts shown in Fig. 32-5*a* are separated by one wavelength $\lambda(= 2\pi/k)$ of the wave. (Waves traveling in approximately the same direction form a *beam*, such as a laser beam, which can also be represented with a ray.)

Drawing the Wave. We can also represent the wave as in Fig. 32-5*b*, which shows the electric and magnetic field vectors in a "snapshot" of the wave at a certain instant. The curves through the tips of the vectors represent the sinusoidal oscillations given by Eqs. 32-1 and 32-2; the wave components $\vec{E}$ and $\vec{B}$ are in phase, perpendicular to each other, and perpendicular to the wave's direction of travel.

Interpretation of Fig. 32-5*b* requires some care. The similar drawings for a transverse wave on a taut string that we discussed in Chapter 16 represented the up and down displacement of sections of the string as the wave passed (*something actually moved*). Figure 32-5*b* is more abstract. At the instant shown, the electric and magnetic fields each have a certain magnitude and direction (but always perpendicular to the x axis) at each point along the x axis. We choose to represent these vector quantities with a pair of arrows for each point, and so we must draw arrows of different lengths for different points, all directed away from the x axis, like thorns on a rose stem. However, the arrows represent field values only at points that are on the x axis. Neither the arrows nor the sinusoidal curves represent a sideways motion of anything, nor do the arrows connect points on the x axis with points off the axis.

Feedback. Drawings like Fig. 32-5 help us visualize what is actually a very complicated situation. First consider the magnetic field. Because it varies sinusoidally, it induces (via Faraday's law of induction) a perpendicular electric field that also varies sinusoidally. However, because that electric field is varying sinusoidally, it induces (via Maxwell's law of induction) a perpendicular magnetic field that also varies sinusoidally. And so on. The two fields continuously create each other via induction, and the resulting sinusoidal variations in the fields travel as a wave—the electromagnetic wave. Without this amazing result, we could not see; indeed, because we need electromagnetic waves from the Sun to maintain Earth's temperature, without this result we could not even exist.

A Most Curious Wave

The waves we discussed in Chapters 16 and 17 require a *medium* (some material) through which or along which to travel. We had waves traveling along a string, through Earth, and through the air. However, an electromagnetic wave (let's use the term *light wave* or *light*) is curiously different in that it requires no medium for its travel. It can, indeed, travel through a medium such as air or glass, but it can also travel through the vacuum of space between a star and us.

Once the special theory of relativity became accepted, long after Einstein published it in 1905, the speed of light waves was realized to be special. One reason is that light has the same speed regardless of the frame of reference from which it is measured. If you send a beam of light along an axis and ask several observers to measure its speed while they move at different speeds along that axis, either in the direction of the light or opposite it, they will all measure the *same speed* for the light. This result is an amazing one and quite different from what would have been found if those observers had measured the speed of any other type of wave; for other waves, the speed of the observers relative to the wave would have affected their measurements.

The meter has now been defined so that the speed of light (any electromagnetic wave) in vacuum has the exact value

$$c = 299\ 792\ 458 \text{ m/s},$$

which can be used as a standard. In fact, if you now measure the travel time of a pulse of light from one point to another, you are not really measuring the speed of the light but rather the distance between those two points.

32.4 | THE TRAVELING ELECTROMAGNETIC WAVE, QUANTITATIVELY

Key Concept

- The speed of any electromagnetic wave in vacuum is c, which can be written as

$$c = \frac{E}{B} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}},$$

where E and B are the simultaneous magnitudes of the fields.

We shall now derive Eqs. 32-3 and 32-4 and, even more important, explore the dual induction of electric and magnetic fields that gives us light.

Equation 32-4 and the Induced Electric Field

The dashed rectangle of dimensions dx and h in Fig. 32-6 is fixed at point P on the x axis and in the xy plane (it is shown on the right in Fig. 32-5b). As the electromagnetic wave moves rightward past the rectangle, the magnetic flux Φ_B through the rectangle changes and—according to Faraday's law of induction—induced electric fields appear throughout the region of the rectangle. We take $\vec{E}$ and $\vec{E} + d\vec{E}$ to be the induced fields along the two long sides of the rectangle. These induced electric fields are, in fact, the electrical component of the electromagnetic wave.

The oscillating magnetic field induces an oscillating and perpendicular electric field.

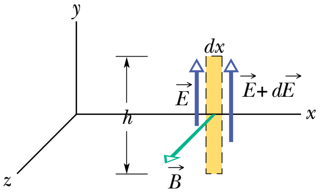


Figure 32-6 As the electromagnetic wave travels rightward past point P in Fig. 32-5b, the sinusoidal variation of the magnetic field $\vec{B}$ through a rectangle centered at P induces electric fields along the rectangle. At the instant shown, $\vec{B}$ is decreasing in magnitude and the induced electric field is therefore greater in magnitude on the right side of the rectangle than on the left.

Note the small red portion of the magnetic field component curve far from the y axis in Fig. 32-5b. Let's consider the induced electric fields at the instant when this red portion of the magnetic component is passing through the rectangle. Just then, the magnetic field through the rectangle points in the positive z direction and is decreasing in magnitude (the magnitude was greater just before the red section arrived). Because the magnetic field is decreasing, the magnetic flux Φ_B through the rectangle is also decreasing. According to Faraday's law, this change in flux is opposed by induced electric fields, which produce a magnetic field $\vec{B}$ in the positive z direction.

According to Lenz's law, this in turn means that if we imagine the boundary of the rectangle to be a conducting loop, a counterclockwise induced current would have to appear in it. There is, of course, no conducting loop; but this analysis shows that the induced electric field vectors $\vec{E}$ and $\vec{E} + d\vec{E}$ are indeed oriented as shown in Fig. 32-6, with the magnitude of $\vec{E} + d\vec{E}$ greater than that of $\vec{E}$. Otherwise, the net induced electric field would not act counterclockwise around the rectangle.

Faraday's Law. Let us now apply Faraday's law of induction,

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}, \tag{32-6}$$

counterclockwise around the rectangle of Fig. 32-6. There is no contribution to the integral from the top or bottom of the rectangle because $\vec{E}$ and $d\vec{s}$ are perpendicular to each other there. The integral then has the value

$$\oint \vec{E}\cdot d\vec{s} = (E+dE)h - Eh = h\,dE. \tag{32-7}$$

The flux Φ_B through this rectangle is

$$\Phi_B = (B)(h\,dx), \tag{32-8}$$

where B is the average magnitude of $\vec{B}$ within the rectangle and $h\,dx$ is the area of the rectangle. Differentiating Eq. 32-8 with respect to t gives

$$\frac{d\Phi_B}{dt} = h\,dx\frac{dB}{dt}. \tag{32-9}$$

If we substitute Eqs. 32-7 and 32-9 into Eq. 32-6, we find

$$h\,dE = -h\,dx\frac{dB}{dt}$$

or

$$\frac{dE}{dx} = -\frac{dB}{dt}. \tag{32-10}$$

Actually, both B and E are functions of *two* variables, coordinate x and time t, as Eqs. 32-1 and 32-2 show. However, in evaluating dE/dx, we must assume that t is constant because Fig. 32-6 is an "instantaneous snapshot." Also, in evaluating dB/dt we must assume that x is constant (a particular value) because we are dealing with the time rate of change of B at a particular place, the point P shown in Fig. 32-5*b*. The derivatives under these circumstances are *partial derivatives*, and Eq. 32-10 must be written

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}. \tag{32-11}$$

The minus sign in this equation is appropriate and necessary because, although magnitude E is increasing with x at the site of the rectangle in Fig. 32-6, magnitude B is decreasing with t.

From Eq. 32-1 we have

$$\frac{\partial E}{\partial x} = kE_m\cos(kx-\omega t)$$

and from Eq. 32-2

$$\frac{\partial B}{\partial t} = -\omega B_m\cos(kx-\omega t).$$

Then Eq. 32-11 reduces to

$$kE_m\cos(kx-\omega t) = \omega B_m\cos(kx-\omega t). \tag{32-12}$$

The ratio ω/k for a traveling wave is its speed, which we are calling c. Equation 32-12 then becomes

$$\frac{E_m}{B_m} = c \quad \text{(amplitude ratio)}, \tag{32-13}$$

which is just Eq. 32-4.

Equation 32-3 and the Induced Magnetic Field

Figure 32-7 shows another dashed rectangle at point P of Fig. 32-5*b*; this one is in the xz plane. As the electromagnetic wave moves rightward past this new rectangle, the electric flux Φ_E through the rectangle changes and—according to Maxwell's law of induction—induced magnetic fields appear throughout the region of the rectangle. These induced magnetic fields are, in fact, the magnetic component of the electromagnetic wave.

The oscillating electric field induces an oscillating and perpendicular magnetic field.

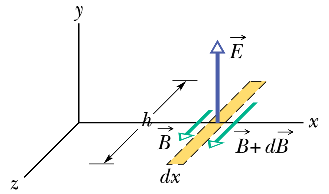


Figure 32-7 The sinusoidal variation of the electric field through this rectangle, located (but not shown) at point *P* in Fig. 32-5*b*, induces magnetic fields along the rectangle. The instant shown is that of Fig. 32-6: $\vec{E}$ is decreasing in magnitude, and the magnitude of the induced magnetic field is greater on the right side of the rectangle than on the left.

We see from Fig. 32-5*b* that at the instant chosen for the magnetic field represented in Fig. 32-6, marked in red on the magnetic component curve, the electric field through the rectangle of Fig. 32-7 is directed as shown. Recall that at the chosen instant, the magnetic field in Fig. 32-6 is decreasing. Because the two fields are in phase, the electric field in Fig. 32-7 must also be decreasing, and so must the electric flux Φ_E through the rectangle. By applying the same reasoning we applied to Fig. 32-6, we see that the changing flux Φ_E will induce a magnetic field with vectors $\vec{B}$ and $\vec{B}+d\vec{B}$ oriented as shown in Fig. 32-7, where field $\vec{B}+d\vec{B}$ is greater than field $\vec{B}$.

Maxwell's Law. Let us apply Maxwell's law of induction,

$$\oint \vec{B}\cdot d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}, \tag{32-14}$$

by proceeding counterclockwise around the dashed rectangle of Fig. 32-7. Only the long sides of the rectangle contribute to the integral because the dot product along the short sides is zero. Thus, we can write

$$\oint \vec{B}\cdot d\vec{s} = -(B+dB)h + Bh = -h\,dB. \tag{32-15}$$

The flux Φ_E through the rectangle is

$$\Phi_E = (E)(h\,dx), \tag{32-16}$$

where E is the average magnitude of $\vec{E}$ within the rectangle. Differentiating Eq. 32-16 with respect to t gives

$$\frac{d\Phi_E}{dt} = h\,dx\frac{dE}{dt}.$$

If we substitute this and Eq. 32-15 into Eq. 32-14, we find

$$-h\,dB = \mu_0\varepsilon_0\left(h\,dx\frac{dE}{dt}\right)$$

or, changing to partial-derivative notation as we did for Eq. 32-11,

$$-\frac{\partial B}{\partial x} = \mu_0\varepsilon_0\frac{\partial E}{\partial t}. \tag{32-17}$$

Again, the minus sign in this equation is necessary because, although B is increasing with x at point P in the rectangle in Fig. 32-7, E is decreasing with t.

Evaluating Eq. 32-17 by using Eqs. 32-1 and 32-2 leads to

$$-kB_m\cos(kx-\omega t) = -\mu_0\varepsilon_0\omega E_m\cos(kx-\omega t),$$

which we can write as

$$\frac{E_m}{B_m} = \frac{1}{\mu_0\varepsilon_0(\omega/k)} = \frac{1}{\mu_0\varepsilon_0 c}.$$

Combining this with Eq. 32-13 leads at once to

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \quad \text{(wave speed)}, \tag{32-18}$$

which is exactly Eq. 32-3.

Let us now explore another question: What is the speed of propagation of an electromagnetic pulse in a dielectric with dielectric constant k? To work out the speed of light in a dielectric, we begin with the modification of the displacement current (Eq. 32-14) in the presence of a dielectric. In a dielectric, as in space, conventional current is zero. So, this equation reduces to

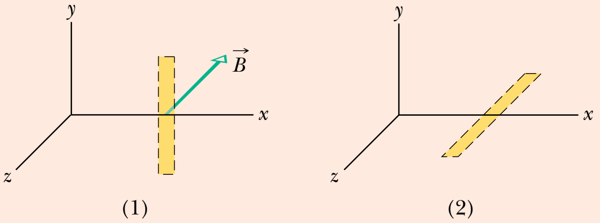
$$\oint \vec{B}\cdot d\vec{s} = \mu\varepsilon_0 k\frac{d\Phi_E}{dt}. \tag{32-19}$$

If we compare this with the analysis before, it is clear that this will yield the same answer for the speed of light except for the multiplying factor of k. We find the following for the speed of light in a dielectric:

$$c' = \frac{1}{\sqrt{k\mu_0\varepsilon_0}} = \frac{c}{\sqrt{k}}. \tag{32-20}$$

CHECKPOINT 1

The magnetic field $\vec{B}$ through the rectangle of Fig. 32-6 is shown at a different instant in part 1 of the figure here; $\vec{B}$ is directed in the xz plane, parallel to the z axis, and its magnitude is increasing. (a) Complete part 1 by drawing the induced electric fields, indicating both directions and relative magnitudes (as in Fig. 32-6). (b) For the same instant, complete part 2 of the figure by drawing the electric field of the electromagnetic wave. Also draw the induced magnetic fields, indicating both directions and relative magnitudes (as in Fig. 32-7).



32.5 | ENERGY TRANSPORT AND THE POYNTING VECTOR

Key Concepts

- The rate per unit area at which energy is transported via an electromagnetic wave is given by the Poynting vector $\vec{S}$:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}.$$

  The direction of $\vec{S}$ (and thus of the wave's travel and the energy transport) is perpendicular to the directions of both $\vec{E}$ and $\vec{B}$.

- The time-averaged rate per unit area at which energy is transported is S_{avg}, which is called the intensity I of the wave:

$$I = \frac{1}{c\mu_0}E_{\text{rms}}^2,$$

  in which $E_{\text{rms}} = E_m/\sqrt{2}$.

- A point source of electromagnetic waves emits the waves isotropically—that is, with equal intensity in all directions. The intensity of the waves at distance r from a point source of power P_s is

$$I = \frac{P_S}{4\pi r^2}.$$

All sunbathers know that an electromagnetic wave can transport energy and deliver it to a body on which the wave falls. The rate of energy transport per unit area in such a wave is described by a vector $\vec{S}$, called the **Poynting vector** after physicist John Henry Poynting (1852–1914), who first discussed its properties. This vector is defined as

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \quad \text{(Poynting vector)}. \tag{32-21}$$

Its magnitude S is related to the rate at which energy is transported by a wave across a unit area at any instant (inst):

$$S = \left(\frac{\text{energy/time}}{\text{area}}\right)_{\text{inst}} = \left(\frac{\text{power}}{\text{area}}\right)_{\text{inst}}. \tag{32-22}$$

From this we can see that the SI unit for $\vec{S}$ is the watt per square meter (W/m²).

The direction of the Poynting vector $\vec{S}$ of an electromagnetic wave at any point gives the wave's direction of travel and the direction of energy transport at that point.

Because $\vec{E}$ and $\vec{B}$ are perpendicular to each other in an electromagnetic wave, the magnitude of $\vec{E}\times\vec{B}$ is EB. Then the magnitude of $\vec{S}$ is

$$S=\frac{1}{\mu_0}EB, \tag{32-23}$$

in which S, E, and B are instantaneous values. The magnitudes E and B are so closely coupled to each other that we need to deal with only one of them; we choose E, largely because most instruments for detecting electromagnetic waves deal with the electric component of the wave rather than the magnetic component. Using $B = E/c$ from Eq. 32-5, we can rewrite Eq. 32-23 in terms of just the electric component as

$$S=\frac{1}{c\mu_0}E^2 \quad \text{(instantaneous energy flow rate).} \tag{32-24}$$

Intensity. By substituting $E=E_m\sin(kx-\omega t)$ into Eq. 32-24, we could obtain an equation for the energy transport rate as a function of time. More useful in practice, however, is the average energy transported over time; for that, we need to find the time-averaged value of S, written S_{avg} and also called the **intensity** I of the wave. Thus from Eq. 32-20, the intensity I is

$$I=S_{\text{avg}}=\left(\frac{\text{energy/time}}{\text{area}}\right)_{\text{avg}}=\left(\frac{\text{power}}{\text{area}}\right)_{\text{avg}}. \tag{32-25}$$

From Eq. 32-24, we find

$$I=S_{\text{avg}}=\frac{1}{c\mu_0}[E^2]_{\text{avg}}=\frac{1}{c\mu_0}[E_m^2\sin^2(kx-\omega t)]_{\text{avg}}. \tag{32-26}$$

Over a full cycle, the average value of $\sin^2\theta$, for any angular variable θ, is 1/2 (see Fig. 31-18). In addition, we define a new quantity E_{rms}, the *root-mean-square* value of the electric field, as

$$E_{\text{rms}}=\frac{E_m}{\sqrt{2}}. \tag{32-27}$$

We can then rewrite Eq. 32-26 as

$$I=\frac{1}{c\mu_0}E_{\text{rms}}^2. \tag{32-28}$$

Because $E=cB$ and c is such a very large number, you might conclude that the energy associated with the electric field is much greater than that associated with the magnetic field. That conclusion is incorrect; the two energies are exactly equal. To show this, we start with the equation which gives the energy density u $(=1/2\varepsilon_0E^2)$ within an electric field, and substitute cB for E; then we can write

$$u_E=\frac{1}{2}\varepsilon_0E^2=\frac{1}{2}\varepsilon_0(cB)^2.$$

If we now substitute for c with Eq. 32-3, we get

$$u_E=\frac{1}{2}\varepsilon_0\frac{1}{\mu_0\varepsilon_0}B^2=\frac{B^2}{2\mu_0}.$$

However, Eq. 30-57 tells us that $B^2/2\mu_0$ is the energy density u_B of a magnetic field $\vec{B}$; so we see that $u_E=u_B$ everywhere along an electromagnetic wave.

Variation of Intensity with Distance

How intensity varies with distance from a real source of electromagnetic radiation is often complex—especially when the source (like a searchlight at a movie premier) beams the radiation in a particular direction. However, in some situations we can assume that the source is a *point source* that emits the light *isotropically*—that is, with equal intensity in all directions. The spherical wavefronts spreading from such an isotropic point source S at a particular instant are shown in cross section in Fig. 32-8.

Let us assume that the energy of the waves is conserved as they spread from this source. Let us also center an imaginary sphere of radius r on the source, as shown in Fig. 32-8. All the energy emitted by the source must pass through the sphere. Thus, the rate at which energy passes through the sphere via the radiation must equal the rate at which energy is emitted by the source—that is, the source power P_s. The intensity I (power per unit area) measured at the sphere must then be, from Eq. 32-25,

$$I = \frac{\text{power}}{\text{area}} = \frac{P_S}{4\pi r^2}, \qquad (32\text{-}29)$$

where $4\pi r^2$ is the area of the sphere. Equation 32-29 tells us that the intensity of the electromagnetic radiation from an isotropic point source decreases with the square of the distance r from the source.

The energy emitted by light source S must pass through the sphere of radius r.

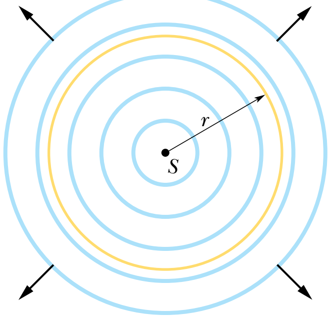
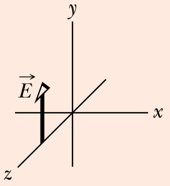


Figure 32-8 A point source S emits electromagnetic waves uniformly in all directions. The spherical wavefronts pass through an imaginary sphere of radius r that is centered on S.

CHECKPOINT 2

The figure here gives the electric field of an electromagnetic wave at a certain point and a certain instant. The wave is transporting energy in the negative z direction. What is the direction of the magnetic field of the wave at that point and instant?



SAMPLE PROBLEM 32.01

Light wave: rms values of the electric and magnetic fields

When you look at the North Star (Polaris), you intercept light from a star at a distance of 431 ly and emitting energy at a rate of 2.2×10^3 times that of our Sun ($P_{\text{sun}} = 3.90 \times 10^{26}$ W. Neglecting any atmospheric absorption, find the rms values of the electric and magnetic fields when the starlight reaches you.

KEY IDEAS

1. The rms value E_{rms} of the electric field in light is related to the intensity I of the light via Eq. 32-28 ($I = E^2_{\text{rms}}/c\mu_0$).
2. Because the source is so far away and emits light with equal intensity in all directions, the intensity I at any distance r from the source is related to the source's power P_s via Eq. 32-29 ($I = P_s/4\pi r^2$).
3. The magnitudes of the electric field and magnetic field of an electromagnetic wave at any instant and at any point in the wave are related by the speed of light c according to Eq. 32-5 ($E/B = c$). Thus, the rms values of those fields are also related by Eq. 32-5.

Electric field: Putting the first two ideas together gives us

$$I = \frac{P_S}{4\pi r^2} = \frac{E^2_{\text{rms}}}{c\mu_0}$$

and

$$E_{\text{rms}} = \sqrt{\frac{P_S c\mu_0}{4\pi r^2}}.$$

By substituting $P_s = (2.2 \times 10^3)(3.90 \times 10^{26}\text{ W})$, $r = 431\text{ ly} = 4.08 \times 10^{18}$ m, and values for the constants, we find

$$E_{\text{rms}} = 1.24 \times 10^{-3}\text{ V/m} \approx 1.2\text{ mV/m}. \qquad \text{(Answer)}$$

Magnetic field: From Eq. 32-5, we write

$$B_{\text{rms}} = \frac{E_{\text{rms}}}{c} = \frac{1.24\times 10^{-3}\ \text{V/m}}{3.00\times 10^{8}\ \text{m/s}}$$
$$= 4.1\times 10^{-12}\ \text{T} = 4.1\ \text{pT}.$$

Cannot compare the fields: Note that E_{rms} (= 1.2 mV/m) is small as judged by ordinary laboratory standards, but B_{rms} (= 4.1 pT) is quite small. This difference helps to explain why most instruments used for the detection and measurement of electromagnetic waves are designed to respond to the electric component. It is wrong, however, to say that the electric component of an electromagnetic wave is "stronger" than the magnetic component. You cannot compare quantities that are measured in different units. However, these electric and magnetic components are on an equal basis because their average energies, which *can* be compared, are equal.

32.6 | RADIATION PRESSURE

Key Concept

- When a surface intercepts electromagnetic radiation, a force and a pressure are exerted on the surface.

Electromagnetic waves have linear momentum and thus can exert a pressure on an object when shining on it. However, the pressure must be very small because, for example, you do not feel a punch during a camera flash.

To find an expression for the pressure, let us shine a beam of electromagnetic radiation—light, for example—on an object for a time interval Δt. Further, let us assume that the object is free to move and that the radiation is entirely **absorbed** (taken up) by the object. This means that during the interval Δt, the object gains an energy ΔU from the radiation. Maxwell showed that the object also gains linear momentum. The magnitude Δp of the momentum change of the object is related to the energy change ΔU by

$$\Delta p = \frac{\Delta U}{c} \quad \text{(total absorption)}, \tag{32-30}$$

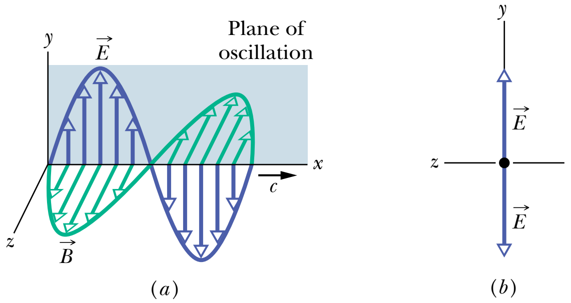
where c is the speed of light. The direction of the momentum change of the object is the direction of the *incident* (incoming) beam that the object absorbs.

32.7 | POLARIZATION

Key Concepts

- Electromagnetic waves are polarized if their electric field vectors are all in a single plane, called the plane of oscillation. Light waves from common sources are not polarized; that is, they are unpolarized, or polarized randomly.
- When a polarizing sheet is placed in the path of light, only electric field components of the light parallel to the sheet's polarizing direction are transmitted by the sheet; components perpendicular to the polarizing direction are absorbed. The light that emerges from a polarizing sheet is polarized parallel to the polarizing direction of the sheet.
- If the original light is initially unpolarized, the transmitted intensity I is half the original intensity I_0:
$$I = \frac{1}{2} I_0.$$
- If the original light is initially polarized, the transmitted intensity depends on the angle θ between the polarization direction of the original light and the polarizing direction of the sheet:
$$I = I_0 \cos^2 \theta.$$

VHF (very high frequency) television antennas in England are oriented vertically, but those in North America are horizontal. The difference is due to the direction of oscillation of the electromagnetic waves carrying the TV signal. In England, the transmitting equipment is designed to produce waves that are **polarized** vertically; that is, their



Vertically polarized light headed toward you—the electric fields are all vertical.

Figure 32-9 (*a*) The plane of oscillation of a polarized electromagnetic wave. (*b*) To represent the polarization, we view the plane of oscillation head-on and indicate the directions of the oscillating electric field with a double arrow.

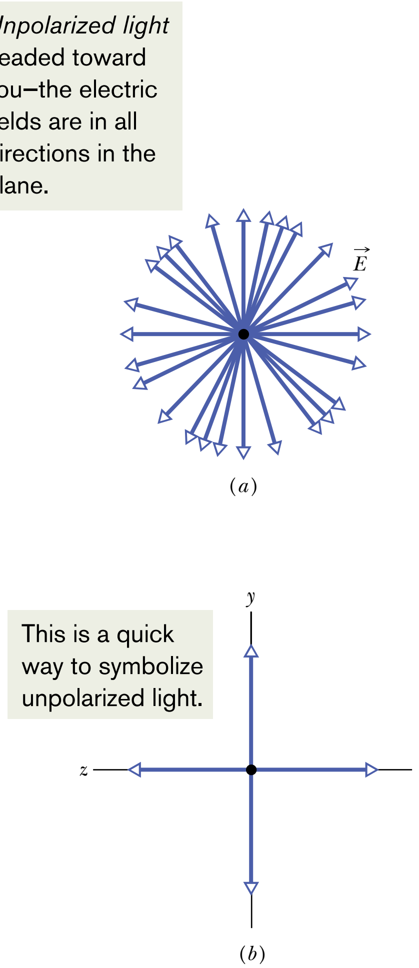
electric field oscillates vertically. Thus, for the electric field of the incident television waves to drive a current along an antenna (and provide a signal to a television set), the antenna must be vertical. In North America, the waves are polarized horizontally.

Figure 32-9*a* shows an electromagnetic wave with its electric field oscillating parallel to the vertical y axis. The plane containing the $\vec{E}$ vectors is called the **plane of oscillation** of the wave (hence, the wave is said to be *plane-polarized* in the y direction). We can represent the wave's *polarization* (state of being polarized) by showing the directions of the electric field oscillations in a head-on view of the plane of oscillation, as in Fig. 32-9*b*. The vertical double arrow in that figure indicates that as the wave travels past us, its electric field oscillates vertically—it continuously changes between being directed up and down the y axis.

Polarized Light

The electromagnetic waves emitted by a television station all have the same polarization, but the electromagnetic waves emitted by any common source of light (such as the Sun or a bulb) are **polarized randomly**, or **unpolarized** (the two terms mean the same thing). That is, the electric field at any given point is always perpendicular to the direction of travel of the waves but changes directions randomly. Thus, if we try to represent a head-on view of the oscillations over some time period, we do not have a simple drawing with a single double arrow like that of Fig. 32-9*b*; instead we have a mess of double arrows like that in Fig. 32-10*a*.

In principle, we can simplify the mess by resolving each electric field of Fig. 32-10*a* into y and z components. Then as the wave travels past us, the net y component oscillates parallel to the y axis and the net z component oscillates parallel to the z axis. We can then represent the unpolarized light with a pair of double arrows as shown in Fig. 32-10*b*. The double arrow along the y axis represents the oscillations of the net y component of the electric field. The double arrow along the z axis represents the oscillations of the net z component of the electric field. In doing all this, we effectively change unpolarized light into the superposition of two polarized waves whose planes of oscillation are perpendicular to each other—one plane contains the y axis and the other contains the z axis. One reason to make this change is that drawing Fig. 32-10*b* is a lot easier than drawing Fig. 32-10*a*.



Unpolarized light headed toward you—the electric fields are in all directions in the plane.

This is a quick way to symbolize unpolarized light.

Figure 32-10 (*a*) Unpolarized light consists of waves with randomly directed electric fields. Here the waves are all traveling along the same axis, directly out of the page, and all have the same amplitude E. (*b*) A second way of representing unpolarised light—the light is the superposition of two polarized waves whose planes of oscillation are perpendicular to each other.

We can draw similar figures to represent light that is **partially polarized** (its field oscillations are not completely random as in Fig. 32-10*a*, nor are they parallel to a single axis as in Fig. 32-9*b*). For this situation, we draw one of the double arrows in a perpendicular pair of double arrows longer than the other one.

Polarizing Direction. We can transform unpolarized visible light into polarized light by sending it through a *polarizing sheet*, as is shown in Fig. 32-11. Such sheets, commercially known as Polaroids or Polaroid filters, were invented in 1932 by Edwin Land while he was an undergraduate student. A polarizing sheet consists of certain long molecules embedded in plastic. When the sheet is manufactured, it is stretched to align the molecules in parallel rows, like rows in a plowed field. When light is then sent through the sheet, electric field components along one direction pass through the sheet, while components perpendicular to that direction are absorbed by the molecules and disappear.

We shall not dwell on the molecules but, instead, shall assign to the sheet a *polarizing direction*, along which electric field components are passed:

> An electric field component parallel to the polarizing direction is passed (*transmitted*) by a polarizing sheet; a component perpendicular to it is absorbed.

Thus, the electric field of the light emerging from the sheet consists of only the components that are parallel to the polarizing direction of the sheet; hence the light is polarized in that direction. In Fig. 32-11, the vertical electric field components are transmitted by the sheet; the horizontal components are absorbed. The transmitted waves are then vertically polarized.

The sheet's polarizing axis is vertical, so only vertically polarized light emerges.

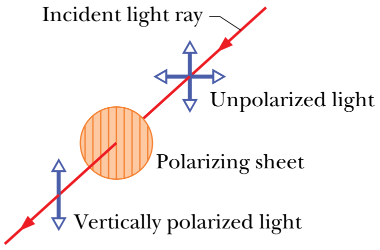


Figure 32-11 Unpolarized light becomes polarized when it is sent through a polarizing sheet. Its direction of polarization is then parallel to the polarizing direction of the sheet, which is represented here by the vertical lines drawn in the sheet.

Intensity of Transmitted Polarized Light

We now consider the intensity of light transmitted by a polarizing sheet. We start with unpolarized light, whose electric field oscillations we can resolve into y and z components as represented in Fig. 32-10*b*. Further, we can arrange for the y axis to be parallel to the polarizing direction of the sheet. Then only the y components of the light's electric field are passed by the sheet; the z components are absorbed. As suggested by Fig. 32-10*b*, if the original waves are randomly oriented, the sum of the y components and the sum of the z components are equal. When the z components are absorbed, half the intensity I_0 of the original light is lost. The intensity I of the emerging polarized light is then

$$I = \frac{1}{2} I_0 \quad \text{(one-half rule).} \qquad (32\text{-}31)$$

Let us call this the *one-half rule*; we can use it *only* when the light reaching a polarizing sheet is unpolarized.

The sheet's polarizing axis is vertical, so only vertical components of the electric fields pass.

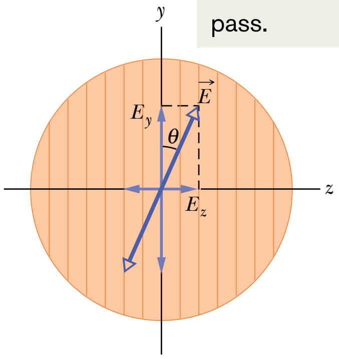


Figure 32-12 Polarized light approaching a polarizing sheet. The electric field $\vec{E}$ of the light can be resolved into components E_y (parallel to the polarizing direction of the sheet) and E_z (perpendicular to that direction). Component E_y will be transmitted by the sheet; component E_z will be absorbed.

Suppose now that the light reaching a polarizing sheet is already polarized. Figure 32-12 shows a polarizing sheet in the plane of the page and the electric field $\vec{E}$ of such a polarized light wave traveling toward the sheet (and thus prior to any absorption). We can resolve $\vec{E}$ into two components relative to the polarizing direction of the sheet: parallel component E_y is transmitted by the sheet, and perpendicular component E_z

is absorbed. Since θ is the angle between $\vec{E}$ and the polarizing direction of the sheet, the transmitted parallel component is

$$E_y = E\cos\theta. \tag{32-32}$$

Recall that the intensity of an electromagnetic wave (such as our light wave) is proportional to the square of the electric field's magnitude (Eq. 32-28, $I = E^2_{\text{rms}}/c\mu_0$). In our present case then, the intensity I of the emerging wave is proportional to E_y^2 and the intensity I_0 of the original wave is proportional to E^2. Hence, from Eq. 32-32 we can write $I/I_0 \cos^2\theta$, or

$$I = I_0\cos^2\theta \quad \text{(cosine-squared rule).} \tag{32-33}$$

Let us call this the *cosine-squared rule*; we can use it *only* when the light reaching a polarizing sheet is already polarized. Then the transmitted intensity I is a maximum and is equal to the original intensity I_0 when the original wave is polarized parallel to the polarizing direction of the sheet (when θ in Eq. 32-33 is 0° or 180°). The transmitted intensity is zero when the original wave is polarized perpendicular to the polarizing direction of the sheet (when θ is 90°).

Two Polarizing Sheets. Figure 32-13 shows an arrangement in which initially unpolarized light is sent through two polarizing sheets P_1 and P_2. (Often, the first sheet is called the *polarizer*, and the second the *analyzer*.) Because the polarizing direction of P_1 is vertical, the light transmitted by P_1 to P_2 is polarized vertically. If the polarizing direction of P_2 is also vertical, then all the light transmitted by P_1 is transmitted by P_2. If the polarizing direction of P_2 is horizontal, none of the light transmitted by P_1 is transmitted by P_2. We reach the same conclusions by considering only the *relative* orientations of the two sheets: If their polarizing directions are parallel, all the light passed by the first sheet is passed by the second sheet (Fig. 32-14*a*). If those directions are perpendicular (the sheets are said to be *crossed*), no light is passed by the second sheet (Fig. 32-14*b*). Finally, if the two polarizing directions of Fig. 32-13 make an angle between 0° and 90°, some of the light transmitted by P_1 will be transmitted by P_2, as set by Eq. 32-33.

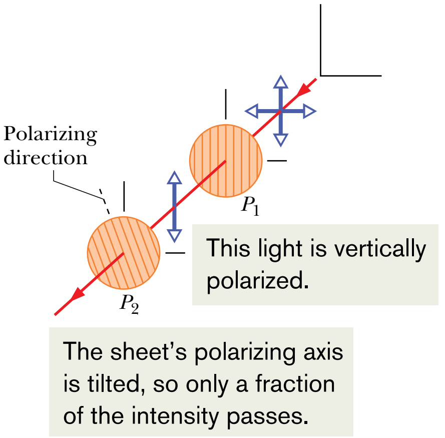
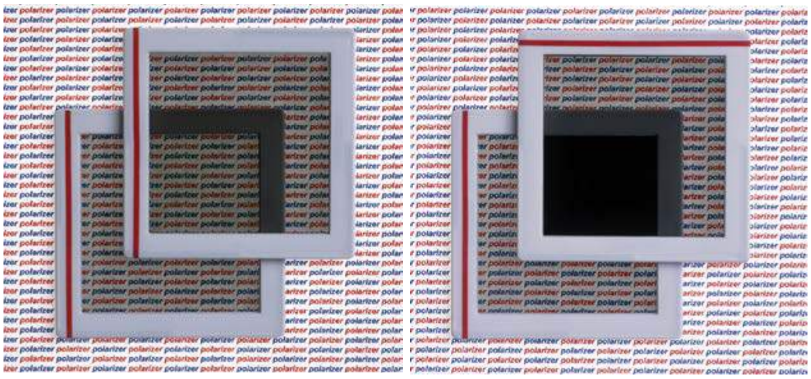

Polarizing direction
P_1
P_2
This light is vertically polarized.
The sheet's polarizing axis is tilted, so only a fraction of the intensity passes.

Figure 32-13 The light transmitted by polarizing sheet P_1 is vertically polarized, as represented by the vertical double arrow. The amount of that light that is then transmitted by polarizing sheet P_2 depends on the angle between the polarization direction of that light and the polarizing direction of P_2 (indicated by the lines drawn in the sheet and by the dashed line).

Other Means. Light can be polarized by means other than polarizing sheets, such as by reflection (discussed in Section 32.9) and by scattering from atoms or molecules. In *scattering*, light that is intercepted by an object, such as a molecule, is sent off in many, perhaps random, directions. An example is the scattering of sunlight by molecules in the atmosphere, which gives the sky its general glow.



Richard Megna/Fundamental Photographs

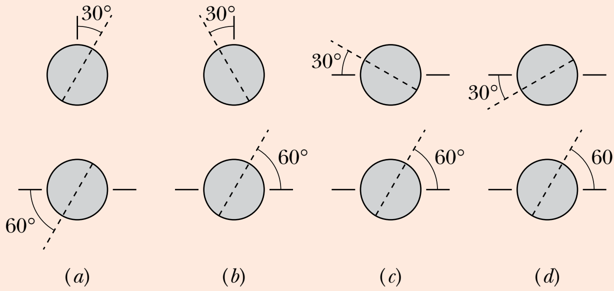
(*a*) (*b*)

Figure 32-14 (*a*) Overlapping polarizing sheets transmit light fairly well when their polarizing directions have the same orientation, but (*b*) they block most of the light when they are crossed.

Although direct sunlight is unpolarized, light from much of the sky is at least partially polarized by such scattering. Bees use the polarization of sky light in navigating to and from their hives. Similarly, the Vikings used it to navigate across the North Sea when the daytime Sun was below the horizon (because of the high latitude of the North Sea). These early seafarers had discovered certain crystals (now called cordierite) that changed color when rotated in polarized light. By looking at the sky through such a crystal while rotating it about their line of sight, they could locate the hidden Sun and thus determine which way was south.

CHECKPOINT 3

The figure shows four pairs of polarizing sheets, seen face-on. Each pair is mounted in the path of initially unpolarized light. The polarizing direction of each sheet (indicated by the dashed line) is referenced to either a horizontal *x* axis or a vertical *y* axis. Rank the pairs according to the fraction of the initial intensity that they pass, greatest first.



SAMPLE PROBLEM 32.02

Polarization and intensity with three polarizing sheets

Figure 32-15*a*, drawn in perspective, shows a system of three polarizing sheets in the path of initially unpolarized light. The polarizing direction of the first sheet is parallel to the *y* axis, that of the second sheet is at an angle of 60° counterclockwise from the *y* axis, and that of the third sheet is parallel to the *x* axis. What fraction of the initial intensity I_0 of the light emerges from the three-sheet system, and in which direction is that emerging light polarized?

KEY IDEAS

1. We work through the system sheet by sheet, from the first one encountered by the light to the last one.
2. To find the intensity transmitted by any sheet, we apply either the one-half rule or the cosine-squared rule, depending on whether the light reaching the sheet is unpolarized or already polarized.
3. The light that is transmitted by a polarizing sheet is always polarized parallel to the polarizing direction of the sheet.

First sheet: The original light wave is represented in Fig. 32-15*b*, using the head-on, double-arrow representation of Fig. 32-10*b*. Because the light is initially unpolarized, the intensity I_1 of the light transmitted by the first sheet is given by the one-half rule (Eq. 32-31):

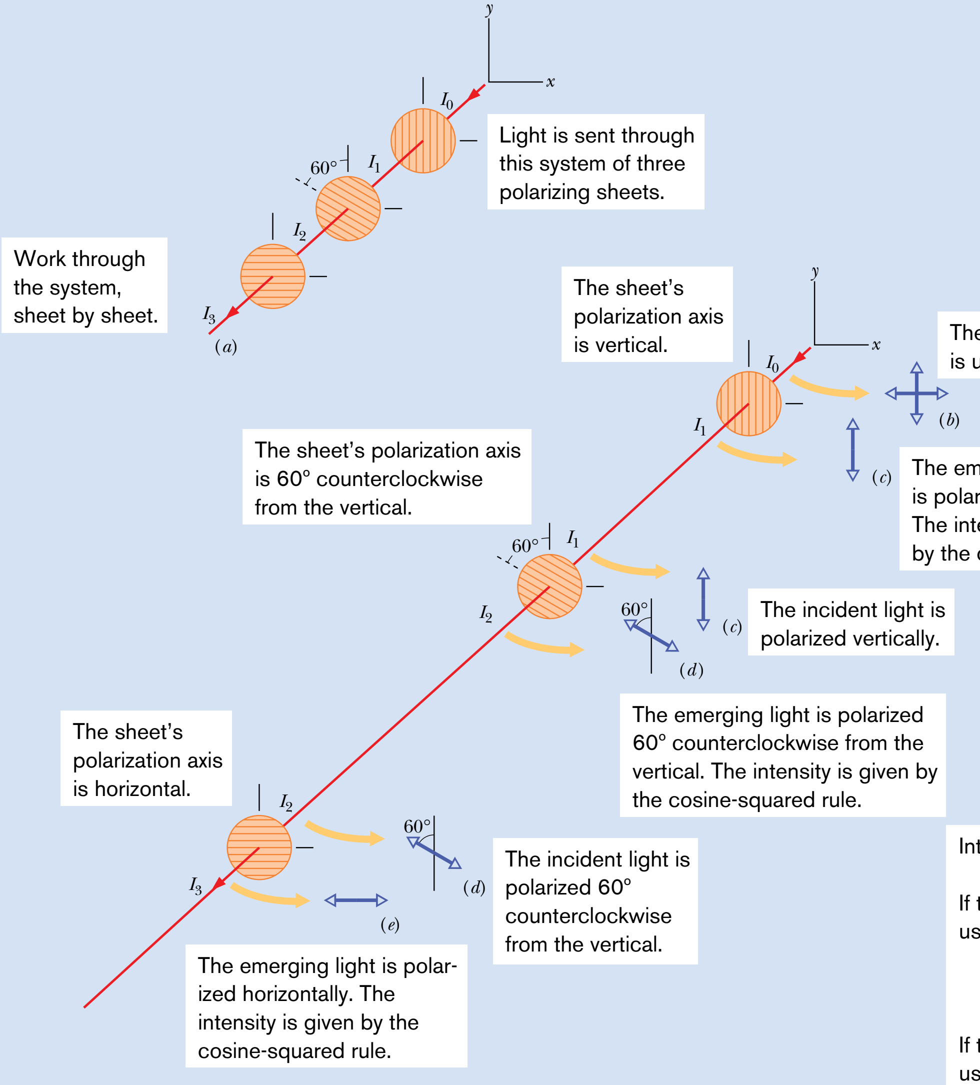
$$I_1 = \frac{1}{2} I_0.$$

Because the polarizing direction of the first sheet is parallel to the *y* axis, the polarization of the light transmitted by it is also, as shown in the head-on view of Fig. 32-15*c*.

Second sheet: Because the light reaching the second sheet is polarized, the intensity I_2 of the light transmitted by that sheet is given by the cosine-squared rule (Eq. 32-33). The angle θ in the rule is the angle between the polarization direction of the entering light (parallel to the *y* axis) and the polarizing direction of the second sheet (60° counterclockwise from the *y* axis), and so θ is 60°. (The larger angle between the two directions, namely 120°, can also be used.) We have

$$I_2 = I_1 \cos^2 60°.$$

The polarization of this transmitted light is parallel to the polarizing direction of the sheet transmitting it—that is, 60° counterclockwise from the *y* axis, as shown in the head-on view of Fig. 32-15*d*.



Light is sent through this system of three polarizing sheets.

Work through the system, sheet by sheet.

The sheet's polarization axis is vertical.

The incident light is unpolarized.

The emerging light is polarized vertically. The intensity is given by the one-half rule.

The sheet's polarization axis is 60° counterclockwise from the vertical.

The incident light is polarized vertically.

The emerging light is polarized 60° counterclockwise from the vertical. The intensity is given by the cosine-squared rule.

The sheet's polarization axis is horizontal.

The incident light is polarized 60° counterclockwise from the vertical.

The emerging light is polarized horizontally. The intensity is given by the cosine-squared rule.

Intensity rules:

If the incident light is unpolarized, use the one-half rule:

$$I_{\text{emerge}} = 0.5 I_{\text{incident}}.$$

If the incident light is already polarized, use the cosine-squared rule:

$$I_{\text{emerge}} = I_{\text{incident}}(\cos\theta)^2,$$

but be sure to insert the angle between the polarization of the incident light and the polarization axis of the sheet.

Figure 32-15 (*a*) Initially unpolarized light of intensity I_0 is sent into a system of three polarizing sheets. The intensities I_1, I_2, and I_3 of the light transmitted by the sheets are labeled. Shown also are the polarizations, from head-on views, of (*b*) the initial light and the light transmitted by (*c*) the first sheet, (*d*) the second sheet, and (*e*) the third sheet.

Third sheet: Because the light reaching the third sheet is polarized, the intensity I_3 of the light transmitted by that sheet is given by the cosine-squared rule. The angle θ is now the angle between the polarization direction of the entering light (Fig. 32-15*d*) and the polarizing direction of the third sheet (parallel to the x axis), and so $\theta = 30°$. Thus,

$$I_3 = I_2 \cos^2 30°.$$

This final transmitted light is polarized parallel to the x axis (Fig. 32-15*e*). We find its intensity by substituting first for I_2 and then for I_1 in the equation above:

$$\begin{aligned} I_3 &= I_2 \cos^2 30° = (I_1 \cos^2 60°)\cos^2 30° \\ &= \left(\frac{1}{2} I_0\right)\cos^2 60° \cos^2 30° = 0.094 I_0. \end{aligned}$$

Thus
$$\frac{I_3}{I_0} = 0.094. \qquad \text{(Answer)}$$

That is to say, 9.4% of the initial intensity emerges from the three-sheet system. (If we now remove the second sheet, what fraction of the initial intensity emerges from the system?)

32.8 | REFLECTION AND REFRACTION

Key Concepts

- Geometrical optics is an approximate treatment of light in which light waves are represented as straight-line rays.
- When a light ray encounters a boundary between two transparent media, a reflected ray and a refracted ray generally appear. Both rays remain in the plane of incidence. The angle of reflection is equal to the angle of incidence, and the angle of refraction is related to the angle of incidence by Snell's law,

$$n_2 \sin \theta_2 = n_1 \sin \theta_1 \quad \text{(refraction)},$$

where n_1 and n_2 are the indexes of refraction of the media in which the incident and refracted rays travel.

Although a light wave spreads as it moves away from its source, we can often approximate its travel as being in a straight line; we did so for the light wave in Fig. 32-5*a*. The study of the properties of light waves under that approximation is called *geometrical optics*. For the rest of this chapter and all of Chapters 33 and 34, we shall discuss the geometrical optics of visible light.

The photograph in Fig. 32-16*a* shows an example of light waves traveling in approximately straight lines. A narrow beam of light (the *incident* beam), angled downward from the left and traveling through air, encounters a *plane* (flat) water surface. Part of the light is **reflected** by the surface, forming a beam directed upward toward the right, traveling as if the original beam had bounced from the surface. The rest of the light travels through the surface and into the water, forming a beam directed downward to the right. Because light can travel through it, the water is said to be *transparent*; that is, we can see through it. (In this chapter we shall consider only transparent materials and not opaque materials, through which light cannot travel.)

The travel of light through a surface (or *interface*) that separates two media is called **refraction**, and the light is said to be *refracted*. Unless an incident beam of light is perpendicular to the surface, refraction changes the light's direction of travel. For this reason, the beam is said to be "bent" by the refraction. Note in Fig. 32-16*a* that the bending occurs only at the surface; within the water, the light travels in a straight line.

Here is a very simple and instructive illustration to show how light refracts. Fold your tablecloth and lay it on the table as shown in Fig. 32-17. Incline the table-top slightly. Then set a couple of wheels on one axle from a broken toy steam engine or some other toy rolling down it. When its path is set at right angles to the tablecloth fold there is no refraction, illustrating the law of refraction, according to which light falling perpendicularly on the boundary between two different media does not bend. However, when its path is set at an angle to the tablecloth fold, the direction of the couple of wheels changes at the boundary.

When passing from that part of the table where velocity is greater (the uncovered part) to that part where velocity is less (the covered part), the direction of the motion of wheels ("the ray") is nearer to the "normal incidence." When

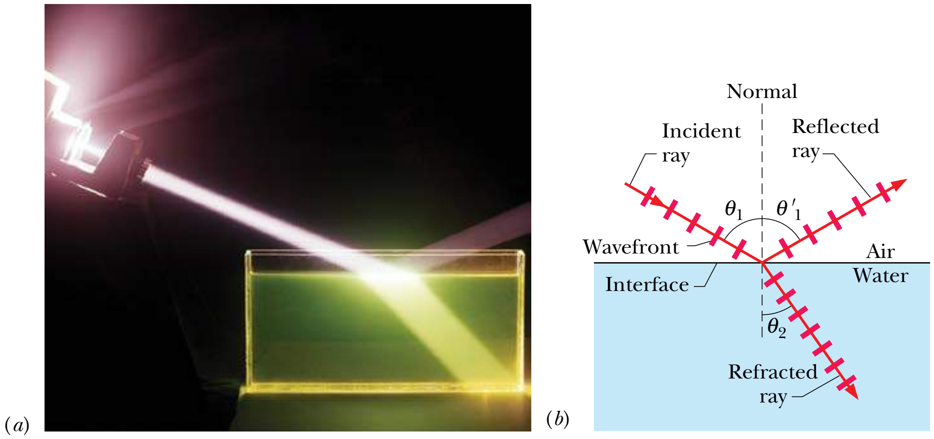


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Figure 32-16 (*a*) A photograph showing an incident beam of light reflected and refracted by a horizontal water surface. (*b*) A ray representation of (*a*). The angles of incidence (θ_1), reflection θ'_1, and refraction (θ_2) are marked.

rolling the other way, the direction is farther away from the normal. This shows that refraction is due to the change in light velocity in the new medium. The greater this change is, the greater is the angle of refraction. The "refractive index," which shows how greatly the direction changes, is nothing but the ratio of the two velocities. If the refractive index in passing from air to water is 4/3, it means that light travels through the air roughly 1.33 times faster than through the water.

In Figure 32-16*b*, the beams of light in the photograph are represented with an *incident ray*, a *reflected ray*, and a *refracted ray* (and wavefronts). Each ray is oriented with respect to a line, called the *normal*, that is perpendicular to the surface at the point of reflection and refraction. In Fig. 32-16*b*, the **angle of incidence** is θ_1, the **angle of reflection** is θ_1', and the **angle of refraction** is θ_2, all measured *relative to the normal*. The plane containing the incident ray and the normal is the *plane of incidence*, which is in the plane of the page in Fig. 32-16*b*.

Experiment shows that reflection and refraction are governed by two laws:

Law of reflection: A reflected ray lies in the plane of incidence and has an angle of reflection equal to the angle of incidence (both relative to the normal). In Fig. 32-16*b*, this means that

$$\theta_1' = \theta_1 \quad \text{(reflection).} \tag{32-34}$$

(We shall now usually drop the prime on the angle of reflection.)

Law of refraction: A refracted ray lies in the plane of incidence and has an angle of refraction θ_2 that is related to the angle of incidence θ_1 by

$$n_2 \sin \theta_2 = n_1 \sin \theta_1 \quad \text{(refraction).} \tag{32-35}$$

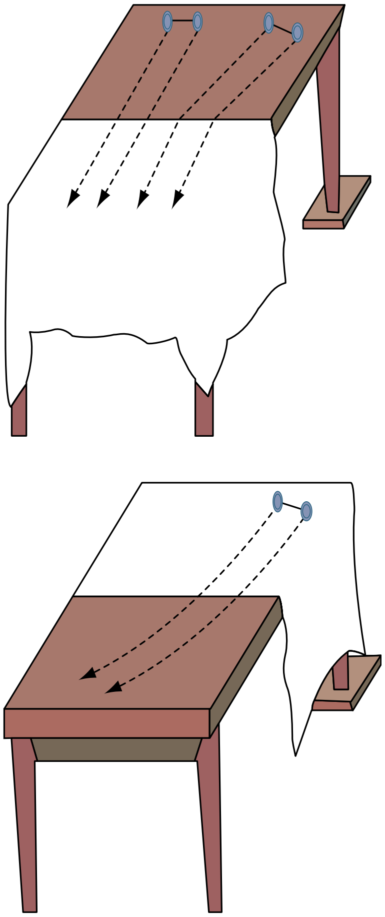


Figure 32-17 Refraction of light explained.

Here each of the symbols n_1 and n_2 is a dimensionless constant, called the **index of refraction**, that is associated with a medium involved in the refraction. We derive this equation, called **Snell's law**, in Chapter 34. As we shall discuss there, the index of refraction of a medium is equal to c/v, where v is the speed of light in that medium and c is its speed in vacuum.

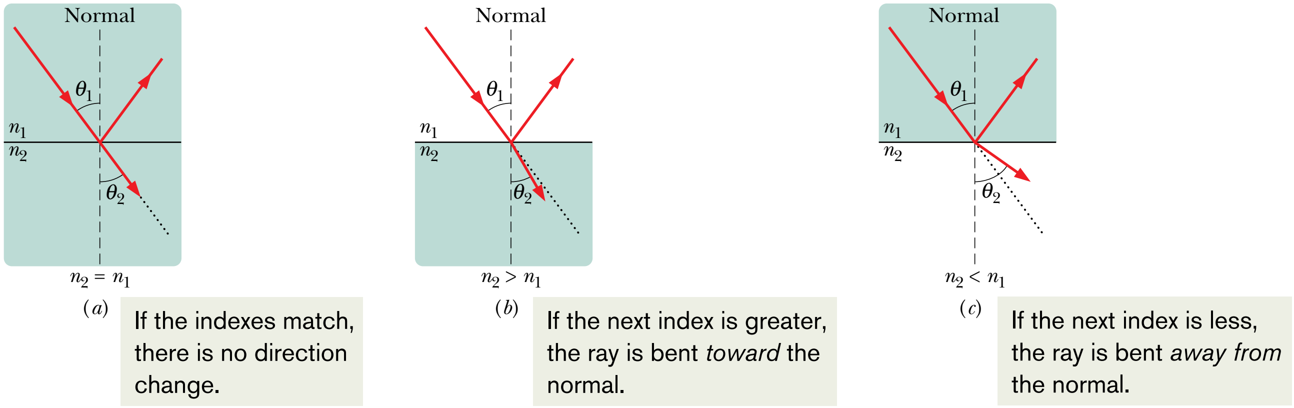
Table 32-1 gives the indexes of refraction of vacuum and some common substances. For vacuum, n is defined to be exactly 1; for air, n is very close to 1.0 (an approximation we shall often make). Nothing has an index of refraction below 1.

Table 32-1 Some Indexes of Refraction[a]

Medium	Index	Medium	Index
Vacuum	Exactly 1	Typical crown glass	1.52
Air (STP)[b]	1.00029	Sodium chloride	1.54
Water (20°C)	1.33	Polystyrene	1.55
Acetone	1.36	Carbon disulfide	1.63
Ethyl alcohol	1.36	Heavy flint glass	1.65
Sugar solution (30%)	1.38	Sapphire	1.77
Fused quartz	1.46	Heaviest flint glass	1.89
Sugar solution (80%)	1.49	Diamond	2.42

[a] For a wavelength of 589 nm (yellow sodium light).

[b] STP mean "standard temperature (0°C) and pressure (1 atm)."



If the indexes match, there is no direction change.

If the next index is greater, the ray is bent *toward* the normal.

If the next index is less, the ray is bent *away from* the normal.

Figure 32-18 Refraction of light traveling from a medium with an index of refraction n_1 into a medium with an index of refraction n_2. (*a*) The beam does not bend when $n_2 = n_1$; the refracted light then travels in the *undeflected direction* (the dotted line), which is the same as the direction of the incident beam. The beam bends (*b*) toward the normal when $n_2 > n_1$ and (*c*) away from the normal when $n_2 < n_1$.

We can rearrange Eq. 32-35 as

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 \tag{32-36}$$

to compare the angle of refraction θ_2 with the angle of incidence θ_1. We can then see that the relative value of θ_2 depends on the relative values of n_2 and n_1:

1. If n_2 is equal to n_1, then θ_2 is equal to θ_1 and refraction does not bend the light beam, which continues in the *undeflected direction*, as in Fig. 32-18*a*.
2. If n_2 is greater than n_1, then θ_2 is less than θ_1. In this case, refraction bends the light beam away from the undeflected direction and toward the normal, as in Fig. 32-18*b*.
3. If n_2 is less than n_1, then θ_2 is greater than θ_1. In this case, refraction bends the light beam away from the undeflected direction and away from the normal, as in Fig. 32-18*c*.

Refraction *cannot* bend a beam so much that the refracted ray is on the same side of the normal as the incident ray.

SAMPLE PROBLEM 32.03

Reflection and refraction of a monochromatic beam

(a) In Fig. 32-19*a*, a beam of monochromatic light reflects and refracts at point A on the interface between material 1 with index of refraction $n_1 = 1.33$ and material 2 with index of refraction $n_2 = 1.77$. The incident beam makes an angle of 50° with the interface. What is the angle of reflection at point A? What is the angle of refraction there?

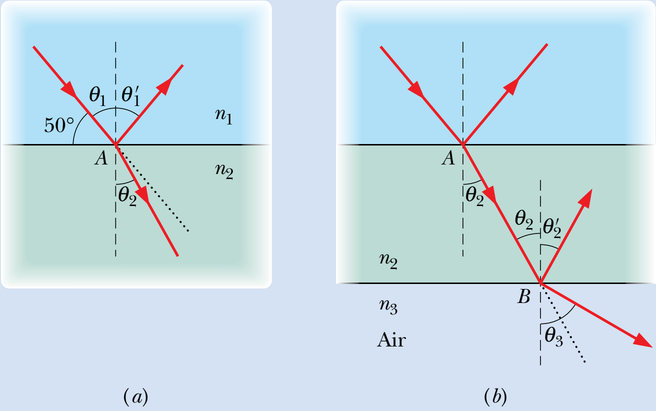
KEY IDEAS

(1) The angle of reflection is equal to the angle of incidence, and both angles are measured relative to the normal to the surface at the point of reflection. (2) When light reaches the interface between two materials with different indexes of refraction (call them n_1 and n_2), part of the light can be refracted by the interface according to Snell's law, Eq. 33-35:

$$n_2 \sin\theta_2 = n_1 \sin\theta_1, \tag{32-37}$$

where both angles are measured relative to the normal at the point of refraction.

Calculations: In Fig. 32-19*a*, the normal at point A is drawn as a dashed line through the point. Note that the angle



(*a*) (*b*)

Figure 32-19 (*a*) Light reflects and refracts at point *A* on the interface between materials 1 and 2. (*b*) The light that passes through material 2 reflects and refracts at point *B* on the interface between materials 2 and 3 (air). Each dashed line is a normal. Each dotted line gives the incident direction of travel.

of incidence θ_1 is not the given 50° but is 90° − 50° = 40°. Thus, the angle of reflection is

$$\theta_1' = \theta_1 = 40°. \qquad \text{(Answer)}$$

The light that passes from material 1 into material 2 undergoes refraction at point *A* on the interface between the two materials. Again we measure angles between light rays and a normal, here at the point of refraction. Thus, in Fig. 32-19*a*, the angle of refraction is the angle marked θ_2. Solving Eq. 32-37 for θ_2 gives us

$$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right) = \sin^{-1}\left(\frac{1.33}{1.77}\sin 40°\right)$$
$$= 28.88° \approx 29°. \qquad \text{(Answer)}$$

This result means that the beam swings toward the normal (it was at 40° to the normal and is now at 29°). The reason is that when the light travels across the interface, it moves into a material with a greater index of refraction.

Caution: Note that the beam does *not* swing through the normal so that it appears on the left side of Fig. 32-19*a*.

(b) The light that enters material 2 at point *A* then reaches point *B* on the interface between material 2 and material 3, which is air, as shown in Fig. 32-19*b*. The interface through *B* is parallel to that through *A*. At *B*, some of the light reflects and the rest enters the air. What is the angle of reflection? What is the angle of refraction into the air?

Calculations: We first need to relate one of the angles at point *B* with a known angle at point *A*. Because the interface through point *B* is parallel to that through point *A*, the incident angle at *B* must be equal to the angle of refraction θ_2, as shown in Fig. 32-19*b*. Then for reflection, we again use the law of reflection. Thus, the angle of reflection at *B* is

$$\theta_2' = \theta_2 = 28.88° \approx 29°. \qquad \text{(Answer)}$$

Next, the light that passes from material 2 into the air undergoes refraction at point *B*, with refraction angle θ_3. Thus, we again apply Snell's law of refraction, but this time we write Eq. 32-35 as

$$n_3 \sin\theta_3 = n_2 \sin\theta_2. \qquad (32\text{-}38)$$

Solving for θ_3 then leads to

$$\theta_3 = \sin^{-1}\left(\frac{n_2}{n_3}\sin\theta_2\right) = \sin^{-1}\left(\frac{1.77}{1.00}\sin 28.88°\right)$$
$$= 58.75° \approx 59°. \qquad \text{(Answer)}$$

Thus, the beam swings away from the normal (it was at 29° to the normal and is now at 59°) because it moves into a material (air) with a lower index of refraction.

Occurrence of Sun Dogs

Sun dogs occur when sunlight passes through atmospheric ice crystals (Fig. 32-20*a*) that have their hexagonal cross sections horizontal. Such a crystal can turn a ray (Fig. 32-20*b*) from its original direction of travel by an angle that depends on the crystal's orientation. However, there is a minimum turning angle θ_{dog} at which the rays bunch up to form an especially bright spot in the sky—a sun dog. The situation is similar to how white wine in a wine glass can bunch up candlelight to form bright lines on a tablecloth. A sunray undergoing such minimum turning follows a path through the crystal that is parallel to one of the hexagonal sides, and its entrance and exit paths are symmetric. The minimum turning angle θ_{dog} is the angle to the left or right of the Sun at which you can see a sun dog.

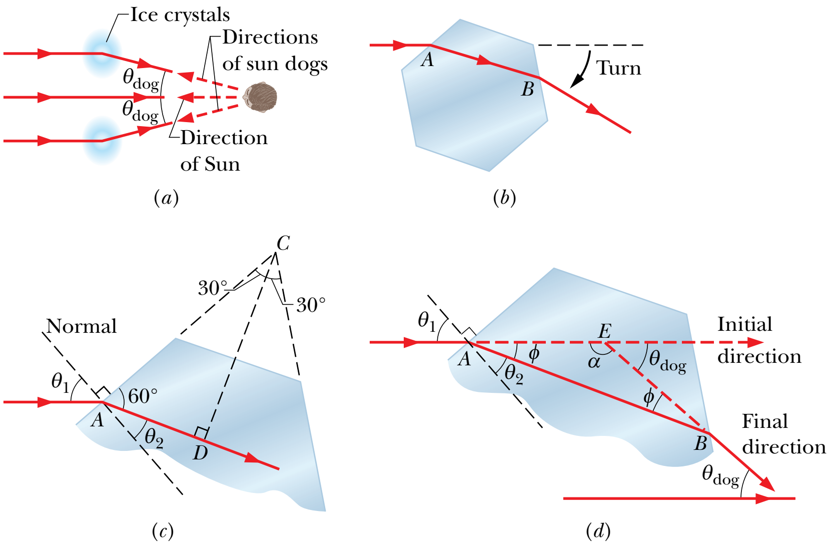


Figure 32-20 (*a*) Overhead view of sunrays redirected by atmospheric ice crystals. (*b*) Overhead view of hexagonal cross section of a crystal. Path of ray (*c*) into and (*d*) out of the cross section.

32.9 | POLARIZATION BY REFLECTION

Key Concept

- A reflected wave will be fully polarized, with its $\vec{E}$ vectors perpendicular to the plane of incidence, if it strikes a boundary at the Brewster angle θ_B, where

$$\theta_B = \tan^{-1}\frac{n_2}{n_1} \quad \text{(Brewster angle).}$$

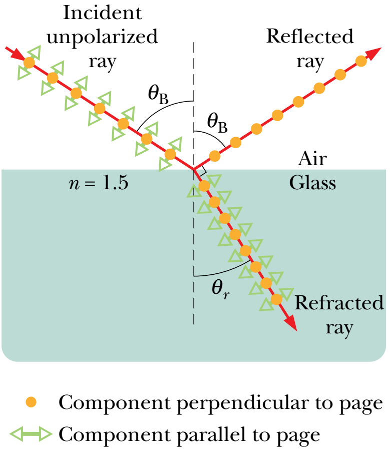


Figure 32-21 A ray of unpolarized light in air is incident on a glass surface at the Brewster angle θ_B. The electric fields along that ray have been resolved into components perpendicular to the page (the plane of incidence, reflection, and refraction) and components parallel to the page. The reflected light consists only of components perpendicular to the page and is thus polarized in that direction. The refracted light consists of the original components parallel to the page and weaker components perpendicular to the page; this light is partially polarized.

You can vary the glare you see in sunlight that has been reflected from, say, water by looking through a polarizing sheet (such as a polarizing sunglass lens) and then rotating the sheet's polarizing axis around your line of sight. You can do so because any light that is reflected from a surface is either fully or partially polarized by the reflection.

Figure 32-21 shows a ray of unpolarized light incident on a glass surface. Let us resolve the electric field vectors of the light into two components. The *perpendicular components* are perpendicular to the plane of incidence and thus also to the page in Fig. 32-21; these components are represented with dots (as if we see the tips of the vectors). The *parallel components* are parallel to the plane of incidence and the page; they are represented with double-headed arrows. Because the light is unpolarized, these two components are of equal magnitude.

In general, the reflected light also has both components but with unequal magnitudes. This means that the reflected light is partially polarized—the electric fields oscillating along one direction have greater amplitudes than those oscillating along other directions. However, when the light is incident at a particular incident angle, called the *Brewster angle* θ_B, the reflected light has only perpendicular components, as shown in Fig. 32-21. The reflected light is then fully polarized perpendicular to the plane of incidence. The parallel components of the incident light do not disappear but (along with perpendicular components) refract into the glass.

Polarizing Sunglasses. Glass, water, and the other dielectric materials discussed in Section 25.5 can partially and fully polarize light by reflection. When you intercept sunlight reflected from such a surface, you see a bright spot (the glare) on the surface where the reflection takes place. If the surface is horizontal as in Fig. 32-21, the reflected light is partially or fully polarized horizontally. To eliminate such glare from horizontal surfaces, the lenses in polarizing sunglasses are mounted with their polarizing direction vertical.

Brewster's Law

For light incident at the Brewster angle θ_B, we find experimentally that the reflected and refracted rays are perpendicular to each other. Because the reflected ray is reflected at the angle θ_B in Fig. 32-21 and the refracted ray is at an angle θ_r, we have

$$\theta_B + \theta_r = 90°. \qquad (32.39)$$

These two angles can also be related with Eq. 32-35. Arbitrarily assigning subscript 1 in Eq. 32-35 to the material through which the incident and reflected rays travel, we have, from that equation,

$$n_1 \sin \theta_B = n_2 \sin \theta_r. \qquad (32\text{-}40)$$

Combining these equations leads to

$$n_1 \sin \theta_B = n_2 \sin (90° - \theta_B) = n_2 \cos \theta_B, \qquad (32\text{-}41)$$

which gives us

$$\theta_B = \tan^{-1} \frac{n_2}{n_1} \quad \text{(Brewster angle)}. \qquad (33\text{-}42)$$

(Note carefully that the subscripts in Eq. 32-42 are *not* arbitrary because of our decision as to their meanings.) If the incident and reflected rays travel *in air*, we can approximate n_1 as unity and let n represent n_2 in order to write Eq. 32-42 as

$$\theta_B = \tan^{-1} n \quad \text{(Brewster's law)}. \qquad (32\text{-}43)$$

This simplified version of Eq. 32-42 is known as **Brewster's law**. Like θ_B, it is named after Sir David Brewster, who found both experimentally in 1812.

Like θ_B, it is named after Sir David Brewster, who found both experimentally in 1812. It is not difficult to understand the physical reason why the light vibrating in the plane of incidence is not reflected at Brewster's angle. The incident light sets the electrons in the atoms of the material into oscillation, and it is the re-radiation from these that generates the reflected beam. When the latter is observed at 90° to the refracted beam, only the vibrations that are perpendicular to the plane of incidence can contribute. Those in the plane of incidence have no component transverse to the 90° direction and hence cannot radiate in that direction.

32.10 | POLARIZATION BY SCATTERING

If a polarizing plate, such as Polaroid, is used to test the blue sky, the light is *partially plane-polarized*. A little exploration will show that maximum polarization occurs at an angle of 90° with the direction of the incoming sunlight. At dusk on a clear day, when the sun has just disappeared over the horizon, one can locate the direction of zero polarization and from it determine the sun's position.

Consider the scattering of light from a single air molecule P, as shown in Fig. 32-22. Suppose ordinary unpolarized light is incident from the left. We assume that it is composed of two plane-polarized components: one in xy plane and another in xz plane (Fig. 32-22). If the incident component is vibrating in the xy plane

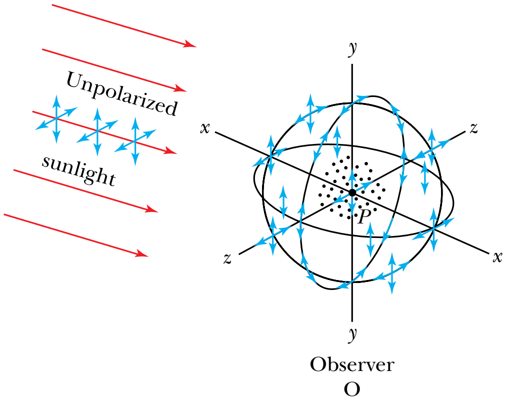


Figure 32-22 The polarization of light by scattering from the particles.

and is absorbed, it sets the particle vibrating in the y direction. In giving up this energy, the same wave can be emitted in any direction except along the y axis. This is because the light waves are transverse, not longitudinal.

Assume that the incident light component is vibrating in the xz plane. The particle at P will be set vibrating along the z axis. Reemission is now allowed in all directions except along the z axis. It can be seen from Fig. 32-22; therefore, an observer at O looking at the blue sky in a direction making 90° with the sun's rays will find the blue light *plane-polarized* with its direction of vibration parallel to the z axis. No particle at P can be set vibrating along the x axis, since this would violate the principle that light has no longitudinal component.

REVIEW AND SUMMARY

Electromagnetic Waves An electromagnetic wave consists of oscillating electric and magnetic fields. The various possible frequencies of electromagnetic form a *spectrum*, a small part of which is visible light. An electromagnetic wave traveling along an x axis has an electric field $\vec{E}$ and a magnetic field $\vec{B}$ with magnitudes that depend on x and t:

$$E = E_m \sin(kx - \omega t)$$

and

$$B = B_m \sin(kx - \omega t), \qquad (32\text{-}1, 32\text{-}2)$$

where E_m and B_m are the amplitudes of $\vec{E}$ and $\vec{B}$. The oscillating electric field induces the magnetic field, and the oscillating magnetic field induces the electric field. The speed of any electromagnetic wave in vacuum is c, which can be written as

$$c = \frac{E}{B} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \qquad (32\text{-}5, 32\text{-}3)$$

where E and B are the simultaneous (but nonzero) magnitudes of the two fields.

Energy Flow The rate per unit area at which energy is transported via an electromagnetic wave is given by the Poynting vector $\vec{S}$:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}. \qquad (32\text{-}21)$$

The direction of $\vec{S}$ (and thus of the wave's travel and the energy transport) is perpendicular to the directions of both $\vec{E}$ and $\vec{B}$. The time-averaged rate per unit area at which energy is transported is S_{avg}, which is called the *intensity* I of the wave:

$$I = \frac{1}{c\mu_0} E_{rms}^2, \qquad (32\text{-}28)$$

in which $E_{rms} = E_m/\sqrt{2}$. A *point source* of electromagnetic waves emits the waves *isotropically*—that is, with equal intensity in all directions. The intensity of the waves at distance r from a point source of power P_s is

$$I = \frac{P_S}{4\pi r^2}. \qquad (32\text{-}29)$$

Polarization Electromagnetic waves are **polarized** if their electric field vectors are all in a single plane, called the *plane of oscillation*. From a head-on view, the field vectors oscillate parallel to a single axis perpendicular to the path taken by the waves. Light waves from common sources are not polarized; that is, they are **unpolarized**, or **polarized randomly**. From a head-on view, the vectors oscillate parallel to every possible axis that is perpendicular to the path taken by the waves.

Polarizing Sheets When a polarizing sheet is placed in the path of light, only electric field components of the light parallel to the sheet's **polarizing direction** are *transmitted* by the sheet; components perpendicular to the polarizing direction are absorbed. The light that emerges from a polarizing sheet is polarized parallel to the polarizing direction of the sheet.

If the original light is initially unpolarized, the transmitted intensity I is half the original intensity I_0:

$$I = \frac{1}{2} I_0. \qquad (32\text{-}31)$$

If the original light is initially polarized, the transmitted intensity depends on the angle θ between the polarization direction of the original light (the axis along which the fields oscillate) and the polarizing direction of the sheet:

$$I = I_0 \cos^2 \theta. \qquad (32\text{-}33)$$

Geometrical Optics *Geometrical optics* is an approximate treatment of light in which light waves are represented as straightline rays.

Reflection and Refraction When a light ray encounters a boundary between two transparent media, a **reflected** ray and a **refracted** ray generally appear. Both rays remain in the plane of incidence. The **angle of reflection** is equal to the angle of incidence, and the **angle of refraction** is related to the angle of incidence by Snell's law,

$$n_2 \sin \theta_2 = n_1 \sin \theta_1 \quad \text{(refraction)}, \qquad (32\text{-}35)$$

where n_1 and n_2 are the indexes of refraction of the media in which the incident and refracted rays travel.

Polarization by Reflection A reflected wave will be fully **polarized**, with its $\vec{E}$ vectors perpendicular to the plane of incidence, if the incident, unpolarized wave strikes a boundary at the **Brewster angle** θ_B, where

$$\theta_B = \tan^{-1} \frac{n_2}{n_1} \quad \text{(Brewster angle)}. \qquad (32\text{-}42)$$

PROBLEMS

1. A beam of polarized light is sent into a system of two polarizing sheets. Relative to the polarization direction of that incident light, the polarizing directions of the sheets are at angles θ for the first sheet and 90° for the second sheet. (a) If 0.20 of the incident intensity is transmitted by the two sheets, what is θ? (b) What percentage of the incident intensity is transmitted if the first sheet angle is reduced to 0°?

2. In Fig. 32-23, initially unpolarized light is sent into a system of three polarizing sheets whose polarizing directions make angles of $\theta_1 = \theta_2 = \theta_3 = 50°$ with the direction of the y axis. What percentage of the initial intensity is transmitted by the system? (*Hint:* Be careful with the angles.)

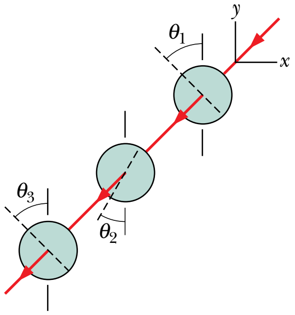


Figure 32-23 Problems 2 and 3.

3. In Fig. 32-23, initially unpolarized light is sent into a system of three polarizing sheets whose polarizing directions make angles of $\theta_1 = 40°$, $\theta_2 = 20°$, and $\theta_3 = 40°$ with the direction of the y axis. What percentage of the light's initial intensity is transmitted by the system? (*Hint:* Be careful with the angles.)

4. In Fig. 32-24, a beam of unpolarized light, with intensity 43 W/m², is sent into a system of two polarizing sheets with polarizing directions at angles $\theta_1 = 70°$ and $\theta_2 = 90°$ to the y axis. What is the intensity of the light transmitted by the system?

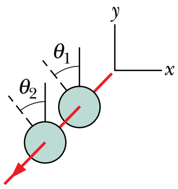


Figure 32-24 Problems 4 and 5

5. In Fig. 32-24, a beam of light, with intensity 43 W/m² and polarization parallel to a y axis, is sent into a system of two polarizing sheets with polarizing directions at angles of $\theta_1 = 70°$ and $\theta_2 = 90°$ to the y axis. (a) What is the intensity of the light transmitted by the two-sheet system? (b) What is the transmitted intensity if, instead, the initial polarization is parallel to the x axis?

6. In Fig. 32-25, unpolarized light is sent into a system of three polarizing sheets. The angles θ_1, θ_2, and θ_3 of the polarizing directions are measured counterclockwise from the positive direction of the y axis (they are not drawn to scale). Angles θ_1 and θ_3 are fixed, but angle θ_2 can be varied. Figure 32-26 gives the intensity of the light emerging from sheet 3 as a function of θ_2. (The scale of the intensity axis is not indicated.) What percentage of the light's initial intensity is transmitted by the system when $\theta_2 = 30°$?

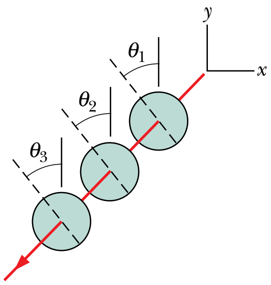


Figure 32-25 Problems 6 and 9.

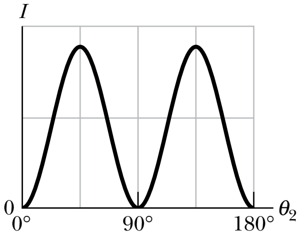


Figure 32-26 Problem 6.

7. In Fig. 32-27, unpolarized light is sent into a system of three polarizing sheets. The angles θ_1, θ_2, and θ_3 of the polarizing directions are measured counterclockwise from the positive direction of the y axis (they are not drawn to scale). Angles θ_1 and θ_3 are fixed, but angle θ_2 can be varied. Figure 32-27 gives the intensity of the light emerging from sheet 3 as a function of θ_2. (The scale of the intensity axis is not indicated.) What percentage of the light's initial intensity is transmitted by the three-sheet system when $\theta_2 = 90°$?

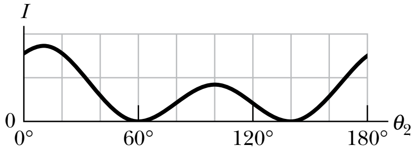


Figure 32-27 Problem 7.

8. We want to rotate the direction of polarization of a beam of polarized light through 90° by sending the beam through one or more polarizing sheets. (a) What is the minimum number of sheets required? (b) What is the minimum number of sheets required if the transmitted intensity is to be more than 65% of the original intensity?

9. In Fig. 32-25, unpolarized light is sent into a system of three polarizing sheets, which transmits 0.0500 of the initial light intensity. The polarizing directions of the first and third sheets are at angles $\theta_1 = 0°$ and $\theta_3 = 90°$. What are the (a) smaller and (b) larger possible values of angle θ_2 ($< 90°$) for the polarizing direction of sheet 2?

10. At a beach the light is generally partially polarized due to reflections off sand and water. At a particular beach on a particular day near sundown, the horizontal component of the electric field vector is 2.3 times the vertical component. A standing sunbather puts on polarizing sunglasses; the glasses eliminate the horizontal field component. (a) What fraction of the light intensity received before the glasses were put on now reaches the sunbather's eyes? (b) The sunbather, still wearing the glasses, lies on his side. What fraction of the light intensity received before the glasses were put on now reaches his eyes?

11. A beam of partially polarized light can be considered to be a mixture of polarized and unpolarized light. Suppose we send such a beam through a polarizing filter and then rotate the filter through 360° while keeping it perpendicular to the beam. If the transmitted intensity varies by a factor of 4.0 during the rotation, what fraction of the intensity of the original beam is associated with the beam's polarized light?

12. In Fig. 32-28, a 2.50 m long vertical pole extends from the bottom of a swimming pool to a point 50.0 cm above the water. Sunlight is incident at angle $\theta = 55.0°$. What is the length of the shadow of the pole on the level bottom of the pool?

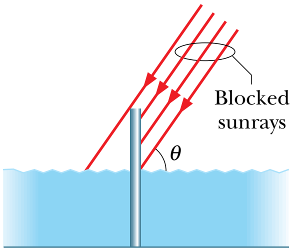


Figure 32-28 Problem 12.

13. Light in vacuum is incident on the surface of a glass slab. In the vacuum the beam makes an angle of 32.0° with the normal to the surface, while in the glass it makes an angle of 16.0° with the normal. What is the index of refraction of the glass?

14. An isotropic point source emits light at wavelength 500 nm, at the rate of 200 W. A light detector is positioned 400 m from the source. What is the maximum rate $\partial B/\partial t$ at which the magnetic component of the light changes with time at the detector's location?

15. The maximum electric field 27 m from an isotropic point source of light is 17 V/m. What are (a) the maximum value of the magnetic field and (b) the average intensity of the light there? (c) What is the power of the source?

16. In a plane radio wave the maximum value of the electric field component is 5.00 V/m. Calculate (a) the maximum value of the magnetic field component and (b) the wave intensity.

17. A plane electromagnetic wave, with wavelength 5.0 m, travels in vacuum in the positive direction of an x axis. The electric field, of amplitude 215 V/m, oscillates parallel to the y axis. What are the (a) frequency, (b) angular frequency, and (c) angular wave number of the wave? (d) What is the amplitude of the magnetic field component? (e) Parallel to which axis does the magnetic field oscillate? (f) What is the time-averaged rate of energy flow in watts per square meter associated with this wave? The wave uniformly illuminates a surface of area 2.0 m^2. If the surface totally absorbs the wave, what are (g) the rate at which momentum is transferred to the surface and (h) the radiation pressure on the surface?

18. A black, totally absorbing piece of cardboard of area A = 2.0 cm^2 intercepts light with an intensity of 10 W/m^2 from a camera strobe light. What radiation pressure is produced on the cardboard by the light?

19. What is the intensity of a traveling plane electromagnetic wave if B_m is 3.0×10^{-4} T?

20. From Fig. 32-2, approximate the (a) smaller and (b) larger wavelength at which the eye of a standard observer has half the eye's maximum sensitivity. What are the (c) wavelength, (d) frequency, and (e) period of the light at which the eye is the most sensitive?

21. Sunlight just outside Earth's atmosphere has an intensity of 1.40 kW/m^2. Calculate (a) E_m, (b) B_m, (c) E_{rms}, and (d) B_{rms} for sunlight there, assuming it to be a plane wave.

22. Assume (unrealistically) that a TV station acts as a point source broadcasting isotropically at 1.0 MW. What is the intensity of the transmitted signal reaching Proxima Centauri, the star nearest our solar system, 4.3 ly away? A light-year (ly) is the distance light travels in one year.

23. A plane electromagnetic wave has a maximum electric field magnitude of 3.20×10^{-4} V/m. Find the magnetic field amplitude.

24. What inductance must be connected to a 25 pF capacitor in an oscillator capable of generating 410 nm (i.e., visible) electromagnetic waves? Comment on your answer.

25. What is the radiation pressure 2.7 m away from a 315 W lightbulb? Assume that the surface on which the pressure is exerted faces the bulb and is perfectly absorbing and that the bulb radiates uniformly in all directions.

26. The intensity I of light from an isotropic point source is determined as a function of distance r from the source. Figure 32-29 gives intensity I versus the inverse square r^{-2} of that distance. The vertical axis scale is set by $I_s = 200$ W/m^2, and the horizontal axis scale is set by $r_s^{-2} = 8.0$ m^{-2}. What is the power of the source?

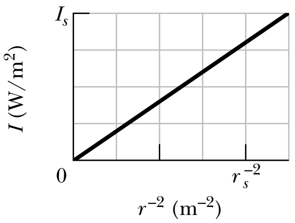
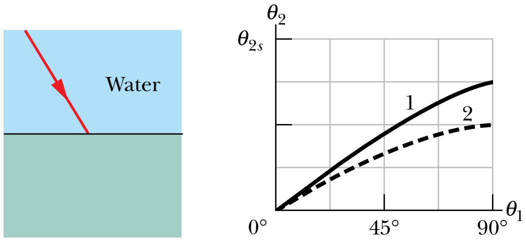


Figure 32-29 Problem 26.

27. An airplane flying at a distance of 10 km from a radio transmitter receives a signal of intensity 28 μW/m^2. What is the amplitude of the (a) electric and (b) magnetic component of the signal at the airplane? (c) If the transmitter radiates uniformly over a hemisphere, what is the transmission power?

28. In Fig. 32-30*a*, a light ray in water is incident at angle θ_1 on a boundary with an underlying material, into which some of the light refracts. There are two choices of underlying material. For each, the angle of refraction θ_2 versus the incident angle θ_1 is given in Fig. 32-30*b*. The vertical axis scale is set by $\theta_{2s} = 90°$. Without calculation, determine whether the index of refraction of (a) material 1 and (b) material 2 is greater or less than the index of water ($n = 1.33$). What is the index of refraction of (c) material 1 and (d) material 2?



(*a*) (*b*)

Figure 32-30 Problem 28.

29. A certain helium–neon laser emits red light in a narrow band of wavelengths centered at 632.8 nm and with a "wavelength width" (such as on the scale of Fig. 32-1) of 5.00 pm. What is the corresponding "frequency width" for the emission?

30. In Fig. 32-31, a light ray in air is incident on a flat layer of material 2 that has an index of refraction $n_2 = 1.5$. Beneath material 2 is material 3 with an index of refraction n_3. The ray is incident on the air–material 2 interface at the Brewster angle for that interface. The ray of light refracted into material 3 happens to be incident on the material 2 – material 3 interface at the Brewster angle for that interface. What is the value of n_3?

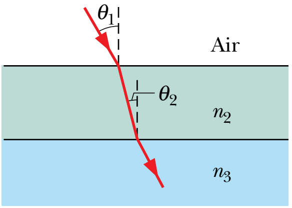


Figure 32-31 Problem 30.

31. A plane electromagnetic wave traveling in the positive direction of an x axis in vacuum has components $E_x = E_y = 0$ and $E_z = (4.0\ \text{V/m})\cos[(\pi \times 10^{15}\ \text{s}^{-1})(t - x/c)]$. (a) What is the amplitude of the magnetic field component? (b) Parallel to which axis does the magnetic field oscillate? (c) When the electric field component is in the positive direction of the z axis at a certain point P, what is the direction of the magnetic field component there? (d) In what direction is the wave moving?

32. What is the wavelength of the electromagnetic wave emitted by the oscillator–antenna system of Fig. 32-3 if $L = 0.253\ \mu\text{H}$ and $C = 25.0$ pF?

33. Unpolarized light of intensity 6.5 mW/m^2 is sent into a polarizing sheet as in Fig. 32-11. What are (a) the amplitude of the electric field component of the transmitted light and (b) the radiation pressure on the sheet due to its absorbing some of the light?

34. Some neodymium–glass lasers can provide 100 TW of power in 2.2 ns pulses at a wavelength of 0.26 mm. How much energy is contained in a single pulse?

35. (a) At what angle of incidence will the light reflected from water be completely polarized? (b) Does this angle depend on the wavelength of the light?

PRACTICE QUESTIONS

Single Correct Choice Type

1. In a plane electromagnetic wave, the electric field oscillates sinusoidally at a frequency of 2.5×10^{10} Hz and amplitude 480 V/m. The amplitude of the oscillating magnetic field will be

(a) 1.52×10^{-8} Wb/m^2 **(b)** 1.52×10^{-7} Wb/m^2
(c) 1.6×10^{-6} Wb/m^2 **(d)** 1.6×10^{-7} Wb/m^2

2. In a plane electromagnetic wave, the electric field oscillates sinusoidally at a frequency of 2.0×10^{10} Hz and amplitude 48 V/m. The wavelength of the wave is

(a) 1.5 m **(b)** 1.5×10^{-1} m
(c) 1.5×10^{-2} m **(d)** 1.5×10^{-3} m

3. The frequency of electromagnetic wave which is best suited to observe a particle of radius 3×10^{-4} cm is of the order of

(a) 10^{15} **(b)** 10^{14}
(c) 10^{13} **(d)** 10^{12}

4. An electric field E and magnetic field B exist in a region. If these fields are not perpendicular to each other, then the electromagnetic wave

(a) Will not pass through the region
(b) Will pass through region
(c) May pass through the region
(d) Nothing is definite

5. The sunlight strikes the upper atmosphere of the Earth with intensity 1.38 kW/m^2. The peak value of electric field at that point will be (in kilovolt/meter)

(a) 2.04 **(b)** 4.08
(c) 8.16 **(d)** 1.02

6. If ε_0 and μ_0 represent the permittivity and permeability of vacuum and ε and μ represent the permittivity and permeability of medium, then the refractive index of the medium is given by

(a) $\sqrt{\dfrac{\varepsilon_0\mu_0}{\varepsilon\mu}}$ **(b)** $\sqrt{\dfrac{\varepsilon\mu}{\varepsilon_0\mu_0}}$

(c) $\sqrt{\dfrac{\varepsilon}{\varepsilon_0\mu_0}}$ **(d)** $\sqrt{\dfrac{\varepsilon_0\mu_0}{\varepsilon}}$

7. Maxwell's modified form of Ampere's circuital law is

(a) $\oint \vec{B}\cdot d\vec{s} = 0$

(b) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$

(c) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I + \dfrac{1}{\varepsilon_0}\dfrac{dq}{dt}$

(d) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0\dfrac{d\phi_E}{dt}$

8. If the velocity of light in free space is 3×10^8 m/s, then value of wavelength of electromagnetic wave of frequency 2×10^6 Hz will be

(a) 6×10^8 m **(b)** 15×10^4 m

(c) 150 m **(d)** 1.5×10^{11} m

9. An electromagnetic wave, going through vacuum, is described by $E = E_0\sin(kx - \omega t)$. Which of the following is/are independent of the wavelength?

(a) k **(b)** k/ω

(c) ω^2 **(d)** k/ω^2

10. If E is an electric field and B is the magnetic induction, then the energy flow per unit area per unit time in an electromagnetic field is given by

(a) $E\times B$ **(b)** $E - B$

(c) $E^2 + B^2$ **(d)** E/B

11. Instantaneous displacement current of 1.0 A in the space between the parallel plates of 1 μF capacitor can be established by changing potential difference of

(a) 10^{-6} V/s **(b)** 10^6 V/s

(c) 10^{-8} V/s **(d)** 10^8 V/s

12. A beam of light of intensity 12 W/cm^2 in incident on a totally reflecting plane mirror of area 1.5 cm^2. The force in Newton's acting on the mirror will be

(a) 2.4×10^{-6} **(b)** 1.2×10^{-7}

(c) 3.6×10^{-8} **(d)** 5.6×10^{-5}

13. A plane electromagnetic wave of wave intensity 6 W/m^2 strikes a small mirror of area 39 cm^2, held perpendicular to the approaching wave. The momentum transferred in kg·m/s by the wave to the mirror each second will be

(a) 1.6×10^{-10} **(b)** 2.4×10^{-9}

(c) 3.6×10^{-8} **(d)** 4.8×10^{-7}

14. The average energy–density of an electromagnetic wave given by $E = (50\text{ N/C})\sin(\omega t - kx)$ will be nearly

(a) 10^{-8} J/m^3 **(b)** 10^{-7} J/m^3

(c) 10^{-6} J/m^3 **(d)** 10^{-5} J/m^3

15. The magnetic field in the plane electromagnetic wave is given by $B_z = 2\times10^{-7}\sin(0.5\times10^3 x + 1.5\times10^{11} t)$ T. The expression for electric field will be

(a) $E_z = 30\sqrt{2}\sin(0.5\times10^3 x + 1.5\times10^{11} t)$ V/m

(b) $E_z = 60\sin(0.5\times10^3 x + 1.5\times10^{11} t)$ V/m

(c) $E_y = 30\sqrt{2}\sin(0.5\times10^{11} x + 0.5\times10^3 t)$ V/m

(d) $E_y = 60\sin(0.5\times10^3 x + 1.5\times10^{11} t)$ V/m

16. Select the correct statement:

(a) Ultraviolet light has a longer wavelength than infrared

(b) Blue light has a higher frequency than x rays

(c) Radio waves have higher frequency than gamma rays

(d) Gamma rays have higher frequency than infrared waves

17. The order of increasing wavelength for blue (b), green (g), red (r), and yellow (y) light is

(a) r, y, g, b **(b)** r, g, y, b

(c) g, y, b, r **(d)** b, g, y, r

18. Of the following, human eyes are most sensitive to

(a) Red light

(b) Violet light

(c) Blue light

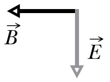
(d) Green light

19. If the wavelength of an electromagnetic wave is about the diameter of an apple, what type of radiation is it?

(a) X ray **(b)** UV

(c) Infrared **(d)** Microwave

20. The given figure shows the electric and magnetic fields of an electromagnetic wave at a certain instant. The wave is traveling



(a) Into the page **(b)** Out of it

(c) In the east direction **(d)** In the west direction

21. Which of the following types of electromagnetic radiation travels at the greatest speed in vacuum?

(a) Radio waves

(b) Visible light

(c) X rays

(d) All of these travel at the same speed

22. An electromagnetic wave is generated by

(a) Any moving charge

(b) Any accelerating charge

(c) Only a charge with changing acceleration

(d) Only a charge moving in a circle

23. Which of these statements correctly describes the orientation of the electric field $(\vec{E})$, the magnetic field $(\vec{B})$, and the velocity of propagation $(\vec{v})$ of an electromagnetic wave?

(a) $(\vec{E})$ is perpendicular to $(\vec{B})$; $(\vec{v})$ may have any orientation relative to $(\vec{E})$

(b) $(\vec{E})$ is perpendicular to $(\vec{B})$; $(\vec{v})$ may have any orientation perpendicular to $(\vec{E})$

(c) $(\vec{E})$ is perpendicular to $(\vec{B})$; $(\vec{B})$ is parallel to $(\vec{v})$

(d) Each of the three vectors is perpendicular to the other two

24. An electromagnetic wave going through vacuum is described by $E = E_0\sin(kx - \omega t)$; $B = B_0\sin(kx - \omega t)$. Then,

(a) $E_0 k = B_0\omega$ **(b)** $E_0 B_0 = \omega k$

(c) $E_0\omega = B_0 k$ **(d)** None of these

25. A free electron is placed in the path of a plane electromagnetic wave. The electron will start moving

(a) Along the electric field

(b) Along the magnetic field

(c) Along the direction of propagation of the wave

(d) In a plane containing the magnetic field and the direction of propagation

26. A beam of light traveling along x axis is described by the electric field as

$$E_x = (600\ \text{V/m})\ \sin\ \omega\,(t - x/c).$$

Then maximum magnetic force on a charge $q = 2e$, moving along y axis with a speed of 3.0×10^7 m/s is ($e = 1.6 \times 10^{-19}$ C):

(a) 19.2×10^{-17} N

(b) 1.92×10^{-17} N

(c) 0.192 N

(d) None of these

27. A plane electromagnetic wave travels in free space along x axis. At a particular point in space, the electric field along y axis is 9.3 V/m. The magnetic induction is

(a) 3.1×10^{-8} T　**(b)** 3×10^{-5} T

(c) 3×10^{-6} T　**(d)** 9.3×10^{-6} T

28. A plane electromagnetic wave traveling along the x direction has a wavelength of 3 mm. The variation in the electric field occurs in the y direction with amplitude 66 V/m. The equation for the electric and magnetic fields as a function of x and t, respectively, is

(a) $E_y = 33\cos\pi \times 10^{11}\left(t - \dfrac{x}{c}\right);$

$B_z = 1.1 \times 10^{-7}\cos\pi \times 10^{11}\left(t - \dfrac{x}{c}\right)$

(b) $E_y = 11\cos 2\pi \times 10^{11}\left(t - \dfrac{x}{c}\right);$

$B_x = 11 \times 10^{-7}\cos 2\pi \times 10^{11}\left(t - \dfrac{x}{c}\right)$

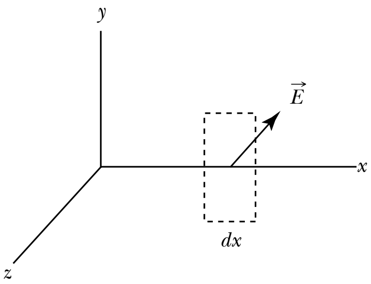
(c) $E_z = 33\cos\pi \times 10^{11}\left(t - \dfrac{x}{c}\right);$

$B_x = 11 \times 10^{-7}\cos\pi \times 10^{11}\left(t - \dfrac{x}{c}\right)$

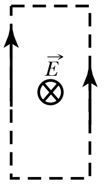
(d) $E_y = 66\cos 2\pi \times 10^{11}\left(t - \dfrac{x}{c}\right);$

$B_z = 2.2 \times 10^{-7}\cos 2\pi \times 10^{11}\left(t - \dfrac{x}{c}\right)$

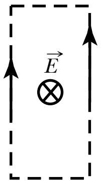
29. A plane electromagnetic wave is traveling in the positive x direction. At the instant shown in the following figure the electric field at the extremely narrow dashed rectangle is in the negative z direction and its magnitude is decreasing. Which diagram correctly shows the direction and relative magnitudes of the magnetic field at the edges of the rectangle?



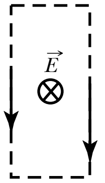
(a)



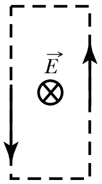
(b)



(c)



(d)



30. An electromagnetic wave is traveling in the positive x direction with its electric filed along the z axis and its magnetic field along the y axis. The fields are related by

(a) $\dfrac{\partial E}{\partial x} = \dfrac{\mu_0\varepsilon_0\partial B}{\partial x}$

(b) $\dfrac{\partial E}{\partial x} = \dfrac{\mu_0\varepsilon_0\partial B}{\partial t}$

(c) $\dfrac{\partial B}{\partial x} = \dfrac{\mu_0\varepsilon_0\partial E}{\partial t}$

(d) $\dfrac{\partial B}{\partial x} = -\dfrac{\mu_0\varepsilon_0\partial E}{\partial t}$

31. An electromagnetic wave of frequency $n = 3.0$ MHz passes from vacuum into a dielectric medium with permittivity $e = 4.0$. Then,

(a) Wavelength is doubled and frequency unchanged

(b) Wavelength is doubled and frequency becomes half

(c) Wavelength is halved and frequency remains unchanged

(d) Wavelength and frequency both remains unchanged

32. A plane electromagnetic wave, $E_z = 100\cos(6 \times 10^8\, t + 4x)$ V/m, propagating in a medium of dielectric constant is

(a) 1.5　**(b)** 2.0

(c) 2.4　**(d)** 4.0

33. The dimensions of $S = (1/\mu_0)\,\vec{E} \times \vec{B}$ are

(a) J/m^2　**(b)** J/s

(c) W/s　**(d)** J/m^3

34. The time-averaged energy in a sinusoidal electromagnetic wave is

(a) Overwhelmingly electrical

(b) Slightly more electrical than magnetic

(c) Equally divided between the electric and magnetic fields

(d) Overwhelmingly magnetic

35. An electromagnetic wave is transposing energy in the negative y direction. At one point and one instant the magnetic field is in the positive x direction. The electric field at that point and instant is

(a) Positive y direction　**(b)** Negative y direction

(c) Positive z direction　**(d)** Negative z direction

36. The light intensity 10 m from a point source is 1000 W/m^2. The intensity 100 m from the same source is

(a) 1000 W/m^2　**(b)** 100 W/m^2

(c) 10 W/m^2　**(d)** 1 W/m^2

37. The amplitude of electric field in a parallel light beam of intensity 4 W/m^2 is

(a) 35.5 N/C **(b)** 45.5 N/C
(c) 49.5 N/C **(d)** 55.5 N/C

38. Consider the following two statements regarding a linearly polarized, plane electromagnetic wave:

(A) The electric field and magnetic field have equal average values.
(B) The electric energy and the magnetic energy have equal average values.

(a) Both A and B are true
(b) A is false but B is true
(c) B is false but A is true
(d) Both A and B are false

39. The rms value of the electric field of the light coming from the Sun is 720 N/C. The average total energy density of the electromagnetic wave is

(a) 4.58×10^{-6} J/m^3 **(b)** 6.37×10^{-9} J/m^3
(c) 81.35×10^{-12} J/m^3 **(d)** 3.3×10^{-3} J/m^3

40. Evidence that electromagnetic waves carry momentum when

(a) The tail of a comet points away from the Sun
(b) Electron flow through a wire generates heat
(c) A charged particle in a magnetic field moves in a circular orbit
(d) Heat can be generated by rubbing two sticks together

41. Light of uniform intensity shines perpendicularly on a totally absorbing surface, fully illuminating the surface. If the area of the surface is decreased

(a) The radiation pressure increases and the radiation force increases
(b) The radiation pressure increases and the radiation force decreases
(c) The radiation pressure stays the same and the radiation force increases
(d) The radiation pressure stays the same and the radiation force decreases

42. A linearly polarized transverse wave is propagating in z direction through a fixed-point P in space. At time t_0, the x component E_x and the y component E_y of the displacement at P are 3 and 4 units, respectively. At a later time t_1, if E_x at P is 2 units, the value of E_y will be

(a) 5 units **(b)** 8/3 units
(c) 3/8 units **(d)** 1/3 units

43. A vertical automobile radio antenna is sensitive to electric fields that are polarized

(a) Horizontally
(b) In circles around the antenna
(c) Vertically
(d) Normal to the antenna in the forward direction

44. Polarized light

(a) Is a different form of light
(b) Can show interference pattern
(c) Travels in any medium with a velocity slightly more than that for unpolarized light
(d) Has direction of vibration restricted in some way

45. For linearly polarized light the plane of polarization is

(a) Perpendicular to both the direction of polarization and the direction of propagation
(b) Perpendicular to the direction of polarization and parallel to the direction of propagation
(c) Parallel to the direction of polarization and perpendicular to the direction of propagation
(d) Parallel to both the direction of polarization and the direction of propagation

46. An unpolarized beam of light has intensity I_0. It is incident on to ideal polarizing sheets. The angle between the axes of polarization of these sheets is θ. Find θ if the emerging light has intensity $I_0/4$

(a) $\sin^{-1}(1/2)$ **(b)** $\sin^{-1}(1/\sqrt{5})$
(c) $\cos^{-1}(1/2)$ **(d)** $\cos^{-1}(1/\sqrt{2})$

47. Three polarizing sheets are placed in a stack with the polarizing directions of the first and third perpendicular to each other. What angle should the polarizing direction of the middle sheet make with the polarizing direction of the first sheet to obtain maximum transmitted intensity when unpolarized light is incident on the stack?

(a) 0 **(b)** 30°
(c) 45° **(d)** 60°

48. A light bulb shines light along the x axis and through two parallel ideal polarizing filters, one with a fixed polarizing axis and the other with an axis that rotates about the x axis and in the yz plane. Looking toward the light bulb through the combined filtering system, you see

(a) A bulb almost disappearing twice per revolution and reaching a maximum intensity twice per revolution. The maximum brightness is not as bright as looking at an unfiltered bulb
(b) As in (a) but once per revolution
(c) As in (a) but with a maximum brightness as bright as an unfiltered bulb
(d) As in (a) but with a bulb that disappears completely

49. Two polaroids are kept crossed to each other. Now one of them is rotated through an angle of 45°. The percentage of incident light now transmitted through the system is

(a) 15% **(b)** 25%
(c) 50% **(d)** 60%

50. A beam of light is linearly polarized. You wish to rotate its direction of polarization by 90° using one or more ideal polarizing sheets. To get maximum transmitted intensity, how many sheets should you use?

(a) 1
(b) 2
(c) 3
(d) There is no way to rotate the direction of polarization 90° using polarizing sheets

51. Electromagnetic waves transverse in nature are evident by

(a) Polarization **(b)** Interference
(c) Reflection **(d)** Diffraction

52. Linear polarized light is incident at Brewster angle on the surface of a medium. If the incident beam is polarized parallel to the plane of incidence, then the parallel component of light is

(a) Completely refracted **(b)** Partly refracted
(c) Completely reflected **(d)** Completely absorbed

53. When light is incident from air to glass at an angle 57°, the reflected beam is completely polarized. If the same beam is incident from water to glass, the angle of incidence at which reflected beam is completely polarized will be
(a) $\theta = 57°$ **(b)** $\theta > 57°$
(c) $\theta < 57°$ **(d)** Cannot be determined

54. Note the different types of electromagnetic radiation:
(1) X-rays
(2) radio waves
(3) gamma rays
(4) visible light
(5) infrared radiation
(6) ultraviolet radiation

Which list correctly ranks the electromagnetic waves in order of *increasing* frequency?
(a) 2, 3, 4, 5, 6, 1
(b) 2, 5, 4, 6, 1, 3
(c) 2, 5, 4, 1, 6, 3
(d) 3, 1, 6, 4, 5, 2

55. When a radio telescope observes a region of space between two stars, it detects electromagnetic radiation that has a wavelength of 0.21 m. This radiation was emitted by hydrogen atoms in the gas and dust located in that region. What is the frequency of this radiation?
(a) 7.1×10^{10} Hz **(b)** 2.1×10^{14} Hz
(c) 1.4×10^{9} Hz **(d)** 6.9×10^{11} Hz

56. An FM radio station generates radio waves that have a frequency of 95.5 MHz. The frequency of the waves from a competing station have a frequency of 102.7 MHz. What is the difference in wavelength between the waves emitted from the two stations?
(a) 0.22 m **(b)** 0.84 m
(c) 0.45 m **(d)** 2.4 m

57. A radio wave sent from the surface of the Earth reflects from the surface of the moon and returns to the Earth. The elapsed time between the generation of the wave and the detection of the reflected wave is 2.6444 s. Determine the distance from the surface of the Earth to the surface of the Moon.

Note: The speed of light is 2.9979×10^8 m/s.
(a) 3.7688×10^8 m
(b) 3.9638×10^8 m
(c) 3.8445×10^8 m
(d) 4.0551×10^8 m

58. The average distance between the surface of the Earth and the surface of the Sun is 1.49×10^{11} m. How much time, in minutes, does it take for light leaving the surface of the Sun to reach the Earth?
(a) zero minutes **(b)** 8.3 min
(c) 500 min **(d)** 74 min

59. Which one of the following statements concerning the energy carried by an electromagnetic wave is true?
(a) The energy is carried only by the electric field.
(b) More energy is carried by the electric field than by the magnetic field.
(c) The energy is carried equally by the electric and magnetic fields.
(d) More energy is carried by the magnetic field than by the electric field.

60. The amplitude of the electric field component of an electromagnetic wave is increased from E to $4E$. What is the corresponding change in the intensity of the wave?
(a) The intensity is unchanged by the increase in E.
(b) The intensity increases by a factor of sixteen.
(c) The intensity increases by a factor of four.
(d) The intensity decreases by a factor of four.

61. The peak value of the electric field component of an electromagnetic wave is E. At a particular instant, the intensity of the wave is of 0.020 W/m^2. If the electric field were increased to $5E$, what would be the intensity of the wave?
(a) 0.020 W/m^2 **(b)** 0.25 W/m^2
(c) 0.10 W/m^2 **(d)** 0.50 W/m^2

62. An electromagnetic wave has an electric field with peak value 250 N/C. What is the average intensity of the wave?
(a) 0.66 W/m^2 **(b)** 83 W/m^2
(c) 0.89 W/m^2 **(d)** 120 W/m^2

63. A laser uniformly illuminates an area with green light that has an average intensity of 550 W/m^2. What is the rms value of the electric field of this light?
(a) 322 N/C **(b)** 455 N/C
(c) 405 N/C **(d)** 643 N/C

64. Electromagnetic waves are radiated uniformly in all directions from a source. The rms electric field of the waves is measured 35 km from the source to have an rms value of 0.42 N/C. Determine the average total power radiated by the source.
(a) 4.1×10^5 W
(b) 3.0×10^6 W
(c) 8.3×10^5 W
(d) 7.2×10^6 W

More than One Correct Choice Type

65. In a plane electromagnetic wave, which of the following have zero average value?
(a) Magnetic field **(b)** Magnetic energy
(c) Electric field **(d)** Electric energy

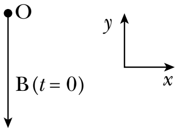
66. Which of the following statements are correct with respect to the electromagnetic waves in an isotropic medium?
(a) Energy due to electric field is equal to that due to magnetic field.
(b) Electric vector E and magnetic vector B are in phase.
(c) For a given amplitude of E vector, the intensity increases as the first power of frequency f.
(d) For cylindrical wave fronts, the amplitude of the wave varies in proportion to $1/r^2$, where r is the radius of the wave front.

67. Which of the following pairs of space and time-varying E and B fields generate a plane electromagnetic wave traveling along z-direction?
(a) E_x, B_y **(b)** E_y, B_x
(c) E_x, B_z **(d)** E_z, B_x

68. Which of the following are the properties of a plane polarized light?
- **(a)** Electric field vectors lie in one plane
- **(b)** Magnetic field vectors lie in one plane
- **(c)** Both and vectors lie in the same plane
- **(d)** Vector is in the direction of propagation

Linked Comprehension

Paragraph for Questions 69–72: The electromagnetic wave of the radiation field from a wire antenna travels toward the plane of the paper (which is in the $-z$ direction). At time $t = 0$ s, the wave strikes the paper at normal incidence. The magnetic field vector at point O in the figure points in the $-y$ direction and has a magnitude of 4.0×10^{-8} T. The frequency of the wave is 1.0×10^{16} Hz.



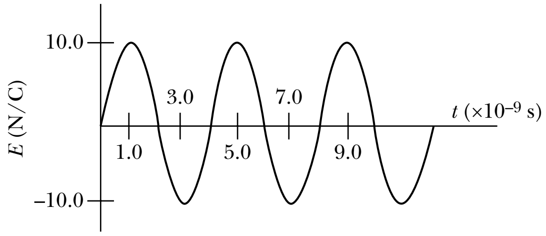
69. What is the magnitude of the associated electric field E at time $t = 0$ s?
- **(a)** 7.5×10^{-16} N/C
- **(d)** 7.5×10^{15} N/C
- **(c)** 1.3 N/C
- **(d)** 12 N/C

70. What is the direction of the electric field?
- **(a)** It points in the negative y direction
- **(b)** It points in the negative x direction
- **(c)** It points in the positive y direction
- **(d)** It points in the positive x direction

71. What is the direction of polarization of this electromagnetic wave?
- **(a)** the x direction
- **(b)** 45° with respect to the x direction
- **(c)** the y direction
- **(d)** the z direction

72. What is the wavelength of this electromagnetic wave?
- **(a)** 0.33 nm
- **(b)** 20 nm
- **(c)** 3.3 nm
- **(d)** 30 nm

Paragraph for Questions 73–76: The figure shows the time variation of the magnitude of the electric field of an electromagnetic wave produced by a wire antenna.



73. Determine the *rms* value of the electric field magnitude.
- **(a)** 7.1 N/C
- **(b)** 14 N/C
- **(c)** 12 N/C
- **(d)** 19 N/C

74. What is the peak value of the magnetic field?
- **(a)** 1.4×10^{-8} T
- **(b)** 3.3×10^{-8} T
- **(c)** 2.3×10^{-8} T
- **(d)** 4.6×10^{-8} T

75. What is the *rms* value of the magnitude of the magnetic field?
- **(a)** 1.4×10^{-8} T
- **(b)** 3.3×10^{-8} T
- **(c)** 2.4×10^{-8} T
- **(d)** 4.6×10^{-8} T

76. Determine the frequency of the wave.
- **(a)** 1.0×10^{9} Hz
- **(b)** 2.5×10^{8} Hz
- **(c)** 1.3×10^{8} Hz
- **(d)** 3.8×10^{8} Hz

Paragraph for Questions 77 and 78: Electromagnetic radiation is emitted by accelerating charges. The rate at which the energy is emitted from an accelerating charge that has charge q and acceleration a is given by $dE/dt = q^2a^2/6\pi\varepsilon_0 c^3$, where c is the speed of light. A proton and an electron of kinetic energy 6 MeV is traveling in a particle accelerator in a circular orbit of radius 0.75 m.

77. What fraction of its energy does a proton emit per second?
- **(a)** 1.1×10^{-11} s^{-1}
- **(b)** 2.1×10^{-11} s^{-1}
- **(c)** 3×10^{-11} s^{-1}
- **(d)** 1.4×10^{-11} s^{-1}

78. What fraction of its energy does the proton emit per second?
- **(a)** 2.8×10^{-8} s^{-1}
- **(b)** 3.2×10^{-8} s^{-1}
- **(c)** 6×10^{-11} s^{-1}
- **(d)** None of these

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

79.

Column I	Column II
(a) $\oint \vec{E}\cdot d\vec{s} = -\dfrac{d\Phi_B}{dt}$	**(p)** Maxwell's law of induction
(b) $\oint \vec{B}\cdot d\vec{s} = \mu_0\varepsilon_0\dfrac{d\Phi_E}{dt}$	**(q)** Ampere-Maxwell law
(c) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$	**(r)** Ampere's Law
(d) $\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0\dfrac{d\Phi_E}{dt} + \mu_0 I_{enc}$	**(s)** Faraday's law of indcution

80.

Column I	Column II
(a) All electromagnetic waves including visible light	**(p)** microwaves
(b) Electromagnetic waves whose frequency is in the range 500 kHz to 1000 MHz	**(q)** have the same speed c, in vacuum
(c) The force exerted by an electromagnetic wave on unit area of surface	**(r)** radio waves
(d) The shortest wavelength radio waves	**(s)** radiation pressure

Directions for Questions 81 and 82: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

81. In the given table, Column I shows different laws in electromagnetic theory, Column II shows the related physical quantities and Column III shows the equations related to the different laws.

Column I	Column II	Column III
(I) Gauss's law	**(i)** Magnetism	**(J)** $\oint \vec{B}\cdot d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 \frac{dq}{dt}$
(II) Ampere's law	**(ii)** Current	**(K)** $\oint \vec{B}\cdot d\vec{A} = 0$
(III) Gauss's law	**(iii)** Induction	**(L)** $\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$
(IV) Faraday's law	**(iv)** Electricity	**(M)** $\oint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$

(1) In which law does the charge create diverging electric fields?

(a) (I) (iv) (M)
(b) (IV) (ii) (M)
(c) (II) (i) (M)
(d) (III) (iii) (M)

(2) In which law there are no magnetic monopoles involved?

(a) (I) (ii) (J)
(b) (III) (i) (K)
(c) (II) (iii) (L)
(d) (I) (i) (M)

(3) In which law currents or changing electric fields(E's) make circulating magnetic fields (B's)?

(a) (II) (ii) (J)
(b) (I) (i) (L)
(c) (IV) (i) (L)
(d) (II) (iii) (M)

82. There are various ways to produce the electromagnetic radiations. In the given table, Column I shows the process to form electromagnetic radiation, Column II shows the range of wavelength of the generated electromagnetic radiation and Column III shows the range of frequency of the generated electromagnetic radiation.

Column I	Column II	Column III
(I) Excitation and ejection of core atomic electrons	**(i)** Wavelengths in the range of 400–700 nanometers (nm), or 4.00×10^{-7} to 7.00×10^{-7} m	**(J)** Frequencies of about 8×10^{14} to 3×10^{16} cycles per second, or hertz (Hz)
(II) Excitation of molecular and atomic valence electrons	**(ii)** Wavelength from 100 nm to 400 nm	**(K)** Frequency range of roughly 430–750 terahertz (THz)
(III) Molecular vibration, plasma oscillation	**(iii)** Wavelength ranging from 0.01 to 10 nanometers	**(L)** Frequency range of approximately 430 THz down to 300 GHz
(IV) Molecular electron excitation	**(iv)** Wavelengths up to 1050 nm	**(M)** Frequencies in the range 30 petahertz to 30 exahertz

(1) What are the characteristics of ultraviolet range of light?

(a) (I) (iv) (M)
(b) (IV) (ii) (L)
(c) (II) (i) (K)
(d) (II) (ii) (J)

(2) What are the characteristics of infrared range of light?

(a) (I) (ii) (J)
(b) (III) (iv) (L)
(c) (II) (iii) (L)
(d) (I) (i) (M)

(3) What are the characteristics of visible range of light?

(a) (II) (ii) (J)
(b) (IV) (i) (K)
(c) (IV) (i) (L)
(d) (II) (iii) (M)

Integer Type

83. The velocity of electromagnetic waves in free space is 3×10^8 m/s. What is the frequency (in MHz) of the wavelength of value 150 m?

84. A parallel capacitor made of circular plates each of radius 10.0 cm has a capacitance of 200 pF. The capacitor is connected to 200 V AC supply with an angular frequency of 200 rad/s. What is the rms value of conduction current (in μA)?

85. A radiation of energy E falls normally on a perfectly reflecting surface. What is the momentum transferred to the surface is kE/c. The value of k is?

ANSWER KEY

Checkpoints

1. (a) Use Eq. 32-5. On the right side of rectangle, $\vec{E}$ is in the negative y direction; on the left side, $\vec{E} + d\vec{E}$ is greater and in the same direction; (b) $\vec{E}$ is downward. On the right side, $\vec{B}$ is in negative z-direction; on the left side, $\vec{B} + d\vec{B}$ is greater and in the same direction.

2. positive direction of x.

3. a, b, c, d (zero)

Problems

1. (a) $\theta = 32°$ or $58°$; (b) the transmitted intensity is zero
2. 0.045%
3. 3.1%
4. $19\ \text{W/m}^2$
5. (a) $4.4\ \text{W/m}^2$; (b) $34\ \text{W/m}^2$
6. 9.4%
7. 7.3%
8. (a) 2 sheets; (b) 6 sheets
9. (a) 19.6°; (b) 70.4°
10. (a) 0.16; (b) 0.84
11. 0.60
12. 1.07 m
13. 1.92
14. 3.44×10^6 T/s
15. (a) 57 nT; (b) $0.38\ \text{W/m}^2$; (c) 3.5 kW
16. (a) 1.67×10^{-8} T; (b) $3.31 \times 10^{-2}\ \text{W/m}^2$
17. (a) 60 MHz; (b) 3.8×10^8 rad/s; (c) $1.3\ \text{m}^{-1}$; (d) $0.72\ \mu\text{T}$; (e) z; (f) $61\ \text{W/m}^2$; (g) $0.41\ \mu\text{N}$; (h) 0.20 mPa
18. 3.31×10^{-8} Pa
19. $1.1 \times 10^7\ \text{W/m}^2 \approx 11\ \text{MW/m}^2$
20. (a) 515 nm; (b) 610 nm; (c) 555 nm; (d) 5.41×10^{14} Hz; (e) 1.85×10^{-15} s
21. (a) 1.03 kV/m; (b) $3.43\ \mu\text{T}$; (c) 726 V/m; (d) $2.42\ \mu\text{T}$
22. $4.8 \times 10^{-29}\ \text{W/m}^2$
23. 1.07×10^{-12} T
24. 1.9×10^{-21} H
25. 1.1×10^{-8} Pa
26. $4\pi(30) \approx 2.5 \times 10^2$ W
27. (a) 0.15 V/m; (b) 0.48 nT; (c) 18 kW
28. (a) $n_2 > n_1$; (b) the value of n_2 for material 2 is also greater than that of water (n_1); (c) $n_2 = 1.4$ for material 1; (d) $n_2 = 1.9$
29. $3.75 \times 10^9\ \text{Hz} \approx 3.75$ GHz
30. $n_3 = 1.0$
31. (a) 13 nT; (b) y; (c) negative direction of y; (d) positive direction of x
32. 4.74 m
33. (a) 1.6 V/m; (b) 1.1×10^{-11} Pa
34. $2.2 \times 10^5\ \text{J} = 0.22$ MJ
35. (a) 53.1°; (b) yes

Practice Questions

Single Correct Choice Type

1. (c)
2. (c)
3. (b)
4. (c)
5. (d)
6. (b)
7. (d)
8. (c)
9. (d)
10. (a)
11. (b)
12. (b)
13. (a)
14. (a)
15. (d)
16. (d)
17. (d)
18. (a)
19. (d)
20. (d)
21. (d)
22. (b)
23. (d)
24. (a)
25. (a)
26. (b)
27. (a)
28. (d)
29. (d)
30. (d)
31. (c)
32. (d)
33. (d)
34. (c)
35. (d)
36. (c)
37. (d)
38. (a)
39. (a)
40. (d)
41. (d)
42. (b)
43. (c)
44. (d)
45. (d)
46. (d)
47. (c)
48. (a)
49. (b)
50. (d)
51. (a)
52. (d)
53. (c)
54. (b)
55. (c)
56. (a)
57. (b)
58. (b)
59. (c)
60. (b)
61. (d)
62. (b)
63. (b)
64. (d)

More than One Correct Choice Type

65. (a), (c)
66. (a), (b)
67. (a), (b)
68. (a), (b), (d)

Linked Comprehension

69. (d)
70. (d)
71. (a)
72. (d)
73. (a)
74. (b)
75. (c)
76. (b)
77. (d)
78. (a)

Matrix-Match

79. (a) → (s); (b) → (p); (c) → (r); (d) → (q)
80. (a) → (q); (b) → (r); (c) → (s); (d) → (p)
81. (1) → (a); (2) → (b); (3) → (a)
82. (1) → (d); (2) → (b); (3) → (b)

Integer Type

83. 2
84. 8
85. 2

CHAPTER

33

Geometrical Optics: Reflection

33.1 | WHAT IS PHYSICS?

One goal of physics is to discover the basic laws governing light, such as the law of reflection. The application of the reflection is ubiquitous. Each one of us uses mirrors everyday, whether it is for dressing, for shaving, or so on. Every vehicle is fitted with a mirror, with the text "Objects in the mirror are closer than they appear." Apart from daily uses, mirrors are used widely in varying applications such as lasers, dentistry, and so on. In this chapter we are going to study the basics of image formation by the mirrors.

Our first step in this chapter is to define and classify images. Then we examine several basic ways in which they can be produced. In Chapter 32, Section 32.8, we laid down the basic laws of reflection and its laws. We will discuss those laws in this chapter.

33.2 | PRINCIPLE OF RECTILINEAR PROPAGATION OF LIGHT

Key Concept

- The light ray travels in a straight line in a homogeneous isotropic medium and changes direction abruptly on encountering a transparent (glass) or polished (mirror) surface, to travel along a new straight line.

There are many principles on which this chapter is based. One of them is **law of rectilinear propagation of light**. The light ray travels in a straight line in a homogenous isotropic medium. When a ray of light encounters the surface of a mirror, or a transparent boundary such as a glass, the ray makes an abrupt change in direction, after which it travels along a new straight line path. This motion of light ray produces shadows, images, and so on.

To understand this business of geometrical optics more clearly, let us try to understand how we see a candle in a room.

From every part of the candle, a ray is coming, rather an infinite number of rays are coming (Fig. 33-1). However, only a small fraction of these enters our eye. Our brain can deduce from this single ray that the flame lies somewhere

Contents

33.1 What is Physics?
33.2 Principle of Rectilinear Propagation of Light
33.3 Laws of Reflection
33.4 Formation of Image by a Mirror
33.5 Formation of Images by Spherical Mirrors

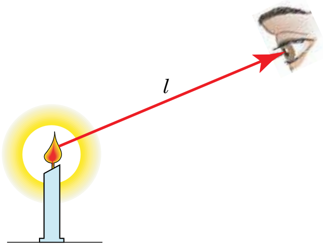


Figure 33-1 Seeing a candle light with a single eye.

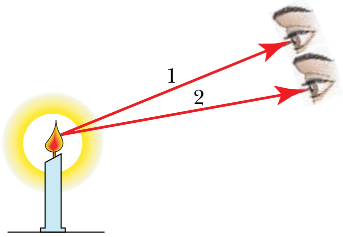


Figure 33-2 Seeing a candle light with a single eye.

on this line. Since we have two eyes, similarly the second eye will see another ray coming from the same point of the flame. The second eye will deduce that the flame lies somewhere on this second line. But the flame has to be at one point only. Therefore, we can say that this flame should be at the intersection of these two lines (Fig. 33-2).

The entire field of geometrical optics deals with tracing back the rays to see their origin. This will obviously involve geometrical considerations. Now you can see why we call this topic as ray optics or geometrical optics.

A very important and useful concept in this context is the **parallactic displacement** or simply **parallax**. The word may sound a little frightening, but as a matter of fact the parallax is a very simple as well as useful thing.

We may start our acquaintance with parallax by trying to put a thread into a needle's eye (Fig. 33-3). Try to do it with one eye closed, and you will find very quickly that it does not work; you will be bringing the end of the thread either too far behind the needle or stopping short in front of it. With only one eye, you are unable to judge the distance between the needle and the thread very clearly. (By the act of focusing, the eye gets only a vague idea of the distance.) However, with two eyes open, you can do it very easily, or at least learn easily how to do it. When you look at the object with two eyes, you automatically focus them both on the object. The closer the object, the more you have to turn your eyes toward each other, and the muscular feeling arising from such adjustment gives you a pretty good idea about the distance.

Now, if instead of looking with both eyes, you first close one and then the other, you will notice that the position of the object (the needle in this case) relative to the distant background (say, the background picture) has changed. This effect is known as parallactic displacement and is certainly familiar to everybody; if you have never heard about it, just try it out or look at Fig. 33-3 showing the needle and background picture as seen by the right eye and the left eye. The farther away the object, the smaller will be its parallactic displacement. Therefore, we can use it for measuring distances. Since parallactic displacement can be measured exactly in the degrees of the arc, this method is more precise than a simple judgment of the distance based on the muscular feeling in the eyeballs. But as the two eyes are set in our head, only about three inches apart, they are not good for the estimate of distances beyond a few feet. If two objects are placed such that their parallactic displacement is same, then we say that they have no parallax.

Formation of Shadow

For understanding the concept of rectilinear propagation of light, let us consider the formation of shadow. We should understand that the shadow is a region and not an area.

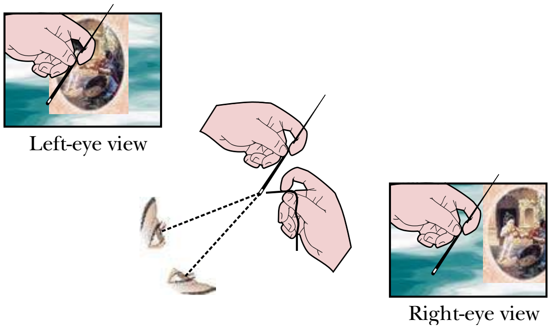


Figure 33-3 Parallactic displacement of the object. Notice how the left eye sees a different background for the needle than the right eye. The brain superimposes these two images to get a clear idea of the distance.

SAMPLE PROBLEM 33.01

Shadow formation

Assume that Sun rises at 6:00 A.M. and sets at 6:00 P.M. Find the angle made by the sunrays with the horizontal at 10:00 A.M. Also find the length of the shadow of a man 6 ft high at this time.

KEY IDEA

We can assume that the sun rays are almost parallel to each other. A part of these rays will be blocked by a man. This blocked region will be the shadow (Fig. 33-4). As the time passes, the Sun revolves around the Earth. We know that the sunrays are horizontal at 6:00 A.M. and at 6:00 P.M. At 10:00 A.M. the Sun would have gone up in the sky.

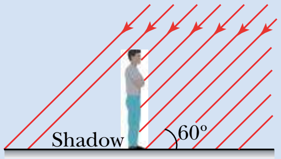


Figure 33-4 The shadow of a man at 10:00 A.M.

Calculation: As the Sun covers 180° in sky in 12 hours from 6:00 A.M. to 12:00 P.M., it covers 15° in 1 hour. (Actually, it is the Earth that is rotating, but the effect is same as seen from the Earth.) So, till 10:00 A.M. the Sun has covered

$$15° \times 4 = 60°.$$

Therefore, the length of the man's shadow will be

$$L = 6 \cot 60° = 2\sqrt{3} \text{ ft.} \quad \text{(Answer)}$$

Learn: Now we can see how the length of the shadow will change as the day progresses. It will be the minimum at the midday. Since the Earth is a sphere, at most of the points on the Earth, even at 12:00 A.M., the shadow length is not zero on any day of the year. Here, we have assumed that the man is standing at such a point that the Sun is at its zenith at noon. Incidentally, this was the way that Eratosthenes first found the radius of the Earth. Can you guess how?

It is easy to explain why sharp shadows are cast by opaque obstacles if there is a point source of light. If there are two sources of light, the shadow cast by an opaque obstacle placed in the field of light of one source, S_1, is illuminated in certain regions by the light from the other source, S_2. Some portions of space will receive no light at all, while still others will be illuminated by both sources. This is shown in Fig. 33-5, where the black portion, which receives no light at all, is called the "umbra." The shaded portion that receives light from only one source is called the "penumbra."

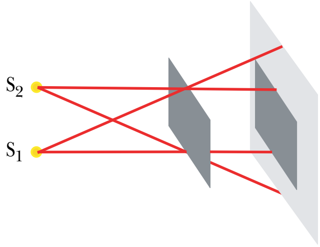


Figure 33-5 If two sources are used to form the shadow; some regions are completely dark while some are partially dark.

☑ CHECKPOINT 1

Keep a small cardboard between the wall and the bulb in your room. Observe the shadow pattern formed for different positions of the cardboard. Explain why and when is the shadow very sharp.

33.3 | LAWS OF REFLECTION

Key Concepts

- The two laws of reflection state that:
  1. The angle of incidence is equal to angle of reflection.
  2. The incident ray, the normal ray and the reflected ray lie in the same plane.
- Laws of reflection in vector form:
  1. Incident ray vector

$$\hat{e}_i = \sin\theta\,\hat{t} - \cos\theta\,\hat{n}.$$

  2. Reflected ray vector

$$\hat{e}_r = \sin\theta\,\hat{t} + \cos\theta\,\hat{n}.$$

  3. Therefore, $\hat{e}_r = \hat{e}_i - 2(\hat{e}_i \cdot \hat{n})\hat{n}$.

A **mirror** is a surface that can reflect a parallel beam of light in one direction instead of either scattering it widely in many directions or absorbing it. A shiny metal surface acts as a mirror whereas a concrete wall does not. In this section, we examine the images that a **plane mirror** (a flat reflecting surface) can produce.

First, let us briefly recollect the laws of reflection we introduced in Section 32.8.

The laws of reflection state that

1. The angle of incidence (i) equals the angle of reflection (r) (Fig. 33-6).
2. Incident ray, the normal ray, and the reflected ray lie in the same plane.

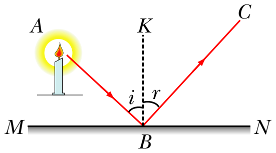


Figure 33-6 The angle of reflection r is equal to the angle of incidence i.

Fermat's Principle of Stationary Time (Optional)

The Fermat's principle of stationary time states that the light ray, while going from one point to another, follows a path which takes either minimum time or maximum time or constant time. This law forms the theoretical basis for geometrical optics. Even the law of rectilinear propagation of light can be deduced from it. We know that the smallest time will be taken by the light in going from A to B in the same medium, if it follows a straight line. Let us deduce the laws of reflection from this principle.

The second law of reflection is easily seen from this principle. If we draw a set of rays going from A to C after reflection from a mirror as shown in Fig. 33-7, the set of rays in the plane of paper will constitute a smaller path and hence time required will be less. If the point of reflection is out of the plane of paper, both incident and reflected rays will have to travel correspondingly longer distance and hence that path is not acceptable.

For proving the first part of the law, let us assume a point F behind the mirror at the same perpendicular distance from the mirror as A. The ray coming from A can be incident on the mirror at any arbitrary point D. It can be easily seen that the time taken by ray in going from A to D will be the same as the time taken by a hypothetical ray in going from F to D. So, all we need to do is choose a point D such that the time taken by the ray in going from C to F is minimum. It is clear that this will be when D is on the straight line joining C and F. Thus, the point of reflection for the ray reaching C is B. A little geometrical insight will then reveal that the angle of incidence is equal to the angle of reflection. Here, it should be noted that the angles are measured from the normal to the surface.

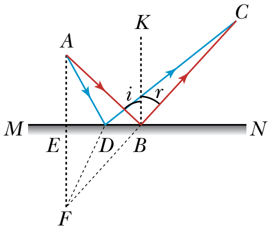


Figure 33-7 Reflecting light chooses the shortest path.

Specular and Diffuse Reflection

Smooth surfaces reflect light **specularly**–as shown in (Fig. 33-8a). At the smooth surface, the normal to the surface does not change its direction abruptly. Therefore, the reflected rays do not change their directions abruptly. If the light is reflected specularly, the eye has to be in a particular position to receive light.

Surfaces that are not smooth reflect light in many directions, as shown in Fig. 33-8b. For such **diffuse reflection**, each individual ray still obeys the laws of reflection, but the direction of the normal to the surface varies over the

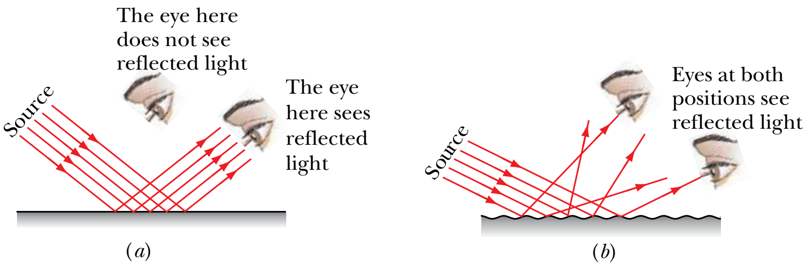

The eye here does not see reflected light
The eye here sees reflected light
Source
(a)
Eyes at both positions see reflected light
Source
(b)

Figure 33-8 (a) Specular reflection of a parallel beam of light and (b) diffuse reflection of a parallel beam of light.

surface and hence the angle of reflection also varies. How smooth is smooth surface? If the height and separation of irregularities on the surface are small compared to the wavelength of the incident light, the surface acts like a smooth surface and most light is reflected specularly. In the case of diffuse reflection, the eye can be positioned at any point and still receive the light. You are reading the pages of this book in this manner only. Do you know that using this argument of diffuse reflection, Galileo showed that the surface of moon should be rough and not smooth as many people of his times thought. Can you see his line of logic?

Many times, we are required to find the angle of the deviation of the light ray on reflection. Recollect that the angle between two vectors is the angle between their positive directions. So, when we talk of the deviation of a light ray, it refers to the angle deviated by a light ray from its original path, and not the angle between the reflected ray and the incident ray. This deviation is shown in Fig. 33-9.

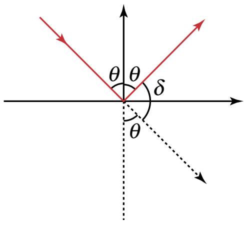


Figure 33-9 Deviation (denoted by δ) of a light ray on reflection.

The reflected ray deviates by an angle $\delta = \pi - 2\,\theta$.

SAMPLE PROBLEM 33.02

Deviation of ray of light after two reflections

Find the deviation of the ray after two reflections as shown in Fig. 33-10.

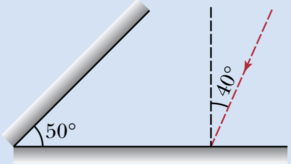


Figure 33-10 Two plane mirrors are inclined at an angle to each other. A ray is incident on one of the mirrors.

KEY IDEA

To solve this problem, we have to find the angle of incidence on each mirror. Then we can use the formula for deviation on each surface. If the deviation is in the same direction, it will be added, otherwise it will be subtracted.

Calculation: Here, the angle of incidence on the first mirror is 40° as seen in Fig. 33-10. So, the angle of reflection is also 40°. However, the angle of incidence on the second mirror is unknown. Consider the triangle formed by the two mirrors and the ray reflected from the first mirror. In this triangle, the angle between the first mirror and the reflected ray is 50°. The angle of reflection is measured from the normal. As the sum of all the angles in a triangle is 180°, the angle made by this ray with the second mirror is 180 – 50 – 50 = 80°. Therefore, the angle of incidence on the second mirror is 90° – 80° = 10°. The angle of reflection will also be 10°. From Fig. 33-11, it is clear that the angle of deviation is the angle between the reflected ray and the initial incident ray. This angle is the sum of the two interior angles made by the incident ray and the two reflected rays. So, the deviation $\delta = 40° + 40° + 10° + 10° = 100°$ in the anticlockwise direction.

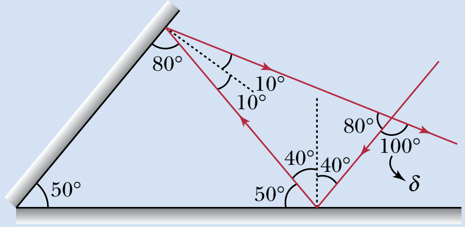


Figure 33-11 The path of incident and the reflected rays.

Here, we should remember that the deviation is an angle between the directions in which the rays are traveling.

We could have also used the formula for the deviation of a ray on reflection. The ray reflected from the first surface undergoes a deviation

$$\delta_1 = 180° - 2 \times 40° \quad \text{(clockwise).}$$

The ray reflected from the second surface undergoes a deviation

$$\delta_2 = 180° - 2 \times 10° \quad \text{(clockwise).}$$

Total deviation will be $\delta_1 + \delta_2 = 100° + 160° = 260°$ clockwise which is same as 100° anticlockwise.

Learn: Deviation after two reflections is 2θ, where θ is the angle between the two mirrors. In fact, a corner reflector is designed so that the light ray goes back in the direction from which it came. If we have three plane mirrors inclined at 90° to each other, the light ray reflects back to the direction from which it came. Such an arrangement is known as a **retro reflector**, a point we will discuss in the next chapter.

SAMPLE PROBLEM 33.03

Reflection of ray incident on a rotating mirror

A light ray is incident on a mirror rotating with a constant angular velocity of $\pi/30$ rad/s in the anticlockwise direction. This light ray is reflected from the mirror and falls on a wall nearby as shown in Fig. 33-12. Find the location of the spot on the wall and its velocity after 5 s.

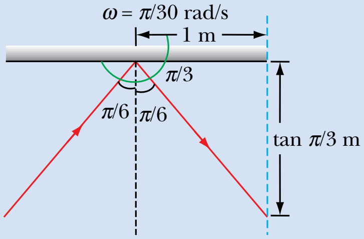


Figure 33-12 Ray incident on a rotating mirror.

KEY IDEAS

(1) If the mirror is rotating, the reflected ray will also rotate. This rotation produces a spot on the screen which moves due to the rotating reflected light.

(2) If mirror is rotated by angle α, the normal to the mirror will also rotate by the same angle. Therefore, the angle of incidence increases by α. This would mean that the angle of reflection also increases by α. But remember that the angle of reflection is measured from the new normal. As seen from its original direction, the reflected ray is rotated by 2α. This is shown in Fig. 33-13. Even if the mirror is not rotated about the point of incidence, the reflected ray still rotates by 2α. If the mirror rotates in the clockwise direction, the reflected ray also rotates in the same direction. Thus, if the mirror rotates with an angular velocity ω, the reflected ray rotates with the same velocity 2ω.

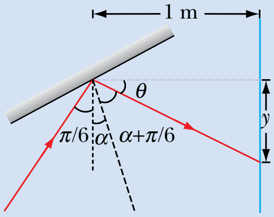


Figure 33-13 The incident ray remains at the same position, when the mirror rotates. The angle made by the reflected ray with the new normal is $\alpha + \pi/6$. Therefore, the angle made by the reflected ray with the original normal would be $2\alpha + \pi/6$. The figure depicts the reflected ray at an intermediate position. At $t = 5$ s, angle θ would be zero.

Calculation: In 5 s, the angle rotated by the mirror

$$\phi = \frac{\pi}{30} \times 5 = \frac{\pi}{6}.$$

Therefore, the reflected ray rotates by = $\pi/3$. The total angle is given by

$$= \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}.$$

As can be seen from Fig. 33-13, y is the position of the spot at angle $\theta = \tan(90 - \theta) = \cot° \theta$. Therefore,

$$y = \frac{dy}{dt} = \operatorname{cosec}^2\theta \frac{d\theta}{dt} = -\operatorname{cosec}^2\theta \cdot 2\omega,$$

$$-\operatorname{cosec}^2(90°) \cdot 2 \times \frac{\pi}{30} = \frac{-\pi}{15} \text{m/s}.$$

Note: Those who are familiar with the Cavendish's experiment of measuring G can relate to this. In that, the twist of the wire is measured with the help of a plane mirror attached to the wire. A small twist of the wire can displace the light spot on a screen by quite a bit of distance.

Laws of Reflection in Vector Form

Suppose that we are given a unit vector in the direction of the incident ray. We wish to find a unit vector in the direction of the reflected ray. For this, we will first represent the incident ray in terms of unit vector notation.

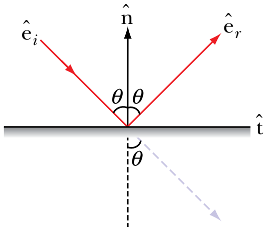


Figure 33-14 The incident ray, the reflected ray, and the normal to the reflecting surface.

As can be seen from Fig. 33-14, the component of the incident ray in the direction of tangent to the reflecting surface would be

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta.$$

The component of the incident ray in the direction of normal to the reflecting surface would be

$$\cos(\pi-\theta)=-\cos\theta.$$

So, the incident ray vector can be written as

$$\hat{e}_i=\sin\theta\ \hat{t}-\cos\theta\ \hat{n}.$$

The component of the reflected ray in the direction of tangent to the reflecting surface would be

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta.$$

The component of the reflected ray in the direction of normal to the reflecting surface would be $\cos\theta$. So, the reflected ray can be written as

$$\hat{e}_r=\sin\theta\ \hat{t}+\cos\theta\ \hat{n}.$$

So, we can say that

$$\hat{e}_r=\hat{e}_i+2\cos\theta\ \hat{n}.$$

But

$$\cos(\pi-\theta)=\frac{\hat{e}_i\cdot\hat{n}}{|\hat{e}_i|\cdot|\hat{n}|}=-\cos\theta.$$

Therefore,

$$\hat{e}_r=\hat{e}_i-2(\hat{e}_i\cdot\hat{n})\hat{n}.$$

Note that the component of the ray parallel to the mirror surface does not change. But the component of the ray perpendicular to the mirror reverses in direction.

CHECKPOINT 2

How many minimum mirrors arranged perpendicular to each other will make a light ray reverse its direction completely?

33.4 | FORMATION OF IMAGE BY A MIRROR

Key Concepts

- An image is a reproduction of an object via light. If the image can form on a surface, it is a real image and can exist even if no observer is present. If the image requires the visual system of an observer, it is a virtual image.
- A plane (flat) mirror can form a virtual image of a light source (said to be the object) by redirecting light rays emerging from the source. The image can be seen where backward extensions of reflected rays pass through one another.

Before we go ahead in geometrical optics, we need to define some key terms commonly used in in optics. These are as follows:

1. ***Object*** is a point of intersection of the incident ray.

Many times, the rays do not actually meet. In such cases, we have to trace the rays backward to make them meet.

Image is a point at which all the reflected rays or refracted rays intersect after passing through an optical system.

Now the object and image can be of two types each. Besides the two mentioned above, we need to understand the following:

An object is said to be **real** if the light rays incident on the surface are diverging in nature. An object is said to be **virtual** if the incident rays are converging in nature.

If the light rays after reflection or refraction are converging in nature, then the image is a ***real image***.

A real image can be photographed directly by placing a photographic plate at the location of image. Since the light rays are actually converging at that point, the point of convergence will be a point of large intensity. So, the photographic plate will record such an image. Real image can be obtained on a screen.

An image is said to be ***virtual*** when the light rays after reflection or refraction are diverging in nature.

A virtual image cannot be directly photographed by placing a photographic plate at that point. Since the rays are not meeting at that point, there is no need to keep photographic plate at that point. The image cannot be obtained on a screen.

In optics, we often encounter situation where the image is formed after multiple reflections or refractions. In order to form the final image, we work event by event. An object forms an image due to reflection or refraction from a surface. This image will act as an object for next surface and so on.

For example, in Fig. 33-15, the rays from O strike the lens first. As the rays incident on the lens are diverging, O is a real object for the lens. This object has no meaning for the mirror because rays from O do not directly reach the mirror. The rays after refraction from the lens converge to point I_1. As the refracted rays are converging in nature, I_1 is a real image formed by the lens. But before converging, the rays strike the mirror. These rays incident on the mirror meet at I_1. So I_1 will act as an object for the mirror. As the incident rays on the mirror are converging in nature, I_1 is a virtual object. After reflection from the mirror, the rays meet at I_2. So I_2 is an image formed by the mirror. As the reflected rays are converging, I_2 is a real image formed by the mirror. Again, these rays strike the lens. The rays incident on the lens intersect at I_2. So I_2 is an object for the lens. As the rays incident on the lens are diverging, I_2 is a real object for the lens. After refraction from the lens, the rays diverge. These diverging rays can be traced back to meet at I_3. As the refracted rays are diverging, I_3 is a virtual image formed by the lens. After this event, no further

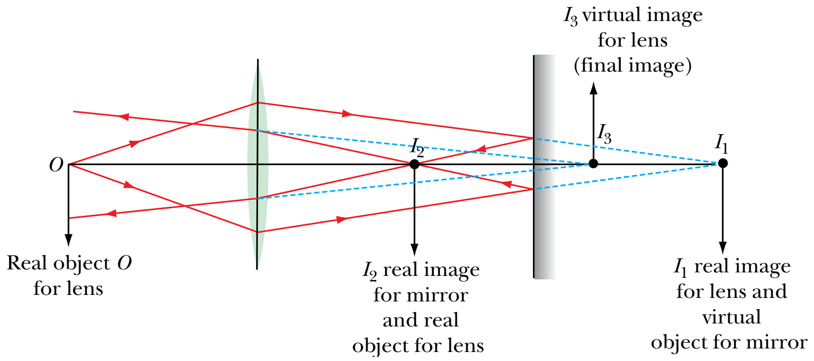

I_3 virtual image for lens (final image)
I_3
I_1
O
I_2
Real object O for lens
I_2 real image for mirror and real object for lens
I_1 real image for lens and virtual object for mirror

Figure 33-15 A point object O is on the axis of a convex lens. After refraction from the lens, reflection from the mirror, and again reflection from the lens, a final image is formed.

event will occur; rays will not return back. So I_3 is the final image formed by the system of lens. If someone asks us about the image formed by this system, the answer will be I_3.

Principle of Optical Reversibility

The principle of optical reversibility states that if a light ray is reversed in its direction it will retrace its path. This essentially implies that the object and image can be interchanged. In other words, if a mirror or a lens forms an image at I when the object is at O, then due to this principle the image is formed at O if the object is instead kept at I (Fig. 33-16).

Any image is formed with many rays, practically infinite. But for the purpose of ray tracing, only two or three rays are enough.

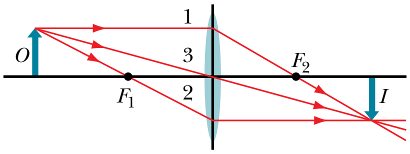


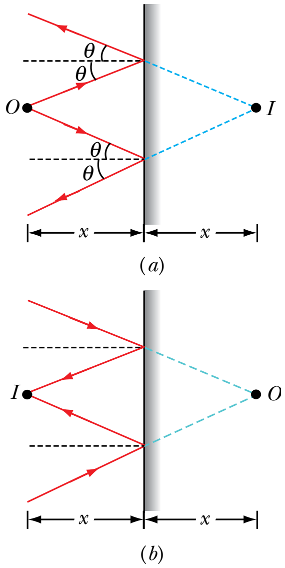
Figure 33-16 Principle of optical reversibility is illustrated.

Image of a Point Object in a Plane Mirror

Suppose that a point object is kept in front of the mirror (Fig. 33-17*a*). The incident rays are diverging (the object is real). The reflected rays from the mirror are also seen to be diverging in nature (the image is virtual). So, we can say that if the object is real, a plane mirror forms its virtual image. On the other hand, if converging rays fall on the mirror as shown in Fig. 33-17*b* (the object is then virtual), they converge after reflection (the image is then real). So, the virtual object forms a real image in a plane mirror.

As can be seen in Fig. 34-17*a*, if the object is at a distance x from the mirror, the image is also at a distance x from the mirror. Here, we should be careful. The distance of the object from the mirror is always the perpendicular distance. Many times, the object is in front of the plane of the mirror but not directly in front of the mirror. Still the image will be formed because some rays will be incident on the mirror whose reflection will form the image.

Only rays that are fairly close together can enter the eye after reflection from a mirror. For the eye position shown in Fig. 33-18, only a small portion of the mirror near point a (a portion smaller than the pupil of the eye) is useful in forming the image. To find this portion, close one eye and look at the mirror image of a small object such as the tip of a pencil. Then move your fingertip over the mirror surface until you cannot see the image. This implies that only the small portion of the mirror under your fingertip produced the image for you.



(*a*)

(*b*)

Figure 33-17 (*a*) Formation of virtual image of a real object. (*b*) Formation of real image of a virtual object.

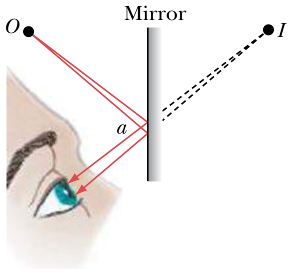


Figure 33-18 A "pencil" of rays from O enters the eye after reflection at the mirror. Only a small portion of the mirror near a is involved in this reflection. The light appears to originate at point I behind the mirror.

Velocity of the Image

If the object moves, its image also moves. To find the velocity of the image, we resolve the velocity of the object into two parts: one is parallel to the mirror and second is perpendicular to the mirror (Fig. 33-19).

The velocity of the object in the direction parallel to the mirror will be equal to the velocity of image in a direction parallel to the mirror. If the mirror is also moving parallel to itself, its velocity does not affect the velocity of the image in the parallel direction.

The speed of the object with respect to the mirror in a direction perpendicular to mirror surface is equal to the velocity of the image with respect to the mirror in the perpendicular direction. The only difference is that the object is in front of the mirror, and the image is behind the mirror. Let us show this. As shown in Fig. 33-17*a*, the

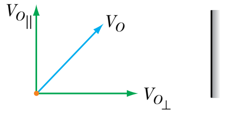


Figure 33-19 A point object moves in front of a mirror. Its velocity is resolved in two directions: parallel and perpendicular to the mirror.

perpendicular distance of the image from the mirror is equal to the perpendicular distance of the object from the mirror. Putting it mathematically,

$$X_{OM} = -X_{IM}.$$

By differentiating this position of the object and the image relative to the mirror, we obtain the velocity of the object and its image relative to the mirror

$$\frac{d}{dt}(X_{OM}) = \frac{d}{dt}(-X_{IM}).$$

Thus, we can say that

$$V_{OM} = -V_{IM}.$$

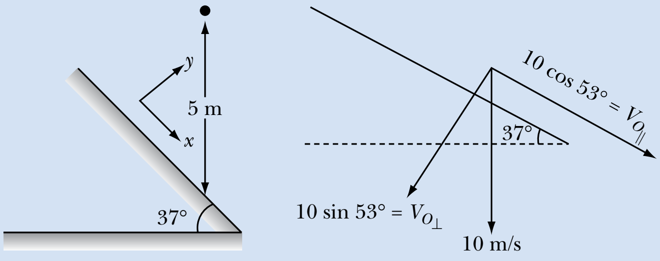
Note that we are referring to the velocities perpendicular to the mirror surface here. By the definition of the relative velocity, we have

$$V_{IM_\perp} = V_{I_\perp} - V_{M_\perp} = -(V_{O_\perp} - V_{M_\perp}). \tag{33-1}$$

SAMPLE PROBLEM 33.04

Velocity of the image and the object in the vector form

Find the velocity of image and the object in the vector form when the ball is just about to hit the inclined plane as shown in Fig. 33-20*a*. The inclined plane is at an angle of 37° to the horizontal. The surface of the inclined plane acts like a mirror.



(*a*) (*b*)

Figure 33-20 (*a*) An object is dropped from a height on an inclined mirror. (*b*) Resolving the velocity of the object in a direction parallel to and perpendicular to the mirror.

KEY IDEA

As indicated in Eq. 33-1, the velocity of the object can be resolved in directions parallel and perpendicular to the mirror. Since the mirror is stationary, we need not bother about the motion of the mirror.

In this situation, the velocity of the striking object is in the vertical direction. Since the object moves, image moves such that the distance of the image perpendicular to the mirror remains the same as the distance of the object perpendicular to the mirror.

Calculation: The velocity of the object before striking the mirror can be found by equation of kinematics.

$$\begin{aligned} v^2 &= u^2 + 2gs \\ v^2 &= 0 + 2 \times 10 \times 5 \\ v &= 10 \text{ m/s} \\ v &= +10 \text{ m/s} \end{aligned}$$

Resolving this in a direction perpendicular and parallel to the mirror as shown in Fig. 33-20*b*, we have

$$V_{O_\parallel} = 10\cos 53° = 6\,\text{m/s},$$

$$V_{O_\perp} = -10\cos 37° = -10 \times \frac{4}{5} = -8 \text{ m/s}.$$

Since the inclined plane is not moving,

$$V_{I_\perp} = -V_{O_\perp} = 8\hat{\jmath}.$$

In the parallel direction,

$$V_{I_\parallel} = V_{O_\parallel} = 10\cos 53° = 6 \text{ m/s}$$

$$v = v_x\hat{\imath} + v_y\hat{\jmath} = 6\hat{\imath} + 8\hat{\jmath}. \quad \text{(Answer)}$$

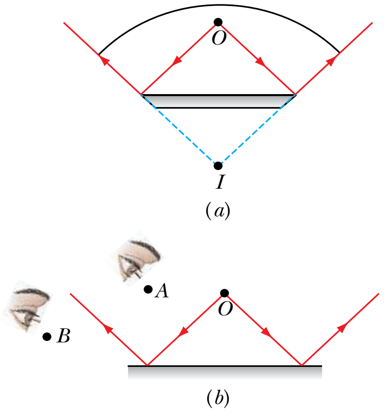
We can see that the velocity of the image makes the same angle with the mirror as the velocity of the object with the mirror.

Field of View

It is the region in which an observer can see the image of the object. For finding the field of view of a mirror, we should consider the mirror as a window through which we are viewing the image of the object. The region in which the observer is present to see this image is the field of view.

From Fig. 33-21*a* we can see that the reflected rays are present in a certain region only. These reflected rays can be traced back to see the image. We conclude this with the help of Fig. 33-21*b*. To see I, the observer can be at A but not at B, because no reflected rays reach B. The same point can be analyzed by treating the mirror as a window. If the mirror is a window, we cannot see I through the window by standing at any point.

We should also note that a human eye can form a sharp image on the retina only when diverging rays are incident on it. So even if an eye receives converging rays, it is not in the field of view as shown in Fig. 33-22.



(*a*)

(*b*)

Figure 33-21 (*a*) Field of view. (*b*) The observer can be at A to view the image but not at B.

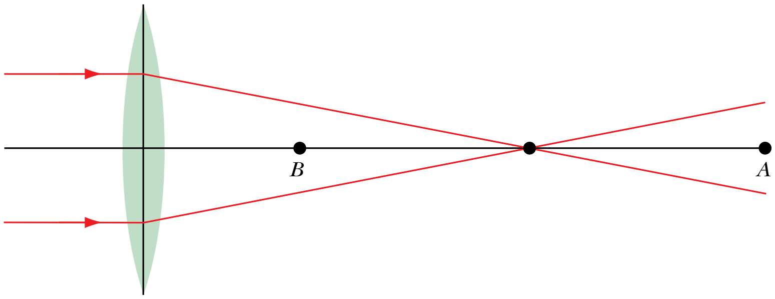


Figure 33-22 The observer at A is in the field of view, but not at B.

SAMPLE PROBLEM 33.05

Image of a moving object formed on a mirror kept in front of a road

A mirror of length 2*d* is kept in front of a road as in Fig. 33-23. A point object is kept at a distance *d* from the mirror. An observer is able to see the image till distance 12*d* while moving along the road. What is the distance *x* between the road and the mirror?

KEY IDEA

We can first form the image of O, and then treat the mirror as a window through which we can view the image. Let us see that the region in which the image is formed. This will define the region in which the observer can see the image.

Calculation: Here we can see that the image, the mirror, and the length of the road when the image is visible from two triangles as shown in Fig. 33-23. We can say that the mirror acts like a window through which we can see the image of the object. This makes two triangles which are similar to each other.

By the property of similar triangles, we can say that

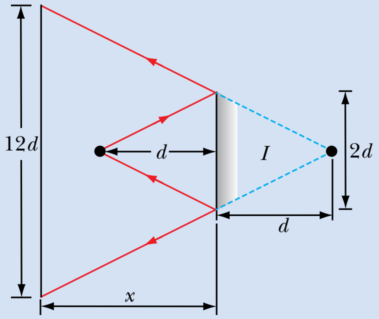


Figure 33-23 An observer is walking on a road parallel to a mirror.

$$\frac{d}{d+x} = \frac{2d}{12d}$$

$$x = 5d.$$

Learn: When we stand in front of a mirror, we can see only a part of the image depending on the length of the mirror. It is not that the image of the rest of the body is not formed by the mirror, only these reflected rays do not reach our eyes.

Image of an Extended Object

To form the image of an extended object, we should divide the object into many points and form the image of each point in the mirror. Then we should join all these point images to make the image of the object. If the object is a straight line, the image in a plane mirror will also be a straight line as can be seen in Fig. 33-24. To see the field of view of an extended object, we can make the image of the object as described above. Then we can see the field of view of the left end of the image by drawing rays reflected from the left end and the rays reflected from the right end. Similarly, we can draw the field of view of the right end of the image. The overlapping region for both the fields of view will give us the field of view of the complete image. This is clear from Fig. 33-24.

To see the image of oneself in the mirror, the mirror should be half of the height of person and the top of the mirror should be between the eye and head level. Similarly, the bottom of the mirror should be between the eye and the foot. This is clear from Fig. 33-25.

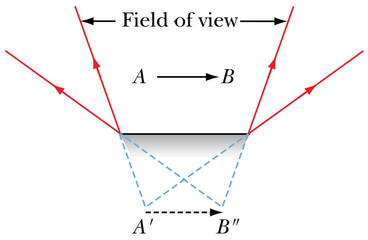


Figure 33-24 The field of view of an extended object.

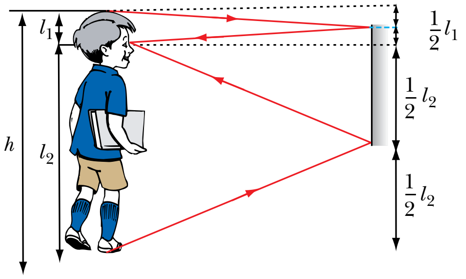


Figure 33-25 The height and the position of the mirror to view one's full image.

SAMPLE PROBLEM 33.06

Image of boy and his father in the mirror

Find the minimum length of the mirror required so that the boy can see the image of himself as well as his father standing behind. The father is of height H and the boy is of height $H/2$ (Fig. 33-26). Assume that the eyes of the boy are at his head.

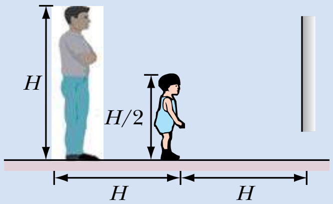


Figure 33-26 A boy standing in front of a mirror such that he can see his own image as well as his father's image.

KEY IDEA

The boy is of height $H/2$. The image of both the father and the boy is shown in Fig. 33-27. It is clear that the lowest point of the mirror is decided by the location of the boy's feet. If he can see his own feet, he can also see his father's feet. He can see his own head if the mirror is at the level of his head. But to see his father's head, he has to tilt his head to some angle above. So, if he can see his father's head, he can see his own head as well.

Here we can assume the mirror to act like a window by which the boy can see the image of the father as well as himself.

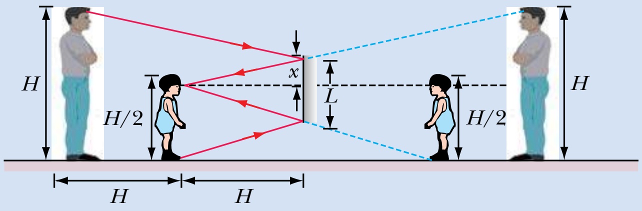


Figure 33-27 Image of the father and the boy.

Calculation: Let us assume that the mirror is at a height x above his own head. From the concept of similar triangles, we can say that

$$\frac{x}{\frac{H}{2}} = \frac{H}{3H}.$$

Therefore, $x = H/6$.

From Fig. 33-27, it can be seen that the boy can see his own feet if the bottom of the mirror is midway between the eye and the foot.

Therefore, the total length of mirror required is

$$\frac{H}{2} - \frac{H}{4} + \frac{H}{6} = \frac{5H}{12}.$$

Formation of Multiple Images

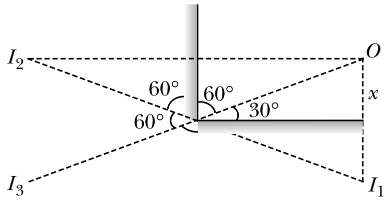


Figure 33-28 Formation of three images by two mirrors held perpendicular to each other.

If an object is placed between two inclined mirrors at an angle to each other, multiple images will be formed. To consider the formation of images, let us take an example where there are two mirrors at 90° to each other. The object is at position O as shown in Fig. 34-28. Each of the mirrors will form one image (I_1 and I_2 in Fig. 33-28). These images will act as an object for the next mirror.

We know that if the object is at perpendicular distance x from the mirror, the image will also be at a distance x behind the mirror. In other words, the angular position of image will be the same as the angular position of the object. So I_1 will be 30° behind the mirror. Similarly, I_2 will be 60° behind the mirror. This image will also be in the front of plane of vertical mirror. Therefore, rays will reflect from the vertical mirror to form image I_3. For the horizontal mirror, I_2 acts as an object. I_2 is 150° ahead of the mirror. Therefore, its image will be 150° behind the mirror. Interestingly, these two images will coincide. No further images will be formed because reflected rays will not strike the mirrors again. Also, we can see that to form I_3, left half of the horizontal mirror and bottom half of vertical mirror only will be useful.

From Fig. 33-28, we can understand that

1. If the object is at angle θ in front of the mirrors, the image will also be at an angle θ behind mirrors.
2. All these images lie on a circle whose center is the point of intersection of the mirrors. The radius of the circle is equal to the distance between the point of intersection of the mirrors and the object.
3. The reflection will continue to take place till the image lies in front of the plane of the mirror. In other words, if reflected rays do not reach the next mirror again, further images will not form.

General rule for finding the number of image when the mirrors are inclined at an angle θ.
Let $m = 360/\theta$. Then

1. If m is an even integer, the number of images will be $(m - 1)$ irrespective of the position of the object.
2. If m is an odd integer, we have the following cases:

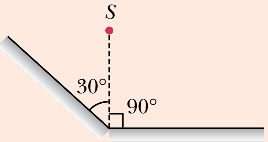
Case-I: If the object is on angle bisector of the mirror, the number of images will be $(m - 1)$.

Case-II: If the object is at any other place, the number of images will be m. We cannot apply this rule when $360/\theta$ is not an integer.

Specifically, if the mirrors are parallel to each other, the number of images will be infinite. But they will be of successively lesser intensity because some reflected rays will escape from the sides and some will be absorbed by the mirror.

CHECKPOINT 3

If an object is at point S as shown in the figure and the mirrors are at an angle of 120° to each other, find the total number of images and their angular position from the x axis.



33.5 | FORMATION OF IMAGES BY SPHERICAL MIRRORS

Key Concepts

- A spherical mirror is in the shape of a small section of a spherical surface and can be concave (the radius of curvature r is a positive quantity), convex (r is a negative quantity), or plane (flat, r is infinite).
- If parallel rays are sent into a (spherical) concave mirror parallel to the central axis, the reflected rays pass through a common point (a real focus F) at a distance f (a positive quantity) from the

mirror. If they are sent toward a (spherical) convex mirror, backward extensions of the reflected rays pass through a common point (a virtual focus F) at a distance f (a negative quantity) from the mirror.

- A concave mirror can form a real image (if the object is outside the focal point) or a virtual image (if the object is inside the focal point).
- A convex mirror can form only a virtual image.
- The mirror equation relates an object distance u, the mirror's focal length f and radius of curvature r, and the image distance v:

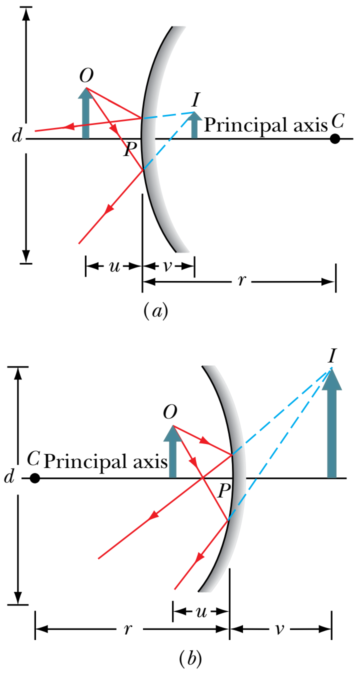
$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}=\frac{2}{r}$$

- The magnitude of the lateral magnification m of an object is the ratio of the image height h_I to object height h_O,

$$|m|=\frac{h_I}{h_O}$$

and is related to the object distance u and image distance v by

$$m=-\frac{v}{u}$$



(a)

(b)

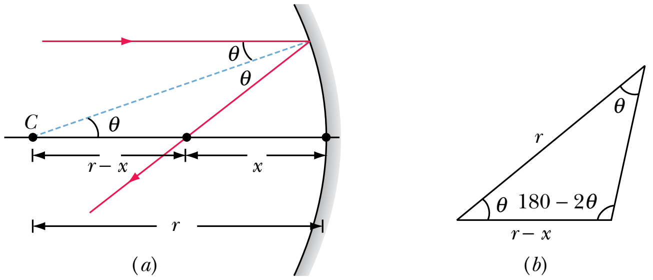
Figure 33-29 Image formation by (*a*) a convex mirror and (*b*) a concave mirror. Note that both the images are virtual.

We turn now from images produced by plane mirrors to images produced by mirrors with curved surfaces. In particular, we consider spherical mirrors, which are simply mirrors in the shape of a small section of the surface of a sphere. A plane mirror is in fact a spherical mirror with an infinitely large *radius of curvature* and thus an approximately flat surface.

You must have seen the image of an object on the back of a spoon. It seems funny, inverted, and distorted. The surface of the spoon acts as a mirror and not plane. In fact, it acts like a curved whose reflecting surface is the part of a sphere. If the reflection occurs from the outer surface of the sphere, the mirror is called a **convex mirror** (Fig. 33-29*a*). If the reflection occurs from the inner surface of the sphere, then mirror is called a **concave mirror** (Fig. 33-29*b*).

There are some terminologies that we should be familiar with: **Aperture** is the diameter of the mirror disk (d in Figs. 33-29*a* and *b*). Center of the spherical surface is called the **center of curvature** (point C in Figs. 33-29*a* and *b*). The line joining center of curvature and the pole P is called the **principal axis**. r is the **radius of curvature** of the mirror surface. It is the distance between the pole and the center of curvature. O is an object whose image is I. The distance of the object from the pole is denoted by u and the distance of the image from the pole is denoted by v.

Focus is a point on the principal axis where a beam parallel to the principal axis converges after reflection from the mirror. To find the location of focus, let us assume that a beam of light rays parallel to the principal axis gets reflected from the mirror and crosses the principal axis after reflection (Fig. 33-30*a*). We know that the line from center of curvature to the mirror will always be along the normal. So, the angle of incidence will be θ. From the laws of reflection, we can say that angle of reflection will also be θ. Let us say that the reflected ray crosses the principal axis at a distance x from the pole.



(a)

(b)

Figure 33-30 (*a*) Reflection of a ray parallel to the principal axis from a spherical mirror. Note that the laws of reflection are valid for spherical as well as plane mirrors. (*b*) Tracing the reflected ray with help of sine rule.

Since the incident ray is parallel to the principal axis, the angle made by the normal at the point of incidence with the principal axis will also be θ.

In the triangle formed by C, focus and the point of incidence (Fig. 33-30*b*), on applying sine rule, we get

$$\frac{r-x}{\sin\theta} = \frac{r}{\sin(\pi-2\theta)}$$

$$x = r - \frac{r}{2}\sec\theta.$$

If θ is small, sec $\theta = 1$, and so

$$x = \frac{r}{2}.$$

Let us now define the **focal length of a mirror**. It is the distance of a point on the principal axis from the pole where a beam of light converges after reflection from the mirror. We can say that the focal length of a spherical mirror is

$$f = \frac{r}{2}.$$

The concept of focus is defined only for the angle of incidence close to 0°. Such rays are called **paraxial rays**. Paraxial rays approximation means that the angle of incidence should be very small ($\theta < 10°$). In such a case, parallel beam of light either converges at the focus after reflection from the mirror or diverges from it (Figs. 33-31*a* and *b*).

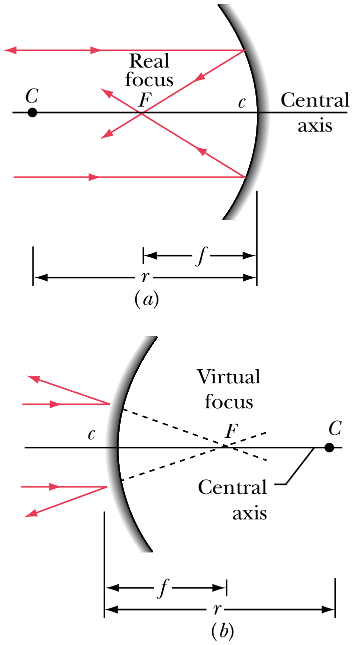


Figure 33-31 (*a*) In a concave mirror, incident parallel light rays are brought to a real focus at F, on the same side of the mirror as the incident light rays. (*b*) In a convex mirror, incident parallel light rays seem to diverge from a virtual focus at F, on the side of the mirror opposite the light rays.

If the rays are not paraxial, the image is not sharp. This defect of image formation is known as **spherical aberration**. You know why? This defect is related to the spherical nature of the surface. If the same surface were parabolic in shape, all the rays parallel to the principal axis will converge at the focal point. Any standard text on mathematics will tell you how.

When two parallel rays, not parallel to principal axis, get reflected from the mirror, they meet at some point above the focus or below the focus. That point is also called **secondary focus**.

Focal plane is a plane passing through focus and perpendicular to the principal axis. A parallel beam of light meets in the focal plane after reflection.

Let us return to our original quest: finding the location of the image if we know the location of the object on the principal axis. To find the location of the image, we have to define a sign convention. In this chapter, we are going to follow **Cartesian sign convention**, in which

1. The principal axis of the mirror is taken as x axis and the pole as origin.
2. The direction of the incident light is taken as positive direction of x axis.
3. The upward direction is taken as positive and downward direction as negative.

All the axial distances are always measured from the pole. This is represented in Fig. 33-32. Therefore, we can see that in the case of mirrors, real objects, u is always negative and for virtual objects, u is positive. For real image v is negative and for virtual image v is positive. Another useful concept is of **lateral magnification**. It is defined as the ratio of the height of the image and the height of the object from the principal axis and is denoted by m.

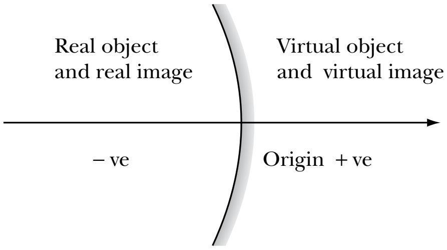


Figure 33.32 Cartesian sign convention.

In Fig. 33-33, *ab* is the height of the object h_O and *de* is the height of the image h_I. Therefore, by our definition,

$$m = \frac{h_I}{h_O}.$$

Let us try to calculate its value. From the figure, it can be seen that h_I is negative and h_O is positive according to our sign convention. Also, *abc* and *cde* form two similar triangles. From the property of the similar triangles, we get

$$\frac{-h_I}{h_O} = \frac{cd}{ac},$$

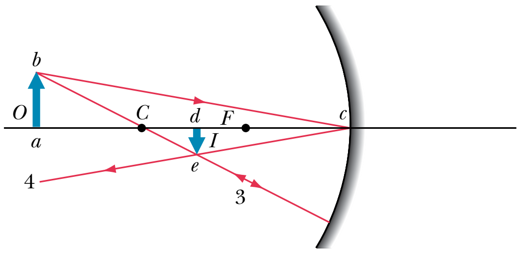


Figure 33-33 Lateral magnification.

where negative sign is introduced to take the modulus of h_I. Now according to the sign convention, $-ac = v$ and $-cd = u$. Therefore, we can say that

$$m = \frac{h_I}{h_O} = -\frac{v}{u}. \tag{33-2}$$

Be careful to substitute v and u with proper signs. This magnification has interesting implications. If the magnification is positive, it means that the image is formed on the same side of the principal axis as the object and vice versa. If the magnification is greater than 1 in magnitude, this would mean that the image is magnified; otherwise the image is diminished in size.

Case I: If object and image both are real, according to Eq. 33-2, we get

$$m = \frac{-v}{u} < 0.$$

Therefore, v and u are both negative. The image is inverted. This means that if the object is above the principal axis (h_O is positive), the image will be below the principal axis (h_I is negative).

Case II: If the object is real and the image virtual, according to Eq. 33-2, we get

$$m = -\left(\frac{v}{u}\right) > 0.$$

Therefore, v is positive and u is negative. This implies that the image is erect. This also means that if the object is above the principal axis (h_O is positive), the image will also be above the principal axis (h_I is positive).

Case III: If the object is virtual and the image is real, according to Eq. 33-2, we get

$$m = -\left(\frac{v}{u}\right) > 0.$$

Therefore, v is positive, and u is positive. This implies that the image is erect.

Case IV: If both the object and the image are virtual, according to Eq. 33-2, we get

$$m = -\left(\frac{v}{u}\right) < 0.$$

This implies that the image is inverted.

Although here we are discussing the case of image formation in a mirror, this happens in case of all the optical systems. This will be shown in each case: the combination of real object and real image (and virtual object and virtual image) is always inverted; and the combination of real object and virtual image (and virtual object and real image) is always erect. The same is shown in Figs. 33-34*a* and *b*.

Mirror Formula

The Spherical Mirror Formula (Eq. 33-6)

Figure 33-35 shows a point object O placed above the central axis of a concave spherical mirror, between its center of curvature C and focus F. A large number of rays starting from O converge at point I to form an image. Here, we

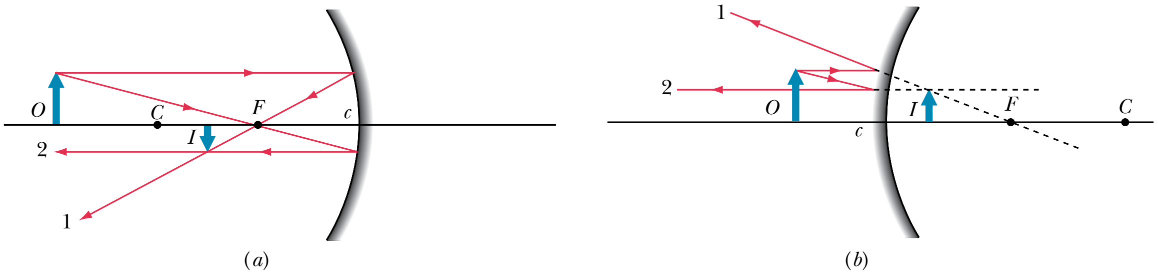


Figure 33-34 (*a*) and (*b*) The real image of a real object is inverted but the virtual image of a real object is erect.

have shown only two of the rays for simplicity. Thus, *I* is the image of *O*; it is a *real* image. Let us find the image distance *v*.

There are two pairs of similar triangles in Fig. 33-35. The first pair is *OaF* and *FPd*. From the property of the similar triangles, we have

$$\frac{h_O}{f-u} = \frac{+h_I}{+f}$$

$$\frac{h_I}{h_O} = \frac{f}{f-u}. \tag{33-3}$$

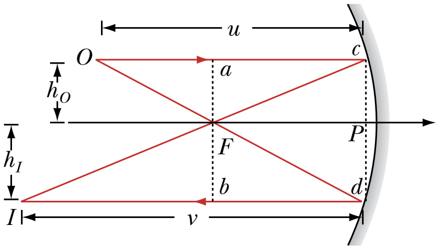


Figure 33-35 Formation of image by a concave mirror. Can you guess the value of *u* and *v* from the ray diagram?

Also, from the triangles made by the *IbF* and *FcP*, we get

$$\frac{-h_I}{f-v} = \frac{h_O}{-f}$$

$$\frac{h_I}{h_O} = \frac{f-v}{f}. \tag{33-4}$$

Comparing Eqs. 33-3 and 33-4, we get

$$\frac{f}{f-u} = \frac{f-v}{f} \tag{33-5}$$

$$\frac{1}{1-(u/f)} = 1 - \frac{v}{f}$$

$$f^2 = f^2 + uv - uf - vf$$

$$uv = uf + vf$$

$$\frac{uv}{f} = u + v$$

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}. \tag{33-6}$$

This is commonly known as the mirror formula.

From Eq. 33-5, we have

$$f^2 = (f-v)(f-u).$$

If we write x_i = distance of image from focus and x_o = distance of object from focus, then

$$x_o x_i = f^2. \tag{33-7}$$

This is what is known as the **Newton's formula**.

CHECKPOINT 4

If the object is on the right side of the focus, on what side of the focus will the image be? If the object is on the left side of the focus, on what side of the focus will the image be?

Finding the Details of the Image from 1/v versus 1/u Graph

Let us consider a concave mirror for which according to our sign convention, the focal length is negative. The mirror equation yields

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}.$$

Substituting $1/v = y$ and $1/u = x$, we obtain the equation of a straight line:

$$x + y = \frac{1}{f}.$$

This straight line has been plotted in Fig. 33-36. We note that for a real object $u < 0$, which means that $1/u$ is negative. Similarly, for a real image $v < 0$ and hence $1/v$ is negative. On the contrary, virtual object means $1/u$ is positive and virtual image means $1/v$ is positive.

Virtual image of virtual object would imply both x and y coordinates are positive. But no portion of the graph lies in that region. Hence, we can conclude that in a concave mirror, virtual object and virtual image combinations are not possible. Arguing along similar lines, we can say that in a convex mirror, real object cannot form a real image.

To see the magnification, we note that

$$m = -\frac{v}{u} = -\left(\frac{1/u}{1/v}\right) = -\frac{x}{y}.$$

In the second quadrant, where $1/u < 1/f$ or $|u| < |f|$ (focal length is negative), the image is virtual. Here, y coordinate is less than x coordinate. Therefore, $|m| > 1$.

Also, when the object lies between pole and focus, the image is virtual, magnified, and erect.

Similarly, we can deduce the information about other location of the object. Let us tabulate this finding. We divide the entire principal axis into four parts as shown in Fig. 33-37 and tabulate our findings in Table 33-1. Similarly, for a convex mirror, the entire principal axis is divided into four parts as shown in Fig. 33-38 and the findings have been given in Table 33-2.

Some conclusions can be easily drawn from Tables 33-1 and 33-2. If the object moves along the principal axis, the image also moves along the principal axis. The object and image will cross each other at the center of curvature and the pole. At the point of crossing, the size of the object and the image are the same. At the center of curvature, the image is inverted ($m = -1$) and at the pole, the image is erect ($m = +1$). This can be understood in the following manner: At the pole, the spherical mirror will look like a plane mirror. Therefore, it will give erect image of the same size. Also, $m = -v/u$. When u and

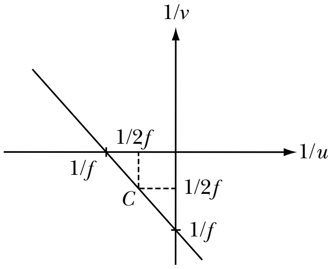


Figure 33-36 $1/v$ versus $1/u$ graph for concave mirror.

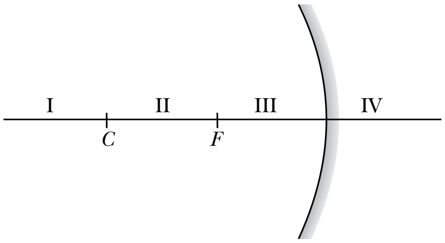


Figure 33-37 The entire principal axis can be divided into four regions as shown in the figure. The region I, II, and III correspond to real side, and region IV refers to the virtual side.

Table 33-1 Concave Mirror: Nature of Object and Image

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Real object and real image	Diminished	Inverted
II	I	Real object and real image	Enlarged	Inverted
III	IV	Real object and virtual image	Enlarged	Erect
IV	III	Virtual object and real image	Diminished	Erect

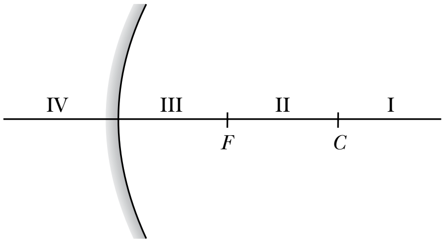


Figure 33-38 The entire principal axis for the convex mirror can also be divided into four regions as shown in the figure. The region I, II, and III correspond to virtual side, and region IV refers to the real side.

Table 33-2 Convex Mirror: Nature of Object and Image

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Virtual object and virtual image	Diminished	Inverted
II	I	Virtual object and virtual image	Enlarged	Inverted
III	IV	Virtual object and real image	Enlarged	Erect
IV	III	Real object and virtual image	Diminished	Erect

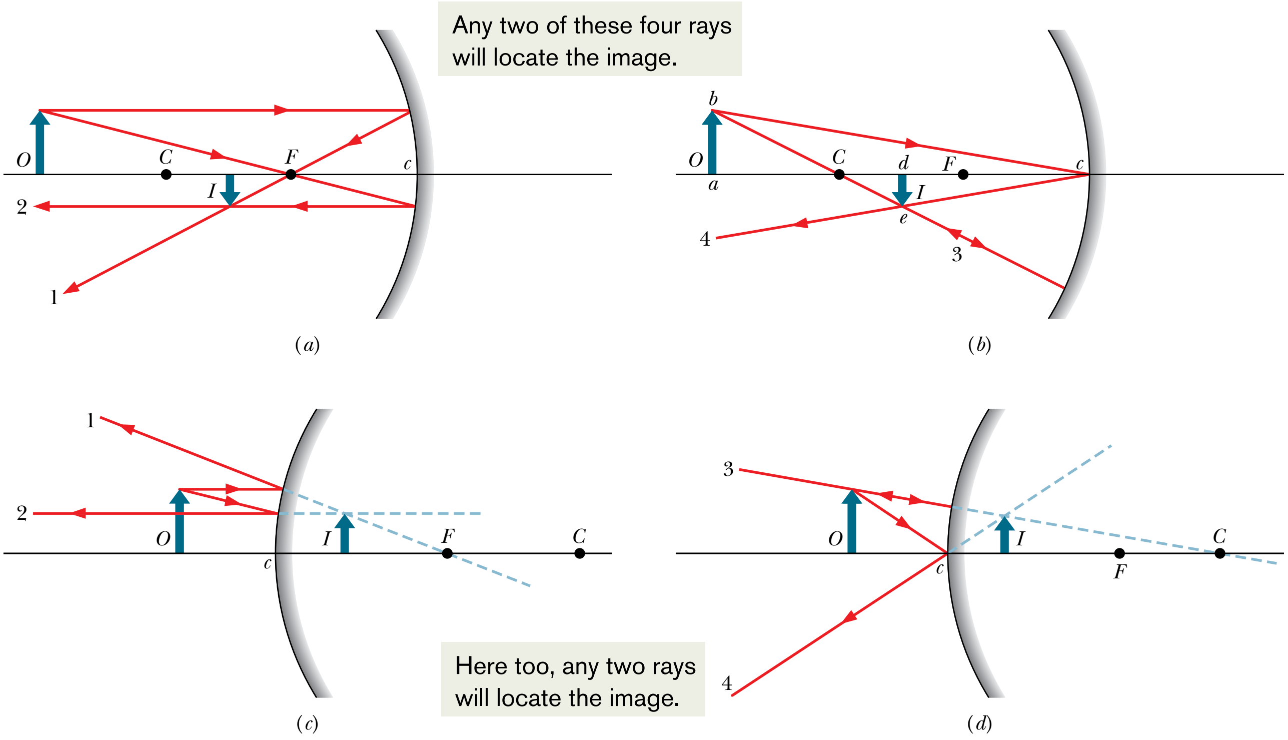

Any two of these four rays will locate the image.
O
C
F
c
I
2
1
(a)
b
O
a
C
d
F
c
I
e
4
3
(b)
1
2
O
c
I
F
C
(c)
3
O
I
C
c
F
4
(d)
Here too, any two rays will locate the image.

Figure 33-39 (*a, b*) Four rays that may be drawn to find the image formed by a concave mirror. For the object position shown, the image is real, inverted, and smaller than the object. (*c, d*) Four similar rays for the case of a convex mirror. For a convex mirror, the image is always virtual, oriented like the object, and smaller than the object. [In (*c*), ray 2 is initially directed toward focal point *F*. In (*d*), ray 3 is initially directed toward center of curvature *C*.]

v both approach 0, we should apply the limits to find that $m = +1$. This condition, where the object and the image coincide with each other is known as ***autocollimation***. It occurs when the object is at the center of curvature of the mirror or at its pole.

Locating Images by Drawing Rays

Figures 33-39*a* and *b* show an object *O* in front of a concave mirror. We can graphically locate the image of any off-axis point of the object by drawing a *ray diagram* with any two of four special rays through the point:

1. A ray that is initially parallel to the central axis reflects through the focus *F* (ray 1 in Fig. 33-39*a*).
2. A ray that reflects from the mirror after passing through the focus emerges parallel to the principal axis (ray 2 in Fig. 33-39*a*).
3. A ray that reflects from the mirror after passing through the center of curvature *C* returns along itself (ray 3 in Fig. 33-39*b*). This essentially means that the line joining the object and the image passes through the center of curvature. The line joining the object and the image is always perpendicular to the mirror.
4. A ray that reflects from the mirror at point *c* is reflected symmetrically about that axis (ray 4 in Fig. 33-39*b*). From this, we can say that a parallel beam of light incident on the mirror will be forming an image of height $f\theta$ from the focus (in the focal plane) as shown in Fig. 33-40.

The image of the point is at the intersection of the two special rays you choose. The image of the object can then be found by locating the images of two or more of its off-axis points (say, the point most off axis) and then sketching in the rest of the image. You need to modify the descriptions of the rays slightly to apply them to convex mirrors, as in Figs. 33-39*c* and *d*.

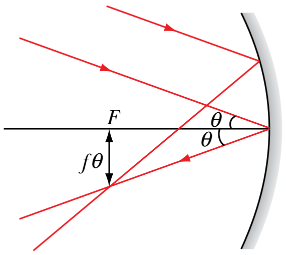


Figure 33-40 Formation of an image in focal plane.

CHECKPOINT 5

A Central American vampire bat, dozing on the central axis of a spherical mirror, is magnified by $m = -4$. Is its image (a) real or virtual, (b) inverted or of the same orientation as the bat, and (c) on the same side of the mirror as the bat or on the opposite side?

SAMPLE PROBLEM 33.07

Image produced by a spherical mirror

A tarantula of height h sits cautiously before a spherical mirror whose focal length has absolute value $|f| = 40$ cm. The image of the tarantula produced by the mirror has the same orientation as the tarantula and has height $h' = 0.20h$.

(a) Is the image real or virtual, and is it on the same side of the mirror as the tarantula or the opposite side?

Reasoning: Because the image has the same orientation as the tarantula (the object), it must be virtual and on the op-posite side of the mirror. (You can easily see this result if you have filled out Table 34-1.)

(b) Is the mirror concave or convex, and what is its focal length f, sign included?

KEY IDEA

We *cannot* tell the type of mirror from the type of image because both types of mirror can produce virtual images. Similarly, we cannot tell the type of mirror from the sign of the focal length f, as obtained from Eq. 34-3 or Eq. 34-4, be-cause we lack enough information to use either equation. However, we can make use of the magnification information.

Calculations: From the given information, we know that the ratio of image height h' to object height h is 0.20. Thus, from Eq. 34-5 we have

$$|m| = \frac{h_i}{h_O} = 0.20.$$

Because the object and image have the same orientation, we know that m must be positive: $m = +0.20$. Substituting this into Eq. 34-6 and solving for, say, i gives us

$$v = -0.20u,$$

which does not appear to be of help in finding f. However, it is helpful if we substitute it into Eq. 34-4. That equation gives us

$$\frac{1}{f} = \frac{1}{i} + \frac{1}{p} = \frac{1}{-0.20p} + \frac{1}{p} = \frac{1}{p}(-5+1),$$

from which we find

$$f = -p/4.$$

Now we have it: Because p is positive, f must be negative, which means that the mirror is convex with

$$f = -40 \text{ cm}. \qquad \text{(Answer)}$$

Learn: In a concave mirror, an erect image will always be magnified. So, if an erect image is formed which is diminished, it must be a convex mirror. The same thing can be understood by using a graph of u versus v (Figs. 33-41 and 33-42).

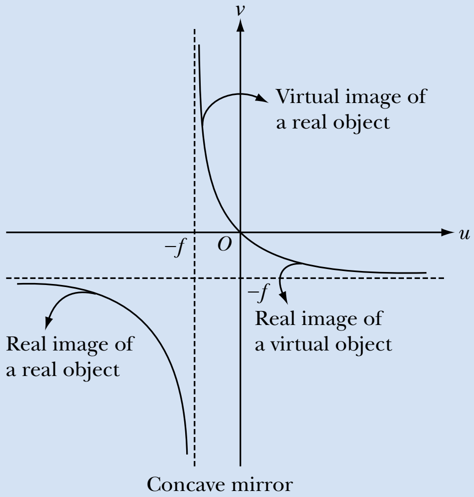


Figure 33-41 u–v graph for concave mirror.

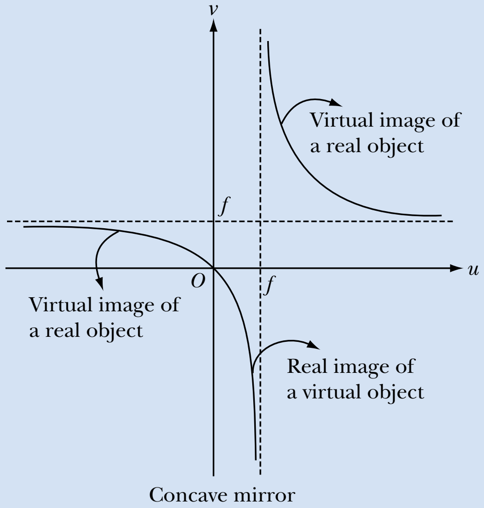


Figure 33-42 u–v graph for convex mirror.

SAMPLE PROBLEM 33.08

Finding angle of deviation

A light ray is incident on a concave mirror at an angle θ at a height h above the principal axis as shown in Fig. 33-43. Find the angle of deviation of the ray after reflection.

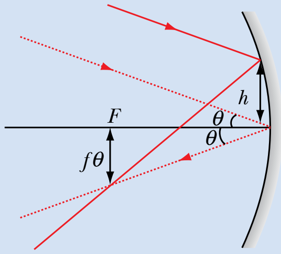


Figure 33-43 Finding the path of a ray incident on a concave mirror. We draw an imaginary ray parallel to it incident at the pole for visualizing the image.

KEY IDEAS

(1) There are a number of ways in which a ray diagram can be made. Here, we are showing only one method. Other methods should yield the same answer. We know that ray is neither passing through the focus nor the center of curvature. Therefore, we assume a ray parallel to it incident at the pole.

(2) This will make a set of rays parallel to each other. As discussed above, the reflected rays will meet at a point in the focal plane at a height $f\theta$ as shown as shown in Fig. 33-43.

(3) Since here we are required to draw a ray which is not one of the principal rays, we have to make an imaginary ray to draw it.

Calculation: So, the angle made by the reflected ray with the horizontal is

$$\tan\phi = \frac{h + f\theta}{f} \approx \phi \quad (\theta \text{ is small angle}).$$

So, the angle of deviation is $\pi - (\phi + \theta)$.

Learn: Many times, to trace the image of an object, we draw imaginary rays to find the path followed by actual rays. This especially happens when we want to trace the image of a point object lying on the principal axis.

CHECKPOINT 6

Of the three types of mirrors: plane, concave, and convex,
(a) which will have larger field of view, if aperture of all the three mirrors is the same?
(b) Can you think of a practical application of this?

Longitudinal Magnification

We have already stated that the lateral magnification or transverse magnification is the ratio of the height of image and the height of object. **Longitudinal magnification** is considered for an object lying along the principal axis or parallel to it.

We know in optics that to find the image of an extended object, we find the image of its end points. The image of the object will be the line joining the end points of the image. Suppose that the object is lying along the principal axis and its ends are at distances u_1 and u_2 from the pole, as seen in Fig. 33-44. Let us assume that the image of these end points is at v_1 and v_2, respectively. So, the length of the object $l_O = (u_2 - u_1)$ and the length of the image $l_I = (v_2 - v_1)$.

Longitudinal magnification is defined as the ratio of the length of the object and the length of its image. Hence

$$\frac{l_I}{l_O} = m_L = \frac{v_2 - v_1}{u_2 - u_1}.$$

Note that in the case of mirror, longitudinal magnification will always come out to be negative because of **lateral inversion**. If we point

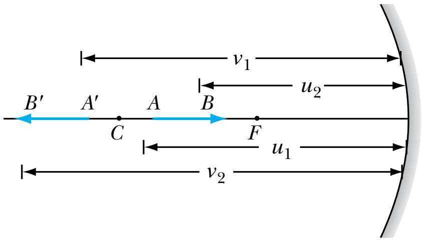


Figure 33-44 The image of an object kept along the principal axis is laterally inverted.

our finger toward a mirror, the image will always point a finger at us. According to our sign convention, this means that if the object is oriented along the positive direction, the image will be oriented along the negative direction.

If the object is very small, we can replace $u_2 - u_1$ with du and $v_2 - v_1$ with dv, then m_L will be

$$m_L = \frac{dv}{du} = \frac{d}{du}\left(\frac{uf}{u-f}\right),$$

$$\frac{du}{dv} = f\frac{[(u-f)-u]}{(u-f)^2} = \frac{-f^2}{(u-f)^2} = -m^2.$$

Therefore,

$$\frac{l_I}{l_O} = \text{Longitudinal magnification} = -m^2.$$

Remember that this is valid only for small objects.

SAMPLE PROBLEM 33.09

Image of an L-shaped object kept on the axis of a mirror

An L-shaped object with dimensions 1 cm each is kept on the principal axis of a concave mirror of focal length 10 cm as shown in Fig. 33-45. Its edge is 15 cm away from the pole. Find the location, shape, and size of the image.

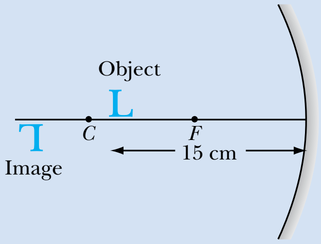


Figure 33-45 An object kept on the axis of a mirror.

KEY IDEA

Here, we can see that the object can be divided into two parts. For the part which is perpendicular to the principal axis, we can find the image by mirror formula and its magnification. The part of the image which lies on the principal axis has to be found with the help of concept of longitudinal magnification.

Calculations: This problem can be solved by two methods:

Approximate method:

In the situation, we can take the object to be small.

$$\frac{h_I}{h_O} = -\frac{v}{u} = -\frac{f}{u-f} = \frac{10}{-15+10}.$$

So, $h_I = -2$ cm. Also

$$\frac{l_I}{l_O} = -m^2 = -\left(\frac{f}{u-f}\right)^2 = -4$$

So, $l_I = -4$ cm.

Exact method:

Here one end of L is at $u_1 = -15$ cm and another end at $u_2 = -14$ cm as seen in Fig. 33-45. Solving by mirror formula for the image of these two points: $v_1 = -30$ cm and $v_2 = -35$ cm. So, length of the image is 5 cm. Here, we can see that the length of the object is comparable to the focal length. So, there is substantial error in length of the image.

Learn: If we want to find the length of an image placed along the principal axis, we should not resort to the approximate method for longitudinal magnification unless the length of the object is very small in comparison to the focal length.

Velocity of Image with Respect to Mirror

Suppose the object and the mirror are both moving. Then naturally, the image might move. If we wish to find out the velocity of the image, we can apply the concepts that we have learnt till now. As usual, we take the x direction to be the principal axis.

In x direction, the location of the image is given by

$$v = \frac{uf}{u-f}.$$

We can say that v is the distance between the image and the mirror. If we differentiate both sides to find dv/dt, this derivative refers to velocity of image relative to the mirror in the x direction. Similarly, du/dt is the velocity of the object relative to mirror in the x direction.

$$v_{IM} = \frac{dv}{dt} = \frac{-f^2}{(u-f)^2} \times \frac{du}{dt}$$

$$v_{IM} = \frac{-f^2}{(u-f)^2} \times v_{OM}$$

But

$$-\frac{v}{u} = \frac{-f}{u-f} = m,$$

$$v_{IM} = -m^2 \times v_{OM}. \tag{33-8}$$

To find the velocity of the image in a direction perpendicular to the principal axis, we calculate

$$\frac{h_I}{h_O} = m = \frac{-f}{u-f}.$$

In y direction

$$h_I = h_O \times \frac{-f}{u-f}.$$

$$v_{Iy} = \frac{d}{dt}\left(\frac{-f\, h_O}{u-f}\right). \tag{33-9}$$

In kinematics, we have studied that the velocity of the objects in x and y direction are independent of each other. But here, the velocity of the image in y direction will depend on its x coordinate and velocity in x direction also. Remember that these velocities are with sign convention. So, the velocity along positive x direction will be positive and so on.

SAMPLE PROBLEM 33.10

Image of an object moving parallel to the principal axis of a concave mirror

An object is traveling toward a concave mirror of focal length 10 cm. It moves with a velocity of 0.1 m/s parallel to the principal axis (Fig. 33-46). When it is at a distance of 15 cm from the mirror at a height of 1 cm from the principal axis, find the velocity of image in x and y direction?

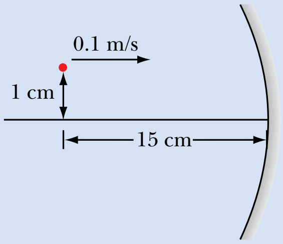


Figure 33-46 An object is moving parallel to the principal axis with a speed of 10 cm/s.

KEY IDEA

Here we note that the object is moving parallel to the principal axis. So, although its height from the principal axis is constant, the image will be moving in x as well as y direction because the magnification is changing.

Calculation: Given $v_O = 0.1$ m/s = 10 cm/s. As we saw in Eq. 33-8

$$v_{Ix} = \frac{-f^2}{(u-f)} \times 10$$

$$= \frac{(10)^2}{(-15-10)^2} \times 0.10$$

$$= \frac{-100 \times 0.10}{25} = -0.40 \text{ m/s}.$$

For finding the velocity in y direction, we use Eq. 33-9:

$$v_{Iy} = \frac{d}{dt}\left(\frac{-fh_O}{u-f}\right) = \frac{-[f(u-f)(dh_O/dt) - fh_O(du/dt)]}{(u-f)^2}$$

$$= -f\frac{-[-(u-f)(dh_O/dt) + h_O(du/dt)]}{(u-f)^2}$$

$$= -\frac{(-10)\times 1\times 0.1}{(-15-10)^2} = -\frac{1}{5^2} = -0.04 \text{ m/s}.$$

Note: The negative sign for the velocity of the image means that the image is moving away from the principal axis and farther from the pole. This is to be expected because the image is approaching the focus where its magnification is going to be infinite and its location is also infinity.

SAMPLE PROBLEM 33.11

Images formed by a mirror broken into two parts

A mirror is broken into two parts and these parts are separated by a distance of 1 cm as shown in Fig. 33-47. Find the location of the images formed by the two parts of mirror. The object is midway between the two principal axes (Fig. 33-47).

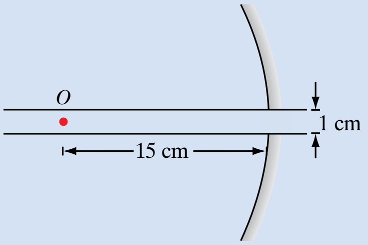


Figure 33-47 A point object is kept midway between the principal axes of the two parts of the mirror. For the upper mirror, the object is below the principal axis and for the lower mirror, the object is above the principal axis.

KEY IDEA

Here, we can see that for both the mirrors, the object is away from their respective principal axis. We can treat each individual mirror to form the image separately.

Calculation: Here for both the parts of the mirror, $u = -15$ cm. Focal length of both the parts of the mirror $= -10$ cm. So,

$$v = \frac{uf}{u-f} = -30 \text{ cm}.$$

For the upper part, the object is 0.5 cm below the principal axis. The height of image can be found by the formula for magnification as

$$\frac{h_I}{h_O} = -\frac{v}{u} = -2.$$

So,

$$h_I = -2\times -0.5 = 1 \text{ cm}.$$

Similarly, the second image will be 1 cm below the principal axis.

Learn: Dividing a mirror into two parts does not mean that the image is also halved. Remember that each point of the mirror reflects rays to form an image. We use only principal rays to trace the image because they are easy to handle. It is obvious to see that if we use only two rays to form an image, the intensity of the image will be too little for us to perceive the image. Actually, halving the mirror area will lead to the power of the light rays forming the image being halved.

REVIEW AND SUMMARY

Real and Virtual Images An *image* is a reproduction of an object via light. If the image can form on a surface, it is a *real image* and can exist even if no observer is present. If the image requires the visual system of an observer, it is a *virtual image.*

Image Formation *Spherical* can form images of a source of light—the object—by redirecting rays emerging from the source. The image occurs where the redirected rays cross (forming a real image) or where backward extensions of those rays cross (forming a virtual image). If the rays are sufficiently close to the *principal axis* through the spherical mirror we have the following relation between the *object distance* u (which is positive) and the *image distance* v (which is positive for real images and negative for virtual images):

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}=\frac{2}{r}$$

where f is the mirror's focal length and r is its radius of curvature. A *plane mirror* is a special case for which $r \to \infty$, so that $u = -v$. Real images form on the side of a mirror where the object is located, and virtual images form on the opposite side.

Lateral Magnification The magnitude of lateral magnification m of an object is the ratio of the image height h_I to the object height h_O

$$|m| = \frac{h_I}{h_O}$$

and is related to the object distance u and image distance v by

$$m = -\frac{v}{u}$$

Longitudinal Magnification It is defined as the ratio of the length of the object and the length of its image.

$$\frac{l_I}{l_O} = m_L = \frac{v_2 - v_1}{u_2 - u_1}.$$

For very small objects, longitudinal magnification is given by

$$\frac{l_I}{l_O} = -m^2$$

Concave Mirror: Nature of object and image

Object	Image	Nature	Magnification	Erect/ Inverted
I	II	Real object and real image	Diminished	Inverted
II	I	Real object and real image	Enlarged	Inverted
III	IV	Real object and virtual image	Enlarged	Erect
IV	III	Virtual object and real image	Diminished	Erect

Convex Mirror: Nature of object and image

Object	Image	Nature	Magnification	Erect/ Inverted
I	II	Virtual object and virtual image	Diminished	Inverted
II	I	Virtual object and virtual image	Enlarged	Inverted
III	IV	Virtual object and real image	Enlarged	Erect
IV	III	Real object and virtual image	Diminished	Erect

PROBLEMS

1. A moth at about eye level is 10 cm in front of a plane mirror; you are behind the moth, 30 cm from the mirror. What is the distance between your eyes and the apparent position of the moth's image in the mirror?
2. A car moving at a speed of 110 km/h approaches a bus traveling at a speed of 80 km/h. What is the velocity of the image of the car in the plane rear view mirror as seen by a passenger on the bus?
3. Figure 33-48 shows an overhead view of a corridor with a plane mirror M mounted at one end. A burglar B sneaks along the corridor directly toward the center of the mirror. If $d = 3.0$ m, how far from the mirror will she be when the security guard S can first see her in the mirror?

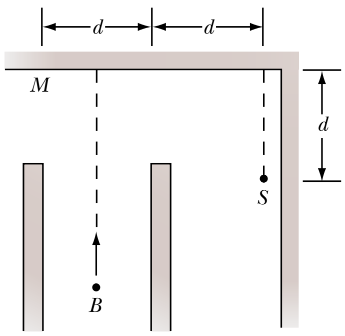


Figure 33-48 Problem 3.

4. (a) A luminous point is moving at speed v_O toward a spherical mirror with radius of curvature r, along the principal axis of the mirror. Show that the image of this point is moving at speed

$$v_I = -\left(\frac{r}{2u-r}\right)^2 v_O,$$

where u is the distance of the luminous point from the mirror at any given time. Now assume the mirror is concave, with $r = 15$ cm, and let $v_O = 5.0$ cm/s. Find v_I when (b) $u = 40$ cm (far outside the focal point), (c) $u = 7.8$ cm (just outside the focal point), and (d) $u = 10$ mm (very near the mirror).

5. A ray of light makes an angle of 10° with the horizontal and strikes the mirror. Find the angle of inclination of this mirror to the horizontal if the ray of light is to be reflected vertically?
6. The plane mirrors are parallel to each other and spaced 20 cm apart. A luminous point is placed in between and 5 cm from one mirror. Determine the distance from each mirror of three nearest images in each.

7. Two plane mirrors are inclined at an angle of 100° to each other. If a point object is kept on their angle bisector, (a) find the number of images and (b) their angular coordinates with respect to one of the mirrors.

8. A fluorescent lamp of length 1 m is placed horizontally at a depth of 1.2 m below a ceiling. A plane mirror of length 0.60 m is placed below the lamp parallel to and symmetric to the lamp at a distance 2.4 m from it. Find the length of the reflected patch of light on the ceiling.

9. You look through a camera toward an image of a hummingbird in a plane mirror. The camera is 4.30 m in front of the mirror. The bird is at camera level, 8.00 m to your right and 3.30 m from the mirror. What is the distance between the camera and the apparent position of the bird's image in the mirror?

10. Two bodies A and B are moving toward a plane mirror with speeds v_A and v_B, respectively, as shown in Fig. 33-49. What is the speed of image of A with respect to the body B?

11. A concave shaving mirror has a radius of curvature of 35.0 cm. It is positioned so that the (upright) image of a man's face is 2.50 times the size of the face. How far is the mirror from the face?

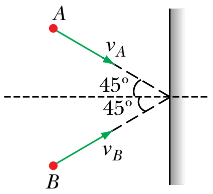


Figure 33-49 Problem 10.

12. An object is 30.0 cm from a spherical mirror, along the mirror's central axis. The mirror produces an inverted image with a lateral magnification of absolute value 0.500. What is the focal length of the mirror?

13. A millipede sits 1.0 m in front of the nearest part of the surface of a shiny sphere of diameter 0.70 m. (a) How far from the surface does the millipede's image appear? (b) If the millipede's height is 2.0 mm, what is the image height? (c) Is the image inverted?

14. Converging rays fall on a convex spherical mirror with radius of curvature 56 cm so that the reflected rays converge on the principal axis of the mirror. The distance from the intersection of these reflected rays to the mirror is 20 cm. Find the intersection point of the rays without the mirror.

15. An object is placed before a concave mirror normal to its principal axis so that the magnification produced is 6/5. After the object has moved away from the mirror by 25 cm, the magnification becomes 2/5. Calculate the focal length of the mirror.

16. A concave shaving mirror has a radius of curvature of 35.0 cm. It is positioned so that the (upright) image of a man's face is 1.70 times the size of the face. How far is the mirror from the face?

17. What will be the radius of the curvature of a concave mirror to form an image of the Sun 2 cm in diameter? The diameter of the Sun may be assumed to be 1/100 of its distance from the Earth.

18. A small piece of wire bent into an L shape with upright and horizontal portions of equal lengths is placed with the horizontal portion along the axis of the concave mirror whose radius of curvature is 10 cm. If the bend is 20 cm from the pole of the mirror, then find the ratio of the lengths of the images of the upright and horizontal portions of the wire.

19. Two spherical mirrors (convex and concave) having the same focal length of 36 cm are arranged so that their optical axes coincide. The separation between the mirrors is 1 m (Fig. 33-50). At what distance from the concave mirror should an object be placed so that its images formed by the concave and convex mirror independently are identical in size?

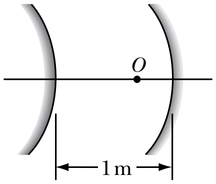


Figure 33-50 Problem 19.

20. A point source of light is 60 cm from a screen and is kept at the focus of a concave mirror which reflects light on the screen. The focal length of the mirror is 20 cm. What is the ratio of average intensities of the illumination on the screen when the mirror is present and when the mirror is removed?

21. An object is placed exactly midway between a concave mirror of radius of curvature 40 cm and a convex mirror of radius of curvature 30 cm. The mirrors face each other and are 50 cm apart. Determine the nature and position of the image formed by the successive reflections (a) first at the concave mirror and (b) then at the convex mirror.

22. A police inspector is chasing a thief who is running away in a car with a speed of 3 m/s. The speed of police jeep is 12 m/s, then what is the speed of image of police jeep as seen by thief in the rear-view mirror when police jeep is at a distance of 30 m. (Given that focal length of rear-view mirror is 15 m.)

23. A point object is 10 cm away from a plane mirror, and the eye of an observer (with pupil diameter 5.0 mm) is 20 cm away. Assuming the eye and the object to be on the same line perpendicular to the mirror surface, find the area of the mirror used in observing the reflection of the point.

24. Two rays are incident on a spherical mirror of radius of R = 5 cm parallel to its principal axis at the distance h_1 = 0.5 cm and h_2 = 3 cm. Determine the approximate distance Δx between the points at which these rays intersect the optical axis after being reflected at the mirror.

25. A small block of mass m and a concave mirror of radius 30 cm fitted with a stand lie on a smooth horizontal table with a separation 20 cm between them. The mirror together with its stand has mass m. The block is pushed at $t = 0$ toward the mirror at a constant speed 10 cm/s and collides with it. Then collision is perfectly elastic. Find the velocity of the image (a) at $t = 1$ s and (b) at $t = 3$ s.

PRACTICE QUESTIONS

Single Correct Choice Type

1. A child walks toward a fixed plane mirror at a speed of 5 km/h. The velocity of the image is

(a) 5 km/h **(b)** –5 km/h
(c) 10 km/h **(d)** –10 km/h

2. An object 0.5 m tall is in front of a plane mirror at a distance of 0.2 m. The size of the image formed is

(a) 0.2 m **(b)** 0.5 m
(c) 0.1 m **(d)** 1 m

3. One the axis of a spherical mirror of focal length f, a short linear object L lies on the axis at a distance u from the mirror. Its image has an axial length L' equal to

(a) $L\left[\dfrac{f}{(u-f)}\right]^{1/2}$ **(b)** $L\left[\dfrac{(u+f)}{f}\right]^{1/2}$

(c) $L\left[\dfrac{(u-f)}{f}\right]^{2}$ **(d)** $L\left[\dfrac{f}{(u-f)}\right]^{2}$

4. A plane mirror is approaching you at 10 cm/s. Your image shall approach you with a speed of

(a) +10 cm/s **(b)** –10 cm/s
(c) + 20 cm/s **(d)** –20 cm/s

5. To get three images a single object, one should have two plane mirrors at an angle of

(a) 60° **(b)** 90°
(c) 120° **(d)** 30°

6. A ray is incident on a plane surface. If $\hat{i}+\hat{j}-\hat{k}$ represents a vector along the direction of incident ray. $\hat{i}+\hat{j}$ is a vector along normal on incident point in the plane of incident and reflected ray, then vector along the direction of reflected ray is

(a) $-\dfrac{1}{\sqrt{19}}(-3\hat{i}+3\hat{j}+\hat{k})$ **(b)** $\dfrac{1}{\sqrt{19}}(3\hat{i}+3\hat{j}-\hat{k})$

(c) $-\dfrac{1}{\sqrt{19}}(\hat{i}+\hat{j}+\hat{k})$ **(d)** $\hat{k}$

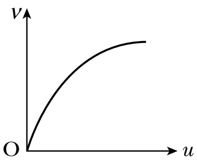
7. A bullet of mass m_2 is fired from a gun of mass m_1 with horizontal velocity v. A plane mirror is fixed at gun facing toward bullet. The velocity of the image of bullet formed by the plane mirror with respect to bullet is

(a) $\left(1+\dfrac{m_2}{m_1}\right)$ **(b)** $\left(\dfrac{m_1+m_2}{m_1}\right)v$

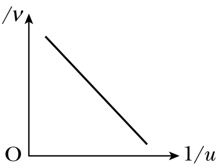
(c) $\dfrac{2(m_1+m_2)}{m_1}v$ **(d)** None of these

8. For a spherical mirror, which is the correct graph of $1/v$ versus $1/u$ among the four graphs shown below?

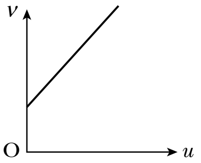
(a)



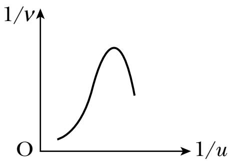
(b)



(c)



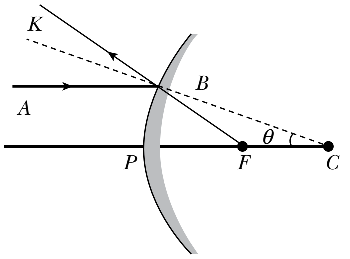
(d)



9. An object placed in front of concave mirror of focal length 0.15 m produces a virtual image, which is twice the size of the object. The position of the object with respect to the lens is

(a) –5.5 cm **(b)** –6.5 cm
(c) –7.5 cm **(d)** –8.5 cm

10. In the following figure, AB represents the incident ray and BK the reflected ray. If angle $BCF = \theta$, then $\angle BFP$ is given by



(a) θ **(b)** 2θ
(c) 3θ **(d)** 2.5θ

11. At an instant, a watch shows time 3:25. When seen through a mirror, time appeared will be

(a) 8:35 **(b)** 9:35
(c) 7:35 **(d)** 8:25

12. A convex mirror and a concave mirror of radius 10 cm each are placed 15 cm apart facing each other. An object is placed midway between them. If the reflection first takes place in the concave mirror and then in the convex mirror, the position of the final image is

(a) On the pole of the convex mirror
(b) On the pole of the concave mirror
(c) At a distance of 10 cm from convex mirror
(d) At a distance of 5 cm from concave mirror

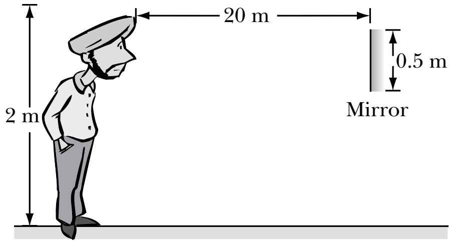
13. Two plane mirrors are inclined to each other at an angle 60°. If a ray of light incident on the first mirror is parallel to the second mirror, it is reflected from the second mirror

(a) Perpendicular to the first mirror
(b) Parallel to the first mirror
(c) Parallel to the second mirror
(d) Perpendicular to the second mirror

14. A plane mirror is placed at origin parallel to y axis, facing the positive x axis. An object starts from (2 m, 0, 0) with a velocity of $(2\hat{i}+2\hat{j})$ m/s. The relative velocity of image with respect to object is along the

(a) Positive x axis **(b)** Negative x axis
(c) Positive y axis **(d)** Negative y axis

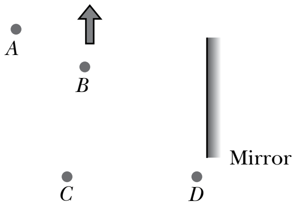
15. The person in the following figure is 2 m tall. A mirror is located 20 m away and is 0.5 m tall. The top of the mirror is at the same height as the top of the person's head. Assume for simplicity that the person's eyes are at the height of the top of his head. What part of his body does the person see in the mirror?



(a) Top 0.25 m **(b)** Top 0.5 m
(c) Top 1 m **(d)** Entire 2 m

16. The number of roman capital letters such as A, B, C, D, … which are not laterally inverted by a plane mirror is
(a) 6 **(b)** 7
(c) 11 **(d)** 13

17. From which point indicated in the following figure, if any, should you position your eyesight if you wish to see an image of an arrow in the mirror?



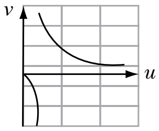
(a) Point *A* **(b)** Point *B*
(c) Point *C* **(d)** Point *D*

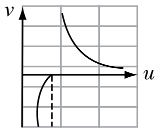
18. You are standing upright in a room in front of a vertical mirror. In this mirror, you can see from your position only the upper two-third of your body. You wish to see the entire length of your body reflected in the mirror. Which combination of the following three courses of action will achieve this?
(I) Move away from the mirror
(II) Move toward the mirror
(III) Use a mirror whose height will allow you to see your whole image
(a) (I) Only **(b)** (II) Only
(c) (III) Only **(d)** Either (I) or (III)

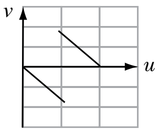
19. Two plane mirrors are placed parallel to each other at a distance L apart. A point object O placed between them, at a distance $L/3$ from one mirror. Both mirrors form multiple images. The distance between any two images cannot be
(a) $3L/2$ **(b)** $2L/3$
(c) $2L$ **(d)** None of these

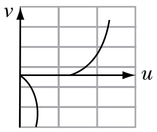
20. The light reflected by a plane mirror may form a real image,
(a) If rays incident on the mirror are converging
(b) If the rays incident on the mirror are diverging
(c) If the object is placed very close to the mirror
(d) Under no circumstances

21. As the position of an object (u) measured from a convex mirror varies, the position of the image (v) also varies. If u changes from 0 to $+\infty$, the graph between v and u will be

(a)


(b)


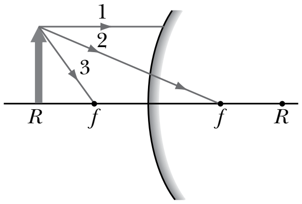
(c)


(d)


22. You walk slowly toward a large concave mirror. At first, you see your inverted image moving toward you. After you pass a certain point, you no longer see your image clearly. Moving still closer, you see a clear, enlarged and erect image of yourself behind the mirror. During the time when you cannot see a clear image
(a) You are closer to the mirror than the focal point and the image is now virtual and invisible
(b) You are between the center of curvature and the focal point, and the image is now behind you
(c) You are between the center of curvature and the focal point, and the image is now virtual and invisible
(d) You are closer to the mirror than the focal point and the image is now behind you

23. What can be the largest distance of an image of a real object from a convex mirror of radius of curvature 20 cm
(a) 10 cm **(b)** Infinity
(c) 20 cm **(d)** None

24. Which pairs of rays (shown in figure) from the object in the drawing are used to construct the image location produced by the convex spherical mirror of focal length f and radius R?



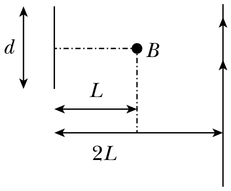
(a) 1 and 3
(b) 1 and 2
(c) 2 and 3
(d) Any pair of rays shown will work

25. A particle moves toward a concave mirror of focal length 30 cm along its axis and with a constant speed of 4 cm/s. What is the speed of its image when the particle is at 90 cm from the mirror?
(a) 2 cm/s **(b)** 8 cm/s
(c) 1 cm/s **(d)** 4 cm/s

26. A concave mirror is placed on a horizontal table with its axis directed vertically upwards. Let O be the pole of the mirror and C its centre of curvature. A point object is placed at C. It has a real image, also located at C. If the mirror is now filled with water, the image will be

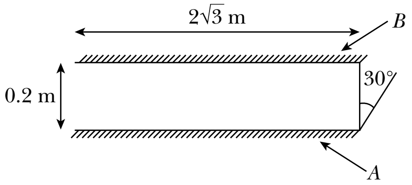
(a) Real and will remain at C.
(b) Real and located at a point between C and ∞.
(c) Virtual and located at a point between C and 0.
(d) Real and located at a point between C and 0.

27. A point source of light B placed at a distance L in front of the centre of a mirror of width d, hangs vertically on a wall. A man walks in front of the mirror along a line parallel to the mirror at a distance $2L$ from it as shown. The greatest distance over which he can see the image of the light source in the mirror is



(a) $d/2$ **(b)** d
(c) $2d$ **(d)** $3d$

28. Two plane mirrors A and B are aligned parallel to each other, as shown in the figure. A light ray is incident at an angle of 30° at a point just inside one end of A. The plane of incidence coincides with the plane of the figure. The maximum number of times the ray undergoes reflections (including the first one) before it emerges out is



(a) 28 **(b)** 30
(c) 32 **(d)** 34

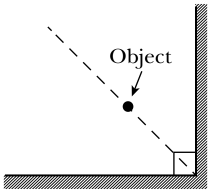
29. **Statement-1:** The formula connecting u, v and f for a spherical mirror is valid only for mirrors whose sizes are very small compared to their radii of curvature.

because

Statement-2: Laws of reflection are strictly valid for plane surfaces, but not for large spherical surfaces.

(a) Statement-1 is True, Statement-2 is True; Statement-2 **is** a correct explanation for Statement-1.
(b) Statement-1 is True, Statement-2 is True; Statement-2 is **not** a correct explanation for Statement-1.
(c) Statement-1 is True, Statement-2 is False.
(d) Statement-1 is False, Statement-2 is True.

30. An object is placed near two perpendicular plane mirrors as shown in the figure. How many images will be formed?

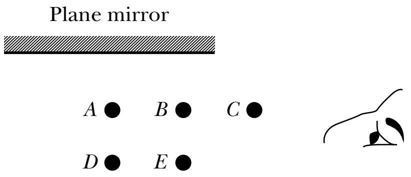


(a) 1 **(b)** 2
(c) 3 **(d)** 4

31. Daniel walks directly toward a plane mirror at a speed of 0.25 m/s. Determine the speed of the image relative to him.

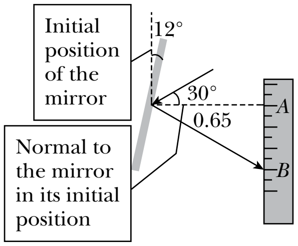
(a) 0.13 m/s **(b)** 0.50 m/s
(c) 0.25 m/s **(d)** 0.75 m/s

32. Five balls labeled A, B, C, D, and E are placed in front of a plane mirror as shown in the figure. Which ball(s) will the observer see reflected in the mirror?



(a) A only **(b)** C only
(c) A and B **(d)** A, B, D and E

33. The view in the figure is from above a plane mirror suspended by a thread connected to the center of the mirror at point C. A scale is located 0.65 m (the distance from point C to point A) to the right of the center of the mirror. Initially, the plane of the mirror is parallel to the side of the scale; and the angle of incidence of a light ray which is directed at the center of the mirror is 30°. A small torque applied to the thread causes the mirror to turn 12° away from its initial position. The reflected ray then intersects the scale at point B. What is the distance from point A to point B on the scale?


Initial position of the mirror
12°
30°
A
0.65
Normal to the mirror in its initial position
B

(a) 0.37 m **(b)** 0.89 m
(c) 0.58 m **(d)** 1.0 m

34. Which one of the following statements concerning a convex mirror is true?

(a) A convex mirror can form a real image.
(b) A convex mirror must be spherical in shape.
(c) The image produced by a convex mirror will always be inverted relative to the object.
(d) The image a convex mirror produces is closer to the mirror than it would be in a plane mirror for the same object distance.

35. An object is placed 30.0 cm from a convex spherical mirror with radius of curvature 40.0 cm. Which one of the following phrases best describes the image?

(a) Virtual and located at infinity
(b) Virtual and located 12 cm from the mirror
(c) Real and located 12 cm from the mirror
(d) Virtual and located 17 cm from the mirror

More than One Correct Choice Type

36. A student performed the experiment of determination of focal length of a concave mirror by u–v method using an optical bench of length 1.5 m. The focal length of the mirror used is 24 cm. The maximum error in the location of the image can be 0.2 cm. The 5 sets of (u, v) values recorded by the student (in cm) are: (42, 56), (48, 48), (60, 40), (66, 33), (78, 39). The data set(s) that cannot come from experiment and is (are) incorrectly recorded, is (are)

(a) (42, 56) **(b)** (48, 48)
(c) (66, 33) **(d)** (78, 39)

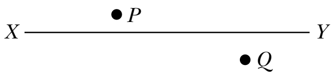
37. In case of three plane mirrors meeting at a point to form a corner of a cube, if incident light suffers one reflection on each mirror,

(a) The emergent ray is anti-parallel to incident one
(b) The emergent ray is perpendicular to incident one
(c) The emergent ray is in phase with incident one
(d) The emergent ray is in opposite phase with incident one

38. A plane mirror, reflecting a ray of incident light, is rotated through an angle θ about an axis through the point of incidence in the plane of the mirror perpendicular to the plane of incidence, then

(a) The reflected ray does not rotate
(b) The reflected ray rotates through an angle θ
(c) The reflected ray rotates through an angle 2θ
(d) The incident ray is fixed

39. Two points P and Q lie on either side of an axis XY as shown in the following figure. It is desired to produce an image of P at Q using a spherical mirror, with XY as the optic axis. The mirror must be



(a) Converging
(b) Diverging
(c) Positioned to the left of P
(d) Positioned to the right of Q

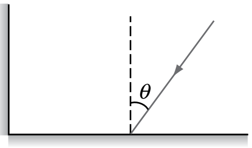
40. If a converging beam of light is incident on a concave mirror, the reflected light

(a) May form a real image
(b) Must form a real image
(c) May form a virtual image
(d) May be a parallel beam

41. A point object P moves toward a convex mirror with a constant speed v, along its optic axis. The speed of the image

(a) Is always less than v
(b) May be greater than and equal to v or less than v depending on the position of P
(c) Increases as P comes closer to the mirror
(d) Decreases as P comes closer to the mirror

42. Two plane mirrors are arranged at right angles to each other as shown in the following figure. A ray of light is incident on the horizontal mirror at angle θ to the normal. For what value of θ the ray emerges parallel to the incoming ray after reflection from the vertical mirror.



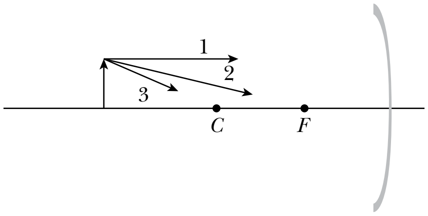
(a) 60° **(b)** 30°
(c) 45° **(d)** 70°

43. Two concave mirrors of the same focal length f are kept facing each other. A point source is kept symmetrically between them. What should be the distance between them so that only one image is formed at the source?

(a) f **(b)** $2f$
(c) $3f$ **(d)** $4f$

Linked Comprehension

Paragraph for Questions 44–46: An object is placed in front of a concave spherical mirror as shown below. The three rays **1**, **2**, and **3**, leave the top of the object and, after reflection, converge at a point on the top of the image. Ray **1** is parallel to the principal axis, ray **2** passes through F, and ray **3** passes through C.



44. Which ray(s) will pass through F after reflection?

(a) **1** only
(b) **3** only
(c) **2** only
(d) both **1** and **2**

45. Which ray(s) will reflect back on itself (themselves)?

(a) **1** only
(b) **3** only
(c) **2** only
(d) both **1** and **2**

46. Which one of the following groups of terms best describes the image?

(a) Real, upright, enlarged
(b) Real, inverted, enlarged
(c) Real, inverted, reduced
(d) Virtual, inverted, reduced

Paragraph for Questions 47 and 48: A 3.0 cm object is placed 8.0 cm in front of a mirror. The virtual image is 4.0 cm further from the mirror when the mirror is concave than when it is planar.

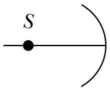
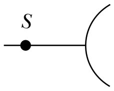
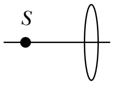
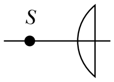
47. Determine the focal length of the concave mirror.

(a) 6.0 cm **(b)** 24 cm
(c) 12 cm **(d)** 48 cm

48. Determine the image height in the concave mirror.
(a) 0.5 (b) 4.5
(c) 1.5 (d) 3.0

Matrix-Match

49. An optical component and an object S placed along its optical axis are given in Column I. The distance between the object and the component can be varied. The properties of images are given in Column II. Match all the properties of images from Column II with the appropriate components given in Column I.

Column I	Column II
(a) 	**(p)** Real image
(b) 	**(q)** Virtual image
(c) 	**(r)** Magnified image
(d) 	**(s)** Image at infinity

Directions for Question 50: In the given question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

50. In the given table, Column I shows the position of objects, Column II shows the position of the image and Column III shows the size of the image formed.

Column I	Column II	Column III
(I) Object is at infinity	**(i)** $v < \infty$	**(J)** very small in dimensions
(II) Object is in between a distance f and $2f$	**(ii)** $v = -f$	**(K)** small image
(III) Object is at center of curvature	**(iii)** v is positive	**(L)** same size
(IV) Object is kept within focus	**(iv)** $v = -2f$	**(M)** magnified image

(1) In which case is the magnitude of magnification zero?
(a) (I) (iii) (L) (b) (III) (ii) (K)
(c) (II) (iv) (K) (d) (I) (ii) (J)

(2) When is the value of magnification greater than 1?
(a) (II) (i) (M) (b) (IV) (iii) (L)
(c) (II) (iii) (K) (d) (II) (iv) (M)

(3) When is the value of magnification equal to 1?
(a) (III) (i) (L) (b) (IV) (i) (J)
(c) (III) (iv) (K) (d) (I) (iii) (J)

Integer Type

51. A girl with two normal eyes wants to see full width of her face by a plane mirror. The eye-to-eye and ear-to-ear distances of her face are 10 cm and 14 cm, respectively. Find the minimum width of required mirror.

52. A man stands in front of a large vertical plane mirror. His height is 1.8 m. The distance of image from the man is 4 m. At what distance is the man standing from the mirror?

53. A construction engineer wants to distinguish between two pillars located at a distance of 11 km. Find the minimum distance between the pillars.

ANSWER KEY

Checkpoints

1. When the cardboard is held close to the wall, the shadow formed is sharp because the light coming from different directions does not spread much behind the cardboard. Think about it: At a large distance, you will see an extended source as a point object only.
2. 3
3. Three images at 150°, –90°, and at –150° to the x axis.
4. The object and image are always on the same side of the focus
5. (a) real; (b) inverted; (c) same
6. (a) Convex mirrors; (b) therefore they are used in rear view mirrors

Problems

1. 40 cm
2. 50 km/h
3. 1.5 m
4. (b) 0.27 cm/s; (c) 3.1×10^3 cm/s; (d) 6.7 cm/s
5. 40°
6. 5 cm, 35 cm, 45 cm and 15 cm, 25 cm, 55 cm
7. (a) Four images; (b) 150°, 210°, 250°, 310°

8. 3 m
9. 11.0 m
10. $V_A + V_B$
11. 10.5 cm
12. 10.0 cm
13. (a) 0.15 m; (b) 0.30 m; (c) since $m > 0$, the image is erect
14. 11.67 cm
15. $f = 15$ cm
16. 7.21 cm
17. 400 cm
18. 3:1
19. 86 cm in front of concave mirror
20. 10:1
21. (a) 100 cm; (b) 21.43 cm
22. 1m/s
23. 2.2 mm^2
24. 5/8 = 0.625 cm
25. (a) 90 cm/s; (b) 100 cm/s

Practice Questions

Single Correct Choice Type

1. (b)
2. (b)
3. (d)
4. (b)
5. (b)
6. (c)
7. (c)
8. (b)
9. (c)
10. (b)
11. (c)
12. (a)
13. (b)
14. (b)
15. (c)
16. (a)
17. (c)
18. (c)
19. (a)
20. (a)
21. (a)
22. (d)
23. (a)
24. (b)
25. (c)
26. (d)
27. (d)
28. (b)
29. (c)
30. (c)
31. (b)
32. (a)
33. (b)
34. (d)
35. (b)

More Than One Correct Choice Type

36. (c), (d)
37. (a), (d)
38. (c), (d)
39. (a), (c)
40. (a), (c), (d)
41. (a), (c)
42. (a), (b), (c), (d)
43. (b), (d)

Linked Comprehension

44. (a)
45. (b)
46. (c)
47. (b)
48. (b)

Matrix-Match

49. (a) → (p), (q), (r), (s); (b) → (q); (c) → (p), (q), (r), (s); (d) → (p), (q), (r), (s)
50. (1) → (d); (2) → (a); (3) → (c)

Integer Type

51. 2
52. 2
53. 3

CHAPTER

34

Geometrical Optics: Refraction

Contents

34.1 What is Physics?
34.2 Snell's Laws of Refraction
34.3 Total Internal Reflection
34.4 Spherical Refracting Surfaces
34.5 Thin Spherical Lenses
34.6 Lens Mirror Combination
34.7 Prism
34.8 Chromatic Dispersion

34.1 | WHAT IS PHYSICS?

Refraction applications are abundant. From the day we open our eyes, we see the world by refraction through our eye lens. When we are viewing very small objects, or very far ones, we make use of optical instruments which are based on laws of refraction. Even measuring the apparent depth of glass of water one is holding requires knowledge of laws of refraction laws. In this chapter, we are going to study the application of laws of refraction we had introduced in Chapter 32.

34.2 | SNELL'S LAWS OF REFRACTION

Key Concept

- According to law of refraction, a refracted ray lies in the plane of incidence and has an angle of refraction θ_2 that is related to the angle of incidence θ_1 by $n_1 \sin \theta_1 = n_2 \sin\theta_2$, where n_1 and n_2 are the refractive index of the mediums.

In Chapter 32, Section 32.8, we introduced the idea of refraction of light. To briefly recapitulate the idea, as light ray travels from one transparent medium to another transparent medium, it bends away from or toward the normal. Snell's law predicts a relationship between the angle of incidence and the angle of refraction.

Law of refraction: A refracted ray lies in the plane of incidence and has an angle of refraction θ_2 that is related to the angle of incidence θ_1 by

$$n_1 \sin \theta_1 = n_2 \sin \theta_2. \tag{34-1}$$

Here, each of the symbols n_1 and n_2 is a dimensionless constant, known as the **index of refraction (or the refractive index)**, which is associated with a medium involved in the refraction.

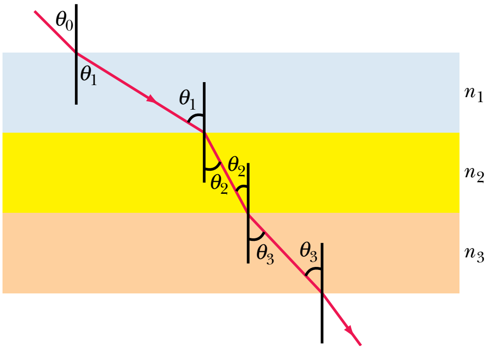


Figure 34-1 Refraction of light through multiple slabs.

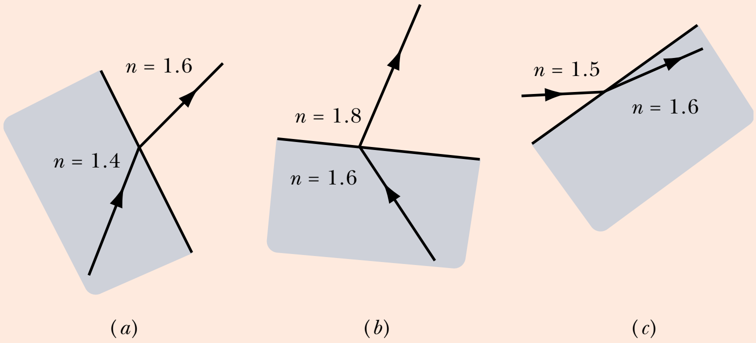
If a light ray passes through many successive boundaries that are parallel to each other as shown in Fig. 34-1, then

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 = n_3 \sin \theta_3. \tag{34-2}$$

This relationship is valid only if all the boundaries are parallel to each other. This is easy to see. The angle of refraction into one medium will be the angle of incidence into the next medium. So, applying Snell's law successively from one medium to another, we arrive at Eq. 34-2.

CHECKPOINT 1

Which of the three drawings here (if any) show physically possible refraction?



SAMPLE PROBLEM 34.01

Lateral displacement of ray passing through a glass slab

A ray of light is incident on a parallel sided glass slab of refractive index n and thickness t as shown in Fig. 34-2. Find the lateral displacement of the ray as it passes through the slab.

KEY IDEA

Once we apply Snell's law, all we need to do is apply geometry to the problem. Let us start by drawing two triangles. From the second point from where the refracted ray enters back into air, we drop a perpendicular on the incident ray extended.

Calculation: From Snell's law

$$1 \sin \theta = n \sin \phi$$

Here, we try to find the common hypotenuse (h) of both the triangle. From the definition of cosine and sine,

$$\cos \phi = \frac{t}{h} \Rightarrow h = \frac{t}{\cos \theta}.$$

Also,

$$\sin(\theta - \phi) = \frac{x}{h}.$$

Here, x is the lateral displacement as is clear from the figure. Therefore,

$$x = \frac{t\sin(\theta-\phi)}{\cos\phi}.$$

Note: If you look at the stars in the sky, they appear to be twinkling. This is due to slight variation in refractive index which changes the lateral displacement of rays coming from them.

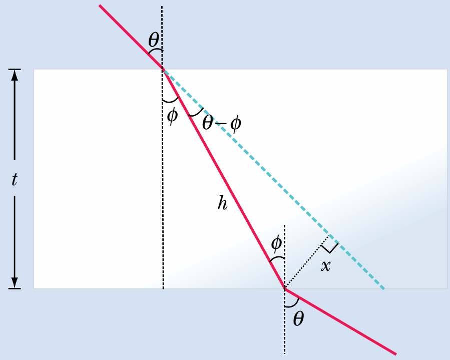


Figure 34-2 A ray of light passes through a glass slab whose sides are parallel to each other. We wish to find the lateral displacement of the ray in passing through the slab.

SAMPLE PROBLEM 34.02

Refraction of ray passing through tank filled with water

An observer is viewing along the line shown in Fig. 34-3. When there is no water filled in the tank, he can see only wall AB and no other part of the base of the tank. Then water is filled in till height h and he is just able to see the point E as shown in Fig. 34-3. Find the angle of refraction into air (θ) and angle of incidence from water (ϕ). Also find the depth to which water has been filled.

KEY IDEA

Here, we can see that refraction of ray enables the observer to see E. So, we can apply Snell's law and geometry.

Calculation: From Fig. 34-3, we can see that the normal and wall AB are parallel to each other. So, angle of refraction into air is

$$\tan\theta = \frac{\frac{160}{3}}{40} \Rightarrow \theta = 53°.$$

From Snell's law, we can say that

$$1\times\sin\theta = \frac{4}{3}\sin\phi \Rightarrow \phi = 37°.$$

From Fig. 34-3, we can say that

$$\tan 53° = \frac{x}{40-h} \quad \text{and} \quad \tan 37° = \frac{30-x}{h}.$$

Solving, we get

$$h = 40 \text{ cm}.$$

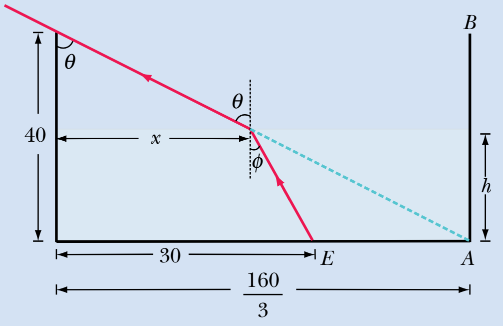


Figure 34-3 When the water is filled in a tank, we can see the hidden part of the tank as well with the help of refracted rays.

SAMPLE PROBLEM 34.03

Refraction of ray incident on a boundary of another plane

A ray given by $-\hat{i}-2\hat{j}$ is incident on xz plane as shown in Fig. 34-4. Above the xz plane, refractive index is $n_1 = 2$ and below the xz plane refractive index is $n_2 = \sqrt{5/2}$. Find a unit vector in the direction of the refracted ray.

KEY IDEAS

(1) Note that although the incident ray is drawn in the first quadrant, its direction is in the third quadrant. This is because the direction of a vector is given by the

direction in which its arrow points. To represent this vector in a vector form, we have to redraw it in such a way that its tail is at the origin. Note that refraction deviates a ray from its path; it does not change the quadrant. For the complete direction of the refracted ray, we can use Snell's law.

(2) First, we convert the incident ray vector into a unit vector by dividing it by its magnitude. Then we can apply Snell's law. As seen from Fig. 34-4, we can say that the tangential component of the incident ray is 1 sin θ because the magnitude of the incident ray vector has been made unity.

Similarly, the tangential component of the refracted ray is given by 1 sin ϕ. By Snell's law,

$$n_1 \sin\theta = n_2 \sin\phi.$$

So, we can say that

$$n_1 \times \text{tangential component of incident ray} = n_2 \times \text{tangential component of refracted ray.}$$

Once we get the tangential component of the refracted ray, we get the normal component by using the fact that the refracted ray is also a unit vector. So, its magnitude is 1.

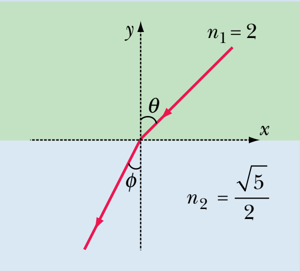


Figure 34-4 A ray travelling in x–y plane is incident on a boundary which separates the two media. The boundary of the media is x–z plane.

Calculation: Converting the incident ray vector to a unit vector:

$$\hat{e}_i = \frac{(-\hat{i}-2\hat{j})}{\sqrt{(-1)^2+(-2)^2}} = -\frac{\hat{i}}{\sqrt{5}} - \frac{2}{\sqrt{5}}\hat{j}.$$

From Fig. 34-4, it can be seen that the normal is in the y direction.

The component of this vector in the tangential direction (x direction in this case) is $-\hat{i}/\sqrt{5}$. So, by Snell's law, we can say that

$$-\frac{\hat{i}}{\sqrt{5}} \times 2 = \frac{\sqrt{5}}{2} \times \text{tangential component of refracted ray.}$$

$$\text{Tangential component of refracted ray} = -\frac{4\hat{i}}{5}.$$

Let the normal component of the refracted ray be $-c\hat{j}$. (The refracted ray should also travel in the third quadrant.) Therefore,

$$\hat{e}_r = -\frac{4\hat{i}}{5} - c\hat{j}.$$

So,

$$1 = \sqrt{\left(-\frac{4}{5}\right)^2 + (-c)^2}$$

$$c = \frac{3}{5}.$$

Therefore,

$$\hat{e}_r = -\frac{4}{5}\hat{i} - \frac{3}{5}\hat{j}.$$

Learn: In more involved problems, we can also involve the fact that incident ray, refracted ray, and normal, all three lie in the same plane. Hence, their scalar triple product is zero. If one finds it encouraging, then one must try this out: $\hat{e}(\hat{e}_r \times \hat{n}) = 0$.

Apparent Depth

It is a common experience that when we view an object kept in water from above, the object seems to be shifted up. This happens due to refraction at the water surface. Thus, the real depth is different from apparent depth. This is clearly shown in Figs. 34-5*a*–*d*.

To understand this more clearly, consider Fig. 34-6. In it, rays are diverging from the real object O. For simplicity, we consider only two rays as shown in the figure. One of the rays is incident normally on the surface and therefore, does not deviate ($\theta = 0$; $\phi = 0$). The second ray goes from denser to rarer medium, so it bends away from the normal. These refracted rays meet at a point I. So, I is the image. Since refracted rays are diverging, the image is virtual in nature. Let us find the value of the apparent depth.

In this figure, we have assumed that the rays go from denser to rarer medium. But the formula is equally valid for the other combinations also. The angle of incidence is θ and angle of refraction is ϕ. The point at which refracted rays meet is called the image and the point at which incident rays meet is called the object.

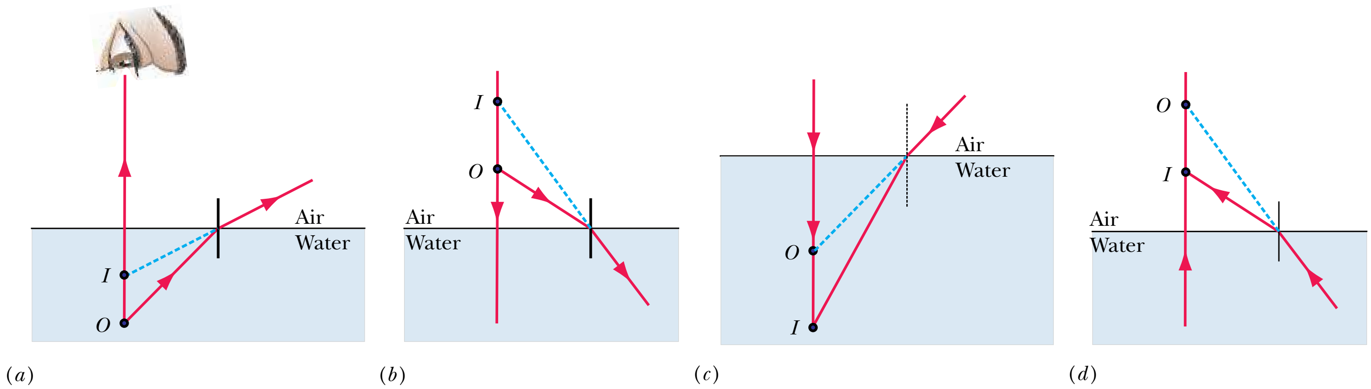


Figure 34-5 (*a*) Real object at O, virtual image is formed at I. The observer is in air. (*b*) Real object is at O, virtual image is formed at I. The observer is in water. (*c*) Virtual object at O, real image formed at I. The observer is in water. (*d*) Virtual object at O, real image is formed at I. The observer is in air.

Since the object is viewed normally and the pupil diameter is very small, only those rays enter our eyes which are very close to normal. So, we can safely assume the angles involved (θ and ϕ) to be very small. For small angles, we can say that

$$\theta = \frac{AB}{h_O} \text{ and } \phi = \frac{AB}{h_I}.$$

According to Snell's law, we can say that

$$n_r \sin \phi = n_i \sin \theta,$$

where n_i is the refractive index for the incident rays and n_r is the refractive index for the refracted rays.

For small angles, we can say that

$$n_r\, \phi = n_i\, \theta.$$

Solving, we have

$$\frac{h_I}{n_r} = \frac{h_O}{n_i}. \tag{34-3}$$

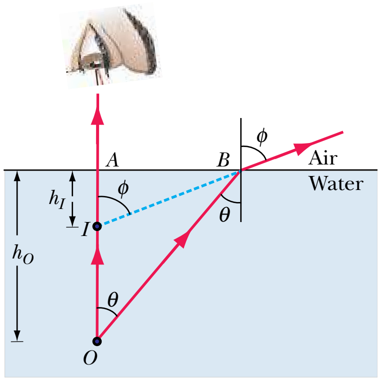


Figure 34-6 An observer in air views an object O in water. The rays form an image at point I. In the figure, we have shown only two rays for simplicity. In reality, infinite rays can be backtracked to form an image at I.

If the object is viewed normally, there is no lateral shift. The shift is only in the normal direction. Here, we should note that the object and the image are always on the same side of the surface. So, we should not take n_i as the refractive index corresponding to the object. Rather, it is the refractive index for the incident rays and n_r is the refractive index for the refracted rays. Be careful of the fact that the distances are always measured from surface. Here, we can note that a real object forms a virtual image and a virtual object forms a real image. Therefore, Eq. 34-3 is valid only when the object is viewed normally.

We have discussed in Chapter 33 that real–virtual combination will always be erect. Here, one can again see that this comes true.

SAMPLE PROBLEM 34.04

Apparent distance of the object and its image formed by plane mirror

A fish is floating in a tank filled with water to depth h. The bottom is silvered. A boy looks at the fish from above (Fig. 34-7).

(a) Find the apparent distance at which the boy sees the fish.

KEY IDEA

Notice that the boy is viewing the fish normally. The rays emanating from fish do not directly reach the boy; they undergo refraction at the water–air interface. It seems

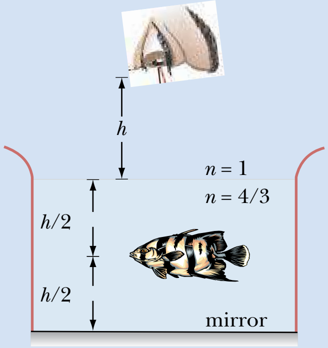


Figure 34-7 A boy is viewing a fish kept in an aquarium whose bottom is made up of a plane mirror.

that the rays should not deviate at all when we are viewing normally. However, it is not one single ray that we are going to use for the formation of image. There is a narrow pencil of rays which refract to form an image. So, we can use the formula for the apparent depth (Eq. 34-3) that we derived just before.

Calculation: Given that the rays are incident from water to air. So, n_i is 4/3 and n_r is 1. The depth is measured from the water surface, so the depth is taken to be $h/2$. Now,

$$\frac{d_I}{\mu_r} = \frac{d_O}{\mu_i}$$

$$\frac{d_{\text{app}}}{1} = \frac{\frac{h}{2}}{\frac{4}{3}}.$$

As discussed before, this answer is the distance from the water surface.

Distance from the observer is

$$\frac{3h}{8} + h = \frac{11h}{8}.$$

(b) Find the apparent distance at which the boy sees the image of the fish (reflected in the mirror at the bottom).

KEY IDEAS

(1) Here, we can see that first the rays from fish are reflected from the mirror. These reflected rays will again strike the water surface and after refraction from there, final image will be formed. Here two events occur one after another.

(2) Reflection from the mirror: For this, the object is at a distance $h/2$ from the mirror. So, the image will be formed at a distance $h/2$ behind the mirror. Note that although this image will be formed behind the mirror, no refraction will occur to form this image. Therefore, we need not consider the apparent depth while considering this reflection. This image will act as an object for refraction from water surface. For this refraction, the distance is measured from the water surface.

Calculation: The distance of this object from the water surface will be $h + (h/2) = 3h/2$. Therefore,

$$\frac{d_{\text{app}}}{1} = \frac{\frac{3h}{2}}{\frac{4}{3}} = \frac{9h}{8}.$$

So, distance from observer is

$$\frac{9h}{8} + h = \frac{17h}{8}.$$

Although the object here is in air (image behind the mirror), we need not consider the refractive index of air because in the formula for the apparent depth, we consider the refractive index for the incident rays and that for the refracted rays.

Note: If you are still having a doubt as to what refers to n_i and what refers to n_r, draw a ray diagram to see the formation of image.

CHECKPOINT 2

Suppose now that the fish sees the boy. Find (a) apparent distance of eye as seen by fish. (b) apparent distance of image of eye as seen by fish.

SAMPLE PROBLEM 34.05

Apparent distance of the image formed by a concave mirror dipped in water

Find the location of the final image of O formed by the system (Fig. 34-8). Assume that O is on the principal axis of the concave mirror.

KEY IDEAS

(1) Here, the image is formed after three events: refraction from water; reflection from mirror, and again refraction from water.

(2) As we discussed in Chapter 33, image of one event will act as an object for the next event. Thus, first event we consider is refraction from water for which we can apply formula for apparent depth. Then next event will be reflection from concave mirror. The next event would be again refraction from water surface which will be the last event.

Calculation: When the mirror sees the object O, it does not appear to be at 10 cm from the water surface. Rather, its apparent distance is

$$\frac{d_{\text{app}}}{n_{\text{refracted}}} = \frac{d_{\text{act}}}{n_{\text{incident}}}$$

$$d_{\text{app}} = 10 \times \frac{4}{3} = \frac{40}{3} \text{ cm from water surface.}$$

The image formed by this refraction will act as an object for the next surface (mirror):

$$u = -\left(\frac{40}{3} + \frac{5}{3}\right) = -15\,\text{cm}.$$

Note that focal length of mirror ($r/2$) does not depend on the refractive index of surrounding medium. The image distance is given by

$$v = \frac{uf}{u-f} = \frac{-15 \times -10}{-15 - 10} = -30 \text{ cm}.$$

These reflected rays will not actually be able to meet at 30 cm in front of the mirror. However, this image will act as an object for the next event: refraction from the water surface.

For this event, we have to measure distance of the object from the water surface:

$$\frac{d_{\text{app}}}{n_{\text{refracted}}} = \frac{d_{\text{act}}}{n_{\text{incident}}}.$$

In this event, the rays are incident from water and refracted into air:

$$\frac{d_{\text{app}}}{1} = \frac{\frac{85}{3}}{\frac{4}{3}}$$

$$d_{\text{app}} = \frac{85}{4} \text{ cm from water surface.}$$

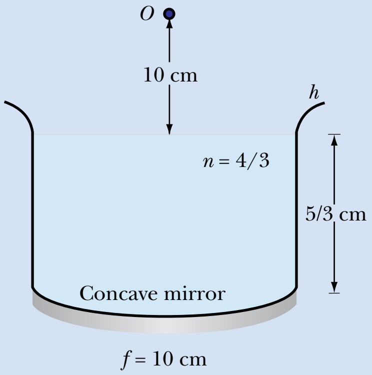


Figure 34-8 Forming an image of object O by a concave mirror dipped in water.

Apparent Length

From the above two examples, we can see that, because of refraction the depth of the object will seem to change but its transverse dimensions will not change. So, if an object is kept normal to the surface, its length will appear to be different from the actual length. Let us consider that an observer is observing an object of length y dipped in water (Fig. 34-9).

Let us assume that water has refractive index n and the person is viewing it normally. The image of the upper end of the object is viewed by the observer to be at

$$\frac{d_{\text{app}}}{1} = \frac{d}{n}.$$

The image of the lower end of the object is viewed by the observer to be at

$$\frac{d_{\text{app}}}{1} = \frac{d+y}{n}.$$

Hence, the apparent length of the object will seem to be

$$\frac{d+y}{n} - \frac{d}{n} = \frac{y}{n}.$$

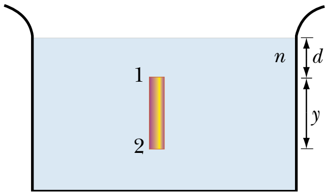


Figure 34-9 Finding the apparent length of an object dipped in water.

SAMPLE PROBLEM 34.06

Apparent velocity of fish floating in tank

A tank is filled with water up to a height of 15 cm and a fish is floating inside at a depth of 10 cm. The tank is now being drained so that the surface moves down with a velocity of 1 cm/s. An observer is viewing the fish from a height of 10 cm.

(a) Find the apparent velocity of the fish as seen by the observer.

(b) Find the apparent velocity of the observer as seen by the fish.

KEY IDEA

If the object moves with a velocity V inside the medium, its apparent velocity can be found by rate of change of apparent depth.
Let us assume that x_{os} = depth of x with respect to surface. Then

$$\frac{d_{\text{app}}}{n_{\text{refracted}}} = \frac{d_{\text{actual}}}{n_{\text{incident}}}.$$

Here, the depth of image is changing because the depth of the object is changing with respect to surface.

Differentiating both sides, we get

$$\frac{V_{IS}}{n_r} = \frac{V_{OS}}{n_i} \Rightarrow \frac{V_I - V_S}{n_r} = \frac{V_O - V_S}{n_i}, \tag{34-4}$$

where V_{IS} is the velocity of image with respect to the surface and V_{OS} is the object of image with respect to the surface. These refer to the velocities in the normal direction. In a direction parallel to the surface, the velocity of the image will be the same as the velocity of the object.

Calculation: Assuming downward direction to be positive, the velocity of the object (fish) is zero (fish is floating). Now, $V_S = +1$ cm/s. Therefore,

$$\frac{V_{IS}}{n_r} = \frac{V_{OS}}{n_i}$$

$$V_{IS} = \frac{n_r}{n_i} V_{OS} = \frac{1}{\frac{4}{3}} \times (V_O - V_S) - \frac{3}{4}\,\text{m/s}$$

$$V_{IS} = V_I - V_S.$$

$$-\frac{3}{4} + 1 = V_I \Rightarrow V_I = \frac{1}{4}\,\text{m/s}. \tag{34-5}$$

Again

$$\frac{V_{IS}}{\frac{4}{3}} = \frac{V_{OS}}{1} = V_S \times \frac{4}{3} = \frac{-4}{3}\,\text{m/s}.$$

$$V_I - V_S = \frac{4}{3}$$

$$V_I = 1 - \frac{-4}{3} = \frac{-1}{3}\,\text{cm/s}. \tag{34-6}$$

Learn: The fish seems to be at a lesser depth than it actually is. So, when depth of water decreases, the image of fish will approach its actual position. So, V_I should be downward in Eq. 34-5. A similar argument can be applied to Eq. 34-6.

CHECKPOINT 3

We know that the light ray bends toward the normal when it goes from a rarer to a denser medium. This adjacent photograph seems to be wrong! The straw dipped in water seems to be bent away from the normal. Explain.

Normal Shift

If there are more than one medium separating the object and the observer (Fig. 34-10), then

$$\frac{d_{\text{app}}}{n} = \frac{d_1}{n_1} + \frac{d_2}{n_2}. \tag{34-7}$$

Here n is the medium for final refracted rays as shown in Fig. 34-10.

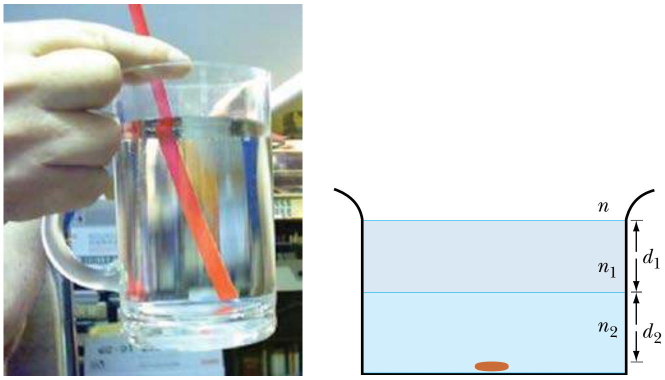


Figure 34-10 Finding the apparent depth in the case where there are more than one medium.

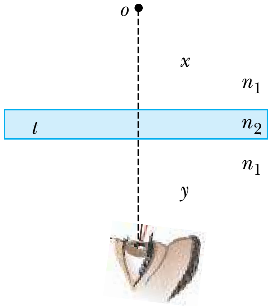


Figure 34-11 Viewing an object through a glass slab makes the object look closer.

Let us now try to visualize another situation related to refraction at plane surfaces. You might have observed that when you wear goggles, the whole world seems closer than it was before. This shift is due to refraction at the two plane surfaces of glass.

The shift of the object due to a thin parallel-sided slab placed on the path of light rays is called **normal shift**. The concept of normal shift is applicable only if the surface of the slab is perpendicular to the line joining the observer and the object. To understand this, let us assume that an object O is at a distance x from a slab of glass (Fig. 34-11). It is separated from the observer by a slab of refractive index n_2 and thickness t. Actual distance of the object from the observer is $x + t + y$.

Looking from medium II, the object seems to be closer than actual due to refraction at the two surfaces. From Eq. 34-7,

$$\frac{d_{\text{app}}}{n_1} = \frac{t}{n_2} + \frac{x}{n_1}$$

or

$$d_{\text{app}} = \frac{n_1 t}{n_2} + x.$$

This will be the depth of the object as seen from the surface of slab. According to the definition of normal shift,

$$\text{Normal shift } = d_{\text{act}} - d_{\text{app}} = x + t - \left(x + \frac{n_1 t}{n_2} \right)$$

$$= t\left(1 - \frac{n_1}{n_2} \right). \tag{34-8}$$

Here t is the thickness of the slab, n_1 is the refractive index of the surroundings, and n_2 is the refractive index of the slab.

Let us try to understand Eq. 34-8 here. The equation says that the object seems to be shifted from its actual position by a distance called the normal shift. This formula is valid for any position of the slab between the observer and the object. (x does not appear in the equation.) Another interesting point is that the normal shift is positive and hence in the direction of propagation of the light if refractive index of slab is greater than that of the surrounding medium. Recall the Cartesian sign convention we introduced in Chapter 33. The direction of propagation of light is assumed to be the positive direction (Figs. 34-12*a* and *b*). On the contrary, the normal shift is opposite to direction of propagation of light if the slab is having lesser refractive index than the medium because normal shift will become negative.

Therefore, O is a virtual object for the glass slab. After refraction, the rays converge at I. Therefore, I is a real image.

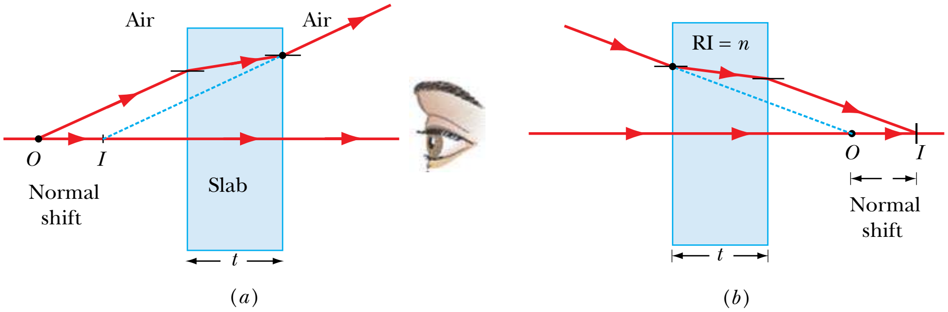


Figure 34-12 (*a*) A real object at *O* is being viewed by an observer through a slab as shown. Due to refraction, the object is perceived by the eye to be at *I*. (*b*) Converging rays from a convex lens or a concave mirror are incident on the slab as shown. The incident rays would converge at *O*.

SAMPLE PROBLEM 34.07

Location of image after multiple events

Find the location of final image as shown in Fig. 34-13. Take the refractive index of the slab as 4/3.

KEY IDEA

(1) This is again an example of multiple events before final image is formed. The rays coming from *O* will pass through the slab, get refracted by it and then reflected by the mirror, and again refracted by the slab. This refraction will form the final image. Whenever we refer to the image of the system, it refers to this image.

(2) The mirror will not see the object at 15 cm but shifted toward it. After reflection, image will not be directly formed at the point where the mirror focuses the light but will be shifted in the direction of light incident on the slab.

Calculations: The very first event is the refraction from the glass slab. It causes a shift in the direction of the incident rays. This normal shift is given by Eq. 34-8.

$$x = t\left(1-\frac{1}{n}\right)$$

$$= 4\times\left(1-\frac{3}{4}\right) = 1\text{ cm}.$$

Thus, the object will seem to be shifted toward the mirror (in the direction of the incident rays). So, the mirror will see the object at 15 − 1 = 14 cm. So, for the mirror $u = -14$ cm. After reflection from the mirror,

$$v = \frac{14\times 10}{14-10} = -35\,\text{cm}.$$

One more time, refraction will occur as light after reflecting from mirror will pass through the slab.

Again, normal shift will be 1 cm, but in the direction of propagation of light. Now the light rays are propagating in the left direction. Therefore, final image will be at a distance of 36 cm from the mirror.

Learn: If there are more than one slabs, the total normal shift is

$$\text{Normal shift} = t_1\left[1-\frac{n}{n_1}\right]+t_2\left[1-\frac{n}{n_2}\right]+\cdots$$

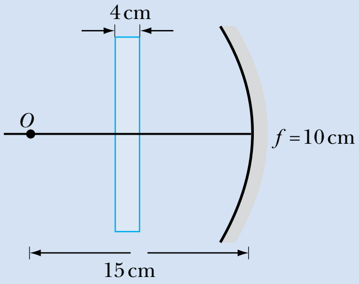


Figure 34-13 An object *O* is located on the principal axis of a concave mirror. We wish to find the image of this optical system.

CHECKPOINT 4

All along we have been considering the apparent depth of an object viewed normally. Suppose that the object is being viewed at an angle θ to the surface. If the actual depth is d and angle of ray inside water is ϕ, what is the apparent depth?

34.3 | TOTAL INTERNAL REFLECTION

Key Concept

- A wave encountering a boundary across which the index of refraction decreases will experience total internal reflection if the angle of incidence exceeds a critical angle θ_c, where

$$\theta = \sin^{-1}\frac{n_2}{n_1} \quad \text{(critical angle).}$$

We mentioned at the beginning of the chapter that when light is incident on a transparent surface, a part of the energy is reflected. Let us now consider the amount of reflected energy. The image of the face of a person in a good mirror is known to be brighter than that on the water surface in a lake or in a well. This is because the entire energy of light incident on the interface between two media is not reflected from it: a part of light penetrates through the interface into the second medium and passes through it or is partially absorbed by it. The fraction of the reflected energy depends on the optical properties of media and on the angle of incidence. If, for example, light is incident on a glass plate at right angles to its surface (the angle of incidence is zero), only about 5% of the energy is reflected, while 95% passes through the interface. As the angle of incidence increases, the fraction of the reflected energy increases.

At a particular angle of incidence, the fraction of reflected energy may be even 100% as can be seen in Fig. 34-14. Let us study this in some greater detail.

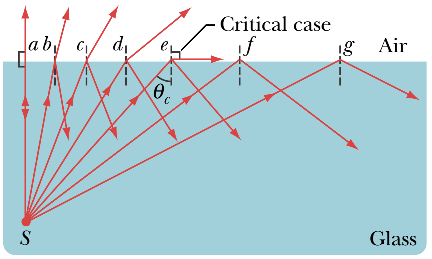


Figure 34-14 (*a*) Total internal reflection of light from a point source S in glass occurs for all angles of incidence greater than the critical angle θ_c. At the critical angle, the refracted ray points along the air—glass interface. (*b*) A source in a tank of water.

When a light ray travels from denser to a rarer medium, the light ray will bend away from normal, thus ray will bend at an angle $\phi > \theta$ (angle of incidence) to the normal (Fig. 34-15*a*). If we keep on increasing the angle of incidence, the angle of refraction will also increase. At a particular value of angle θ, angle of refraction may even go to 90º (something we call as grazing emergence). This is depicted in Fig. 34-15*b*.
In this situation,

$$n_1 \sin\theta = n_2 \sin 90^\circ. \tag{34-9}$$

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right). \tag{34-10}$$

This phenomenon is called total internal reflection.

Because the sine of an angle cannot exceed unity, n_2 cannot exceed n_1 in this equation. This restriction tells us that total internal reflection cannot occur when the incident light is in the medium of lower index of refraction. If source S

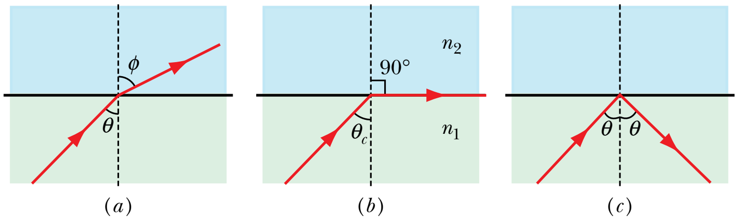


Figure 34-15 (*a*) Refraction of a ray incident from denser to rarer medium. Note that the ray bends away from the normal. (*b*) The situation of total internal reflection of a ray incident from a medium of refractive index n_1 into a medium of refractive index n_2. Here, n_1 is a denser medium and n_2 is a rarer medium. (*c*) No light energy enters the rarer medium if the angle of incidence is greater than the critical angle θ_c.

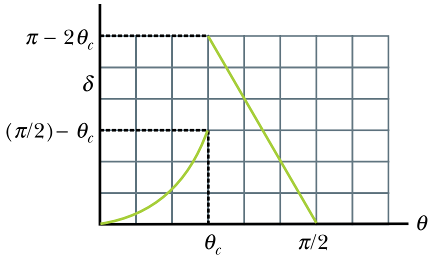


Figure 34-16 The angle of deviation versus the angle of incidence graph of a ray incident from a denser medium to a rarer medium. Beyond the critical angle, the deviation F is due to reflection.

Figure 34-17 An endoscope used to inspect an artery. (© Laurent/phototake)

were in the air as shown in Fig. 34-14, all its rays that are incident on the air–glass interface (including f and g) would be both reflected and refracted at the interface. In this situation (ray incident from denser to rarer medium), let us plot the graph of angle of deviation versus angle of incidence. The graph is shown in Fig. 34-16.

Here, we can say that when the ray is incident at 0º, it will be refracted at 0º. So, the angle of deviation is 0º. However, as we increase the angle of incidence, the angle of refraction also increases:

$$n \sin\theta = 1 \sin\phi.$$

So, the angle of deviation is $\phi - \theta$. At the critical angle, angle of deviation becomes $(\pi/2) - \phi$ Beyond the critical angle, the ray is totally reflected so the angle of deviation follows the formula for reflection:

$$\delta = \pi - 2\,\theta.$$

This is the equation of a straight line. The maximum angle of incidence can be $\pi/2$. At that point, the angle of deviation becomes 0. Here, in Fig. 34-16, we can see that the deviation from 0 to $(\pi/2) - \theta_c$ can be obtained at two angles of incidence. The smaller angle corresponds to refraction and the larger angle corresponds to total internal reflection.

Total internal reflection has found many applications in medical technology. For example, a physician can view the interior of an artery of a patient by running two thin bundles of *optical fibers* through the chest wall and into an artery (Fig. 34-17). Light introduced at the outer end of one bundle undergoes repeated total internal reflection within the fibers so that, even though the bundle provides a curved path, most of the light ends up exiting the other end and illuminating the interior of the artery. Some of the light reflected from the interior then comes back up, the second bundle in a similar way, to be detected and converted to an image on a monitor's screen for the physician to view.

SAMPLE PROBLEM 34.08

Sparkle in the diamond

The purpose of a diamond is, of course, to sparkle. Part of the art of cutting a diamond is to ensure that all the light entering through the top face or side facets leaves through those surfaces, to participate in the sparkle. Figure 34-18 shows part of a cross-sectional slice through a brilliant-cut diamond, with a ray entering at point A on the top face. In this type of cut, the top and bottom surfaces have normal lines that intersect at the indicated 48.84º. At point B, at least part of the light reflects and leaves the diamond properly, but part could refract and thus leak out of the diamond. Consider a light ray incident at angle $\theta_1 = 40º$ at A. Does light leak at B if air ($n_4 = 1.00$) lies next to the bottom surface? Does light leak if greasy grime ($n_4 = 1.63$) coats the surface? The index of refraction of diamond is $n_{\text{dia}} = 2.419$.

KEY IDEA

When light reaches the interface between two materials with different indexes of refraction (call them n_1 and n_2), part or all of the light is reflected. The reflection is total if (1) the incident light is in the material with the *higher* index of refraction ($n_1 > n_2$) and (2) the incident angle exceeds a critical value given by Eq. 34-10:

$$\theta = \sin^{-1}\left(\frac{n_1}{n_2}\right).$$

If those two conditions are not met, part of the light is refracted across the interface according to Snell's law.

Case 1: Clean diamond

We need to follow the light from point A to point B to see if it can leak at point B. The light incident at point A is in the material (air, $n_1 = 1.00$) with a *lower* index of refraction than the material on the other side of the interface (diamond, $n_{\text{dia}} = 2.419$). Thus, some of the light is refracted across the interface (the reflected portion is not shown in Fig. 34-18), and we find the angle of refraction from Snell's law.

Calculation:

$$\theta_2 = \sin^{-1}\left(\frac{\sin 40°}{2.419}\right) = 15.41°.$$

Now, note that the given angle of 48.84° at point C is an *exterior angle* of triangle ABC and thus (from Appendix E), we can write

$$\theta_3 + \theta_2 = 48.84°$$

or

$$\theta_3 = 48.84° - \theta_2 = 33.43°.$$

This is the angle of incidence of the ray at point B. Now the incident light is in the material (diamond) with the *greater* index of refraction than the material (air) on the other side of the interface. However, if we apply Snell's law at point B, we find no answer. The reason is that the light is incident at an angle greater than the critical value of

$$\theta_c = \sin^{-1}\left(\frac{n_4}{n_{\text{dia}}}\right) = \sin^{-1}\left(\frac{1.00}{2.419}\right) = 24.4°.$$

Thus, all the light reaching B reflects and none leaks out in the air.

Case 2: Grimy diamond

We again follow the light from A to B, with the only difference lying in the last two calculations.

Now at B, the material on the other side of the interface is grime with an index $n_4 = 1.63$. The incident light is again in the material with the greater index, but now the critical angle is

$$\theta_c = \sin^{-1}\left(\frac{n_4}{n_{\text{dia}}}\right) = \sin^{-1}\left(\frac{1.63}{2.419}\right) = 42.4°.$$

Therefore, the incident angle $n_3 = 33.43°$ is *less* than the critical angle and light leaks through the bottom of the diamond.

From Snell's law, the angle of refraction is

$$\theta_4 = \sin^{-1}\left(\frac{2.419}{1.63}\sin 33.43°\right) = 54.8°.$$

Learn: Thus, to keep a diamond sparkling, clean both the top and bottom surfaces.

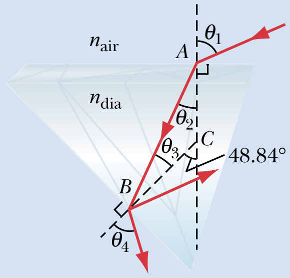


Figure 34-18 Light entering the top of a brilliant-cut diamond. Can the light leak from the bottom surface at point B?

SAMPLE PROBLEM 34.09

Total internal reflection of light ray from cylindrical optical fiber

An optical fiber consists of a cylindrical core of unknown refractive index kept in air. A ray of light is incident on the flat face of the core so that it gets totally reflected at the curved surface (Fig. 34-19). What is the possible refractive index if the ray is totally reflected at any angle of incidence ϕ from air?

KEY IDEA

When a ray refracts from air to glass, total internal reflection cannot occur. However, when this ray strikes curved surface of the fiber, the incidence is from denser to rarer medium. If n is such that total internal reflection occurs at this point, it will continue to occur as the ray strikes the curved surface repeatedly at the same angle.

Calculation: Let us assume that the ray is incident at an angle ϕ from air and is refracted into the cylinder at an angle θ. It is then again incident on the air cylinder boundary at the curved surface as shown in Fig. 34-19. The condition is that the ray should undergo total internal reflection at the face.

For this, the angle of incidence here should be greater than critical angle.

$$90^\circ - \theta > \theta_c.$$

Since sine is an increasing function, taking sine of both sides will not change the inequality.

$$\sin(90^\circ - \theta) > \sin\theta_c = \frac{1}{n}\,(n\sin\theta_c = 1\times\sin 90^\circ).$$

$$\cos\theta \geq \frac{1}{n}.$$

Again, applying Snell's law at flat face, we get

$$1 \times \sin\phi = n \times \sin\theta.$$

Squaring and adding, we get

$$n^2\sin^2\theta + n^2\cos^2\theta \geq 1 + \sin^2\phi$$

$$n^2 \geq 1 + \sin^2\phi.$$

But the maximum value of sin ϕ is 1 when the angle of incidence is 90°. For this condition,

$$n^2 \geq 2$$

or

$$n \geq \sqrt{2}.$$

Learn: Interestingly, this yields $\mu \geq 1.41$. Thus, even ordinary glass can be used in optical fiber. But for practical reasons, this core is coated with a cylindrical shell called **cladding**. If cladding has refractive index of μ_1, the condition on μ_2 is

$$\mu_2 \geq \sqrt{\mu_1^2 + 1}.$$

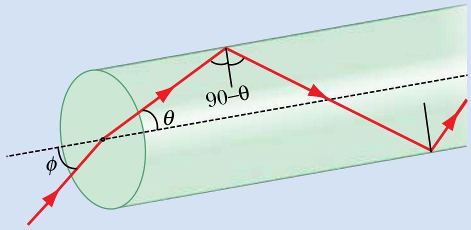


Figure 34-19 A light ray is incident from air on a cylindrical optical fiber.

SAMPLE PROBLEM 34.10

Refraction and total internal reflection of light from circle of illuminance

A point source of light is kept in a vessel which is filled with liquid (of refractive index 1.25) up to a depth of 1m (Fig. 34-20*a*). The light rays come out from the surface of water and make a circular patch on the surface (Fig. 34-20*b*).

(a) The radius of the circular patch through which the light can escape.

KEY IDEA

Just vertically above the point source, the angle of incidence is 0º. As we move sideways, the angle of incidence will increase. So, the angle of refraction will also increase. Since the rays are going from denser to the rarer medium, the rays will bend away from the normal. At the position where the angle of refraction becomes 90º, the angle of incidence will be critical angle. A little visualization says that this condition will be achieved at all the points on a circle just above the source.

Calculation: From Eq. 34-10 [$\theta_c = \sin^{-1}(n_1/n_2)$], we get

$$1.25\sin\theta_c = 1\sin 90^\circ.$$

Therefore,

$$\theta_c = 53^\circ.$$

From the triangle made by the radius of the circle and the vertical line, we can say that

$$\tan\theta = \frac{r}{1}$$

or

$$r = 1\tan\theta = \frac{4}{3}.$$

(b) If the power of source is P find the power coming out of surface ($n_{\text{water}} = 4/3$).

KEY IDEAS

(1) Here, we are required to find the part of the power which is incident on the circular patch whose radius we found just now. An associated concept here is the concept of intensity. If a source of power P is kept at a certain point, the light energy emitted by it will be transmitted equally in all the directions. So, we can say that

after some time, this light energy will be distributed on the surface of a sphere of radius $R = ct$.
Intensity at that point is defined as

$$\frac{\text{Power}}{\text{Area over which light is distributed}} = \frac{P}{4\pi R^2},$$

where R is the distance from source at which we are measuring the intensity.

(2) It is clear here that if a disk is kept in the path of the light rays (this disk may be a mirror, a slab, etc.), a part of the light energy will be incident on the disk. If we wish to find the amount of light energy incident on the disk, we should note that disk is obstructing that part of the light energy which would have been distributed on the part of the sphere denoted by AB (see Fig. 34-20*b*).

Here we can see that since the light energy is equally distributed on all parts of the sphere, the amount of light energy incident on the disk is proportional to area A of the spherical portion above the water surface:

$$\frac{P_s}{P} = \frac{A}{4\pi R^2}.$$

(3) Recall the concept of solid angle discussed in Chapter 23. There we showed how solid angle

$$\Omega = \frac{A}{R^2} = 2\pi(1-\cos\theta).$$

To find the power passing through this circular patch, we should find the power emitted by the source in the cone of half angle $\theta_c = 53°$.

From the concept discussed above, the power contained in this cone is in proportion of solid angle subtended by the cone at the source. Therefore,

$$\Omega\ (\text{solid angle}) = 2\pi(1-\cos\theta) = \frac{A}{r^2}.$$

The solid angle is measured in steredians. Now,

$$P_S = P \times \frac{\Omega}{4\pi}$$

$$= \frac{P}{4\pi} 2\pi(1-\cos\theta) = \frac{3P}{10}.$$

Note: The actual power will be somewhat less than this value because we have neglected the partial reflection at the water surface.

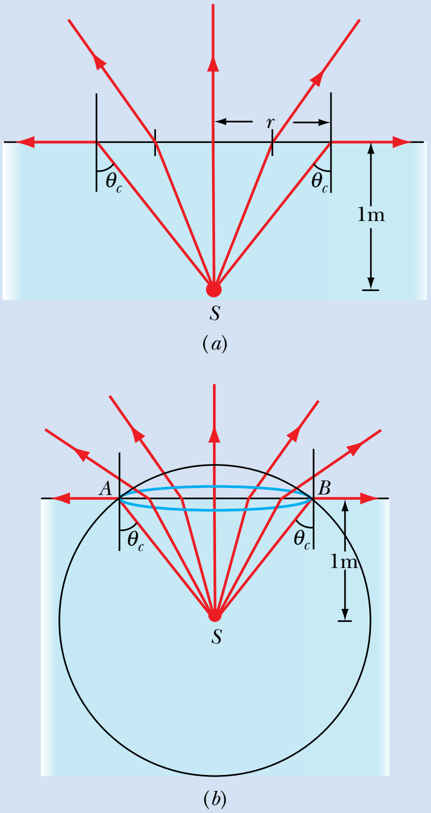


Figure 34-20 (*a*) A source of light S is kept under water. Light is incident from it on water–air interface and gets refracted and totally reflected. (*b*) Light beam comes out of the water surface from a circular path known as the circle of illuminance.

A Common Mirage

A common example of a virtual image is a pool of water that appears to lie on the road some distance ahead of you on a sunny day, but that you can never reach. The pool is a *mirage* (a type of illusion), formed by light rays coming from the low section of the sky in front of you (Fig. 34-21*a*). As the rays approach the road, they travel through progressively warmer air that has been heated by the road, which is usually relatively warm. With an increase in air temperature, the density of the air—and hence the index of refraction of the air—decreases slightly. Thus, as the rays descend, encountering progressively smaller indexes of refraction, they continuously bend toward the horizontal (Fig. 34-21*b*).

Once a ray is horizontal, somewhat above the road's surface, it still bends because the lower portion of each associated wavefront is in slightly warmer air and is moving slightly faster than the upper portion of the wavefront (Fig. 34-21*c*). This nonuniform motion of the wavefronts bends the ray upward. As the ray then ascends, it continues to bend upward through progressively greater indexes of refraction (Fig. 34-21*d*).

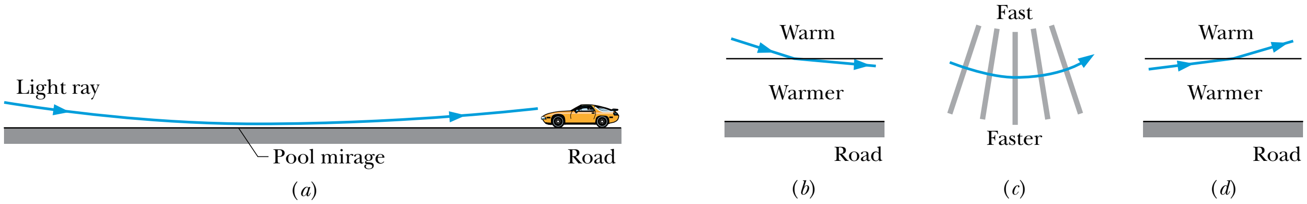


Figure 34-21 (*a*) A ray from a low section of the sky refracts through air that is heated by a road (without reaching the road). An observer who intercepts the light perceives it to be from a pool of water on the road. (*b*) Bending (exaggerated) of a light ray descending across an imaginary boundary from warm air to warmer air. (*c*) Shifting of wavefronts and associated bending of a ray, which occur because the lower ends of wavefronts move faster in warmer air. (*d*) Bending of a ray ascending across an imaginary boundary to warm air from warmer air.

If you intercept some of this light, your visual system automatically infers that it originated along a backward extension of the rays you have intercepted and, to make sense of the light, assumes that it came from the road surface. If the light happens to be bluish from blue sky, the mirage appears bluish, like water. Because the air is probably turbulent due to the heating, the mirage shimmies, as if water waves were present. The bluish coloring and the shimmy enhance the illusion of a pool of water, but you are actually seeing a virtual image of a low section of the sky. As you travel toward the illusionary pool, you no longer intercept the shallow refracted rays and the illusion disappears.

SAMPLE PROBLEM 34.11

Variable refractive index—Trajector of ray of light

A ray of light traveling in air is incident at grazing angle (incident angle = 90°) on a long rectangular slab of a transparent medium of thickness $t = 1.0$ m (Fig. 34-22*a*). The point of incidence is the origin *A* (0, 0). The medium has a variable index of refraction given by $n = (ky^{3/2} + 1]^{1/2}$, where $k = 1.0\ (\text{m})^{-3/2}$. The refractive index of air is 1.0.

(a) Obtain a relation between the slope of the trajectory of the ray at a point $B(x, y)$ in the medium and the incident angle at that point.

(b) Obtain an equation for the trajectory $y(x)$ of the ray in the medium.

(c) Determine the coordinates (x_1, y_1) of the point *P*, where the ray intersects the upper surface of the slab–air boundary.

(d) Indicate the path of the ray subsequently.

KEY IDEA

(1) In situation where refractive index changes only in one direction, medium can be considered as a collection of large number of the thin layers. Let refractive index be a function of *y*, that is, $n = f(y)$ then medium can be considered as to made up of a large number of thin slabs placed parallel to *x* axis and normal at any surface is parallel to *y* axis.

(2) Although the refractive index is not constant, we cannot apply Snell's laws here, but still we can divide the whole medium into thin layers as shown in Fig. 34-22. Within a layer, refractive index is constant. So, here we have a case of many parallel slabs. We can apply Snell's law here.

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 = n_3 \sin \theta_3 = \ldots$$

$$n \sin \theta = \text{constant}.$$

Calculations:

(a) From Fig. 34-22*b*,

$$i + \theta = 90°$$

Slope of the tangent = $\tan \theta$

$$= \tan(90 - i) = \cot i$$

(b) We know that at any point in the medium with variable refractive index $n \sin i$ = constant, where n = refractive index at any point y and i = angle of incidence at any point y. Therefore,

$$n \sin i = 1 \sin 90°$$

$$\sin i = \frac{1}{\operatorname{cosec} i} = \frac{1}{\sqrt{1 + \cot^2 i}}.$$

Substituting the values $(ky^{3/2}+1)^{1/2}$, we get

$$\frac{1}{\sqrt{1+m^2}}=1$$

$$ky^{3/2}+1=1+m^2.$$

$$\text{Slope } m=\frac{dy}{dx}=\sqrt{k}\,y^{3/4}$$

$$y^{-3/4}dy=\sqrt{k}\;dx.$$

On integrating both sides, we get

$$\int_0^y y^{-3/4}dy=\int_0^x \sqrt{k}\;dx$$

$$4y^{1/4}=x \qquad (\text{since } k=1)$$

$$y=\frac{x^4}{256}.$$

(c) For $y = 1$ m, $x = 4$ m.

(d) Let α_e be the angle of emergence in the air, then by Snell's law,

$n_e \sin\alpha_e = n_1 \sin\alpha_1 = 1$ (refractive index of air, $n_e = 1$).

This gives

$$\sin\alpha_e = 1 \Rightarrow \alpha_e = 90^\circ.$$

Thus, it emerges parallel to the incident ray; this could have been interpreted by Snell's law too.

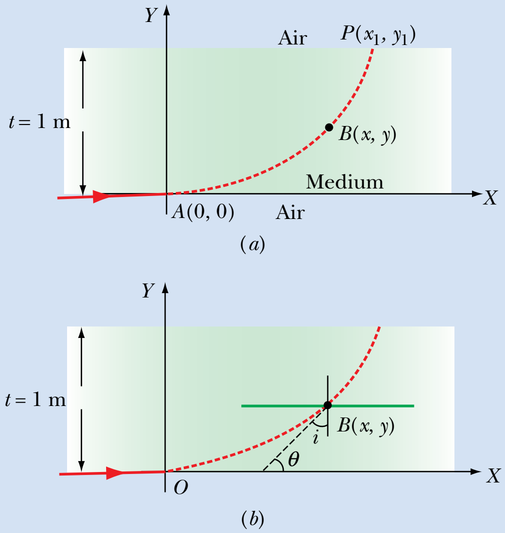


Figure 34-22 (*a*) A light ray enters a medium at grazing incidence. Note that at the *xz* plane, the refractive index of the medium is 1.00, so the ray does not bend there. As the ray proceeds upward, the refractive index of the medium increases and hence the ray progressively bends toward the normal. (*b*) A ray being considered at an arbitrary point *B* (*x*, *y*) in the medium.

34.4 | SPHERICAL REFRACTING SURFACES

Key Concepts

- A single spherical surface that refracts light can form an image.
- The object distance u, the image distance v, and the radius of curvature r of the surface are related by

$$\frac{n_1}{v}+\frac{n_2}{u}=\frac{n_2-n_1}{r},$$

where n_1 is the index of refraction of the material where the object is located and n_2 is the index of refraction on the other side of the surface.

- If the surface faced by the object is convex, r is positive, and if it is concave, r is negative.
- Images on the object's side of the surface are virtual, and images on the opposite side are real.

We now turn from images formed by plane refracting surfaces to images formed by refraction through spherical surfaces of transparent materials, such as glass. We shall consider spherical surfaces, with radius of curvature r and center of curvature C. The light will be emitted by a point object O in a medium with index of refraction n_1; it will refract through a spherical surface into a medium of index of refraction n_2.

Let us consider an object O kept along the principal axis of the spherical refracting surface as shown in Fig. 34-23. The principal axis is defined by the line along the radius of the surface. In the figure, one of the rays is assumed to be incident along the principal axis. This ray will pass undeviated because the angle of incidence is 0°. Another ray is required to trace the image. Applying Snell's law for this ray, we obtain

$$n_1 \sin\theta = n_2 \sin\phi.$$

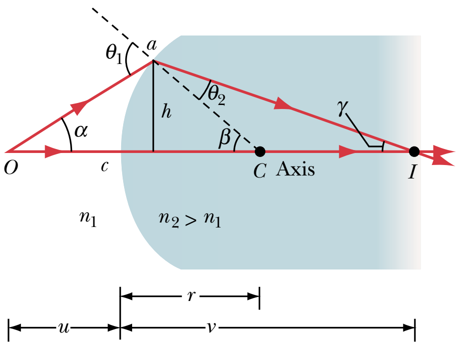


Figure 34-23 A real point object O kept on the principal axis forms a real image I on the principal axis. A large number of paraxial rays form the image, but only two are shown for simplicity.

Dr. Paul A. Zahl/Photo Researchers, Inc.

This insect has been entombed in amber for about 25 million years. Because we view the insect through a curved refracting surface, the location of the image we see does not coincide with the location of the insect (see Fig. 34-24*d*).

We know that if a surface is spherical, the normal will always pass through the center of curvature C. Let the angle made by the normal with the principal axis be β. This angle will be the exterior angle for the triangle formed by I, C, and a. So, the angle at I is $\beta - \theta_2$. From the triangle formed by the image and the vertical height h, we can say, according to small angle approximation, that

$$\gamma = \beta - \theta_2 \approx \tan(\beta - \theta_2) = \frac{h}{v}.$$

Similarly, the angle between the horizontal and normal will also be β. The angle made at the object turns out to be $\alpha = \theta_1 - \beta$. According to small angle of approximation,

$$\theta_1 - \beta \approx \tan(\theta_1 - \beta) = \frac{h}{-u}.$$

Here, u happens to be negative according to our sign convention. Since tangent of an angle is always positive, we take modulus of u. Similarly,

$$\beta \approx \tan(\beta) = \frac{h}{R}.$$

Also, for the small angles, we can say that

$$\sin\theta_1 = \theta_1 \text{ and } \sin\theta_2 = \theta_2$$
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$
$$n_1\theta_1 = n_2\theta_2.$$

By substituting the values of θ_1 and θ_2, we get

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{R}. \qquad (34\text{-}11)$$

Here, n_1 refers to the refractive index of the incident rays and n_2 refers to the refractive index of the refracted rays.

To apply this formula, we should substitute the values of u, v, and R with their proper sign. The sign convention remains the same as that of mirror. Seeing the ray diagram, we can note one interesting point. If the object is real and the radius of curvature of the surface is positive as shown the image cannot be formed between the center of curvature and optical center for any value of the refractive index. A similar conclusion can be drawn for other cases.

Cartesian Sign Convention

1. Principal axis of the spherical surface is taken as x axis and optical center as origin. Here, the principal axis is the diameter extended.
2. The direction of the incident light is taken as positive direction of x axis.
3. Upward direction is taken as positive and the downward direction as negative.

All the axial distances are always measured from the optical center. Object distance is taken to be u and image distance is taken to be v. For real objects, u is always negative and for virtual object u is positive as seen in Figs. 34-24*a*–*e*.

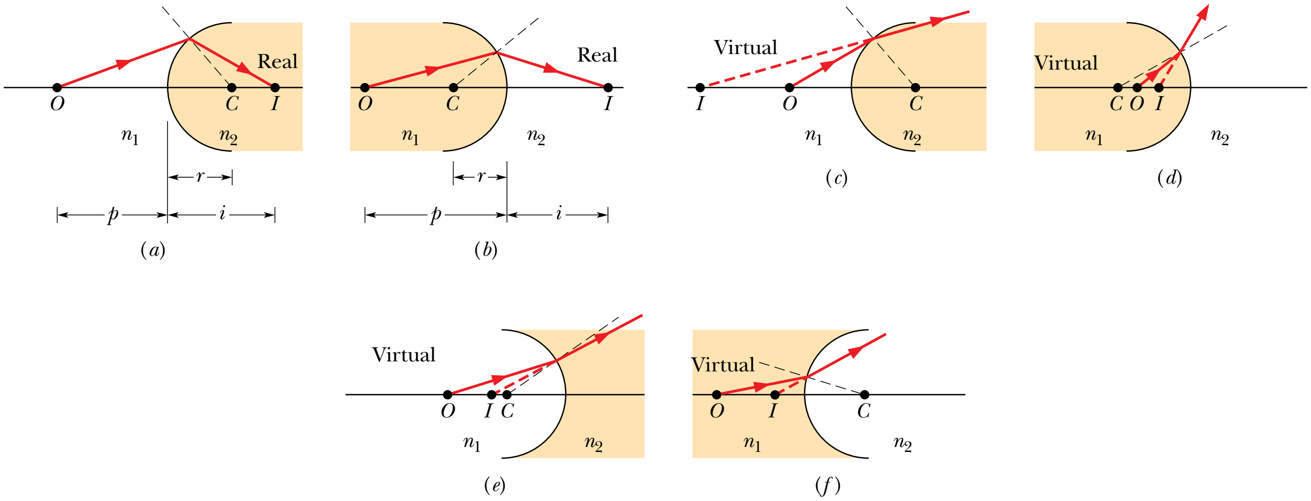


Figure 34-24 Six possible ways in which an image can be formed by refraction through a spherical surface of radius r and center of curvature C. The surface separates a medium with index of refraction n_1 from a medium with index of refraction n_2. The point object O is always in the medium with n_1, to the left of the surface. The material with the lesser index of refraction is unshaded (think of it as being air, and the other material as being glass). Real images are formed in (*a*) and (*b*); virtual images are formed in the other four situations.

For real image v is positive and for virtual image v is negative. Note here the difference between the mirror and the spherical surface. Since in the case of mirror, light is reflected, real image is formed on the negative side whereas in this case, the light is refracted. So, a real image is formed on the positive side.

To find the transverse magnification, consider a ray incident from the extended object at the optical center (Fig. 34-25).

Here, we can see that the angle of incidence is θ and angle of refraction is ϕ. Applying Snell's law, we get

$$n_1 \sin\theta = n_2 \sin\phi.$$

For small angles, we can say that

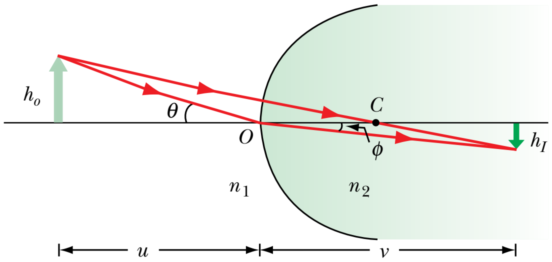


Figure 34-25 If the image is kept perpendicular to the principal axis, the image in general is magnified or diminished.

$$n_1\theta = n_2\phi$$

$$\frac{n_2 \times -h_I}{v} = \frac{n_1 \times h_O}{-u}$$

$$m = \frac{h_I}{h_O} = \frac{n_1 \times v}{n_2 \times u}. \tag{34-12}$$

Thus, we can see that for a real object forming a real image, magnification is negative (inverted image).

CHECKPOINT 5

A bee is hovering in front of the concave spherical refracting surface of a glass sculpture. (a) Which part of Fig. 34-24 is like this situation? (b) Is the image produced by the surface real or virtual, and (c) is it on the same side as the bee or the opposite side?

SAMPLE PROBLEM 34.12

Image produced by a refracting surface

A Jurassic mosquito is discovered embedded in a chunk of amber, which has index of refraction 1.6. One surface of the amber is spherically convex with radius of curvature 3.0 mm (Fig. 34-26). The mosquito's head happens to be on the central axis of that surface and, when viewed along the axis, appears to be buried 5.0 mm into the amber. How deep is it really?

KEY IDEAS

The head appears to be 5.0 mm into the amber only because the light rays that the observer intercepts are bent by refraction at the convex amber surface. The image distance i differs from the object distance p according to Eq. 34-11. To use that equation to find the object distance, we first note:

1. Because the object (the head) and its image are on the same side of the refracting surface, the image must be virtual and so $i = -5.0$ mm.
2. Because the object is always taken to be in the medium of index of refraction n_1, we must have $n_1 = 1.6$ and $n_2 = 1.0$.
3. Because the *object* faces a concave refracting surface, the radius of curvature r is negative, and so $r = -3.0$ mm.

Calculations: Making these substitutions in Eq. 34-11,

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{r},$$

yields

$$\frac{1.6}{p} + \frac{1.0}{-5.0\text{ mm}} = \frac{1.0 - 1.6}{-3.0\text{ mm}},$$

and

$$p = 4.0\text{ mm.} \qquad \text{(Answer)}$$

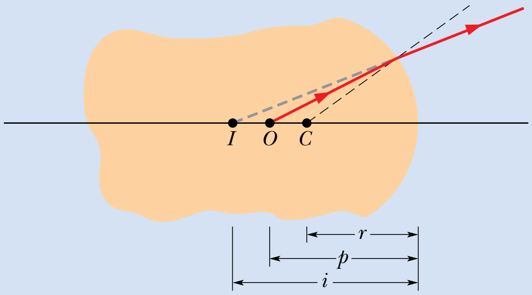


Figure 34-26 A piece of amber with a mosquito from the Jurassic period, with the head buried at point O. The spherical refracting surface at the right end, with center of curvature C, provides an image I to an observer intercepting rays from the object at O.

SAMPLE PROBLEM 34.13

Image of the fish in a spherical aquarium

Assume that a boy views a fish in a spherical aquarium of radius 10 cm kept at a distance of 2 cm from the center toward the left of center. The boy is viewing from the left (Fig. 34-27). Where will the image of fish be formed?

KEY IDEA

The positive direction is decided by the incident rays. Since the rays incident on the boundary of the aquarium is incident are from right to left so that they reach the eyes of the boy sitting to the left of the center, the positive direction will be from right to left. Here, the boundary of the aquarium will be the origin. All the distances to the left of this boundary will be positive and the distances to the right will be negative. So, here we can see that the center of the surface is to the negative side and so the radius R will be taken as negative. Similarly, the distance of the object from the optical center is to the right side and hence it will be negative. Obviously, the refraction is from a spherical surface, so Eq. 34-11 will be used.

Calculations: Rays are incident from water to air. So, in Eq. 34-11, $n_2 = 1$; $n_1 = 4/3$; $R = -10$ cm; $u = -8$ cm. Substituting values in the formula, we get

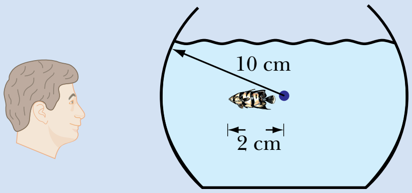


Figure 34-27 A boy views a fish through a spherical aquarium.

$$\frac{1}{v} - \frac{\frac{4}{3}}{-8} = \frac{1-\frac{4}{3}}{-10}.$$

So, $v = -7.5$ cm.

Learn: Here, the image formed is virtual. Students often tend to mistake that since the image is formed behind the water, refraction will occur again. Here, we should understand that if image is at −7.5 cm it means the refracted rays have to be traced to meet at that point. It does not mean that rays have reversed and met at that point.

SAMPLE PROBLEM 34.14

Refraction through eye of Anableps anableps fish

Figure 34-28 shows a vertical cross section through an eye of the *Anableps anableps* fish that swims with each eye half in and half out of the water, with a pigment band separating the two halves at the water surface. The front of the eye (the cornea) is a spherically convex refracting surface of radius $r = 1.95$ mm and index of refraction $n_2 = 1.335$. The refraction at the cornea is the first step in the eye's focusing of a *real* image onto the back of the eye (the retina), where visual processing begins. If the cornea faces an insect (lunch) at object distance $p = 0.20$ m, what is the image distance i of that refraction for the cornea in air ($n_1 = 1.000$) and in water ($n_1 = 1.333$)?

KEY IDEAS

(1) Because the object and its image are on opposite sides of the refracting surface with its convex side facing the object, the situation is like either Fig. 34-24*a* (for a real image) or Fig. 34-24*c* (for a virtual image).

(2) Image distance and object distance are related by

$$\frac{n_1}{p} + \frac{n_1}{i} = \frac{n_2 - n_1}{r}.$$

Calculation: Solving this equation for i gives us

$$i = \frac{n_2}{\dfrac{n_2 - n_1}{r} - \dfrac{n_1}{p}}.$$

Substituting the given values and using the index of refraction $n_1 = 1.000$ for air, we find

$$i = \frac{1.335}{\dfrac{1.335 - 1.000}{0.00195} - \dfrac{1.000}{0.20}}$$

$$= 8.00 \text{ mm.} \qquad \text{(Answer)}$$

Repeating the calculation but using the index of refraction $n_1 = 1.333$ for water, we find

$$i = \frac{1.335}{\dfrac{1.335 - 1.333}{0.00195} - \dfrac{1.333}{0.20}}$$

$$= -0.237 \text{ m.} \qquad \text{(Answer)}$$

Learn: After light is refracted by the cornea, it is further refracted by a lens to give a final real image on the retina. (The function of a lens is discussed in the next section.) Our first answer is positive (indicating a real image, as needed) and about twice the diameter of the eye. Thus, the role of the lens in focusing the light onto the retina is moderate because so much refraction occurs at the cornea. Our second answer is quite different: It is negative (indicating a virtual image) and much larger. So, far more focusing is required by the lens to put a real image onto the retina. To provide moderate focusing of light from the air and much stronger focusing of light from the water, the eye lens in *Anableps anableps* is egg shaped, with much greater curvature in the bottom half than in the top.

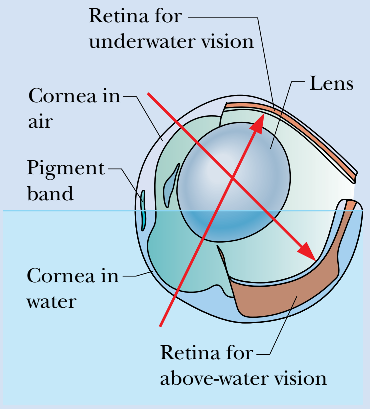


Figure 34-28 Cross section of the eye of *Anableps anableps*.

Origin Shifting

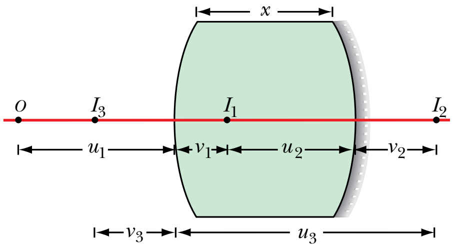


Figure 34-29 Formation of the image of an object placed at *O* by a thick lens.

Many times, we are required to consider that multiple events occur before final image is formed. In such situations, we take the image of one event as an object for another. The caution is that one should not worry about whether the image is real or virtual while applying the sign convention. Instead, one should measure the distance from the optical center or pole and then apply the appropriate formula to form the image.

Suppose that a thick lens is silvered at the back to form an optical system (Fig. 34-29). Here, there will be three events before the final image is formed:

1. Refraction at the front surface of the lens.
2. Reflection at the back surface of the lens.
3. Refraction at the front surface of the lens again.

As we stated earlier, image for first event will act as an object for the second event. But for the second event, the pole is ahead of the optical center of first surface.

So, to measure the value of u_2 (object distance for the mirror), the measurement has to be done from pole of mirror. This is similar to origin shifting we study in coordinate geometry.
From Fig. 34-9 we can say that

$$u_2 = v_1 - x.$$

Note that this includes the sign convention. For example, if image was formed at 5 cm and the thickness of the lens is 10 cm, this formula gives $u_2 = -5$ cm. This can be easily seen from Fig. 34-29 also.

The situation becomes a little tricky if we consider the event after the reflection. Here, the direction of light ray reverses. So what was positive before reflection becomes negative and vice versa. So, here

$$u_3 = v_2 - x.$$

This formula again includes the sign convention. We can see this clearly from Fig. 34-29. If we have $v_2 = +10$ cm and $x = 10$ cm, u_3 will be -20 cm. To understand this clearly, we take an example.

SAMPLE PROBLEM 34.15

Image of an object kept on the principal axis of transparent sphere with silvered rear surface

A sphere of radius 1 m and $n = 1.5$ is silvered at its back. A point object is kept at a distance of 1 m from the front face (Fig. 34-30). Where will the final image be formed?

KEY IDEA

Here, we can clearly see that there will be three events before the final image will be formed. Refraction, reflection, and refraction back into the air.

Calculation: For the first event *O* is the optical center, $n_1 = 1$; $n_2 = 1.5$; $u = -1$ m; $R = +1$ m. Therefore, we get

$$\frac{1.5}{v_1} - \frac{1}{-1} = \frac{1.5-1}{1}$$

$$v_1 = 3 \text{ m}.$$

This image will act as an object for the rear silvered surface. However, for taking the object distance, we have to measure it from the pole of the mirror. As we have discussed before,

$$u_2 = v_1 - x$$

$$u_2 = -3 - 2 = -5 \text{ m}.$$

Applying mirror formula, we get (radius of curvature is –1 m, so focal length of the mirror is –1/2 m)

$$\frac{1}{v_2} + \frac{1}{-5} = \frac{1}{-0.5}$$

$$v_2 = -\frac{5}{9} \text{ m}.$$

These reflected rays will again be refracted from the spherical surface. Since now the direction of the light rays has reversed, therefore

$$u_3 = -v_2 - x$$
$$u_3 = -(-\tfrac{5}{9}) - 2 = -\tfrac{13}{9}\ \text{m}.$$

This can be seen from Fig. 35-29 also. Now,

$$\frac{1}{v_3} - \frac{1.5}{-\frac{13}{9}} = \frac{1-1.5}{-1} = 0.5.$$

Note here that the radius of this surface is negative because the direction of the incident rays has reversed. Solving we get $v_3 = -13/7$ m.

Negative means that the image is virtual formed inside the spherical surface. This is distance from the left edge of the spherical surface.

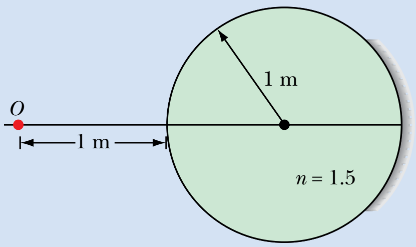


Figure 34-30 A point object O is kept on the principal axis of a transparent sphere. The sphere is silvered at its rear surface.

34.5 | THIN SPHERICAL LENSES

Key Concepts

- This section primarily considers thin lenses with symmetric, spherical surfaces.
- If parallel rays are sent through a converging lens parallel to the central axis, the refracted rays pass through a common point (a real focus F) at a focal distance f (a positive quantity) from the lens. If they are sent through a diverging lens, backward extensions of the refracted rays pass through a common point (a virtual focus F) at a focal distance f (a negative quantity) from the lens.
- A converging lens can form a real image (if the object is outside the focal point) or a virtual image (if the object is inside the focal point).
- A diverging lens can form only a virtual image.
- For an object in front of a lens, object distance u and image distance v are related to the lens's focal length f, index of refraction n, and radii of curvature r_1 and r_2 by

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

- The magnitude of the lateral magnification m of an object is the ratio of the image height h_I to object height h_O,

$$|m| = \frac{h_I}{h_O},$$

and is related to the object distance p and image distance i by

$$m = -\frac{v}{u}.$$

- For a system of lenses with a common central axis, the image produced by the first lens acts as the object for the second lens, and so on, and the overall magnification is the product of the individual magnifications.

A **lens** is a transparent object with two refracting surfaces whose central axes coincide. The common central axis is the central axis of the lens. When a lens is surrounded by air, light refracts from the air into the lens, crosses through the lens, and then refracts back into the air. Each refraction can change the direction of travel of the light.

Lens are of two types: (1) convex lens (Figs. 34-31*a*–*c*) and (2) concave lens (Figs. 34-32*a*–*c*). We will consider only thin spherical lens. This means that the surfaces of lens are spherical in nature and the radius of curvature of the surface is very large in comparison to the thickness of the lens.

Before going ahead, let us define some common terminologies related to thin lens. ***First focus or object focus*** is point at which if an object is kept, the rays become parallel to the principal axis after refraction from the lens (Figs. 34-33*a* and *b*).

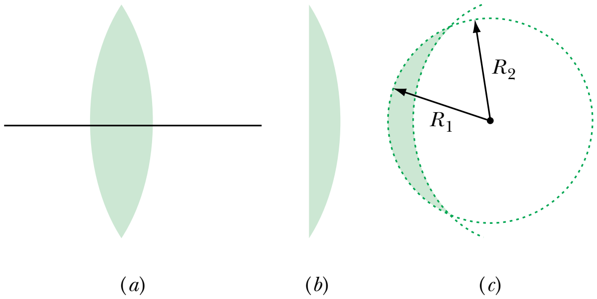


Figure 34-31 (*a*) A typical convex lens. If the radii of curvature of both the refracting surfaces are equal in magnitude, the lens is said to be equiconvex, else it is a biconvex lens. (*b*) If one of the surfaces is plane, the lens is said to be plano-convex lens. (*c*) If the center of curvature of both the surfaces of the lens are on the same side, and the lens is thicker at the center than at the periphery, the lens is said to be concavo-convex lens.

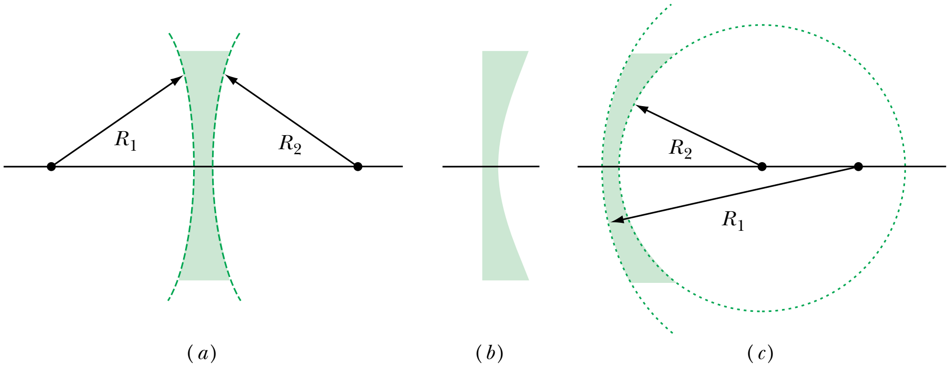


Figure 34-32 (*a*) A typical concave lens. If the radii of curvature of both the refracting surfaces are equal, the lens is said to be equiconcave, else it is a biconcave lens. (*b*) If one of the surfaces is plane, the lens is said to be plano-concave lens. (*c*) If the center of curvature of both the surfaces of the lens are on the same side, and the lens is thinner at the center than at the periphery, the lens is said to be convexo-concave lens.

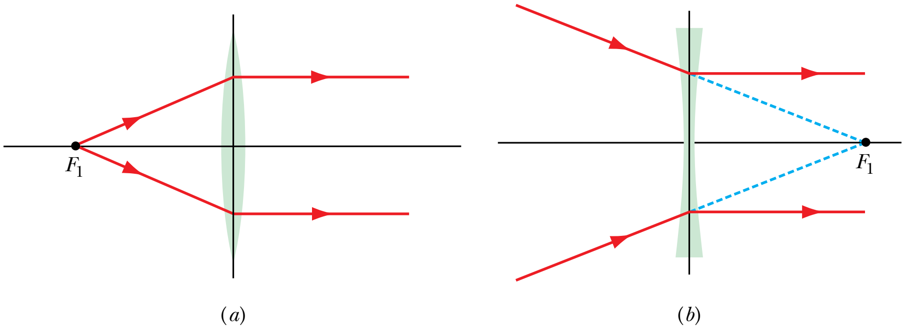
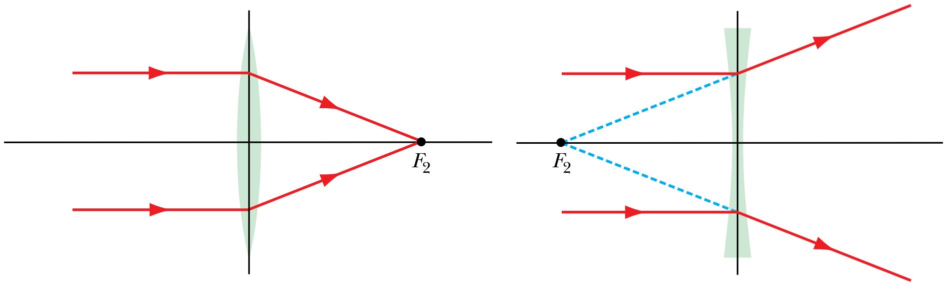


Figure 34-33 First focus of a (*a*) convex lens and (*b*) concave lens.

Second focus or image focus is a point at which rays parallel to principal axis will converge after refraction from the lens (Figs. 34-34*a* and *b*).

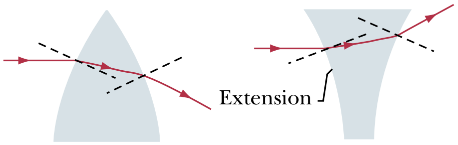
Both these focal lengths are equal in magnitude if the medium on both the sides of lens is same. Refraction in a lens occurs at both the surfaces (Figs. 35-35*a* and *b*).

Normally, focal length refers to the image focus.



(*a*) (*b*)

Figure 34-34 Second focus of a (*a*) convex lens and (*b*) concave lens.



(*a*) (*b*)

Figure 34-35 (*a*) and (*b*) Although we are drawing the ray diagram as if the rays are refracted at the central line, the refraction occurs at both the surfaces. But to draw such a ray diagram is tedious. So, we avoid it in general.

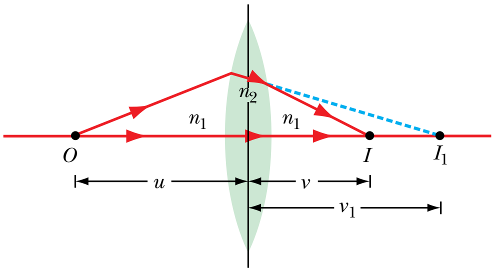


Figure 34-36 Formation of image by a lens. Here, only two rays are drawn for simplicity.

Derivation of Lens Formula and Lens Maker's Formula

Consider a point object O kept on the principal axis of the lens (Fig. 34-36). The lens has a refractive index n_2 and the surroundings have a refractive index n_1. As we did before in the case of refraction from spherical surface, here there will be two events: refraction from the first face of radius of curvature R_1 and from the second face of radius of curvature R_2. For the first surface,

$$\frac{n_2}{v_1} - \frac{n_1}{u} = \frac{n_2 - n_1}{R_1}. \tag{34-13}$$

For the second surface, image of the first surface acts as an object. So, $v_1 \approx u_2$ because the distance between the first surface and the second surface is small:

$$\frac{n_1}{v_2} - \frac{n_2}{u_2} = \frac{n_1 - n_2}{R_2} \qquad (\text{Since } u_2 = v_1 - t \simeq v_1)$$

$$\frac{n_1}{v_2} - \frac{n_2}{v_1} = \frac{n_1 - n_2}{R_2}. \tag{34-14}$$

Adding Eqs. 34-13 and 34-14, we get

$$\frac{n_1}{v} - \frac{n_1}{u} = \frac{(n_2 - n_1)}{R_1} + \frac{(n_1 - n_2)}{R_2}.$$

Since final image is at v_2, we can take $v_2 = v$. Therefore,

$$\frac{1}{v} - \frac{1}{u} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

According to the definition of second focus, if the object is at infinity, the image is formed at the focus, that is, $u = \infty \Rightarrow v = f$. We now have

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

This is known as **lens maker's formula** for obvious reasons.

Moreover,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

This is the **lens formula**.

Note that this f refers to second focus.

By using thin lens formula, for different positions of the object, we can find the location of images. To analyze this, we will divide space into two parts. For the object real side is left and virtual side is to the right of the lens. But for the image, real side is to the right of the lens and virtual side is to the left of the lens. As before (in the case of mirror), we will divide the entire space into four parts (Fig. 34-37). By plugging in different values of object distance in lens formula, we see that the image distance can be represented as in Table 34-1.

From Tables 34-1 and 34-2, we can draw many important conclusions.

I. In convex lens virtual object cannot form virtual image and in concave lens real object cannot form real image.

II. If an object moves from ∞ to $2f$, image moves from f to $2f$. So, velocity of object is greater than velocity of image, when the object is in region I. When the object moves from $2f$ to f, the image moves from $2f$ to ∞ in the same

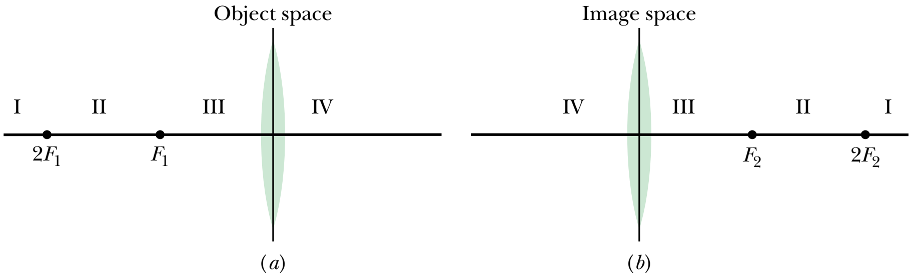


Figure 34-37 (*a*) Object space and (*b*) image space for a convex lens.

Table 34-1 Location, Nature, and Magnification of Image for Convex Lens

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Real object and real image	Diminished	Inverted
II	I	Real object and real image	Enlarged	Inverted
III	IV	Real object and virtual image	Enlarged	Erect
IV	III	Virtual object and real image	Diminished	Erect

Table 34-2 Location, Nature, and Magnification of Image for a Concave Lens

Object	Image	Nature	Magnification	Erect/Inverted
I	II	Virtual object and virtual image	Diminished	Inverted
II	I	Virtual object and virtual image	Enlarged	Inverted
III	IV	Virtual object and real image	Enlarged	Erect
IV	III	Real object and virtual image	Diminished	Erect

time. So, velocity of image is greater than that of object when the object is in region II. The velocity of the object is same as that of the image when the object is at $2f$. Note here that $2f$ does not mean R. In a lens, $2f$ is not equal to the radius of curvature in general.

III. When the object is at $2f$, the relative velocity of object and image is 0. So, at this position, the distance between the real object and real image is minimum ($4f$).

IV. When the object is on the left side of the object focus, the image is on the right side of image focus and vice versa. So, the object and its image are always on the opposite side of their respective focus. Yes, there is an analogous Newton's law for the lens also.

CHECKPOINT 6

Show that for a lens, $x_o x_i = -f^2$, where x_o and x_i are the distance of the object from object focus and the distance of the image from the image focus, respectively.

Converging and Diverging Lens

A **converging lens** is one which converges a parallel beam of light on the other side of the lens. This beam will converge at the second focus or image focus. For a converging lens this focal length (f) is positive. The object focus is located on the negative side.

As discussed before, for a converging lens, focal length should be positive. From the lens maker's formula, f is positive if $n_2 > n_1$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is positive. This is the situation with a convex lens kept in air. But in the case of concave lens also, f is positive if $n_1 > n_2$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is negative. This is the situation when a concave lens is dipped in a medium of refractive index greater than that of the lens.

A **diverging lens** is one which diverges a parallel beam of light passing through it. For such a lens, focal length is negative. So, its image focus is on negative side and object focus is on the positive side.

As stated before, for a diverging lens, focal length should be negative. From the lens maker's formula, f is negative if $n_2 > n_1$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is negative. This is the situation with a concave lens. However, in the case of convex lens also, f is negative if $n_1 > n_2$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ is positive. This is the situation when a convex lens is dipped in a medium of refractive index greater than that of the lens.

Thus, we can say that in the case of lens, the focal length depends on the properties of the medium. It may seem strange, but actually the focal length does not depend on which side is facing the incident light. For the application of the lens maker's formula, we take R_1 to be the radius of the surface on which light is first incident, but actually even if we invert the lens and make the second surface face the light; the answer for the focal length will be the same.

CHECKPOINT 7

A thin symmetric lens provides an image of a fingerprint with a magnification of +0.2 when the fingerprint is 1.0 cm farther from the lens than the focal point of the lens. What are the type and orientation of the image, and what is the type of lens?

SAMPLE PROBLEM 34.16

Focal length of a concavo-convex lens

A concavo-convex lens has radii of its faces 20 cm and 60 cm. If the refractive index of the material of the lens is 1.5, find its focal length.

KEY IDEA

Here, we should be careful to see what values R_1 and R_2 will have. Using Fig. 34-31c, $R_1 = +20$ cm and $R_2 = 60$ cm.

Calculations: Surroundings have a refractive index of 1. So, the lens maker's formula yields

$$\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$
$$=\left(\frac{1.5}{1}-1\right)\left(\frac{1}{20}-\frac{1}{60}\right).$$

Therefore, $f = 60$ cm. Here note that sign convention is very important. As can be seen in Fig. 34-31, R_1 and R_2 are both positive because light is incident from the left and both the surfaces have center of curvature to the right of the optical center.

Let us now assume that light is now incident from the right so that light is first incident on the surface of radius of curvature 60 cm. As can be seen from Fig. 34-31*c*, the side to the left of the optical center becomes the positive side. So, now both the surfaces have their center of curvature on the negative sides. So, now $R_1 = -60$ (light is first incident on it) and $R_2 = -20$ cm. Substituting the values, we get

$$\frac{1}{f}=\left(\frac{1.5}{1}-1\right)\left(\frac{1}{-60}-\frac{1}{-20}\right)=\frac{1}{60}.$$

We can see that the focal length remains the same in both the cases.

Principles for Drawing Ray Diagram

The principles used for drawing ray diagrams are illustrated through Figs. 34-38 *a*–*e*.

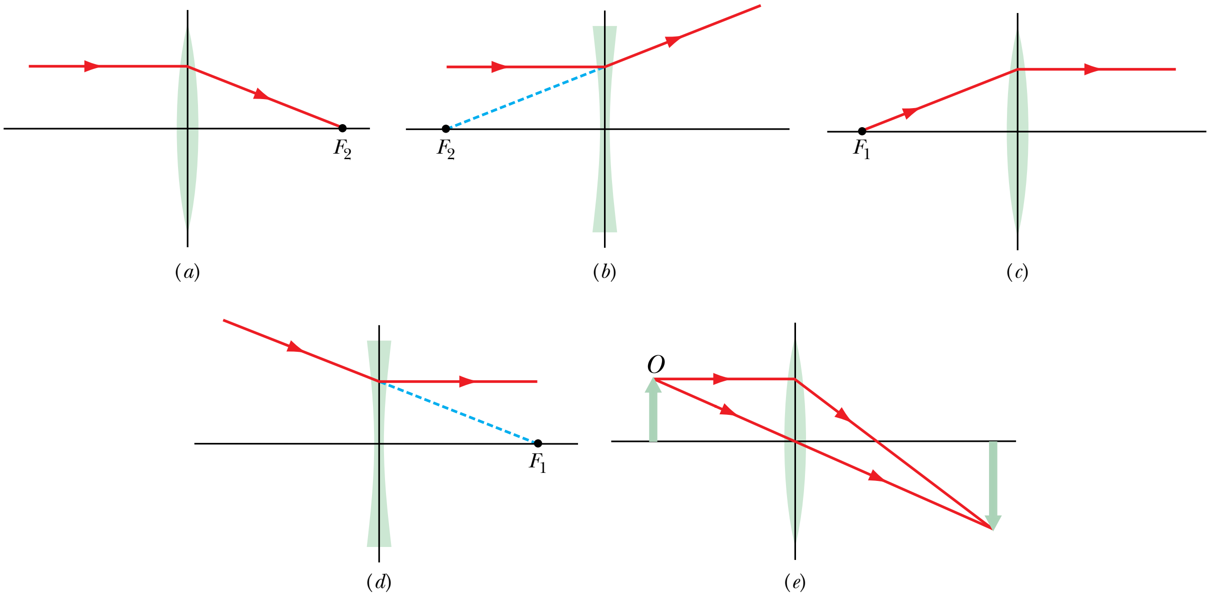


Figure 34-38 (*a*, *b*) A ray parallel to the principal axis will pass through the second focus after refraction from the lens. (*c*, *d*) A ray passing through F_1 will become parallel to the principal axis after refraction. (*e*) A ray passing through the optical center will pass undeviated. Since on the line of incident ray, object will lie and on the line of the refracted ray, image will lie, this line is common to both the object and the image. So, we can also say that the line joining the object and its image will always pass through the optical center.

SAMPLE PROBLEM 34.17

Deviation of a ray incident on a converging lens

Find the point at which the first ray crosses the principal axis after refraction (Fig. 34-39). Also find its deviation.

KEY IDEA

We can see that the two refracted rays will meet at a point in the focal plane. Since the ray passing through the

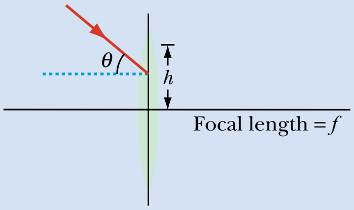


Figure 34-39 Deviation of a ray incident on a converging lens.

optical center will pass undeviated, we can say that this meeting occurs at an angle of θ and hence at a distance $f\theta$ below the focus (Fig. 34-40).

Calculation: By the property of the similar triangles, we can say that

$$\frac{h}{x} = \frac{f\theta}{f - x} = \phi.$$

Solving, we get

$$x = \frac{hf}{h + f\theta} \quad \text{and} \quad \phi = \frac{h + f\theta}{f}.$$

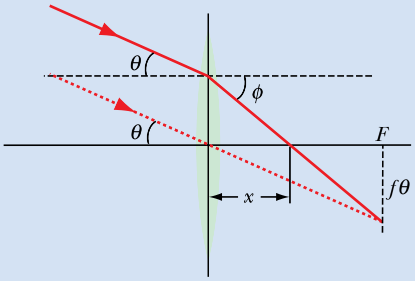


Figure 34-40 We draw an imaginary ray passing through the optical center to find the path of the refracted rays. We know that a parallel beam of light will converge at a point in the focal plane. The ray passing through the optical center will pass undeviated.

Therefore, the deviation is

$$\theta - \phi = \frac{h}{f}.$$

SAMPLE PROBLEM 34.18

Formation of image by a combination of converging and a diverging lens

Find the location of the final image after all the refractions in Fig. 34-41. The radius of both the plano-convex lens and plano-concave lens is 10 cm and refractive index is 1.5.

KEY IDEA

Here again we are seeing an example of multiple events. The image of plano-convex lens will act as an object for the plano-concave lens. But to get there, we need to find the focal length of the lens first.

Calculation: Focal length of the plano-convex lens is

$$\frac{1}{f} = (1.5 - 1)\left(\frac{1}{+10} - \frac{1}{\infty}\right) = \frac{1}{20}.$$

Focal length of plano-concave lens is

$$\frac{1}{f} = (1.5 - 1)\left(\frac{1}{\infty} - \frac{1}{+10}\right) = -\frac{1}{20}.$$

Since parallel beam is incident on the lens, its image from plano-convex lens will be formed at +20 cm from it (at the focus) and will act as an object for the plano-concave lens. Since the two lenses are at a distance of 10 cm from each other, therefore, for the next lens $u = +10$ cm. Hence

$$v = \frac{uf}{u + f} = \frac{10 \times -20}{10 - 20} = 20 \text{ cm}.$$

Learn: We can see that a diverging lens is forming a real image. This is because the object for it is virtual and is between pole and object focus.

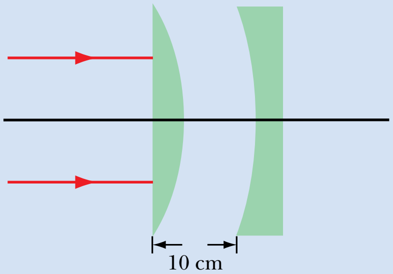


Figure 34-41 Formation of image by a combination of converging and a diverging lens.

SAMPLE PROBLEM 34.19

Condition for autocollimation

Find the distance between the convex lens and convex mirror, both of focal length 20 cm (Figs. 34-42*a* and *b*), so that for an object kept at 30 cm from the lens, the final image is at the object itself. (This is commonly known as ***condition for autocollimation*** that we discussed in Section 33.5.)

KEY IDEA

(1) This condition occurs when the object and image coincide. In the case of mirror, this happens when the object for the mirror is at its pole or at the center of curvature (Figs. 34-42*a* and *b*). In the second case, the angle of incidence is 0º and hence the angle of reflection will also be 0º.

(2) In the case of single lens, image can never coincide with the object because the rays are refracted into the next medium. But if the lens is kept before the mirror, after refraction from the lens, rays will fall on the mirror. If the image of the lens is at the pole or at the center of the curvature of the mirror, the rays will retrace their path and hence by the principle of optical reversibility, we can say that image will again form on the object.

Calculation: For the lens,

$$v = \frac{uf}{u+f} = \frac{20 \times -30}{-30+20} = 60 \text{ cm.}$$

For the mirror, this image should act as an object. The autocollimation will occur if this image is formed at pole. In this situation, the distance between the lens and the pole will be 60 cm.

Autocollimation will also occur if this image is formed at the center of curvature of the mirror. But the center of curvature is 40 cm away from the pole as can be seen in the figure. So, the lens can also be at 60 − 40 = 20 cm from the mirror.

Learn: In fact, a beam of light can be made to retrace its path by using a transparent sphere silvered at its back. For this the condition of autocollimation requires that the intermediate image which acts as an object for the mirror at the back is at the center of the sphere (Fig. 34-43*b*) or at its pole (Fig. 34-43*a*). It is obvious that due to refraction from a sphere, image cannot be formed at its center (angle of refraction will become zero for a finite angle of incidence as can be

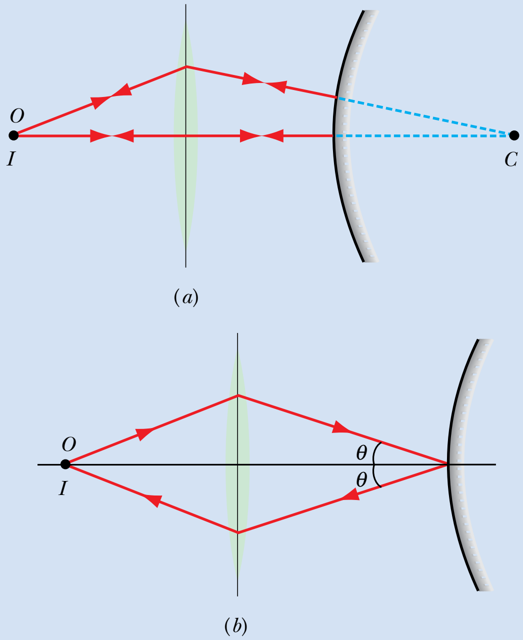


Figure 34-42 (*a*) The principle of optical reversibility is illustrated. Here, the image of the object *O* formed by the lens acts as an object for the mirror. The final image will coincide with the object if the rays retrace their path. The rays retrace their path if they are incident normally on the mirror. (*b*) Another situation where the object and the image coincide with each other. The image of the lens is formed at the pole of the mirror. So, the rays are reflected symmetrically.

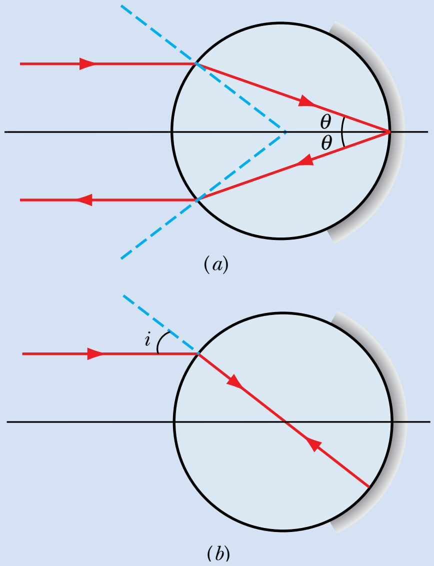


Figure 34-43 A parallel beam of light is incident on a transparent sphere such that the image is formed at (*a*) pole of the mirror, (*b*) the center of curvature of the spherical surface.

seen in Fig. 35-43*b*). But the image due to refraction at the curved surface can be definitely formed at the pole of the mirror.

This would happen if the refractive index of the sphere is exactly two times the refractive index of the medium from which the radiation is incident.

Note: This is conventionally known as a *cat's eye* retroreflector. The term *cat's eye* derives from the resemblance of the cat's eye retroreflector to the optical system that produces the well-known phenomenon of "glowing eyes" or eye shine in cats and other vertebrates (which are only reflecting light, rather than actually glowing).

SAMPLE PROBLEM 34.20

Formation of light spot on screen through refraction

Find the radius of the light spot on the screen as shown in Fig. 34-44. Consider only the light rays refracted from the lens. Find the average intensity on the screen. The source is point source of power 100 W. The lens has an aperture of 6 cm.

KEY IDEAS

(1) Only a real image can be focused on a screen. A virtual image cannot be focused on a screen but can be seen with naked eyes. To see a real image with a naked eye, we should place the eye behind the image.
(2) Here, we will try to locate the image formed by the lens. This image will be formed by all the rays collected by the lens. So, if we consider the extreme rays which are being focused at the aperture (Fig. 35-45), we will get the field of view for the image. This will also show us the region in which the refracted light will be present.

Calculation: If the screen was not present, the image would have formed at

$$v = \frac{uf}{u+f} = \frac{-30\times 20}{-30+20} = 60 \text{ cm}$$

from the lens.

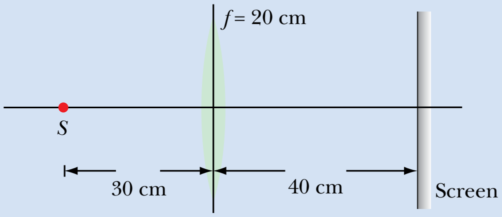


Figure 34-44 A lens is used to form a light spot on the screen.

From similar triangles, we can estimate the radius of the image formed:

$$\frac{R}{3} = \frac{20}{60}$$
$$R = 1 \text{ cm.}$$

The amount of light used to form the image can be estimated with the help of solid angle subtended by the lens at the location of the source:

$$\frac{\text{Power of image}}{100 \text{ W}} = \frac{1-\cos\theta}{2}.$$

Here, tan $\theta = 3/30 = 0.1$, for a small angle and cos $\theta = (1 - \theta^2/2)$. The power of the image $50 \times (0.1)^2/2 = 0.25$ W.

$$\text{Intensity} = \frac{\text{Power}}{\text{area}}$$
$$= \frac{0.25}{\pi \times (0.01)^2} = \frac{2500}{\pi} \text{ W/m}^2.$$

Learn: Here, we can see how the intensity can be quite high even though the power transmitted by the lens is very small.

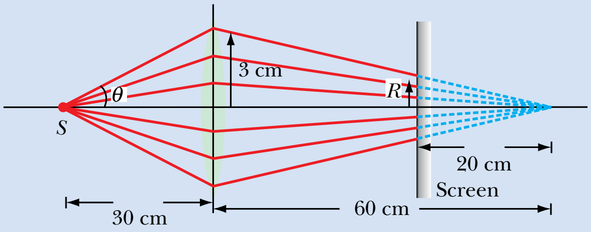


Figure 34-45 The formation of a refracted spot on a screen.

SAMPLE PROBLEM 34.21

Beam expander

Assume that a parallel beam is incident on two convex lens of focal length 20 cm and 40 cm. The beam has a diameter of 2 cm. What should be the distance between them so that the final beam emerges parallel to the principal axis (Fig. 34-46)?

KEY IDEA

Here, again we can see that the image of one will act as an image of another. However, we can also see that for the final beam to be parallel, the object for it should be at its first focus. The image of first lens is formed at its second focus. So, we can say that the distance between them should be equal to sum of their focal length.

Calculation: From Fig. 34-46, it is clear that

$$d = f_1 + f_2 = 60\text{ cm}.$$

Learning: If a parallel beam passes through a series of lens, under suitable arrangement, the width of the beam can be expanded. As can be seen from Fig. 34-46, the radius of the beam is changed. This new radius can be found with the help of similar triangles. Here, we can say that

$$\frac{R_i}{R_f} = \frac{f_i}{f_f}.$$

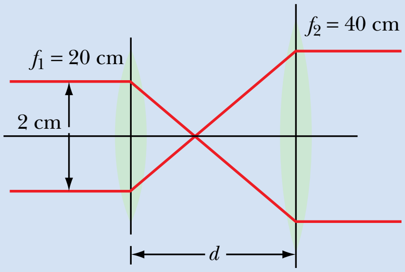


Figure 34-46 Using two lens system to expand or contract a parallel beam of light.

Displacement Method for Finding the Focal Length

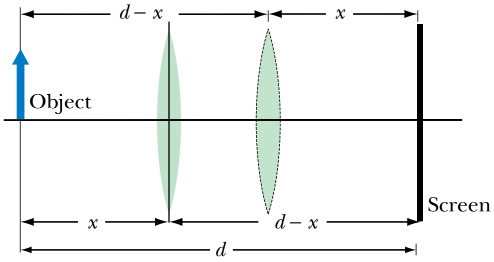


Figure 34-47 Displacement method for finding the focal length of the lens. Here, the same lens is initially located at a distance of x from the object. Then the lens is moved to a distance of $d - x$ from the object so that a sharp image is again obtained on the screen.

In the displacement method, we fix the object and its image is obtained on a fixed screen as shown in Fig. 34-47. We know that only a real image can be obtained on a screen. We can say that real object and real image distance can be

$$d > 4f,$$

where d is the distance between screen and object. So, for this experiment, the object should be kept at a distance of not less than $4f$ from the screen.

Let us assume that the object is at a distance x from the lens. So, the image will be formed at a distance of $d - x$ from the lens. By lens formula, we have

$$\frac{1}{d-x} - \frac{1}{-x} = \frac{1}{f}$$

or

$$\frac{1}{d-x} + \frac{1}{x} = \frac{1}{f}. \tag{34-15}$$

We know by the principle of optical reversibility that the object and its image can be interchanged. Here, although we cannot move the object and screen, we can move the lens. So, if object distance is x, image is at $d - x$. If instead, the object is kept at distance of $d - x$, the image is formed at a distance of x. Hence, there is an alternative position of the lens for which an image can be formed on the screen. We can also apply the lens formula for this case and obtain the same formula as Eq. 34-15. The magnification in the first case is

$$m_1 = \frac{d-x}{-x}.$$

The magnification in the second case is

$$m_2 = \frac{x}{-(d-x)}.$$

Here, we note that both the magnifications are negative as real image of real object is always inverted. Also,

$$m_1 \times m_2 = 1.$$

Therefore,

$$\frac{h_{I_1}}{h_O} \times \frac{h_{I_2}}{h_O} = 1. \tag{34-16}$$

SAMPLE PROBLEM 34.22

Finding focal length by displacement method

In an experiment for finding focal length of a convex lens, the object and screen are kept fixed. There are two position of lens between the object and screen for which an image is obtained on the screen. These positions are separated by 20 cm. Also, the magnification in first situation is –2. Find the distance between the object and the screen, that is the focal length of the lens.

KEY IDEA

As we can see from the analysis presented before, this refers to the situation of displacement method of finding the focal length of lens (Fig. 34-47).

Calculation:

$$\frac{1}{d-x} + \frac{1}{x} = \frac{1}{f}.$$

From Fig. 34-47, it is clear that in the first situation, object is at a distance x from the lens and in second situation screen is at a distance x from the lens. Therefore,

$$d - 2x = 20 \text{ cm}.$$

Also

$$m_1 = \frac{d-x}{-x} = -2$$

$$d = 3x.$$

So, $x = 20$ cm and $d = 60$ cm. Therefore,

$$\frac{1}{40} + \frac{1}{20} = \frac{1}{f}$$

$$f = 40/3 \text{ cm}.$$

We stated previously that d must be greater than $4f$. This can be easily seen from the answer.

Equivalent Focal Length

When two lenses are joined in such a way that the distance between them is negligible, the equivalent focal length is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}.$$

Proof: Suppose that we have two lens of focal length f_1 and f_2 joined together (Fig. 34-48). As we have discussed numerous times, the image of one lens will act as an object for the other. Let us assume that a real object is situated to the left, say at a distance u. The final image is formed at v from the combination.

For the first lens, we can apply the lens formula:

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1}.$$

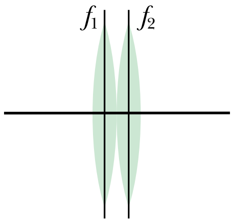


Figure 34-48 Two lens are joined together. Here, we have taken the example of two converging lens. But the analysis will be equally well applied to the case of any other lens combination.

This image at v_1 from the first lens will act as an object for the next lens. The distance of the object from the second lens would be

$$u_2 = v_1 - t \approx u_1,$$

because the thickness t of both the lens is small.

For the second lens, we can apply the lens formula:

$$\frac{1}{v} - \frac{1}{u_2} = \frac{1}{f_2}$$

and

$$\frac{1}{v} - \frac{1}{v_1} = \frac{1}{f_2}.$$

Adding the two equations, we get

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f_{eq}}.$$

To solve such situations, we should first find the focal length of individual lens and then apply the above formula. Note that the focal lengths should be substituted with their sign.

SAMPLE PROBLEM 34.23

Finding equivalent focal length

Two equiconcave lens of focal length –20 cm each having a refractive index of 1.50 are kept together and the space between them is filled with water as shown in Fig. 34-49. Find the equivalent focal length of the arrangement thus formed.

KEY IDEA

(1) Here we can imagine a thin layer of air in between lens and water at both the surfaces (Fig. 34-50). This will give us two concave lenses combined with a convex lens of water. This thin layer of air will not affect the formation of image but rather will simplify the calculations a lot.

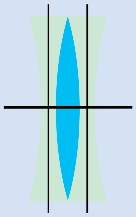


Figure 34-49 Two equiconcave lens with their intermediate space filled with water.

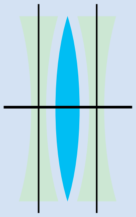


Figure 34-50 Assuming a thin air gap between the lens.

(2) First let us find the radius of curvature of the lens because this will be needed to find the focal length of the convex lens formed by water in between.

Calculations:

$$\frac{1}{-20} = (1.5 - 1)\left(\frac{1}{-R} - \frac{1}{R}\right)$$

$$R = 20 \text{ cm}.$$

So, the focal length of water convex lens (in Fig. 34-50) is

$$\frac{1}{f} = \left(\frac{4}{3} - 1\right)\left(\frac{1}{20} - \frac{1}{-20}\right)$$

$$f = 30 \text{ cm}.$$

Note here carefully that we are taking the focal length of the water lens in air because when we are using the focal length of the concave lens, we are assuming that there is air on both the sides.

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_1}$$

$$\frac{1}{f} = \frac{1}{-20} + \frac{1}{30} + \frac{1}{-20}$$

$$f = -15 \text{ cm}.$$

Learn: Similarly, if a lens is cut into two parts, the focal length of the two parts are given by

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2},$$

where f is the focal length of the original lens and f_1 and f_2 are focal lengths of the two cut parts. The interesting situation is that if we join these lenses back, we will get back the original lens. We have studied previously that the focal length of the lens does not depend on the side facing the light. So, the focal length of all the following four arrangements (Figs. 34-51*a*–*d*) will be the same.

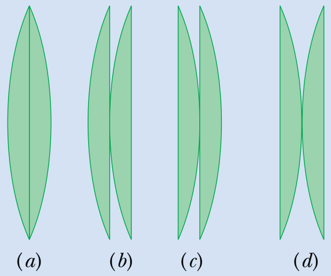


Figure 34-51 (*a*) A convex lens is cut into two parts by a section done along the central line. (*b*)–(*d*) The two parts are recombined. It is found that the focal length in all these three cases is the same as that in (*a*).

Focal Length of a Lens if the Medium on Both the Sides of Lens is Different

Here we have a spherical lens, each of whose surfaces is spherical (Fig. 34-52). So, instead of applying thin lens formula which we derived when the medium on both the sides of the lens is the same, we apply the formula for refraction from the spherical surface one by one. As usual, the image of one surface acts as an object for the next. Let us assume that an object is located at a distance of u from the left side of the lens in medium 1.
For the first surface,

$$\frac{n_2}{v_1}-\frac{n_1}{u}=\frac{n_2-n_1}{R_1}, \tag{34-17}$$

where v_1 is the distance of image from the left surface of the lens. For second surface, image of the first surface acts as an object. So, $v_1 \approx u_2$ because the distance between the first surface and the second surface is small (it is a thin lens).

$$\frac{n_3}{v}-\frac{n_2}{u_2}=\frac{n_3-n_2}{R_2}. \tag{34-18}$$

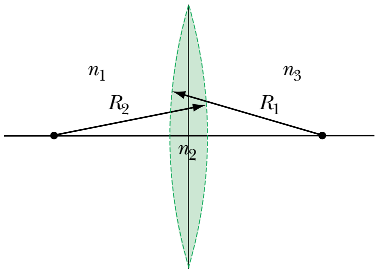


Figure 34-52 A lens of refractive index n_2 is surrounded by a medium of refractive index n_1 on the left and n_3 on the right.

Adding Eqs. 34-17 and 34-18, we get

and $$\frac{n_3}{v}-\frac{n_1}{u}=\frac{(n_2-n_1)}{R_1}+\frac{(n_3-n_2)}{R_2}.$$

34.6 | LENS MIRROR COMBINATION

Key Concept

- A system of combination of a lens and mirror behaves like a mirror whose focal length is given by

$$\frac{1}{f_{\text{eq}}}=-\frac{2}{f_L}+\frac{1}{f_M}.$$

Many times, we come across situations where the rear surface of the lens is silvered (Fig. 34-53) or it may be that a lens and a mirror are kept in close contact. As we stated before, the image of one system acts as an object for the next. However, it turns out that in this situation, there is a simple formula to find out the image.

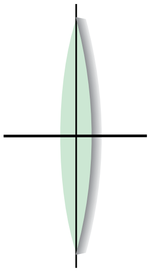


Figure 34-53 A lens silvered at its back to form an equivalent mirror.

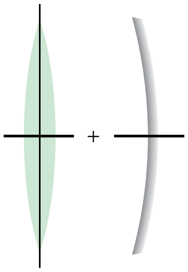


Figure 34-54 A lens silvered at its back is equivalent to a combination of lens and a mirror.

Since the light rays will be reflected, the system behaves like a mirror. We are interested in replacing this with an equivalent mirror and we will work to find the focal length of a mirror with which this system can be replaced.

As before, we will divide the system into three parts. We will assume that a thin layer of air is in between rear silvered surface and the lens (Fig. 34-54). This will not make any difference to the answer but will simplify the calculations a lot.

Let us assume that an object is located at a distance of u from the left side of the lens.For the lens, using the lens formula will yield

$$\frac{1}{v_1}-\frac{1}{u}=\frac{1}{f_L}, \tag{34-19}$$

where v_1 is the distance of image from the lens. For the mirror, image of the lens acts as an object. So, $v_1 \approx u_2$ because the distance between the lens and mirror is zero. Applying mirror formula, we get

$$\frac{1}{v_2}+\frac{1}{u_2}=\frac{1}{f_M}$$

or

$$\frac{1}{v_2}+\frac{1}{v_1}=\frac{1}{f_M}. \tag{34-20}$$

Again, this image at a distance of v_2 from the mirror will act as an object for the lens. We can again apply lens formula to find the location of the final image.

$$\frac{1}{v_3}-\frac{1}{u_3}=\frac{1}{f_L}.$$

But now we must be careful. Light rays have reversed their direction after striking the mirror. So, although the lens is of the same nature and f_L remains the same, the sign convention for the object and the image reverses. Recall that in Cartesian sign convention, the direction of incident rays is assumed to be in positive direction. In fact, what is positive for the mirror will be negative for the lens.

Therefore,

$$u_3=-v_2.$$

Since in the equivalent system, we are replacing with a mirror whose positive direction is same as that of the first lens, therefore, location of final image $v=-v_3$ and so

$$\frac{1}{-v}-\frac{1}{-v_2}=\frac{1}{f_L}. \tag{34-21}$$

By adding Eqs. 34-19, 34-20, and 34-21 according to the formula –(34-19) – (34-21) + (34-20), we get

$$\frac{1}{v}+\frac{1}{u}=-\frac{2}{f_L}+\frac{1}{f_M}.$$

So, we can say that this system behaves like a mirror whose focal length is given by

$$\frac{1}{f_{\text{eq}}}=-\frac{2}{f_L}+\frac{1}{f_M}. \tag{34-22}$$

In this the focal length of a convex lens is considered positive. Similarly, the focal lengths of the mirror should be substituted with the respective sign. If the focal length of the mirror comes out to be positive, the system would be treated as an equivalent convex mirror. If the focal length of the mirror comes out to be negative, the system would be treated as an equivalent concave mirror.

SAMPLE PROBLEM 34.24

Equivalent focal length of convex lens silvered on one surface

A convex lens of radii 20 cm and 30 cm is silvered at the surface which has smaller radius of curvature. Find the equivalent focal length ($n = 1.5$).

KEY IDEA

Here, the focal length of equivalent mirror can be found by Eq. 35-22.

Calculation: We know that

$$\frac{1}{f_L} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= (1.5 - 1)\left(\frac{1}{20} + \frac{1}{30}\right) = 0.5 \times \frac{5}{60}.$$

Therefore,

$$f_L = \frac{120}{5} = 24 \text{ cm}.$$

Also

$$f_M = \frac{-20}{2} = -10 \text{ cm}.$$

From Eq. 34-22, we get

$$\frac{1}{f_{\text{eq}}} = -\frac{2}{f_L} + \frac{1}{f_M}$$

$$= -\frac{2}{24} + \frac{1}{-10} = \frac{-60}{11} \text{ cm}.$$

So, this system behaves like a concave mirror of focal length 60/11 cm.

34.7 | PRISM

Key Concepts

- The primary use of the prism to cause deviation of a ray from its path.
- The angle between the refracting faces is called angle of the prism (angle A).
- The angle of deviation is given by $\delta = i + e - A$, where A is angle of prism, i is angle of incidence on first surface and e is angle of refraction (emergence) at the second surface.

Figure 34-55 shows a ray of monochromatic light passing through a glass prism surrounded by air. When the light enters the prism at the left face, the refracted ray is bent toward the normal, because the refractive index of glass is greater than that of air. When the light leaves the prism at the right face, it is refracted away from the normal. Thus, the net effect of the prism is to change the direction of the ray, causing it to bend downward upon entering the prism, and downward again upon leaving. So, the ray is deviated from its path. This is one of the primary uses of the prism: deviation of a ray from its path. The edge at which the refracting faces meet is called the refracting edge. The angle between the refracting faces is called refracting angle of the prism or angle of the prism (angle A in Fig. 34-55).

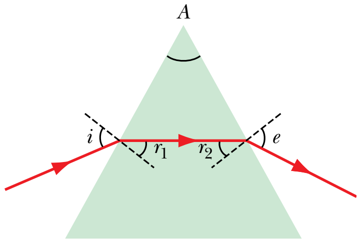


Figure 34-55 A glass prism deviating a monochromatic beam of light.

From Fig. 34-55, it can be seen that

$$A + 90° - r_1 + 90° - r_2 = 180°$$

or

$$A = r_1 + r_2. \tag{34-23}$$

The incident ray is deviated from its path at each surface by the process of refraction. At the first surface, the angle of incidence is i and the angle of refraction is r_1. So, the deviation at that face is

$$\delta_1 = i - r_1 \quad \text{(clockwise)}.$$

At the second surface, the angle of incidence is r_2 and angle of refraction or emergence is e. So, the deviation at that face is

$$\delta_2 = e - r_2 \quad \text{(clockwise)}.$$

So, the total deviation produced is

$$\delta = \delta_1 - \delta_2 = i + e - (r_1 + r_2)$$

or

$$\delta = i + e - A. \tag{34-24}$$

Here, to find the deviation when we know the angle of incidence, we can find the angle r_1 by applying Snell's law:

$$1 \sin i = n \sin r_1.$$

By applying Snell's law at the next surface, we get

$$1 \sin e = n \sin r_2.$$

This value of deviation is obtained for a prism and the graph of deviation δ against angle of incidence i has been plotted as shown in Fig. 34-56. Take a realistic prism and you will obtain a curve as shown in Fig. 34-56.

Some conclusions can be easily drawn by observing the graph. This graph is not symmetric about the minimum. It seems from the graph that the same deviation occurs for two different angles of incidence. One of these angles is the angle of incidence and another is the angle of emergence. But if the ray is incident at the angle of emergence on the prism, by the principle of optical reversibility, the angle of emergence will be the previous angle of incidence. So, the angle of deviation remains the same.

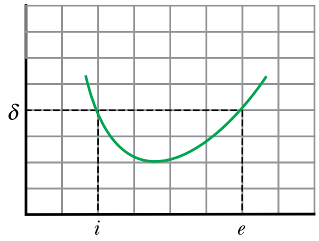


Figure 34-56 A plot of angle of deviation v/s the angle of incidence. At an angle of incidence i as well as e, the deviation is the same.

SAMPLE PROBLEM 34.25

Refractive index of a prism

A ray is incident on an equilateral prism such that the angle of deviation is 30º. It is seen that if the angle of incidence is increased by 30º, then the deviation is again 30º. Find the refractive index of the prism.

KEY IDEA

We can easily see that in one situation the angle of incidence will be i and in the next situation angle of incidence will be $i + 30º$. From the above discussion, we can say that in the first situation, the angle of emergence should be $i + 30º$ and in the second situation, the angle of emergence should be i.

Calculation: In both the cases

$$\begin{aligned}\delta &= i + e - A\\ 30° &= i + i + 30° - 60°\\ i &= 30°.\end{aligned}$$

The angle of emergence thus is

$$i + 30° = 60°.$$

By Snell's law,

$$1 \sin 30 = n \sin r_1.$$

By Snell's law at the second surface, we have

$$\begin{aligned}1 \sin 60 &= n \sin r_2\\ 1 \sin 60 &= n \sin(60 - r_1).\end{aligned}$$

Solving for n, we get

$$n = \sqrt{\frac{4+\sqrt{3}}{3}}.$$

Angle of Minimum Deviation

From Fig. 34-57, we can say that at the minimum deviation, there is only one angle of incidence. So, at the minimum deviation we can say that $i = e$. This gives

$$1 \times \sin i = n \sin r_1$$
$$n \sin r_2 = 1 \times \sin i.$$

Since $i = e$, we can also say that

$$r_1 = r_2 \text{ and } r_1 + r_2 = A. \tag{34-25}$$

Therefore, $r_1 = A/2$. From $\sin i = n \sin r_1$, we get

$$\sin i = n \sin \frac{A}{2}.$$

Now, $\delta = i + e - A$. Therefore,

$$\delta_{\min} = 2i - A$$
$$i = \frac{\delta_{\min} + A}{2}$$

or

$$n = \frac{\sin[(\delta_{\min} + A)/2]}{\sin(A/2)}. \tag{34-26}$$

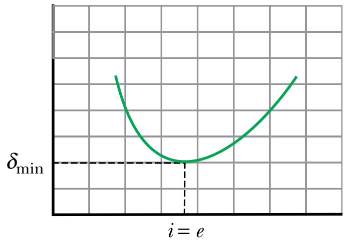


Figure 34-57 At the minimum deviation, angle of incidence is equal to the angle of emergence. So in the situation of minimum deviation, the ray passes through the prism symmetrically.

Angle of Maximum Deviation

From Fig. 34-58, it can be easily seen that the deviation is maximum when the angle of incidence is maximum. But we can also note that this same maximum deviation can be obtained at another angle of incidence which we can call as $i_{\min}$. It is clear that the maximum angle of incidence can be 90º. At this angle of incidence, as we discussed earlier, the angle of emergence will be $i_{\min}$. If we take the angle of incidence as less than $i_{\min}$ then r_2 will be greater than the critical angle for prism–air interface and hence total internal reflection will occur at the second surface. At an angle less than $i_{\min}$ the ray will not emerge from the prism. Normally, $i_{\min}$ can be any value depending on the angle of prism and the refractive index of the prism.

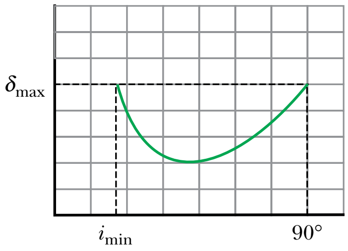


Figure 34-58 At the maximum deviation, angle of incidence is equal to 90°.

SAMPLE PROBLEM 34.26

Minimum and maximum deviation in a prism

A 60º prism has a refractive index of 1.5.

(a) Calculate the angle of incidence for minimum deviation.

KEY IDEA

As discussed in the Section: Angle of Minimum Deviation, the angle of incidence will be equal to the angle of emergence.

Calculation: As the prism is in the position of minimum deviation, $r = A/2 = 60º/2 = 30º$, so that at either face

$$\sin i = 1.5 \sin 30° = 0.75$$
$$i = \sin^{-1}(0.75) = 49°.$$

(b) Calculate the angle of emergence of light at maximum deviation.

KEY IDEA

As we have discussed in the Section: Angle of Maximum Deviation, the angle of incidence will be equal to 90º.

Calculation: For maximum deviation, i_1 = 90º so that $r_1 = \theta_C = \sin^{-1}$ (2/3) = 42º. But as in a prism,

$$r_1 + r_2 = A.$$

Therefore,

$$r_2 = A - r_1 = 60° - 42° = 18°.$$

Now applying Snell's law at the second face,

$$n \sin r_2 = \sin i_2,\ \frac{3}{2}\sin 18° = \sin i_2.$$

that is,

$$i_2 = \sin^{-1}(1.5 \times 0.31) = \sin^{-1}(0.465) = 28°.$$

SAMPLE PROBLEM 34.27

Total internal reflection in a prism

What is the relationship between A and n, so that no rays come out of second face (Fig. 34-59)?

KEY IDEA

(1) Here, we want to choose the refractive index of the prism such that a ray will always undergo total internal reflection. Note that this total internal reflection can occur only when the light ray goes from denser to rarer medium. In other words, this will occur only when the ray is incident on the second surface.

(2) The situation implies that for the smallest angle r_2 also, the total internal reflection should occur.

(3) $r_1 + r_2 = A$. So, when r_2 is minimum, r_1 is maximum. The angle of incidence i is also maximum by Snell's law. But the maximum angle of incidence can be 90º. This problem implies that if a total internal reflection occurs when angle of incidence is 90º, then total internal reflection will occur at all the angles.

Calculation: Applying Snell's law at the first surface:

$$1 \times \sin 90° = n \sin r_1$$

$$r_1 = \sin^{-1}\left(\frac{1}{n}\right)$$

$$r_2 = A - \sin^{-1}\left(\frac{1}{n}\right).$$

But for total internal reflection at the second surface:

$$r_2 > \theta_c$$

$$\sin r_2 > 1 \sin \theta_c \left(\frac{1}{n}\right)$$

$$A - \sin^{-1}\left(\frac{1}{n}\right) > \sin^{-1}\left(\frac{1}{n}\right).$$

Thus, the condition becomes

$$n > \operatorname{cosec}\left(\frac{A}{2}\right).$$

Note: Many optical instruments, such as binoculars, periscopes, and telescopes, use glass prisms and total internal reflection to turn a beam of light through 90º or 180º.

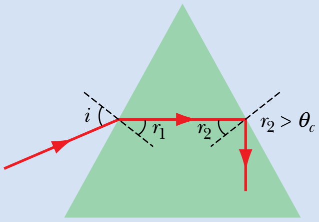


Figure 34-59 The prism has such a refractive index and an angle of prism combination that all the incident rays undergo total internal reflection.

Thin Prism

A very common situation in the laboratory is a thin prism. This refers to a prism whose angle is very small. As you might have studied in your mathematics classes, this would mean that the small angle approximation can be applied in the case of trigonometric ratios. Interestingly, in this situation, the deviation of the light ray comes out to be independent of the angle of incidence, if the angle of incidence is small too. The angle of deviation is given by

$$\delta = i + e - A.$$

From Eq. 34-25

$$\sin i = n \sin r_1.$$

For small angle of incidence

$$i = n r_1.$$

Since i is a small angle, r_1 turns out to be a small angle too. In turn,

$$r_1 + r_2 = A.$$

So, r_2 is also a small angle. This would imply that e is also a small angle.

$$n r_2 = e.$$

Now,

$$i + e = n(r_1 + r_2) = nA$$

Therefore,

$$\delta = nA - A$$

or

$$\delta = (n-1)A.$$

Interestingly, all the rays undergo the same deviation.

SAMPLE PROBLEM 34.28

Image formation by a system of thin prism and lens

A beam parallel to the principal axis is incident on a thin prism (Fig. 34-60). Find the location of final image.

KEY IDEAS

(1) Here, two events occur before the formation of image, the first is refraction through the thin prism which will only deviate the parallel beam of rays. This parallel beam will be incident on the lens which would form the image.

(2) When the rays pass through the prism, all the rays undergo the same deviation. A parallel beam incident on the lens will form an image somewhere on the focal plane.

Calculation: The deviation of the rays while passing through the prism is given by:

$$\delta = (n-1)A = 0.5 \times 2 = 1°.$$

The parallel beam of light will pass through the lens. So, the image will be formed in focal plane at 20 cm from the lens. The image will be located at an angle θ to the principal axis.

$$\theta = \frac{\pi}{180} \times 1°.$$

The distance of the image from the focal plane is given by

$$\frac{h}{20} = \frac{\pi}{180}$$

$$h = \frac{\pi}{9}\,\text{cm}.$$

Learn: If an object is viewed through a thin prism, all the rays coming from the object are deviated by the same angle. So, the image is formed at some point above the object as shown in Fig. 34-61. Note that this is a virtual image.

Shift of image due to prism = $x\delta = x\,(n-1)\,A$.

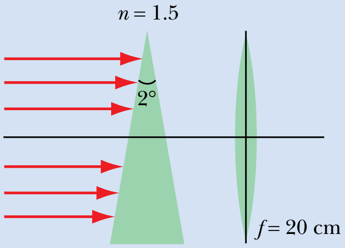


Figure 34-60 A parallel beam of light is incident on a thin prism. After refraction from the prism, the light is incident on a converging lens.

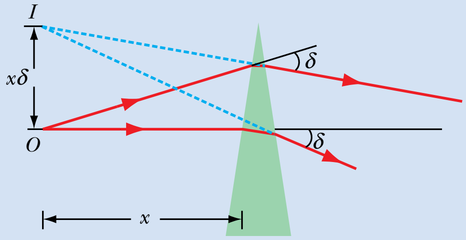


Figure 34-61 Formation of a virtual image by a prism.

34.8 | CHROMATIC DISPERSION

Key Concepts

- Rays of different wavelengths are refracted at different angles by a surface, so, the light gets spread out by refraction. This spreading of light is called chromatic dispersion and refers to the colors associated with the individual wavelengths.
- The relationship between n, of refraction of a given medium and the wavelength of light λ is given by

$$n = A + \frac{B}{\lambda^2}.$$

- For any transparent material, dispersive power ω is defined as

$$\omega = \left(\frac{n_V - n_R}{n_Y - 1}\right).$$

The index of refraction n encountered by light in any medium except vacuum depends on the wavelength of the light. The dependence of n on wavelength implies that when a light beam consists of rays of different wavelengths, the rays will be refracted at different angles by a surface; that is, the light will be spread out by the refraction. This spreading of light is called **chromatic dispersion**, in which "chromatic" refers to the colors associated with the individual wavelengths and "dispersion" refers to the spreading of the light according to its wavelengths or colors. The refractions discussed till now do not show chromatic dispersion because the beams are *monochromatic* (of a single wavelength or color).

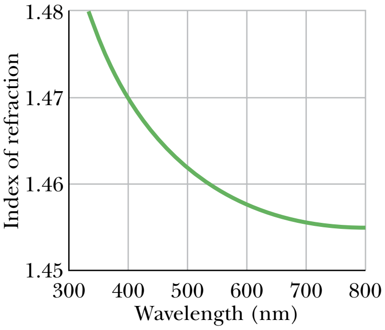


Figure 34-62 The index of refraction as a function of wavelength for fused quartz. The graph indicates that a beam of short-wavelength light, for which the index of refraction is higher, is bent more upon entering or leaving quartz than a beam of long-wavelength light.

Generally, the index of refraction of a given medium is *greater* for a shorter wavelength (corresponding to, say, blue light) than for a longer wavelength (say, red light). As an example, Fig. 34-62 shows how the index of refraction of fused quartz depends on the wavelength of light. Such dependence means that when a beam made up of waves of both blue and red light is refracted through a surface, such as from air into quartz or vice versa, the blue *component* (the ray corresponding to the wave of blue light) bends more than the red component.

A beam of *white light* consists of components of all (or nearly all) the colors in the visible spectrum with approximately uniform intensities. When you see such a beam, you perceive white rather than the individual colors. In Fig. 34-63*a*, a beam of white light in air is incident on a glass surface. (Because the pages of this book are white, a beam of white light is represented with a gray ray here. Also, a beam of monochromatic light is generally represented with a red ray.) Of the refracted light in Fig. 34-63*a*, only the red and blue components are shown. Because the blue component is bent more than the red component, the angle of refraction θ_{2b} for the blue component is *smaller* than the angle of refraction θ_{2r} for the red component. (Remember, angles

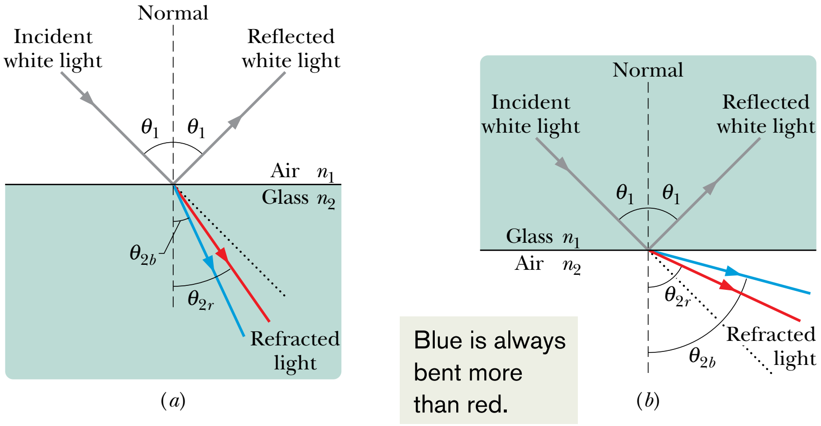


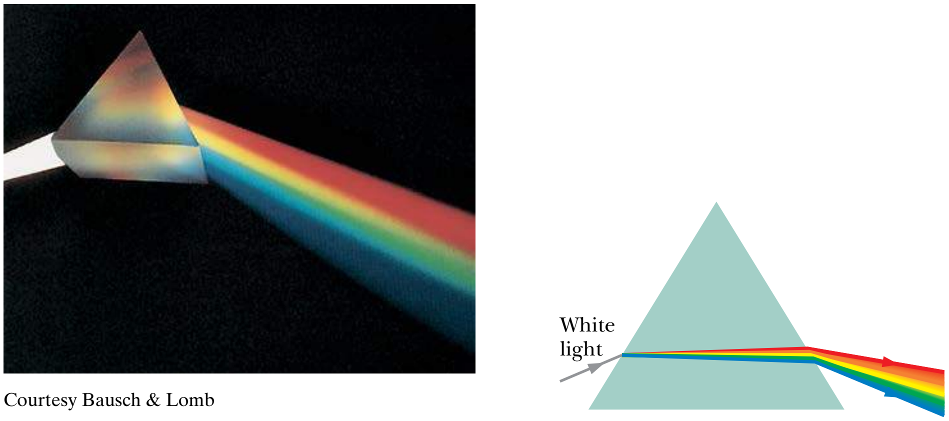
Figure 34-63 Chromatic dispersion of white light. The blue component is bent more than the red component. (*a*) Passing from air to glass, the blue component ends up with the smaller angle of refraction. (*b*) Passing from glass to air, the blue component ends up with the greater angle of refraction. Each dotted line represents the direction in which the light would continue to travel if it were not bent by the refraction.

are measured relative to the normal.) In Fig. 34-63*b*, a ray of white light in glass is incident on a glass–air interface. Again, the blue component is bent more than the red component, but now θ_{2b} is greater than θ_{2r}.

To increase the color separation, we can use a solid glass prism with a triangular cross section, as in Fig. 34-64*a*. The dispersion at the first surface (on the left in Figs. 34-64*a*, *b*) is then enhanced by the dispersion at the second surface.

Rainbows

The most charming example of chromatic dispersion is a rainbow. When sunlight (which consists of all visible colors) is intercepted by a falling raindrop, some of the light refracts into the drop, reflects once from the drop's inner surface, and then refracts out of the drop. Figure 34-65*a* shows the situation when the Sun is on the horizon at the left (and thus when the rays of sunlight are horizontal). The first refraction separates the sunlight into its component colors, and the second refraction increases the separation. (Only the red and blue rays are shown in the figure.) If many falling drops are brightly illuminated, you can see the separated colors they produce when the drops are at an angle of 42° from the direction of the *antisolar point A*, the point directly opposite the Sun in your view.



Courtesy Bausch & Lomb

(*a*) (*b*)

Figure 34-64 (*a*) A triangular prism separating white light into its component colors. (*b*) Chromatic dispersion occurs at the first surface and is increased at the second surface.

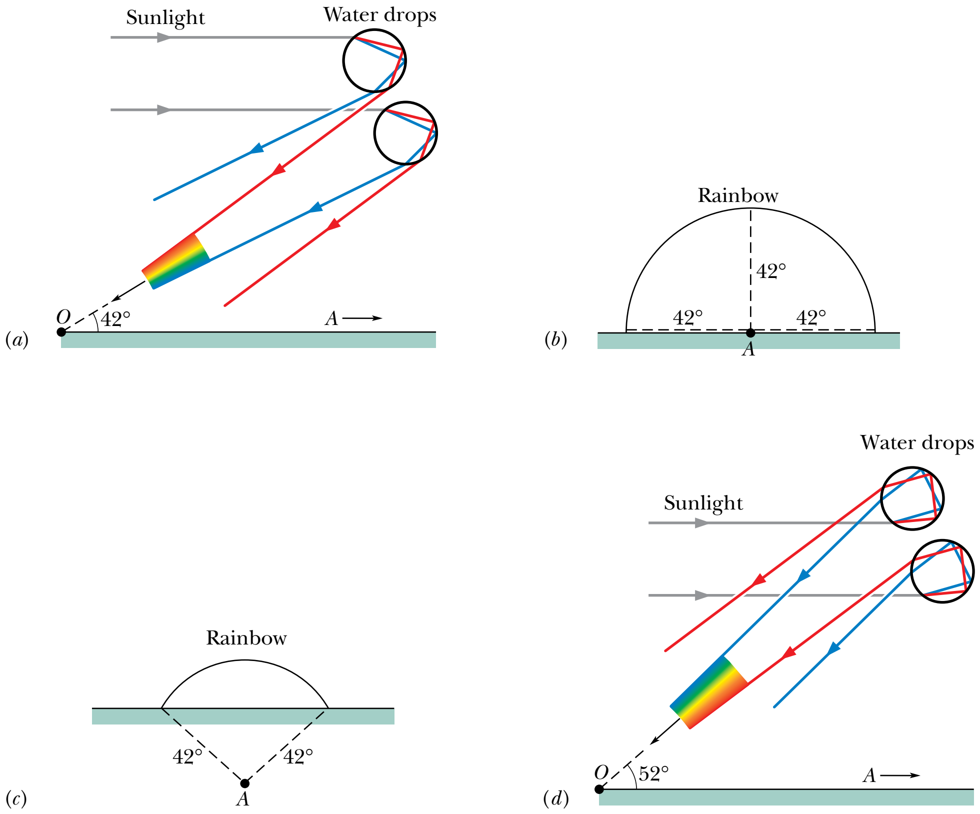


Figure 34-65 (*a*) The separation of colors when sunlight refracts into and out of falling raindrops leads to a primary rainbow. The antisolar point *A* is on the horizon at the right. The rainbow colors appear at an angle of 42° from the direction of *A*. (*b*) Drops at 42° from *A* in any direction can contribute to the rainbow. (*c*) The rainbow arc when the Sun is higher (and thus *A* is lower). (*d*) The separation of colors leading to a secondary rainbow.

To locate the drops, face away from the Sun and point both arms directly away from the Sun, toward the shadow of your head. Then move your right arm directly up, directly rightward, or in any intermediate direction until the angle between your arms is 42°. If illuminated drops happen to be in the direction of your right arm, you see color in that direction.

Because any drop at an angle of 42° in any direction from *A* can contribute to the rainbow, the rainbow is always a 42° circular arc around *A* (Fig. 34-65*b*) and the top of a rainbow is never more than 42° above the horizon. When the Sun is above the horizon, the direction of *A* is below the horizon, and only a shorter, lower rainbow arc is possible (Fig. 34-65*c*).

Because rainbows formed in this way involve one reflection of light inside each drop, they are often called *primary rainbows.* A *secondary rainbow* involves two reflections inside a drop, as shown in Fig. 34-65*d*. Colors appear in the secondary rainbow at an angle of 52° from the direction of *A*. A secondary rainbow is wider and dimmer than a primary rainbow and thus is more difficult to see. Also, the order of colors in a secondary rainbow is reversed from the order in a primary rainbow, as you can see by comparing parts *a* and *d* of Fig. 34-65.

Rainbows involving three or four reflections occur in the direction of the Sun and cannot be seen against the glare of sunshine in that part of the sky but have been photographed with special techniques.

Deviation and Dispersion through a Prism

In white light comprising of all colors, as wavelength increases, refractive index decreases. For a thin prism deviation is given by

$$\delta = (n-1)\,A.$$

From the analysis just presented, we can comment that the deviation decreases as we go from violet to red end of the spectrum. In other words, we can say that the deviation of violet is greater than the deviation of red color. This is why we say ultraviolet and infrared rays and not ultra-red and infra-violet. Let us now try to find difference in the deviation of these rays.

Angle of Dispersion

As shown in Fig. 34-66, we can say that the red ray and violet ray are separated by an angle. This angle is called angle of dispersion θ. It is the angle between emerging red ray and violet ray after a ray of white light is incident on the prism.

So, we can see from Fig. 34-66, the deviation or red and violet rays are different. This difference will cause dispersion.

$$\begin{aligned}\theta &= \delta_V - \delta_R \\ &= (n_V - 1)A - (n_R - 1)A \\ &= (n_V - n_R)A.\end{aligned}$$

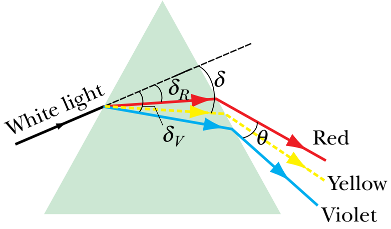


Figure 34-66 The deviation of violet color is greater than the deviation of red color. The deviation of yellow color is the average of red and violet because the refractive index of yellow is the average of red and violet. The prism is thin but the figure has been exaggerated for clarity.

For any transparent material, dispersive power ω is defined as

$$\omega = \left(\frac{n_V - n_R}{n_Y - 1}\right).$$

It is the property of a given material. By rationalizing it by A, we get

$$\omega = \frac{(n_V - n_R)A}{(n_Y - 1)A} = \frac{n_V - n_R}{n_Y - 1}$$

$$\omega = \frac{\theta}{\delta} \quad \text{(for a thin prism).}$$

Normally, ω is much less than 1.

SAMPLE PROBLEM 34.29

Deviation without dispersion and dispersion without deviation

What should be the condition on a combination of prism for the following?

(a) Angle of dispersion is zero but angle of deviation is not zero.

KEY IDEA

(1) Deviation without dispersion is not possible for a single prism but we can combine two or more prisms so that dispersion is zero. Such a combination is known as achromatic combination of prisms.

(2) Mathematically we want that

$$\delta_1 + \delta_2 \neq 0,$$

but in this case

$$\theta_1 + \theta_2 = 0.$$

Calculation: We have

$$(n_{V_1} - n_{R_1})A_1 + (n_{V_2} - n_{R_2})A_2 = 0.$$

Since $n_V - n_R$ is always greater than 0, this is possible only where A_1 and A_2 are of the opposite signs. That is,

$$A_2 = -\left(\frac{n_{V_1} - n_{R_1}}{n_{V_2} - n_{R_2}}\right)A_1.$$

Now,

$$\begin{aligned}\delta &= \delta_1 + \delta_2 \\ &= (n_{Y_1} - 1)A_1 + (n_{Y_2} - 1)A_2 \\ &= (n_{Y_1} - 1)A_1 - (n_{Y_2} - 1)\left(\frac{n_{V_1} - n_{R_1}}{n_{V_2} - n_{R_2}}\right)A_1 \\ &= (n_{Y_1} - 1)A_1\left[1 - \left(\frac{n_{V_1} - n_{R_1}}{n_{Y_1} - 1}\right)\left(\frac{(n_{Y_2} - 1)}{n_{V_2} - n_{R_2}}\right)\right] \\ &= \delta_1\left[1 - \frac{\omega_1}{\omega_2}\right].\end{aligned}$$

(b) Angle of deviation is zero but angle of dispersion is not zero.

KEY IDEA

(1) Dispersion without deviation is not possible for a single prism but we can combine two or more prisms so that deviation is zero. Such a combination is known as direct vision prism.

(2) Mathematically we want that

$$\delta_1 + \delta_2 = 0$$

but in this case

$$\theta_1 + \theta_2 \neq 0.$$

Calculation: We know that

$$\delta = \delta_1 + \delta_2 = 0,$$

where δ is average deviation for each prism, that is,

$$\begin{aligned}(n_{Y_1} - 1)A_1 + (n_{Y_2} - 1)A_2 &= 0 \\ A_2 &= -\frac{(n_{Y_1} - 1)A_1}{(n_{Y_2} - 1)}.\end{aligned}$$

This would mean that in this case also both angles of the prism are of opposite signs.

$$\theta = \theta_1 + \theta_2.$$

Now,

$$\begin{aligned}\theta_1 &= (n_{V_1} - n_{R_1})A_1 \\ \theta_2 &= (n_{V_2} - n_{R_2})A_2\end{aligned}$$

$$\theta = (n_{V_1} - n_{R_1})A_1 - (n_{V_2} - n_{R_2})\frac{(n_{Y_1} - 1)A_1}{(n_{Y_2} - 1)}.$$

Substituting the value of A_2, we get

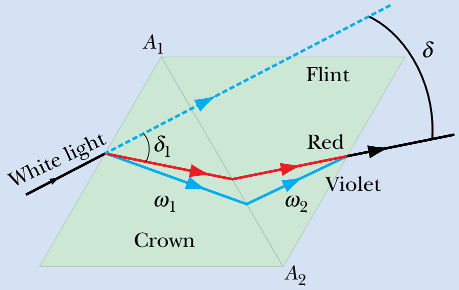
$$(n_{V_1} - n_{R_1})\left[1 - \left[\left(\frac{n_{V_2} - n_{R_2}}{n_{Y_1} - 1}\right)\left(\frac{n_{Y_2} - 1}{n_{V_1} - n_{R_2}}\right)\right]\right].$$

Finally, we have

$$\theta = \theta_1 + \theta_2 = \theta_1\left[1 - \frac{\omega_2}{\omega_1}\right].$$

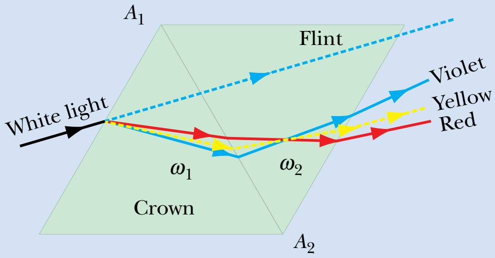
(Only yellow color ray will not deviate, all other colored rays will deviate.)

Learn: This kind of combination of prisms is possible only if the prisms are kept inverted with respect to each other as shown in the Fig. 34-67.



Deviation without dispersion

(*a*)



Dispersion without deviation

(*b*)

Figure 34-67 (*a*) Deviation without dispersion: When the crown and flint glass prism are combined as shown, the crown glass prism disperses white light into constituent colors but flint glass recombines them. The resultant beam is deviated from its path by an angle *d*. (*b*) Dispersion without deviation: When the crown and flint glass prism are combined as shown, the crown glass prism as well as flint glass prism disperse the light. The deviation produced by crown glass prism is positive but the deviation produced by the flint glass prism is negative. The net deviation (that of yellow color) is zero.

Chromatic Aberration in Lens

Just as in case of a prism, where deviation depends on the refractive index in the case of a lens the focal length depends on the refractive index of the lens. So, obviously the previous discussion implies that the focal length of the lens will depend on the color of light used. Considering the lens to be kept in air, we have

$$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Since $n_V > n_R, f_V < f_R$ for a convex lens.

As is clear from Fig. 34-68, the violet beam will focus closer to the lens and the red beam will focus farther away. This distance between the violet focus and the red focus is known as longitudinal chromatic aberration. To find the distance between the red focus and violet focus, we can differentiate this expression

$$\frac{d}{dn}\left[\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]$$

$$-\frac{1}{f^2}\times\frac{df}{dn}=1\times\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$-\frac{df}{f^2}=dn\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\left(\frac{dn}{f(n-1)}\right)$$

$$\frac{df}{f}=-\frac{dn}{n-1}.$$

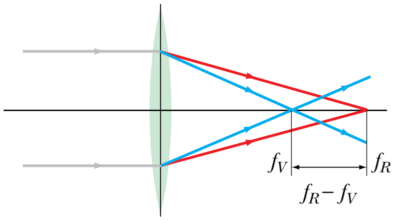


Figure 34-68 A parallel beam of white light represented by gray colored line is incident on a lens. The lens splits the incident ray into its constituent colors of which only red and violet are shown. The red rays will converge farther than the violet rays.

Now,

$$\frac{df}{f^2}=\frac{f_V-f_R}{f_{\text{av}}}=\frac{f_V-f_R}{f_Y}=\frac{n_V-n_R}{n_Y-1}=+\omega$$

$$f_R-f_V=-\omega f=\text{longitudinal chromatic aberration.}$$

So, for a single lens, there will be some longitudinal chromatic aberration. Lens are commonly used in so many daily application. Imagine taking a photograph with a defective lens. If a man is wearing a red T-shirt and blue jeans, at least one of them will be out of focus due to this aberration. This defect has to be removed. As you can easily guess, this is also done by combining two lens of different nature.

For a lens to be free of chromatic aberration, we should use two lenses in conjugation. For such a combination, we know that the equivalent focal length is given by:

$$\frac{1}{f}=\frac{1}{f_1}+\frac{1}{f_2}.$$

Differentiating this equation, we get

$$\frac{-df_{\text{eq}}}{f^2}=\frac{-df_1}{f_1^2}+\frac{-df_2}{f_2^2}.$$

For the aberration of the combination to be 0, df_{eq} should be 0. So,

$$\frac{df_1}{f_1^2}+\frac{df_2}{f_2^2}=0 \Rightarrow \frac{\omega_1}{f_1}+\frac{\omega_2}{f_2}=0.$$

Since ω_1 and ω_2 are always positive; at least one of f_1 and f_2 should be negative. Thus, the two lenses should be of opposite nature and should be kept close to each other.

REVIEW AND SUMMARY

Image Formation *Spherical refracting surfaces,* and *thin lenses* can form images of a source of light—the object—by redirecting rays emerging from the source. The image occurs where the redirected rays cross (forming a real image) or where backward extensions of those rays cross (forming a virtual image). If the rays are sufficiently close to the *principal axis* through the refracting surface, or thin lens, we have the following relations between the *object distance u* (which is positive) and the *image distance v* (which is positive for real images and negative for virtual images).

Refracting Surface

$$\frac{n_1}{u}+\frac{n_2}{v}=\frac{n_2-n_1}{R} \quad \text{(single surface)}, \qquad (34\text{-}11)$$

where n_1 is the index of refraction of the material where the object is located, n_2 is the index of refraction of the material on the other side of the refracting surface, and r is the radius of curvature of the surface. When the object faces a convex refracting surface, the radius r is positive. When it faces a concave surface, r is negative. Real images form on the side of a refracting surface that is opposite the object, and virtual images form on the same side as the object.

Thin Lens

$$\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \quad \text{(Lens maker's formula)}$$

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f} \quad \text{(Lens formula)}$$

where f is the lens's focal length, n is the index of refraction of the lens material, and r_1 and r_2 are the radii of curvature of the two sides of the lens, which are spherical surfaces. A convex lens surface that faces the object has a positive radius of curvature; a concave lens surface that faces the object has a negative radius of curvature. Real images form on the side of a lens that is opposite the object, and virtual images form on the same side as the object.

Lateral Magnification The lateral magnification m produced by a thin lens is

$$m=-\frac{v}{u}$$

The magnitude of lateral magnification m of an object is given by

$$|m|=\frac{h_I}{h_O}$$

where h_I and h_O are the heights (measured perpendicular to the principal axis) of the image and the object, respectively.

Lens Mirror Combination Such a situation is observed when the rear surface of the lens is silvered or a lens and a mirror are kept in close contact. Here the image of one system acts as an object for the next. The system behaves like a mirror whose focal length is given by

$$\frac{1}{f_{\text{eq}}}=-\frac{2}{f_L}+\frac{1}{f_M}. \qquad (34\text{-}22)$$

Deviation Produced by a Prism The net effect of the prism is to change the direction of ray or cause deviation of a ray from its path. Total deviation produced is given by

$$\delta=i+e-A \qquad (34\text{-}24)$$

where i is the angle of incidence, e is the angle of emergence and A is the angle of the prism.

For a thin prism, deviation is given by

$$\delta=(n-1)A$$

Chromatic Aberration in Lens In a lens, the distance between the violet focus and the red beam is known as longitudinal chromatic aberration and is given by

$$f_R-f_V=-\omega f$$

Chromatic aberration in a lens is overcome by using two lenses in conjugation such that

$$\frac{\omega_1}{f_1}+\frac{\omega_2}{f_2}=0.$$

PROBLEMS

1. A lens is made of glass having an index of refraction of 1.5. One side of the lens is flat, and the other is convex with a radius of curvature of 20 cm. (a) Find the focal length of the lens. (b) If an object is placed 85 cm in front of the lens, where is the image?

2. When the rectangular metal tank in Fig. 34-69 is filled to the top with an unknown liquid, observer O, with eyes level with the top of the tank, can just see corner E. A ray that refracts toward O at the top surface of the liquid is shown. If D = 85.0 cm and L = 85 cm, what is the index of refraction of the liquid?

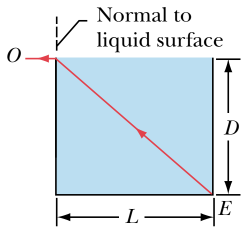


Figure 34-69 Problem 2.

3. Light in vacuum is incident on the surface of a glass slab. In the vacuum the beam makes an angle of 37° with the normal to the surface, while in the glass it makes an angle of 23.5° with the normal. What is the index of refraction of the glass?

4. A small piece of wood is floating in the water of depth 1 m. The refractive index of water is 4/3 and the sun rays are incident at an angle of 53° to the vertical. Find the location of the shadow of the wood piece on the bottom of the ocean.

5. A ray of light falls on a transparent sphere with center at C as shown in Fig. 34-70. The ray gets refracted from the sphere and emerges from the sphere parallel to the line AB. With the help of ray diagram, (a) find out the angle of refraction. (b) What is the refractive index of the sphere?

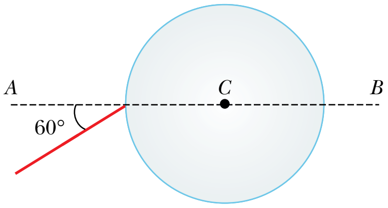


Figure 34-70 Problem 5.

6. Light rays travel from liquid (refractive index = 2) to air. (a) What is the maximum angle of deviation? (b) What is the angle of incidence corresponding to this maximum angle of deviation? (c) At what angle(s) of incidence can the angle of deviation be 30°?

7. A double-convex lens is to be made of glass with an index of refraction of 1.5. One surface is to have 1.5 times the radius of curvature of the other and the focal length is to be 60 mm. What is the (a) smaller and (b) larger radius?

8. A 20-mm-thick layer of water (n = 1.33) floats on a 40-mm-thick layer of carbon tetrachloride (n = 1.46) in a tank. A coin lies at the bottom of the tank. At what depth below the top water surface do you perceive the coin?

9. In Fig. 34-71, a fish watcher at point P watches a fish through a glass wall of a fish tank. The watcher is level with the fish; the index of refraction of the glass is 8/5, and that of the water is 4/3. The distances are $d_1 = 8.0$ cm, $d_2 = 3.0$ cm, and $d_3 = 6.8$ cm. To the fish, how far away does the watcher appear to be?

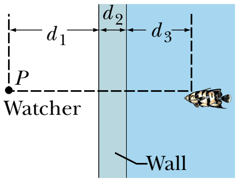


Figure 34-71 Problem 9.

10. A rectangular slab of refractive index m is placed over another slab of refractive index 3. Both slabs have identical dimensions. If a small coin is placed below the lower slab, for what value of m will the coin appear at the interface between the two slabs, when viewed from the top?

11. A concave mirror of radius of curvature 1 m, placed at the bottom of a water tank, forms an image of the Sun when it is directly overhead. Determine the distance of the image from the mirror for a depth of 80 cm of water in the tank. Refractive index of water = 4/3.

12. A slab of glass of surface index 1.5 and thickness of 3 cm is placed with the faces perpendicular to the principal axis of the concave mirror. If the radius of the curvature of the mirror is 10 cm, find where an object must be placed so that the image coincides with the object?

13. A goldfish in a spherical fish bowl of radius R is at the level of the center C of the bowl and at distance $R/2$ from the glass (Fig. 34-72). What magnification of the fish is produced by the water in the bowl for a viewer looking along a line that includes the fish and the center, with the fish on the near side of the center? The index of refraction of the water is 1.33. Neglect the glass wall of the bowl. Assume the viewer looks with one eye.

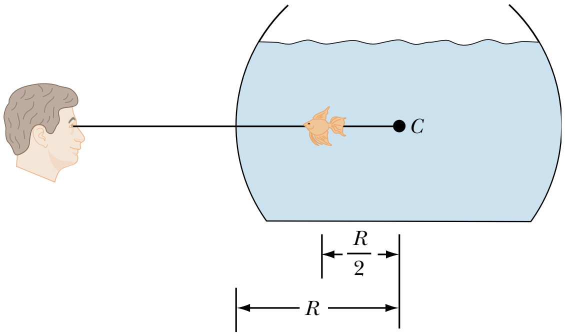


Figure 34-72 Problem 13.

14. A light ray, going through a prism with the angle of prism 60° is found to deviate by 30°. What limit on the refractive index can be put from these data?

15. In Fig. 34-73, a beam of parallel light rays from a laser is incident on a solid transparent sphere of index of refraction n. (a) If a point image is produced at the back of the sphere, what is the index of refraction of the sphere? Assume paraxial rays. (b) What index of refraction, if any, will produce a point image at the center of the sphere?

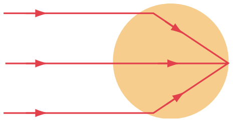


Figure 34-73 Problem 15.

16. A movie camera with a (single) lens of focal length 75 mm takes a picture of a person standing 27 m away. If the person is 180 cm tall, what is the height of the image on the film?

17. The index of refraction of benzene is 1.8. What is the critical angle for a light ray traveling in benzene toward a flat layer of air above the benzene?

18. A point source of light is 80.0 cm below the surface of a body of water. Find the diameter of the circle at the surface through which light emerges from the water.

19. Two spherical-shaped equal-sized glass fishbowls filled with water are standing on the table. In the middle of each fishbowl a small fish is swimming. What will be the distance of the image as seen by another? The refraction index of water is 4/3. Take radius = R and distance between the centers of fishbowls as $3R$.

20. When viewed normally through the flat surface the apparent thickness of a plano-convex lens appears to be 2 mm. If the radius of curvature of the spherical surface is 10 cm then find thickness of the lens when viewed from the curved surface. Refractive index of the glass is 1.5.

21. One end of a glass rod 8 cm in diameter has a hemispherical surface of 4 cm radius. Determine the position of the image of an object placed on the axis at following distance from the hemispherical end (a) at infinity, (b) 16 cm, (c) 4 cm, refractive index of the glass is 1.5.

22. There are two spherical surfaces of radii $R_1 = 30$ cm and $R_2 = 60$ cm. (a) In how many ways these surfaces may be arranged to get different lenses. (b) If all the lenses are made of glass ($\mu = 1.5$), find the focal length of each lens.

23. What area of the ground below can be photographed at one time by a camera of focal length 45 cm and plate size 3 cm × 3 cm kept in a satellite at height of 1500 km?

24. In the ray diagram of Fig. 34-74, where the angles are not drawn to scale, the ray is incident at the critical angle on the interface between materials 2 and 3. Angle $\phi = 60.0°$, and two of the indexes of refraction are $n_1 = 1.70$ and $n_2 = 1.60$. Find (a) index of refraction n_3 and (b) angle θ. (c) If θ is decreased, does light refract into material 3?

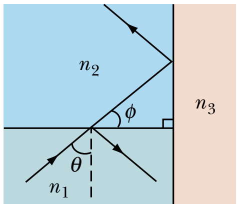


Figure 34-74 Problem 24.

25. Fig. 34-75, a ray of light is perpendicular to the face *ab* of a glass prism ($n = 1.50$). Find the largest value for the angle f so that the ray is totally reflected at face *ac* if the prism is immersed (a) in air and (b) in water.

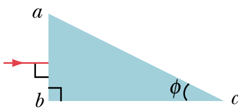


Figure 34-75 Problem 25.

26. Two watch glasses of radii of curvature 10 cm and 30 cm are cemented at the edges to form an air convex lens. (a) What is its focal length in air and water? (b) Is it convergent or divergent in water?

27. A glass sphere has radius $R = 5.0$ cm and index of refraction 1.6. A paperweight is constructed by slicing through the sphere along a plane that is 2.0 cm from the center of the sphere, leaving height $h = 3.0$ cm. The paperweight is placed on a table and viewed from directly above by an observer who is at distance $d = 8.0$ cm from the tabletop (Fig. 34-76). When viewed through the paperweight, how far away does the tabletop appear to be to the observer?

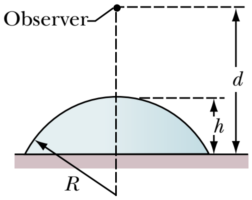


Figure 34-76 Problem 27.

28. The boundary surface of the media of refractive indices n_1 and n_2 is spherical with radius R (Fig. 34-77). Find the location of a point on axis in medium *I* where we can keep a point object such that after refraction, the rays become parallel to the axis. (use only paraxial rays).

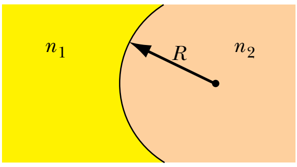


Figure 34-77 Problem 28.

29. Figure 34-78*a* shows the basic structure of a human eye. Light refracts into the eye through the cornea and is then further redirected by a lens whose shape (and thus ability to focus the light) is controlled by muscles. We can treat the cornea and eye lens as a single effective thin lens (Fig. 34-78*b*). A "normal" eye can focus parallel light rays from a distant object *O* to a point on the retina at the back of the eye, where processing of the visual information begins. As an object is brought close to the eye, however, the muscles must change the shape of the lens so that rays form an inverted real image on the retina (Fig. 34-78*c*).

(a) Suppose that for the parallel rays of Figs. 34-78*a* and *b*, the focal length f of the effective thin lens of the eye is 2.50 cm. For an object at distance $u = 60.0$ cm, what focal length f' of the effective lens is required for the object to be seen clearly?

(b) Must the eye muscles increase or decrease the radii of curvature of the eye lens to give focal length f'?

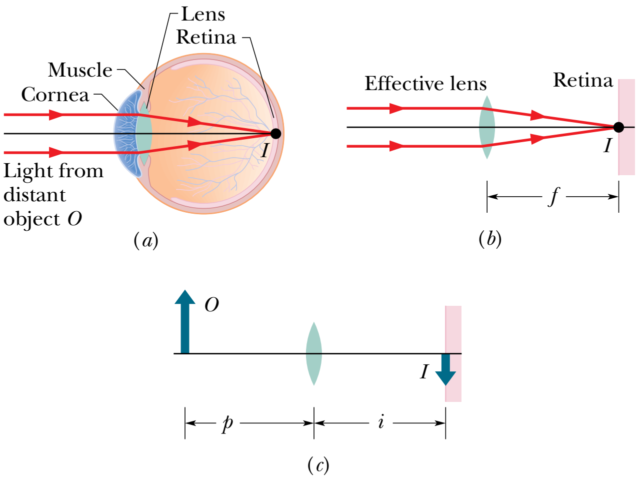


Figure 34-78 Problem 29.

30. An illuminated slide is held 68 cm from a screen. How far from the slide must a lens of focal length 11 cm be placed (between the slide and the screen) to form an image of the slide's picture on the screen?

31. A beam of light converges to a point on a screen. If a lens is placed in the path of the beam at 10 cm away from the screen, the screen has to be moved through 8 cm to obtain a sharp image. (a) Find the focal length of the lens and (b) determine whether it is concave or convex.

32. A convex lens focuses an object 40 cm from it on a screen placed 10 cm away from it. A glass plate ($\mu = 1.5$) and of thickness 3 cm is inserted between the lens and the screen. Where should the object be placed so that its image is again focused on the screen?

33. A plano-convex lens of focal length 40 cm and refractive index = 1.5 is silvered at plane surface. (a) What will be its focal length in air? (b) What will be the equivalent mirror behave like? (c) What will be its focal length when it is silvered at curved surface instead of being silvered at the plane surface?

34. Two identical thin watch glasses are joined together with air inside as shown in Fig. 34-79. The surface *B* is silvered.

A sharp image of object placed on optic axis at 20 cm from the lens is formed at distance 10 cm from lens. Find the position of the sharp image when the space between the glasses is filled with water ($\mu_w = 4/3$).

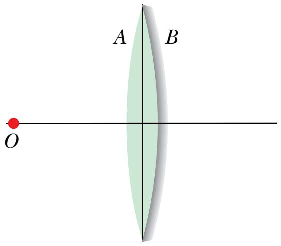


Figure 34-79 Problem 34.

35. A fish hides under a floating plank that is circular in shape with radius R (Fig. 34-80). If the index of refraction of water is $\sqrt{2}$. Find the volume of water in which it can swim without being detected from above.

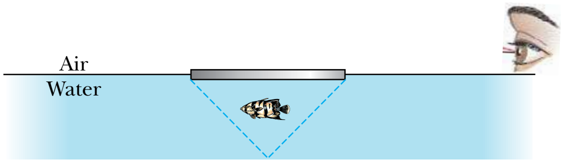


Figure 34-80 Problem 35.

36. A beam of light diverges from P on the axis of a convex lens and the after passing through the lens is reflected from the surface of a convex mirror. The reflected beam is brought to a focus by the lens at P itself. (a) Find the focal length of the lens. Given that the distance of the lens and the mirror is 10 cm. The distance of P from the mirror is 30 cm and the focal length of the mirror is 10 cm.

37. Find the nature and focal length of a lens which must be placed in contact with a concave lens of focal length 25 cm in order that the lens combination may produce a real image of an object placed at infinity.

38. A thin equiconvex lens of refractive index 3/2 and radius of curvature 50 cm is placed on a reflecting convex surface of radius of curvature 100 cm. A point object is placed on the principle axis of the system such that its final image coincides with itself. Now few drops of a transparent liquid are placed between the mirror and lens such that final image of the object is at infinity. Find (a) the refractive index of the liquid used and (b) the position of the object.

39. Figure 34-81 shows a *beam expander* made with two coaxial converging lenses of focal lengths f_1 and f_2 and separation $d = f_1 + f_2$. The device can expand a laser beam while keeping the light rays in the beam parallel to the central axis through the lenses. Suppose a uniform laser beam of width $W_i = 2.5$ mm and intensity $I_i = 9.0$ kW/m^2 enters a beam expander for which $f_1 = 12.5$ cm and $f_2 = 30.0$ cm. What are (a) W_f and (b) I_f of the beam leaving the expander? (c) What value of d is needed for the beam expander if lens 1 is replaced with a diverging lens of focal length $f_1 = -26.0$ cm?

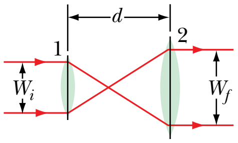


Figure 34-81 Problem 39.

40. In Fig. 34-82, an object is placed in front of a converging lens at a distance equal to twice the focal length f_1 of the lens. On the other side of the lens is a concave mirror of focal length f_2 separated from the lens by a distance $2(f_1 + f_2)$. Light from the object passes rightward through the lens, reflects from the mirror, passes leftward through the lens, and forms a final image of the object. What are (a) the distance between the lens and that final image and (b) the overall lateral magnification m of the object? Is the image (c) real or virtual (if it is virtual, it requires someone looking through the lens toward the mirror), (d) to the left or right of the lens, and (e) inverted or noninverted relative to the object?

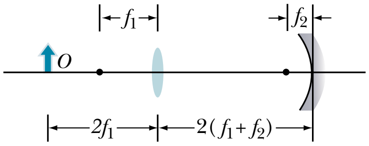


Figure 34-82 Problem 40.

41. The cross section of a glass prism $\mu = 2$ has the form of an isosceles right-angled triangle. It is submerged in a liquid of refractive index 1.2. A ray is incident on to one of the equal faces perpendicularly. What is the angle between the direction of incident ray and the ray that emerges from the prism?

42. A prism having refractive index $\sqrt{2}$ and refracting angle 30° has one of the refracting surfaces polished. A beam of light incident on the other refracting surface is seen to trace its path. What is the angle of incidence?

43. A quartz glass prism of refracting angle 4° is to be combined with a crown glass prism to create a direct-vision prism combination. (a) Find the refracting angle of the crown prism. (b) Find the angular width of the spectrum found by the combination. (For crown glass $\mu_R = 1.52$, $\mu_v = 1.56$; and for quartz glass $\mu_R = 1.70$, $\mu_v = 1.74$.)

44. Focal lengths of two lenses are 10 cm and −20 cm and dispersive powers of their materials are 0.03 and w. To form achromatic combination from these, (a) what is the value of ω? (b) What is the resulting focal length of the combination?

45. A small fish, 0.4 m below the surface of a lake, is viewed through a simple converging lens of focal length 3 m. The lens is kept at 0.2 m above the water surface such that the fish lies on the optical axis of the lens. Find the distance of the image of the fish from the lens as seen by the observer. The refractive index of water is 4/3.

46. The dispersive powers of crown and flint glasses are 0.03 and 0.05, respectively. The refractive index for yellow light for these glasses are 1.518 and 1.62, respectively. If we want to make an achromatic combination of these prisms that produces a mean deviation of 1° what are the angles of the two prisms needed?

47. White light falls on one face of crown glass prism of refracting angle 50°. The angle of incidence is 30°. Find the (a) mean deviation produced by the prism and (b) the angle of dispersion ($\mu_v = 1.538, \mu_R = 1.52$).

48. A flint glass prism and a crown glass prism are to be combined in such a way that the deviation of the mean ray is zero. The refractive index of the flint glass for the mean ray is 1.62 and for the crown glass it is 1.518. If the angle of the flint glass prism is 6°, what would be refracting angle of the crown prism.

49. A fish is rising up vertically inside a pond with velocity of 4 cm/s, and notices a bird, which is diving vertically downward and its velocity appears to be 16 cm/s (to the fish). What is the real velocity of the diving bird, if refractive index of water is 4/3?

50. A ray of light is incident at angle i on a surface of a prism of small angle A and emerges normally from the opposite surface. If the refractive index of the material of the prism is m, what is the approximate value of the angle of incidence?

51. A prism of refractive index $\sqrt{2}$ has refracting angle = 60°. What should be the angle of incidence so that a light ray suffers minimum deviation?

52. Find the (a) angle of minimum and (b) the maximum deviation for an equilateral prism having refractive index $n = \sqrt{3}$.

53. Suppose the prism of Fig. 34-83 has apex angle $\phi = 60.0°$ and index of refraction $n = 1.60$. (a) What is the smallest angle of incidence θ for which a ray can enter the left face of the prism and exit the right face? (b) What angle of incidence θ is required for the ray to exit the prism with an identical angle θ for its refraction, as it does in Fig. 34-83.

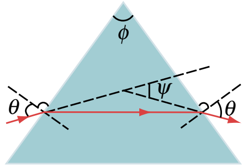


Figure 34-83 Problem 53.

54. In Fig. 34-84, a light ray enters a glass slab at point A at incident angle $\theta_1 = 45.0°$ and then undergoes total internal reflection at point B. What minimum value for the index of refraction of the glass can be inferred from this information?

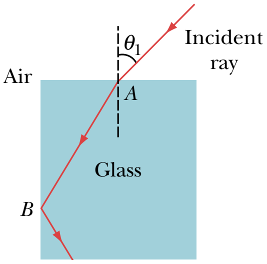


Figure 34-84 Problem 54.

55. An object is placed at a certain distance from a screen. A convex lens of focal length 20 cm is placed between the screen and the object. A real image is formed on the screen for two positions of the lens, which differ by a distance of 10 cm. What is the distance of the object from the screen?

56. A convex lens A of focal length 20 cm and a concave lens B of focal length 5 cm are kept along the same axis with a distance d between them. If a parallel beam of light falling on A leaves B as a parallel beam then what will be the distance d?

57. Two thin symmetrical lenses of different nature and of different material have equal radii of curvature $R = 15$ cm. The lenses are put close together and immersed in water ($\mu_w = 4/3$). The focal length of the system in water is 30 cm. What is the difference between refractive indices of the two lenses?

58. An object is placed 1 cm above the principal axis of a convex lens of focal length of 40 cm. Object distance is 60 cm. If the object now starts moving perpendicularly away from optical axis with a speed of 10 cm/s, then what is the speed of the image?

PRACTICE QUESTIONS

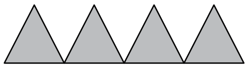
Single Correct Choice Type

1. A concave mirror has a local length of 2 m in vacuum. What would be its focal length when used in a medium of refractive index 2.759?

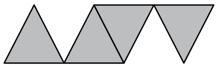
(a) 2 m
(b) 2 × 2.759 m
(c) 2/2.759 m
(d) Data inadequate

2. Four similar prisms of same material having same angle of prism are arranged. Which among the four of the arrangements shown in the following figures gives no net angular deviation?

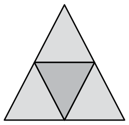
(a)



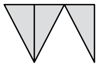
(b)



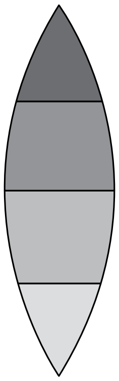
(c)



(d)

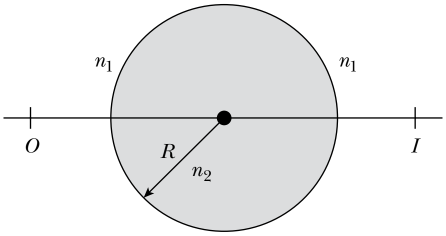


3. A layered lens is made of materials indicated by shades in the following figure. The number of images formed is



(a) 1
(b) 2
(c) 3
(d) 4

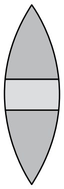
4. Where would an object be placed in a medium of refractive index n_1 so that its real image is formed at equidistant from the sphere of radius R and refractive index n_2 which is also placed in the medium of refractive index n_1 as shown in the below figure?



(a) $\left(\dfrac{n_2 - n_1}{n_2 + n_1}\right)R$ **(b)** $\left(\dfrac{n_2}{n_2 - n_1}\right)R$

(c) $\left(\dfrac{n_1}{n_2 - n_1}\right)R$ **(d)** None of these

5. How will the image formed by a convex lens be affected, if the central portion of the lens is wrapped in black paper, as shown in the below figure?



(a) No image will be formed
(b) Full image will be formed but it is less bright
(c) Full image will be formed but without the central portion
(d) Two images will be formed, one due to each exposed half

6. In a calm swimming pool, a person is viewing outside objects by keeping an eye at a depth h inside water. If the critical angle of water is θ_c then the value of the diameter of the circle of view for outside objects will be

(a) $2h \sin\theta_c$ **(b)** $2h \cos\theta_c$
(c) $2h \tan\theta_c$ **(d)** $2h \cot\theta_c$

7. If f_V and f_R are the focal lengths of a convex lens for violet and red light, respectively, and f'_V and f'_R are the focal lengths of a concave lens for violet and red light, respectively, we must have

(a) $f_V = f_R$ and $f'_V < f'_R$ **(b)** $f_V > f_R$ and $f'_V > f'_R$
(c) $f_V < f_R$ and $f'_V < f'_R$ **(d)** $f_V < f_R$ and $f'_V > f'_R$

8. The angle of prism is 60° and its refractive index is 1.5. There will be no emergent light if the angle of incidence on the first face is

(a) equal to 30° **(b)** less than 27°
(c) more than 30° **(d)** equal to 60°

9. A thin prism of glass is placed in air and water successively. If ${}^a n_g = 3/2$ and ${}^a n_g = 4/3$, then the ratio of the deviation produced by the prism for a small angle of incidence when placed in air and water is

(a) 9:8 **(b)** 4:3
(c) 3:4 **(d)** 4:1

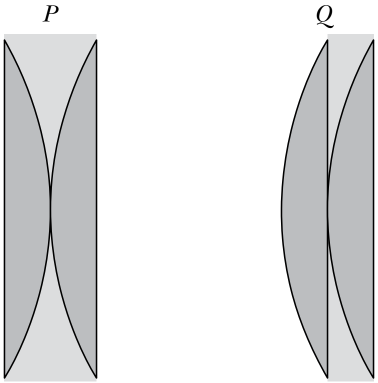
10. The dispersion of a medium for wavelength λ is D. Then the dispersion for the wavelength 2λ will be

(a) $(D/8)$ **(b)** $(D/4)$
(c) $(D/2)$ **(d)** D

11. A concave lens of focal length f produces an image $(1/x)$ of the size of the object, the distance of the object from the lens is

(a) $(x - 1)f$ **(b)** $(x + 1)f$
(c) $\{(x - 1)/x\}f$ **(d)** $\{(x + 1)/x\}f$

12. Two convex lenses of powers 4 D and 6 D are separated by a distance of 1/6 m as shown in the below figure. The power of the optical system so formed is



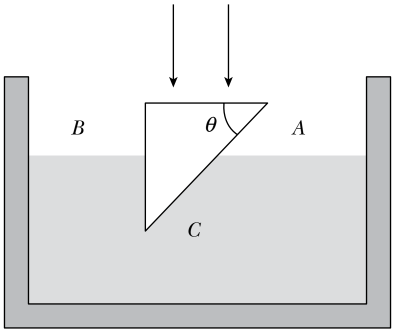
(a) –6 D **(b)** +6 D
(c) 10 D **(d)** 2 D

13. A lens is placed between the source of light and a wall. It forms images of area A_1 and A_2 on the wall for its two displaced positions. The area of source of light is

(a) $\dfrac{A_1 + A_2}{2}$ **(b)** $\sqrt{A_1 A_2}$

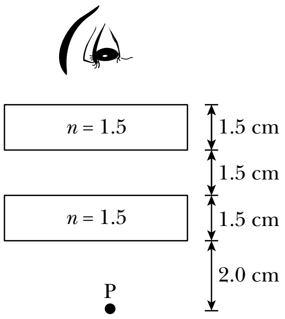
(c) $\left[\dfrac{1}{A_1} + \dfrac{1}{A_2}\right]$ **(d)** $\left(\dfrac{\sqrt{A_1} + \sqrt{A_2}}{2}\right)^2$

14. A glass prism of refractive index 1.5 is immersed in water of refractive index 4/3 (see the following figure). A light ray incident normally on face AB is totally reflected at face AC if



(a) $\sin\theta > 8/9$ **(b)** $\sin\theta < 2/3$
(c) $\sin\theta = \sqrt{3}/2$ **(d)** $2/3 < \sin\theta \le 8/2$

15. The image of point P when viewed from top of the slabs shown in the following figure will be



(a) 2.0 cm above P **(b)** 1.5 cm above P
(c) 2.0 cm below P **(d)** 1 cm above P

16. A thin biconvex lens of focal length $f = 25$ cm forms a real image of a real object on a screen. The object is separated by 5 cm from the lens. The screen is drawn closer to the lens by a distance 18 cm. Through what distance (approximately) must the object be shifted so that its image is again formed on the screen?

(a) 0.5 mm toward left **(b)** 0.25 mm toward left
(c) 2.5 mm toward left **(d)** 1.4 mm toward left

17. A laser beam and a sound wave from directional sources both enter a liquid at an angle of 60° from the horizontal surface of the liquid. The speed of sound in the liquid is 5 times than that of the speed of the sound in air. For light, the refractive index of the liquid is 1.8. What happens?

(a) Both the light and the sound refract at an angle of 74° from the horizontal
(b) The light refracts to an angle of 26° from the horizontal and the sound refracts to an angle of 74° from the horizontal.
(c) The light refracts to an angle of 26° from the horizontal and the sound reflection completely off the surface.
(d) The light refracts to an angle of 74° from the horizontal and the sound refracts to an angle of 26° from the horizontal.

18. A tiny air bubble in a glass slab ($n = 1.5$) appears from one side to be 6 cm from the glass surface and from other side, 4 cm. The thickness of the glass slab is

(a) 10 cm **(b)** 6.67 cm
(c) 15 cm **(d)** None of these

19. A bird is flying 3 m above the surface of water. To a fish underwater, the height of the bird from the water surface appears to be

(a) 2.25 m **(b)** 3.33 m
(c) 4 m **(d)** 4/3 m

20. In the previous question, if the bird is diving vertically down with speed = 6 m/s, his apparent speed as seen by a stationary fish underwater is

(a) 8 m/s **(b)** 6 m/s
(c) 12 m/s **(d)** 4 m/s

21. A bucket of total height 60 cm is half filled with a liquid of index 1.5 and half with another liquid of index 2. The apparent depth of the bucket for an observer directly above the bucket

(a) 45 cm
(b) 30 cm
(c) 35 cm
(d) 45 cm

22. A flat glass slab of thickness 6 cm and index 1.5 is placed in front of a plane mirror. An observer is standing behind the glass slab and looking at the mirror. The actual distance of the observer from the mirror is 50 cm. The distance of his image from himself, as seen by the observer is

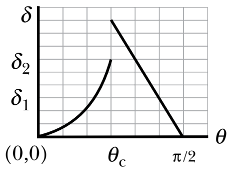
(a) 94 cm
(b) 96 cm
(c) 98 cm
(d) 100 cm

23. Critical angle of glass is θ_1 and that of water is θ_2. The critical angle for water and glass surface would be ($n_g = 3/2, n_w = 4/3$)

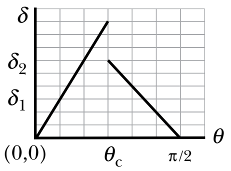
(a) Less than θ_2
(b) Between θ_1 and θ_2
(c) Greater than θ_2
(d) Greater than θ_1

24. A ray of light travels from a medium of refractive index n into air. If the angle of incidence at the plane surface of separation is θ. and the corresponding angle of deviation is δ, the variation δ with θ is shown correctly in the figure

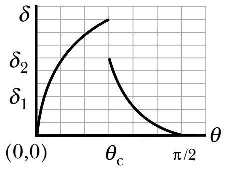
(a)



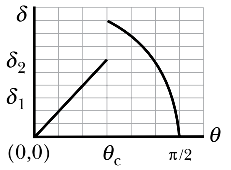
(b)



(c)



(d)



25. A real object is kept in air and refraction occurs at the convex boundary of a spherical glass surface. For the image to be real, the real object distance ($n_g = 3/2$)

(a) Should be greater than three times the radius of curvature of the refracting surface
(b) Should be lesser than two times the radius of curvature of the refracting surface
(c) Should be greater than the radius of curvature of the refracting surface
(d) Is independent of the radius of curvature of the refracting surface

26. Half the surface of a transparent sphere of refractive index 2 is silvered. A narrow, parallel beam of light is incident on the unsilvered surface, symmetrically with respect to the silvered part. The light finally emerging from the sphere will be a

(a) Parallel beam
(b) Converging beam
(c) Slightly divergent beam
(d) Widely divergent beam

27. A solid, transparent sphere has a small, opaque dot at its center. When observed from outside, the apparent position of the dot will be
(a) Closer to the eye than its actual position
(b) The same as its actual position
(c) Farther away from the eye than its actual position
(d) Independent of the refractive index of the sphere

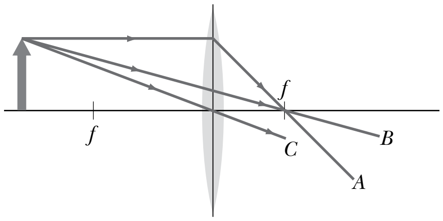
28. An object was placed 25 cm in front of a converging lens with a focal length of 20 cm. A concave mirror with a focal length of 15 cm was placed 120 cm behind the lens. Which of these describes the final image?
(a) Real, erect **(b)** Virtual, erect
(c) Real, inverted **(d)** Virtual, inverted

29. A lens behaves as a converging lens in air and as a diverging lens in water. If refractive index of water is 1.33, the refractive index of the material of the lens will be
(a) Equal to unity
(b) Equal to 1.33
(c) Between unity and 1.33
(d) Greater than 1.33

30. A convergent lens of focal length 20 cm and made of a material with refractive index 1.1 is immersed in water. The lens will behave as a
(a) Converging lens of focal length 20 cm
(b) Converging lens of focal length less than 20 cm
(c) Converging lens of focal length more than 20 cm
(d) Divergent lens

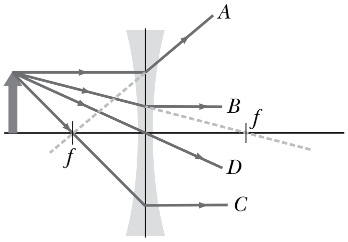
31. A parallel beam of light is incident on a converging lens parallel to its principal axis. As one moves away from the lens on the other side on its principal axis, the intensity of light
(a) Remains constant
(b) Continuously increases
(c) Continuously decreases
(d) First increases then decreases

32. Which one of the rays of the converging lens shown in the following figure is not correct?



(a) *A* **(b)** *B*
(c) *C* **(d)** None of these

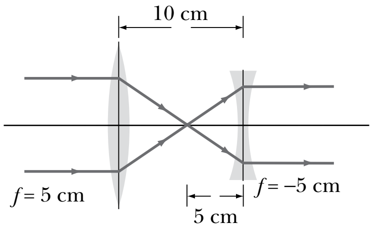
33. Which of the rays in the following figure for the diverging lens shown is not correct?



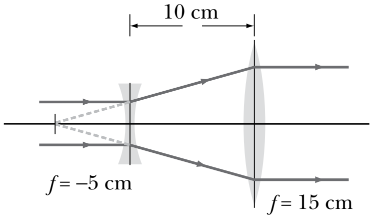
(a) *A* **(b)** *B*
(c) *C* **(d)** *D*

34. Which one of these ray diagrams is most accurate?

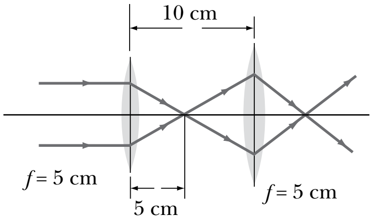
(a)



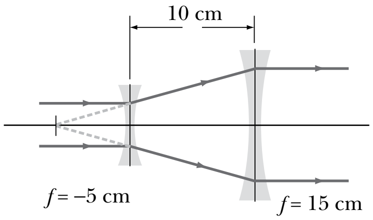
(b)



(c)



(d)



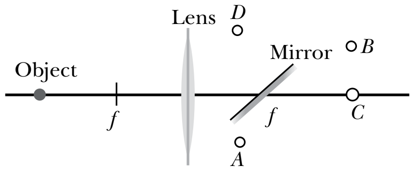
35. A point source of light is placed at a distance of 2*f* from a converging lens of focal length *f*. The intensity on the other side of the lens is maximum at a distance
(a) *f* **(b)** Between *f* and 2*f*
(c) 2*f* **(d)** More than 2*f*

36. A converging lens can from a virtual image if the object is placed
(a) Between the lens and its focus
(b) At the focus of the lens
(c) Between *f* and 2*f*
(d) At infinity

37. A concave mirror and a convex lens are of the same focal length in air. When they are immersed in water
(a) The concave mirror will have its focal length increased
(b) The convex lens will have its focal length increased
(c) They will have equal focal lengths, different from those in air
(d) They will have equal focal lengths, same as those in the air

38. As shown in the following figure, an object is placed on the left side of a lens with its focal points labeled as *F*.

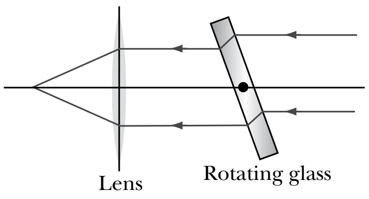
A flat mirror is placed at the image focus, with its reflecting surface facing the lens. The image of the object is at



(a) Point *A* **(b)** Point *B*
(c) Point *C* **(d)** Point *D*

39. A section of a spherical shell of outer radius R_0 and inner radius R_1 is cut, and used as a lens. The light first strikes what was originally the outer surface of the glass shell. Which of the following statements about the resulting lens is true?
(a) The lens is converging
(b) The lens is diverging
(c) The lens has zero focal length
(d) The lens has infinite focal length

40. A slab of high quality flat glass, with parallel faces, is placed in the path of a parallel light beam before it is focused to a spot by a lens. The glass is rotated slightly back and forth from the vertical orientation, about an axis coming out of the page, as shown in the following figure. According to ray optics the effect on the focused spot is



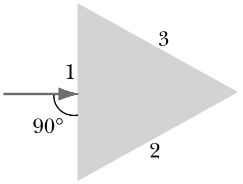
(a) There is no effect on the spot
(b) The spot moves toward then away from the lens
(c) The spot moves up and down parallel to the lens
(d) The spot blurs out of focus

41. A biconcave glass lens having refractive index 1.5 has both surfaces of same radius of curvature *R*. On immersion in a medium of refractive index 1.75, it will behave as a
(a) Convergent lens of focal length 3.5 *R*
(b) Convergent lens of focal length 3.0 *R*
(c) Divergent lens of focal length 3.5 *R*
(d) Divergent lens of focal length 3.0 *R*

42. When an object is at a distance u_1 and u_2 from the optical center of a lens, a real and virtual image are formed, respectively, with the same magnification. The focal length of lens is
(a) $u_1 + u_2$ **(b)** $(u_1 + u_2)/2$
(c) $u_1 + u_2/2$ **(d)** $\sqrt{u_1 \cdot u_2}$

43. A concavo-convex glass (index = 1.5) lens has radii of curvatures 60 cm and 40 cm, respectively. Its convex surface is silvered, and it is placed on a horizontal table with concave surface up. The concave surface is then filled with a liquid of index 2.0. The combination behaves like
(a) Concave mirror **(b)** Convex mirror
(c) Flat mirror **(d)** Convex lens

44. A beam of light enters side 1 of a triangular piece of plastic (a prism) of index of refraction 1.5 as shown in the following figure. The three sides of the prism are of equal length. Which of the following statements is correct?

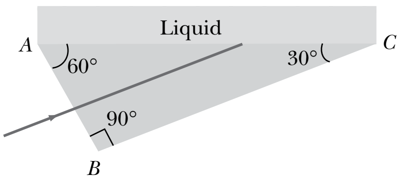


(a) The beam will emerge from side 1
(b) The beam will emerge from side 2
(c) The beam will emerge from side 3
(d) The beam will not emerge

45. Angle of minimum deviation is equal to the angle of prism *A* of an equilateral glass prism. The angle of incidence at which minimum deviation will be obtained is
(a) 60° **(b)** 45°
(c) 30° **(d)** $\sin^{-1}(2/3)$

46. A ray of light passes through an equilateral prism such that the angle of incidence is equal of emergence and later is equal to 3/4th the angle of prism. The angle of deviation is
(a) 45° **(b)** 20°
(c) 39° **(d)** 30°

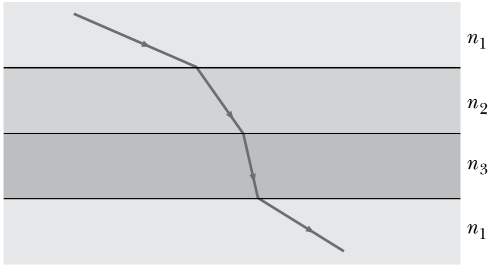
47. Light is incident normally on face *AB* of a prism as shown in the following figure. A liquid of refractive index *n* is placed on face *AC* of the prism. The prism is made of glass of refractive index 3/2. The limits of *n* for which total internal reflection takes place on face *AC* is



(a) $n > \frac{3\sqrt{3}}{2}$ **(b)** $n > \sqrt{3}$
(c) $n < \frac{3\sqrt{3}}{4}$ **(d)** $n < \frac{3\sqrt{3}}{8}$

48. The critical angle for glass to air refraction is least for which color?
(a) Orange **(b)** Blue
(c) Violet **(d)** Red

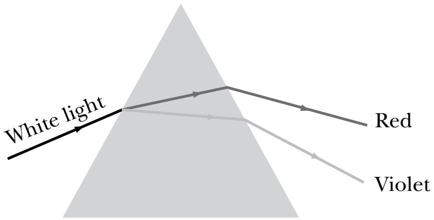
49. A ray of red light passes through a series of media with different refractive indices, n_1, n_2, and n_3. It then enters a medium with an index of refraction n_1 as shown in the following figure. If the incident ray of light in the figure is blue (instead of red), it will emerge from the interface between n_3 and n_1



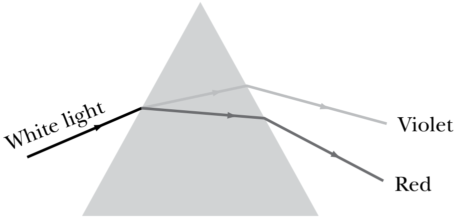
(a) To the right of where the red ray emerged
(b) At the same point as where the red ray emerged
(c) To the left of where the red ray emerged
(d) Any of the above is possible

50. A prism is made of glass which has a higher index of refraction for violet light than for red light. Which diagram best indicates the paths of red and violet light rays through the prism?

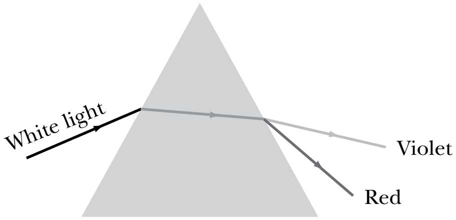
(a)



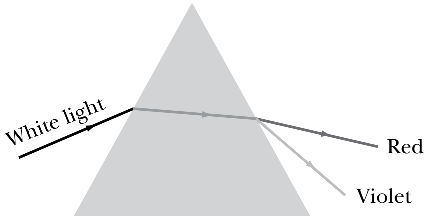
(b)



(c)



(d)



51. The focal lengths of a convex lens for blue and red colors of light are f_B and f_R, respectively, and those of a concave lens are f'_B and f'_R then

(a) $f_B > f_R$ and $f'_B < f'_R$
(b) $f_B < f_R$ and $f'_B > f'_R$
(c) $f_B > f_R$ and $f'_B > f'_R$
(d) $f_B < f_R$ and $f'_B < f'_R$

52. A secondary rainbow is formed when light rays coming from the Sun undergo the following through spherical water droplets

(a) A refraction, internal reflection, and then refraction
(b) Two refractions only
(c) A refraction, internal reflection, again internal reflection, and then refraction
(d) A refraction, internal reflection, and again internal reflection

53. The dispersive powers of two lenses are 0.01 and 0.02. If focal length of first lens is +10 cm, then what should be the focal length of the second lens, so that they form an achromatic combination?

(a) Diverging lens having focal length 20 cm
(b) Converging lens having focal length 20 cm
(c) Diverging lens having focal length 10 cm
(d) Converging lens having focal length 10 cm

54. Focal lengths of two lenses are f and f' and dispersive powers of their materials are ω and ω'. To form achromatic combination from these, which relation is correct?

(a) $\omega = \omega_0, \omega' = 2\omega_0, f' = 2f$
(b) $\omega = \omega_0, \omega' = 2\omega_0, f' = f/2$
(c) $\omega = \omega_0, \omega' = 2\omega_0, f' = -2f$
(d) $\omega = \omega_0, \omega' = 2\omega_0, f' = f/2$

55. The table lists the index of refraction for various substances at 20 °C for light with a wavelength of 589 nm in a vacuum. Through which substance will light with a vacuum wavelength of 589 nm travel with the greatest speed?

Substance	***n***
Fused quartz	1.458
Ethyl alcohol	1.362
Crown glass	1.520
Carbon tetrachloride	1.461
Crystalline quartz	1.544

(a) Fused quartz
(b) Ethyl alcohol
(c) Crown glass
(d) Carbon tetrachloride

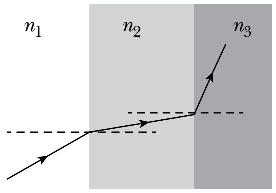
56. What is the frequency of light that has a wavelength in water of 6.00×10^2 nm if the refractive index for this light is 1.33?

(a) 3.76×10^{14} Hz
(b) 6.65×10^{14} Hz
(c) 5.00×10^{14} Hz
(d) 7.25×10^{14} Hz

57. Blue light with a wavelength of 425 nm passes from a vacuum into a glass lens; and the index of refraction is found to be 1.65. The glass lens is replaced with a plastic lens. The index of refraction for the plastic lens is 1.54. In which one of the two lenses does the light have the greatest speed and what is that speed?

(a) glass, 2.28×10^8 m/s
(b) plastic, 2.13×10^8 m/s
(c) glass, 1.82×10^8 m/s
(d) plastic, 1.95×10^8 m/s

58. The figure shows the path of a portion of a ray of light as it passes through three different materials. What can be concluded concerning the refractive indices of these three materials?



(a) $n_1 < n_2 < n_3$ **(b)** $n_2 < n_1 < n_3$
(c) $n_1 > n_2 > n_3$ **(d)** $n_3 < n_1 < n_2$

59. A child is looking at a reflection of the Sun in a pool of water. When she puts on a pair of Polaroid sunglasses with a vertical transmission axis, she can no longer see the reflection. At what angle is she looking at the pool of water?
(a) 45.0° **(b)** 53.1°
(c) 48.8° **(d)** 61.6°

60. A ray of light originating in oil (n = 1.21) is incident at the Brewster angle upon a flat surface of a quartz crystal (n = 1.458). Determine the angle of incidence for this ray.
(a) 0.82° **(b)** 40°
(c) 1.2° **(d)** 50°

61. A converging lens with a focal length of 12 cm produces a 3 cm high virtual image of a 1 cm high object. Which entry in the table below is correct?

	Image distance	**Location of image**
(a)	8 cm	Same side of lens as object
(b)	8 cm	Opposite side of lens from object
(c)	12 cm	Opposite side of lens from object
(d)	24 cm	Same side of lens as object

62. A camera with a focal length of 0.0500 m (a 50 mm lens) is focused for an object at infinity. To focus the camera on a subject which is 4.00 m away, how should the lens be moved?
(a) 1.0 cm closer to the film
(b) 0.06 cm farther from the film
(c) 0.06 cm closer to the film
(d) 4.94 cm farther from the film

More than One Correct Choice Type

63. A virtual image larger than the object can be produced by
(a) Convex mirror **(b)** Concave mirror
(c) Diverging lens **(d)** Converging lens

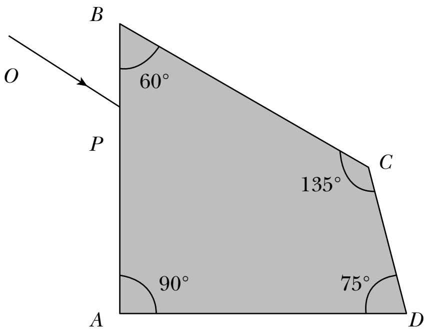
64. If a convergent beam of light passes through a diverging lens, the result
(a) May be a convergent beam
(b) May be a divergent beam
(c) May be a parallel beam
(d) Must be a parallel beam

65. Which of the following form virtual and erect image for all positions of a self-luminous subject
(a) Convex lens **(b)** Concave lens
(c) Convex mirror **(d)** Concave mirror

66. When the object is moved slightly closer to a converging lens, the image may
(a) Increase in size and move closer to the lens
(b) Increase in size and move farther away from the lens
(c) Decrease in size and move closer to the lens
(d) Decrease in size and move farther away from the lens

67. A convex lens forms an image of an object on a screen. The height of the image is 9 cm. The lens is now displaced until an image is again obtained on the screen. The height of this image is 4 cm. The distance between the object and the screen is 90 cm
(a) The distance between the two positions of the lens is 30 cm
(b) The distance of the object from the lens in its first position is 36 cm
(c) The height of the object is 6 cm
(d) The focal length of the lens is 21.6 cm

68. A ray OP of monochromatic light is incident on the face AB of prism $ABCD$ near vertex B at an incident angle of 60° (below figure). If the refractive index of the material of the prism is $\sqrt{3}$, which of the following is (are) correct?



(a) The ray gets totally internally reflected at face CD
(b) The ray comes out through face AD
(c) The angle between the incident ray and the emergent ray is 90°
(d) The angle between the incident ray and the emergent ray is 120°

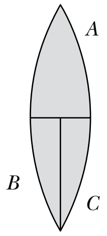
69. A ray of light traveling in a transparent medium falls on a surface separating the medium from air, at an angle of incidence of 45°. The ray undergoes total internal reflection. If n is the refractive index of the medium with respect to air, select the possible values of n from the following.
(a) 1.3 **(b)** 1.4
(c) 1.5 **(d)** 1.6

70. A solid, transparent sphere has a small, opaque dot at its center. When observed from outside, the apparent position of the dot will be
(a) Closer to the eye than its actual position
(b) Farther away from the eye than its actual position
(c) The same as its actual position
(d) Independent of the refractive index of the sphere

71. A thin concavo-convex lens has two surfaces of radii of curvature R and $2R$. The material of the lens has a refractive index n. When kept in air, the focal length of the lens

(a) Will depend on the direction from which light is incident on it
(b) Will be the same, irrespective of the direction from which light is incident on it
(c) Will be equal to $R/n - 1$
(d) Will be equal to $2R/n - 1$

72. A thin, symmetric double convex lens of power P is cut into three parts A, B, and C as shown in the below figure. The power of



(a) A is P **(b)** A is $2P$
(c) B is $P/2$ **(d)** B is $P/4$

73. If a convergent beam of light passes through a diverging lens, the result
(a) May be a convergent beam
(b) May be a divergent beam
(c) May be a parallel beam
(d) Must be a parallel beam

74. A diverging lens of focal length f_1 is placed in front of and coaxially with a concave mirror of focal length f_2. Their separation is d. A parallel beam of light incident on the lens returns as a parallel beam from the arrangement.
(a) The beam diameters of the incident and reflected beams must be the same
(b) $d = 2|f_2| - |f_1|$
(c) $d = |f_2| - |f_1|$
(d) If the entire arrangement is immersed in water, the conditions will remain unaltered

75. A converging lens of focal length f_1 is placed in front of and coaxially with a convex mirror of focal length f_2. Their separation is d. A parallel beam of light incident on the lens returns as a parallel beam from the arrangement.
(a) The beam diameters of the incident and reflected beams must be the same.
(b) $d = f_1 - 2|f_2|$
(c) $d = f_1 - |f_2|$
(d) If the entire arrangement is immersed in water, the conditions will remain unaltered.

76. A ray of white light passes through a rectangular glass slab, entering and emerging at parallel faces. The angle of incidence, measured from the normal to the glass surface, is large. Then
(a) White light will emerge from the slab
(b) The light emerging from the slab will have a number of parallel, colored rays
(c) The emergent rays will not form a spectrum on a screen
(d) Colors will be seen if the emergent rays enter the eye directly

77. A light of wavelength 6000 Å in air enters a medium of refractive index 1.5. Inside the medium, its frequency is v and its wavelength is λ.
(a) $v = 5 \times 10^{14}$ Hz **(b)** $v = 7.5 \times 10^{14}$ Hz
(c) $\lambda = 4000$ Å **(d)** A = 9000 Å

78. When lights of different colors move through water, they must have different
(a) Wavelengths **(b)** Frequencies
(c) Velocities **(d)** Amplitudes

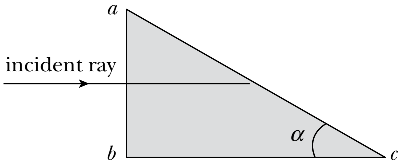
Linked Comprehension

Paragraph for Questions 79 and 80: Light in air is incident on a plastic plate at the Brewster angle. The angle of refraction is 35.0°.

79. Determine the Brewster angle.
(a) 35.0° **(b)** 46.5°
(c) 55.0° **(d)** 43.5°

80. What is the index of refraction of the plastic plate?
(a) 1.58 **(b)** 1.36
(c) 1.43 **(d)** 1.61

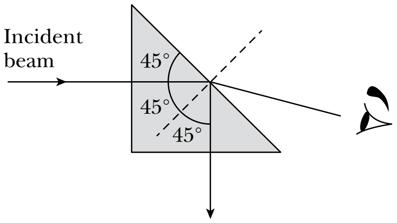
Paragraph for Questions 81 and 82: A ray of light is normally incident on face ab of a plastic prism with an index of refraction $n = 1.20$ as shown.



81. Determine the largest value of the angle α so that the ray is totally reflected at the face ac if the prism is immersed in air.
(a) 28° **(b)** 45°
(c) 34° **(d)** 56°

82. Determine the largest value of the angle α so that the ray is totally reflected at the face ac if the prism is immersed in a liquid with refractive index 1.12.
(a) 21° **(b)** 69°
(c) 34° **(d)** 78°

Paragraph for Questions 83–86: A beam of light that consists of a *mixture* of **red**, **green** and **violet** light strikes a prism (surrounded by air) as shown. Indices of refraction for this prism for the various colors are indicated in the table. An observer is located to the right of the prism as shown.



Color	n
Red	1.43
Green	1.40
Violet	1.37

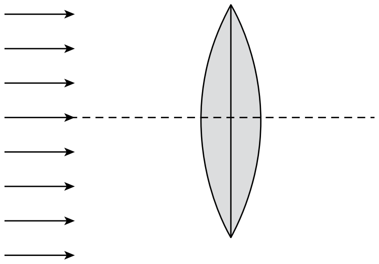
83. Which color(s) could, in principle, be seen by the observer?
(a) Only red light
(b) Only green and violet light
(c) Only green light
(d) Only green and red light

84. Which physical phenomenon is illustrated by the fact that the observer will not see all three colors of light?
(a) Doppler effect **(b)** Diffraction
(c) Dispersion **(d)** Total internal reflection

85. Which physical phenomenon is illustrated by the fact that the prism has different refractive indices for different colors?
(a) Doppler effect **(b)** Diffraction
(c) Dispersion **(d)** Total internal reflection

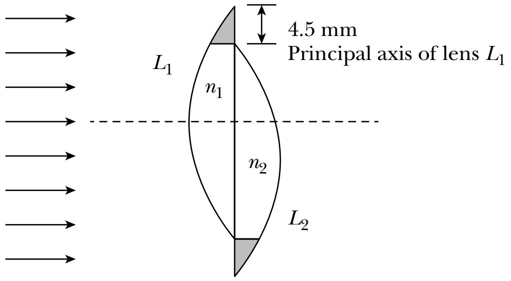
86. Which physical phenomenon is illustrated by the fact that the emerging rays are spread into the component colors of the beam?
(a) Doppler effect
(b) Diffraction
(c) Dispersion
(d) Total internal reflection

Paragraph for Questions 87 and 88: Two identical plano-convex lenses L_1 ($n_1 = 1.4$) and L_2 ($n_1 = 1.5$) of radii of curvature $R = 20$ cm are placed as shown in the following figure.



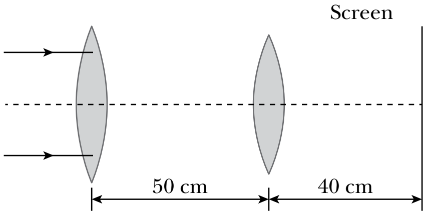
87. Find the position of the image of the parallel beam of light relative to the common principal axis.
(a) 100/7 cm **(b)** 200/9 cm
(c) 31.2 cm **(d)** 21.8 cm

88. Now the second lens is shifted vertically downward by a small distance 4.5 mm and the extended parts of L_1 and L_2 are blackened as shown in the following figure. Find the new position of the image of the parallel beam.



(a) (200/9) cm behind the lens 2.5 mm below the principal axis of L_1.
(b) (100/9) cm in front of the lens 2 mm below the principal axis of L_1.
(c) (200/9) cm in front of the lens 2.5 mm below the principal axis of L_1.
(d) (100/9) cm behind the lens 2 mm below the principal axis of L_1.

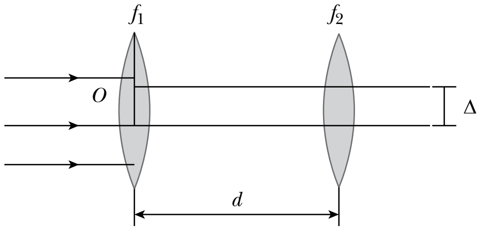
Paragraph for Questions 89 and 90: A parallel beam of light falls successively on a thin convex lens of focal length 40 cm and then on a thin convex lens of focal length 10 cm as shown in the following figure. In another case the second lens is an equi-concave lens of focal length 10 cm and made of material of refractive index 1.5. In both the cases, the second lens has an aperture equal to 1 cm.



89. Compare the area illuminated by the beam of light on the screen, which passes through the second lens in the two cases. The ratio (A_2/A_1) will be
(a) 72 **(b)** 81
(c) 56 **(d)** 29

90. Now a liquid of refractive index n is filled to the right of the second lens in case B such that the area illuminated in both the cases is the same. Determine the refractive index of the liquid.
(a) 1 **(b)** 2.5
(c) 3 **(d)** 0.72

Paragraph for Questions 91 and 92: Two thin convex lenses of focal length f_1 and f_2 are separated by a horizontal distance d ($d < f_1$ and $d < f_2$) and their centers are displaced by a vertical separation as shown in the below figure. Taking the origin of coordinates O at the center of first lens.



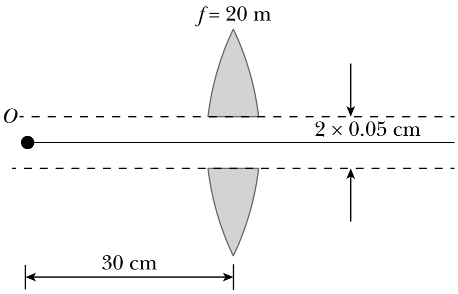
91. Find the y-coordinate of the focal point of this lens system for a parallel beam of rays coming from left?
(a) $\dfrac{(f_1+d)\Delta}{(f_1+f_2-d)}$ **(b)** $\dfrac{(f_1+d)}{(f_1+f_2-d)}$
(c) $\dfrac{2(f_1-d)\Delta}{(f_1+f_2+d)}$ **(d)** $\dfrac{(f_1-d)\Delta}{(f_1+f_2-d)}$

92. Similarly, find the x-coordinate in the same problem from the focal point of this lens system for a parallel beam of rays coming from left?

(a) $\dfrac{d(f_1-d)+f_1f_2}{(f_1+f_2-d)}$ **(b)** $\dfrac{f_1f_2}{(f_1+f_2-d)}$

(c) $\dfrac{d(f_1-d)}{(f_1+f_2-d)}$ **(d)** $\dfrac{2d(f_1+d)-f_1f_2}{(f_1+f_2-d)}$

Paragraph for Questions 93–95: A point object O is placed at a distance of 0.3 m from a convex lens (focal length 0.2 m) cut into two halves each of which is displaced by 0.0005 m as shown in the below figure.

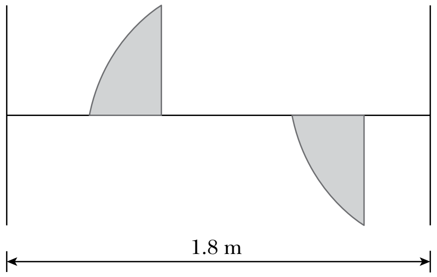


93. What will be the location of the image?
- **(a)** 30 cm, right of lens
- **(b)** 60 cm, right of lens
- **(c)** 70 cm, left of lens
- **(d)** 40 cm, left of lens

94. If this arrangement will generate more than one image then what will be the total number of images?
- **(a)** 2 **(b)** 4
- **(c)** 6 **(d)** 5

95. Find the spacing between the images so formed is
- **(a)** 0.1 cm **(b)** 0.5 cm
- **(c)** 0.3 cm **(d)** 1 cm

Paragraph for Questions 96 and 97: A thin plano-convex lens of focal length f is split into two halves. One of the halves is shifted along the optical axis. The separation between the object and image planes is 1.8 m (below figure). The magnification of the image formed by one of the half lenses is 2.

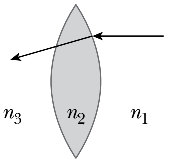
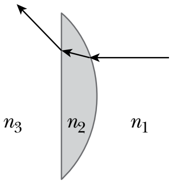
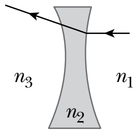
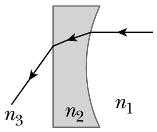
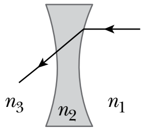


96. Find the focal length of the lens used.
- **(a)** 0.4 m **(b)** 0.6 m
- **(c)** 1 m **(d)** 2 m

97. Find the separation between the two halves of the thin plano-convex lens.
- **(a)** 0.4 m **(b)** 0.6 m
- **(c)** 0.2 m **(d)** 0.8 m

Matrix-Match

98. Two transparent media of refractive indices n_1 and n_3 have a solid lens shaped transparent material of refractive index n_2 between them as shown in the figures shown in Column II. A ray traversing these media is also shown in the figures shown in Column I. Different relationships between n_1, n_2, and $n_3$are given. Match them to the ray diagram shown in Column II.

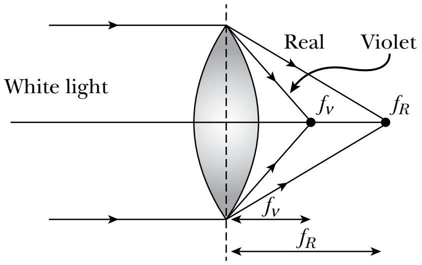
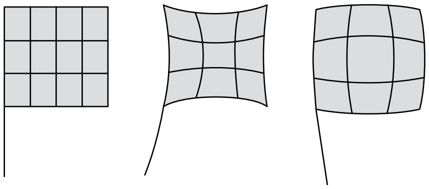
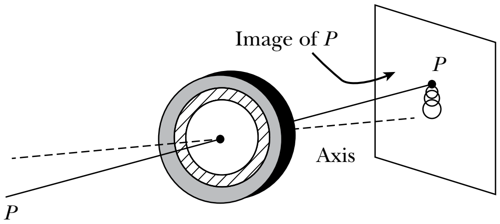
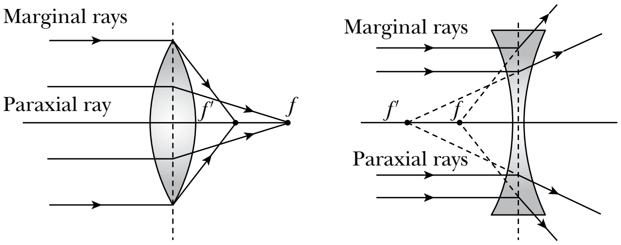
Column I	Column II
(a) $n_1 < n_2$	**(p)** 
(b) $n_1 > n_2$	**(q)** 
(c) $n_2 = n_3$	**(r)** 
(d) $n_2 > n_3$	**(s)** 
	(t) 

99.

Column I	Column II
(a) Light entering per unit area of telescope	**(p)** Radius of aperture of objective lens
(b) Length of telescope	**(q)** Dispersion of lenses
(c) Magnification	**(r)** Focal length of objective lens (f_o) and eye lens (f_e)
(d) Sharpness of image	**(s)** Spherical aberration

Directions for Questions 100–102: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

100. In the given table, Column I describe the defect in lens, Column II shows how we resolve this defect from lens and Column III shows some figures about defects and there resolving techniques.

Column I	Column II	Column III
(I) For a point object placed off the axis, the image is spread both along and perpendicular to the principal axis. The best image is obtained not on a plane but on a curved surface	**(i)** Reduced by using proper stops placed at proper locations along the axis	**(J)**  Real Violet White light f_V f_R f_V f_R
(II) Inability of a lens to form the point image of a point object on the axis	**(ii)** To remove this defect, we use two or more lenses in contact in place of single lens	**(K)** 
(III) Image of a white object is colored and blurred	**(iii)** Reduced by properly designing radii of curvature of the lens surfaces	**(L)** 
(IV) When the point object is placed away from the principle axis and the image is received on a screen perpendicular to the axis, the shape of the image is like a comet	**(iv)** To reduce this defect we use a stop before and in front of the lens	**(M)**  Marginal rays Paraxial ray f' f Marginal rays f' f Paraxial rays

(1) What are the characteristics of chromatic aberration?

(a) (III) (ii) (J) **(b)** (IV) (ii) (L)
(c) (II) (i) (K) **(d)** (II) (ii) (J)

(2) What are the distinct features of comatic aberration?

(a) (I) (ii) (J) **(b)** (III) (iv) (L)
(c) (IV) (iii) (L) **(d)** (I) (i) (M)

(3) What are the characteristics of spherical aberration?

(a) (II) (ii) (J) **(b)** (II) (iv) (M)
(c) (IV) (iii) (L) **(d)** (II) (iii) (M)

101. In the given table, Column I shows the refractive index of different materials, Column II shows the velocity of light in different materials and Column III gives names of different materials.

Column I	Column II	Column III
(I) Refractive index of a material is 1.33	**(i)** Velocity of light in space is 3×10^8 m/s	**(J)** Air
(II) Refractive index of a material is 2.5	**(ii)** Velocity of light in vacuum is 3×10^8 m/s	**(K)** Glass
(III) Refractive index of a material is 1.75	**(iii)** Velocity of light in material with refractive index 1 is 3×10^8 m/s	**(L)** Water
(IV) Refractive index of a material is 1.5	**(iv)** Velocity of light in air is 3×10^8 m/s	**(M)** Diamond

(1) Determine the material with a velocity of 2.25×10^8 m/s?

(a) (II) (iv) (M) **(b)** (IV) (ii) (L)
(c) (II) (i) (K) **(b)** (I) (i) (L)

(2) Determine the material with a velocity of 2×10^8 m/s?

(a) (I) (ii) (J) **(b)** (IV) (iv) (K)
(c) (II) (iii) (L) **(d)** (I) (i) (M)

(3) Determine the material with a velocity of 1.2×10^8 m/s?

(a) (II) (ii) (J) **(b)** (II) (iii) (M)
(c) (IV) (i) (L) **(d)** (III) (iii) (M)

102. Snell's law gives the relation between angle of incidence, angle of refraction and refractive index. In the given table, Column I shows the different angles of incidence, Column II shows the different angles of refraction and Column III shows the two different mediums through which refraction takes place.

Column I	Column II	Column III
(I) Angle of incidence of 30°	**(i)** Angle of refraction 48°	**(J)** Vacuum–block of quartz
(II) Angle of incidence of 37°	**(ii)** Angle of refraction of 20°	**(K)** Vacuum–a piece of crown glass
(III) Angle of incidence of 45°	**(iii)** Angle of refraction 27.7°	**(L)** Water into polyethylene
(IV) Angle of incidence of 57.5°	**(iv)** Angle of refraction 26.3	**(M)** Vacuum–ethanol

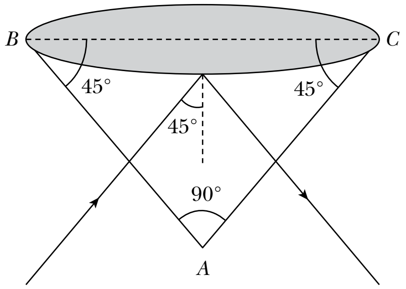
(1) What are the conditions for index of refraction (n_2) of 1.46?
- **(a)** (I) (iii) (M)
- **(b)** (III) (ii) (K)
- **(c)** (I) (ii) (J)
- **(d)** (II) (iii) (J)

(2) What are the conditions for index of refraction (n_3) of 1.5?
- **(a)** (I) (ii) (M)
- **(b)** (IV) (i) (L)
- **(c)** (II) (ii) (K)
- **(d)** (II) (iv) (M)

(3) What are the conditions for index of refraction (n_4) of 1.36?
- **(a)** (III) (i) (L)
- **(b)** (IV) (i) (J)
- **(c)** (III) (iv) (J)
- **(d)** (II) (iv) (M)

Integer Type

103. As shown in the below figure, a drop of liquid is spread over the hypotenuse of a right angled isosceles prism and a ray of light is incident normally on face AB of the prism. If refractive index of liquid is $\sqrt{2}$ and if internal reflection to occur, what is the refractive index of material of prism?



104. Let us consider simple microscope of a concave lens of power –10 D and a convex lens of power +30 D in contact. If the image is formed at infinity, what is the magnifying power of the microscope? Consider distance of the distinct vision as 25 cm.

105. Radius of curvature of two surfaces of a glass convex lens is 20 cm each. If refractive index of glass is 3/2, what is the power of lens in D?

106. A thin glass prism of angle 6° of refractive index 1.5 is combined with another glass prism of refractive index 1.6 to produce dispersion without deviation. What is the angle of the second prism?

107. In a beaker filled with a liquid of refractive index 5/3, a point source of light is placed 2 m below the $\sqrt{3}$ surface of water. To cut off all light coming out of water from the source, what should be the minimum diameter of a disc that should be placed over the source on the surface of water?

ANSWER KEY

Checkpoints

1. (*a*) **2.** (a) 11*h*/6 from fish itself; (b) 17 *h*/6

3. Straw is not a ray! Note that the straw has a lateral shift as well as a bend. Actually, the rays from the straw reach our eyes after refraction from the water and the apparent depth of different parts of the straw is different from its actual value. So, the straw appears to be bent.

4. $d_a = \dfrac{d\cos^3\theta}{n\cos^3\phi}$ **5.** (a) *e*; (b) virtual, same **7.** Virtual, same as object, diverging

Problems

1. (a) + 40 cm; (b) 76 cm **2.** $\sqrt{2}$ **3.** 1.56

4. 0.75 m away from the point vertically below the wood piece **5.** 30°; (b) $\sqrt{3}$

6. (a) 90°; (b) 45°; (c)42.4° and 75° **7.** (a) 4.5 cm; (b) 9 cm **8.** 42 mm **9.** 20 cm; 10

10. 1.5 **11.** 22.5 cm below water **12.** 11 cm away from mirror **13.** 1.14

14. μ lies between $2/\sqrt{3}$ and $\sqrt{2}$
15. (a) $n = 2$; (b) not possible
16. 0.5 cm
17. 34°
18. 182 cm
19. $9R$ from itself
20. 2.22 mm
21. (a) 12 cm; (b) 24 cm; (c) 12 cm
22. (a) four ways; (b) 40 cm, 120 cm, –40 cm; –120 cm
23. 100 × 100 km
24. (a) 60°; (b) 28.1°; (c) no
25. (a) 48.9°; (b)29.0°
26. (a) infinite, 30 cm; (b) divergent lens
27. 75/31 cm
28. $\dfrac{n_1 R}{n_2 - n_1}$
29. (a) 2.40 cm; (b) decrease
30. 54 cm or 14 cm
31. (a) $f = 22.5$ cm; (b) concave lens
32. 72 cm away from the lens
33. (a) 20 cm; (b) concave mirror; (c) 20/3 cm
34. –5 cm
35. $\pi R^3/3$
36. 12 cm and 20/3 cm
37. Convex lens of focal length less than 25 cm
38. (a) $\mu = 7/6$; 100 cm from the lens
39. (a) 6.0 mm; (b) 1.6 kW/m^2; (c) 4.0 cm
40. (a) $2f_1$; (b) –1.0; (c) real; (d) it is at a distance i_1' to the left of the lens; (e) inverted
41. 90°
42. 45°
43. (a) $\alpha = 3°$; (b) $\theta = 0.04°$
44. (a) $\omega = 0.06$; (b) $f = 20$ cm (convex lens)
45. 0.6 m
46. 4.8° and 2.4°
47. (a) 31°45′; (b)1°18′
48. 7.2°
49. 9 cm/s
50. μA
51. 45°
52. (a) minimum deviation = 60°; (b)maximum deviation = 76.5°
53. (a) 35.6°(b) 53.1°
54. 1.22
55. 0.9 m
56. 15 cm
57. 1/3
58. 20 cm/s

Practice Questions

Single Correct Choice Type

1. (a)
2. (b)
3. (d)
4. (c)
5. (b)
6. (c)
7. (d)
8. (b)
9. (d)
10. (b)
11. (a)
12. (b)
13. (b)
14. (a)
15. (d)
16. (a)
17. (a)
18. (b)
19. (c)
20. (a)
21. (c)
22. (b)
23. (c)
24. (c)
25. (b)
26. (a)
27. (b)
28. (c)
29. (c)
30. (d)
31. (d)
32. (b)
33. (c)
34. (c)
35. (c)
36. (a)
37. (b)
38. (d)
39. (b)
40. (a)
41. (a)
42. (b)
43. (a)
44. (c)
45. (a)
46. (d)
47. (c)
48. (c)
49. (c)
50. (a)
51. (b)
52. (c)
53. (a)
54. (d)
55. (b)
56. (a)
57. (d)
58. (d)
59. (b)
60. (d)
61. (d)
62. (b)

More than One Correct Choice Type

63. (b), (d)
64. (a), (b), (c)
65. (b), (c)
66. (b), (d)
67. (b), (c), (d)
68. (a), (b), (d)
69. (c), (d)
70. (c), (d)
71. (b), (d)
72. (a), (c)
73. (a), (b), (c)
74. (a), (b)
75. (a), (b)
76. (a), (b), (c)
77. (a), (c)
78. (a), (c)

Linked Comprehension

79. (c)
80. (c)
81. (c)
82. (a)
83. (b)
84. (d)
85. (c)
86. (c)
87. (b)
88. (a)
89. (a)
90. (c)
91. (d)
92. (a)
93. (b)
94. (a)
95. (c)
96. (a)
97. (b)

Matrix-Match

98. (a) → (q), (r); (b) → (q), (s), (t); (c) → (q), (r), (t); (d) → (q), (s)

99. (a) → (p); (b) → (r); (c) → (p), (r); (d) → (p), (q), (s)

100. (1) → (a); (2) → (c); (3) → (b)

101. (1) → (d); (2) → (b); (3) → (b)

102. (1) → (c); (2) → (b); (3) → (d)

Integer Type

103. 2

104. 5

105. 5

106. 5

107. 3

CHAPTER

35

Interference and Diffraction

Contents

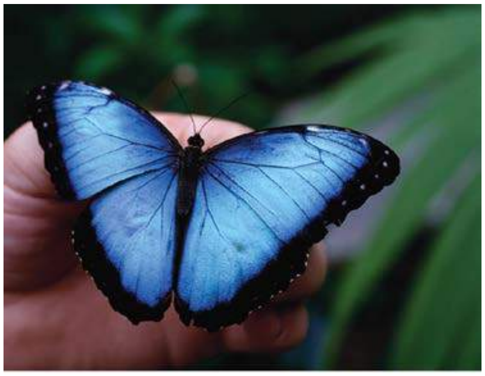
35.1 What is Physics?
35.2 Light as a Wave
35.3 Diffraction
35.4 Young's Interference Experiment
35.5 Coherence
35.6 Intensity in Double-Slit Interference
35.7 Modifications of Young's Double-Slit Experiment
35.8 Interference from Thin Films
35.9 Diffraction and the Wave Theory of Light
35.10 Diffraction by a Single Slit: Locating the Minima
35.11 Intensity in Single-Slit Diffraction, Quantitatively
35.12 Diffraction by a Circular Aperture
35.13 Diffraction by a Double Slit

35.1 | WHAT IS PHYSICS?

One of the major goals of physics is to understand the nature of light. This goal has been difficult to achieve (and has not yet fully been achieved) because light is complicated. However, this complication means that light offers many opportunities for applications, and some of the richest opportunities involve the interference of light waves—**optical interference**.

Nature has long used optical interference for coloring. For example, the wings of a *Morpho* butterfly are a dull, uninspiring brown, as can be seen on the bottom wing surface, but the brown is hidden on the top surface by an arresting blue due to the interference of light reflecting from that surface (Fig. 35-1). Moreover, the top surface is color-shifting; if you change your perspective or if the wing moves, the tint of the color changes. Similar color shifting is used in the inks on many currencies to thwart counterfeiters, whose copy machines can duplicate color from only one perspective and therefore cannot duplicate any shift in color caused by a change in perspective.

To understand the basic physics of optical interference, we must largely abandon the simplicity of geometrical optics (in which we describe light as rays) and return to the wave nature of light.



Philippe Colombi/PhotoDisc/Getty Images, Inc.

Figure 35-1 The blue of the top surface of a *Morpho* butterfly wing is due to optical interference and shifts in color as your viewing perspective changes.

35.2 | LIGHT AS A WAVE

Key Concepts

- The three-dimensional transmission of waves, including light, may often be predicted by Huygens' principle, which states that all points on a wavefront serve as point sources of spherical secondary wavelets. After a time t, the new position of the wavefront will be that of a surface tangent to these secondary wavelets.
- The law of refraction can be derived from Huygens' principle by assuming that the index of refraction of any medium is $n = c/v$, in which v is the speed of light in the medium and c is the speed of light in vacuum.
- The wavelength λ_n of light in a medium depends on the index of refraction n of the medium:

$$\lambda_n = \frac{\lambda}{n},$$

in which λ is the wavelength in vacuum.

- Because of this dependency, the phase difference between two waves can change if they pass through different materials with different indexes of refraction.

The first convincing wave theory for light was in 1678 by Dutch physicist Christian Huygens. Mathematically simpler than the electromagnetic theory of Maxwell, it nicely explained reflection and refraction in terms of waves and gave physical meaning to the index of refraction.

Huygens' wave theory is based on a geometrical construction that allows us to tell where a given wavefront will be at any time in the future if we know its present position. **Huygens' principle** is

> All points on a wavefront serve as point sources of spherical secondary wavelets. After a time t, the new position of the wavefront will be that of a surface tangent to these secondary wavelets.

Here is a simple example. At the left in Fig. 35-2, the present location of a wavefront of a plane wave traveling to the right in vacuum is represented by plane ab, perpendicular to the page. Where will the wavefront be at time Δt later? We let several points on plane ab (the dots) serve as sources of spherical secondary wavelets that are emitted at $t = 0$. At time Δt, the radius of all these spherical wavelets will have grown to $c\,\Delta t$, where c is the speed of light in vacuum. We draw plane de tangent to these wavelets at time Δt. This plane represents the wavefront of the plane wave at time Δt; it is parallel to plane ab and a perpendicular distance $c\,\Delta t$ from it.

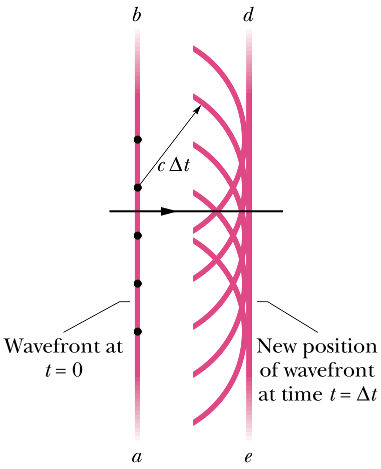


Figure 35-2 The propagation of a plane wave in vacuum, as portrayed by Huygens' principle.

Laws of Reflection

We now use Huygen's principle to prove the laws of reflection.

Let us consider a plane wave AB incident at an angle i on a plane mirror as shown in Fig. 35-3. We consider the reflection of the plane wave and try to obtain the shape of the reflected wavefront. Let AB be the position of the wavefront at $t = 0$. If the mirror was not present, then at a late time τ the position of the wavefront would have been CB', where $BB' = PP' = AC = v\tau$ and v is the speed of propagation of the wave. In order to determine the shape of the reflected wavefront at the instant $t = \tau$, we consider an arbitrary point P on the wavefront AB and let τ_1 be the time taken by a disturbance to reach the point P_1 from P. According to Huygen's principle, the point P_1 will act as a source of secondary spherical wavefronts. From point P_1, we draw a sphere of radius $v(\tau - \tau_1)$. We draw a tangent plane on this sphere from the point B'. Since $BB_1 = PP_1 = v\tau_1$, the distance B_1B'

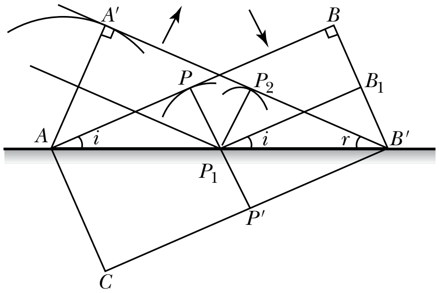


Figure 35-3 Reflection of a plane wavefront AB incident on a plane mirror. $A'B'$ is the reflected wavefront; i and r correspond to angles of incidence and reflection, respectively.

will be equal to $P_1P_2 = [v(\tau - \tau_1]$. If we consider triangles P_2P_1B' and B_1P_1B', then the side P_1B' is common to both and since $P_1P_2 = B'B_1$ and also since both the triangles are right-angled triangles, we have $\angle P_2B'P_1 = \angle PAP_1$. The former is the angle of reflection and the latter is the angle of incidence. Thus, we have the law of reflection, when a plane wavefront gets reflected from a plane surface, the angle of reflection is equal to the angle of incidence and the reflected wave is a plane wave.

Reflection of Light from a Point Source near a Mirror

As a second example, let us consider spherical waves (emanating from a point source P) incident on a plane mirror MM', as shown in Fig. 35-4. We know that a point source produces a point image behind the mirror. Let us try to see this by Huygen's principle. Let ABC denote the shape of the wavefront at time $t = 0$. In the absence of the mirror, the shape of the wavefront at a later time t would have been $A_1B_1C_1$ where $AA_1 = BB_1 = CC_1 = v\tau$, Q being an arbitrary point on the wavefront. If the time taken for the disturbance to traverse the distance QQ' be τ_1 then, in order to determine the shape of the reflected wavefront, we draw a sphere of radius $v(\tau - \tau_1)$ whose center is at the point Q'. In a similar manner we can draw the secondary wavelets emanating from other points on the mirror and, in particular, from the point B we have to draw a sphere of radius $v\tau$. The shape of the reflected wavefront is obtained by drawing a common tangent plane to all these spheres, which is shown as $A_1B_1'C_1$ in the figure. It can immediately be seen that $A_1B_1'C_1$ will have an exactly similar shape as $A_1B_1C_1$ except that $A_1B_1'C_1$ will have its center of curvature at the point P' where $PB = BP'$. Thus, the reflected waves will appear to emanate from the point P' which will be the virtual image of the point P.

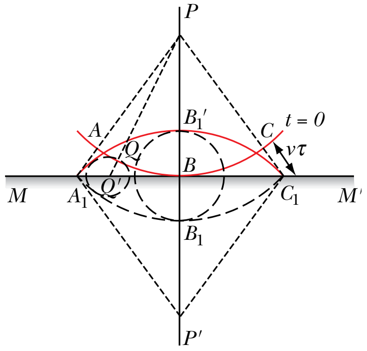


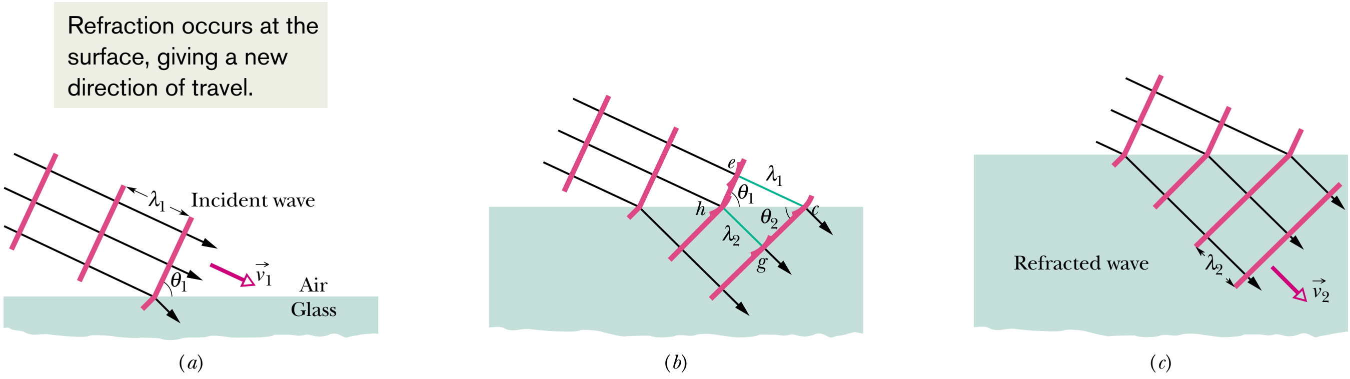
Figure 35-4 P is a point source placed in front of a plane mirror MM'. ABC is the incident wavefront (which is spherical centered at P) and $A_1B_1'C_1$ is the corresponding reflected wavefront (which is spherical and centered at P'). P' is the virtual image of P.

The Law of Refraction

We now use Huygens' principle to derive the law of refraction, Eq. 32-35 (Snell's law). Figure 35-5 shows three stages in the refraction of several wavefronts at a flat interface between air (medium 1) and glass (medium 2). We arbitrarily choose the wavefronts in the incident light beam to be separated by λ_1, the wavelength in medium 1. Let the speed of light in air be v_1 and that in glass be v_2. We assume that $v_2 < v_1$, which happens to be true.

Angle θ_1 in Fig. 35-5a is the angle between the wavefront and the interface; it has the same value as the angle between the *normal* to the wavefront (that is, the incident ray) and the *normal* to the interface. Thus, θ_1 is the angle of incidence.

As the wave moves into the glass, a Huygens wavelet at point e in Fig. 35-5b will expand to pass through point c, at a distance of λ_1 from point e. The time interval required for this expansion is that distance divided by the speed


Refraction occurs at the surface, giving a new direction of travel.

(a) (b) (c)

Figure 35-5 The refraction of a plane wave at an air–glass interface, as portrayed by Huygens' principle. The wavelength in glass is smaller than that in air. For simplicity, the reflected wave is not shown. Parts (a) through (c) represent three successive stages of the refraction.

of the wavelet, or λ_1/v_1. Now note that in this same time interval, a Huygens wavelet at point *h* will expand to pass through point *g*, at the reduced speed v_2 and with wavelength λ_2. Thus, this time interval must also be equal to λ_2/v_2. By equating these times of travel, we obtain the relation

$$\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}, \tag{35-1}$$

which shows that the wavelengths of light in two media are proportional to the speeds of light in those media.

By Huygens' principle, the refracted wavefront must be tangent to an arc of radius λ_2 centered on *h*, say at point *g*. The refracted wavefront must also be tangent to an arc of radius λ_1 centered on *e*, say at *c*. Then the refracted wavefront must be oriented as shown. Note that θ_2, the angle between the refracted wavefront and the interface, is actually the angle of refraction.

For the right triangles *hce* and *hcg* in Fig. 35-5*b* we may write

$$\sin\theta_1 = \frac{\lambda_1}{hc} \quad \text{(for triangle } hce\text{)}$$

and

$$\sin\theta_2 = \frac{\lambda_2}{hc} \quad \text{(for triangle } hcg\text{)}.$$

Dividing the first of these two equations by the second and using Eq. 35-1, we find

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}. \tag{35-2}$$

We can define the **index of refraction** *n* for each medium as the ratio of the speed of light in vacuum to the speed of light *v* in the medium. Thus,

$$n = \frac{c}{v} \quad \text{(index of refraction).} \tag{35-3}$$

In particular, for our two media, we have

$$n_1 = \frac{c}{v_1} \quad \text{and} \quad n_2 = \frac{c}{v_2}.$$

We can now rewrite Eq. 35-2 as

$$\frac{\sin\,\theta_1}{\sin\,\theta_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1}$$

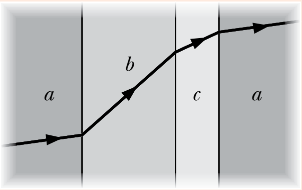
or

$$n_1 \sin\,\theta_1 = n_2 \sin\,\theta_2 \quad \text{(law of refraction),} \tag{35-4}$$

as introduced in Chapter 34.

CHECKPOINT 1

The figure shows a monochromatic ray of light traveling across parallel interfaces, from an original material *a*, through layers of materials *b* and *c*, and then back into material *a*. Rank the materials according to the speed of light in them, greatest first.



Wavelength and Index of Refraction

We have now seen that the wavelength of light changes when the speed of the light changes, as happens when light crosses an interface from one medium into another. Further, the speed of light in any medium depends on the index of refraction of the medium, according to Eq. 35-3. Thus, the wavelength of light in any medium depends on the index of refraction of the medium. Let a certain monochromatic light have wavelength λ and speed c in vacuum and wavelength λ_n and speed v in a medium with an index of refraction n. Now we can rewrite Eq. 35-1 as

$$\lambda_n = \lambda \frac{v}{c}. \tag{35-5}$$

Using Eq. 35-3 to substitute $1/n$ for v/c then yields

$$\lambda_n = \frac{\lambda}{n}. \tag{35-6}$$

This equation relates the wavelength of light in any medium to its wavelength in vacuum: A greater index of refraction means a smaller wavelength.

Next, let f_n represent the frequency of the light in a medium with index of refraction n. Then from the general relation of $v = \lambda f$, we can write

$$f_n = \frac{v}{\lambda_n}.$$

Substituting Eqs. 35-3 and 35-6 then gives us

$$f_n = \frac{c/n}{\lambda/n} = \frac{c}{\lambda} = f,$$

where f is the frequency of the light in vacuum. Thus, although the speed and wavelength of light in the medium are different from what they are in vacuum, *the frequency of the light in the medium is the same as it is in vacuum.*

Phase Difference. The fact that the wavelength of light depends on the index of refraction via Eq. 35-6 is important in certain situations involving the interference of light waves. For example, in Fig. 35-6, the *waves of the rays* (that is, the waves represented by the rays) have identical wavelengths λ and are initially in phase in air ($n \approx 1$). One of the waves travels through medium 1 of index of refraction n_1 and length L. The other travels through medium 2 of index of refraction n_2 and the same length L. When the waves leave the two media, they will have the same wavelength—their wavelength λ in air. However, because their wavelengths differed in the two media, the two waves may no longer be in phase.

The difference in indexes causes a phase shift between the rays.

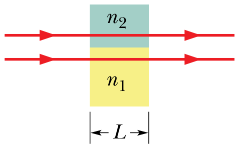


Figure 35-6 Two light rays travel through two media having different indexes of refraction.

The phase difference between two light waves can change if the waves travel through different materials having different indexes of refraction.

As we shall discuss soon, this change in the phase difference can determine how the light waves will interfere if they reach some common point.

To find their new phase difference in terms of wavelengths, we first count the number N_1 of wavelengths there are in the length L of medium 1. From Eq. 35-6, the wavelength in medium 1 is $\lambda_{n1} = \lambda/n_1$; so

$$N_1 = \frac{L}{\lambda_{n1}} = \frac{Ln_1}{\lambda}. \tag{35-7}$$

Similarly, we count the number N_2 of wavelengths there are in the length L of medium 2, where the wavelength is $\lambda_{n2} = \lambda/n_2$:

$$N_2 = \frac{L}{\lambda_{n2}} = \frac{Ln_2}{\lambda}. \tag{35-8}$$

To find the new phase difference between the waves, we subtract the smaller of N_1 and N_2 from the larger. Assuming $n_2 > n_1$, we obtain

$$N_2 - N_1 = \frac{Ln_2}{\lambda} - \frac{Ln_1}{\lambda} = \frac{L}{\lambda}(n_2 - n_1). \tag{35-9}$$

Suppose Eq. 35-9 tells us that the waves now have a phase difference of 45.6 wavelengths. That is equivalent to taking the initially in-phase waves and shifting one of them by 45.6 wavelengths. However, a shift of an integer number of wavelengths (such as 45) would put the waves back in phase; so it is only the decimal fraction (here, 0.6) that is important. A phase difference of 45.6 wavelengths is equivalent to an *effective phase difference* of 0.6 wavelength.

A phase difference of 0.5 wavelength puts two waves exactly out of phase. If the waves had equal amplitudes and were to reach some common point, they would then undergo fully destructive interference, producing darkness at that point. With a phase difference of 0.0 or 1.0 wavelength, they would, instead, undergo fully constructive interference, resulting in brightness at the common point. Our phase difference of 0.6 wavelength is an intermediate situation but closer to fully destructive interference, and the waves would produce a dimly illuminated common point.

We can also express phase difference in terms of radians and degrees, as we have done already. A phase difference of one wavelength is equivalent to phase differences of 2π rad and 360°.

Path Length Difference. As we discussed with sound waves in Section 17.5, two waves that begin with some initial phase difference can end up with a different phase difference if they travel through paths with different lengths before coming back together. The key for the waves (whatever their type might be) is the path length difference ΔL, or more to the point, how ΔL compares to the wavelength λ of the waves. From Eqs. 17-44 and 17-45, we know that, for light waves, fully constructive interference (maximum brightness) occurs when

$$\frac{\Delta L}{\lambda} = 0, 1, 2, \ldots \quad \text{(fully constructive interference),} \tag{35-10}$$

and that fully destructive interference (darkness) occurs when

$$\frac{\Delta L}{\lambda} = 0.5, 1.5, 2.5, \ldots \quad \text{(fully destructive interference).} \tag{35-11}$$

Intermediate values correspond to intermediate interference and thus also illumination.

CHECKPOINT 2

The light waves of the rays in Fig. 35-6 have the same wavelength and amplitude and are initially in phase. (a) If 7.60 wavelengths fit within the length of the top material and 5.50 wavelengths fit within that of the bottom material, which material has the greater index of refraction? (b) If the rays are angled slightly so that they meet at the same point on a distant screen, will the interference there result in the brightest possible illumination, bright intermediate illumination, dark intermediate illumination, or darkness?

SAMPLE PROBLEM 35.01

Phase difference of two waves due to difference in refractive indexes

In Fig. 35-6, the two light waves that are represented by the rays have wavelength 550.0 nm before entering media 1 and 2. They also have equal amplitudes and are in phase. Medium 1 is now just air, and medium 2 is a transparent plastic layer of index of refraction 1.600 and thickness 2.600 μm.

(a) What is the phase difference of the emerging waves in wavelengths, radians, and degrees? What is their effective phase difference (in wavelengths)?

KEY IDEA

The phase difference of two light waves can change if they travel through different media, with different indexes of refraction. The reason is that their wavelengths are different in the different media. We can calculate the change in phase difference by counting the number of wavelengths that fits into each medium and then subtracting those numbers.

Calculations: When the path lengths of the waves in the two media are identical, Eq. 35-9 gives the result of the subtraction. Here we have $n_1 = 1.000$ (for the air), $n_2 = 1.600$, $L = 2.600$ mm, and $\lambda = 550.0$ nm. Thus, Eq. 35-9 yields

$$\begin{aligned} N_2 - N_1 &= \frac{L}{\lambda}(n_2 - n_1) \\ &= \frac{2.600 \times 10^{-6}\,\text{m}}{5.500 \times 10^{-7}\,\text{m}}(1.600 - 1.000) \\ &= 2.84. \qquad \text{(Answer)} \end{aligned}$$

Thus, the phase difference of the emerging waves is 2.84 wavelengths. Because 1.0 wavelength is equivalent to 2π rad and 360°, you can show that this phase difference is equivalent to

$$\text{phase difference} = 17.8 \text{ rad} \approx 1020°. \qquad \text{(Answer)}$$

The effective phase difference is the decimal part of the actual phase difference *expressed in wavelengths*. Thus, we have

$$\text{effective phase difference} = 0.84 \text{ wavelength.} \quad \text{(Answer)}$$

You can show that this is equivalent to 5.3 rad and about 300°. *Caution:* We do *not* find the effective phase difference by taking the decimal part of the actual phase difference as expressed in radians or degrees. For example, we do *not* take 0.8 rad from the actual phase difference of 17.8 rad.

(b) If the waves reached the same point on a distant screen, what type of interference would they produce?

Reasoning: We need to compare the effective phase difference of the waves with the phase differences that give the extreme types of interference. Here the effective phase difference of 0.84 wavelength is between 0.5 wavelength (for fully destructive interference, or the darkest possible result) and 1.0 wavelength (for fully constructive interference, or the brightest possible result), but closer to 1.0 wavelength. Thus, the waves would produce intermediate interference that is closer to fully constructive interference—they would produce a relatively bright spot.

35.3 | DIFFRACTION

Key Concept

- When a wave encounters a barrier (edge, obstacle or aperture) of dimensions similar to the wavelength of the wave, it spreads out or diffracts into the region beyond the barrier.

In this Section we shall discuss the experiment that first proved that light is a wave. To prepare for that discussion, we must introduce the idea of **diffraction** of waves, a phenomenon that we explore much more fully later in the chapter. Its essence is this: If a wave encounters a barrier that has an opening of dimensions similar to the wavelength, the part of the wave that passes through the opening will flare (spread) out—will *diffract*—into the region beyond the barrier. The flaring is consistent with the spreading of wavelets in the Huygens construction of Fig. 35-2. Diffraction occurs for waves of all types, not just light waves; Fig. 35-7 shows the diffraction of water waves traveling across the surface of water in a shallow tank.

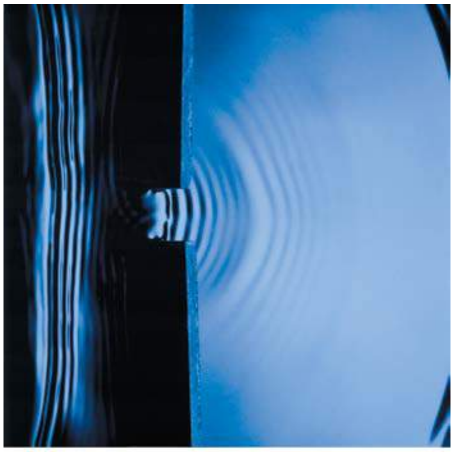

George Resch/Fundamental Photographs

Figure 35-7 Waves produced by an oscillating paddle at the left flare out through an opening in a barrier along the water surface.

Figure 35-8*a* shows the situation schematically for an incident plane wave of wavelength λ encountering a slit that has width $a = 6.0\lambda$ and extends into and out of the page. The part of the wave that passes through the slit flares out on the far side. Figures 35-8*b* (with $a = 3.0\lambda$) and 35-8*c* ($a = 1.5\lambda$) illustrate the main feature of diffraction: the narrower the slit, the greater the diffraction.

Diffraction limits geometrical optics, in which we represent an electromagnetic wave with a ray. If we actually try to form a ray by sending light through a narrow slit, or through a series of narrow slits, diffraction will always defeat our effort because it always causes the light

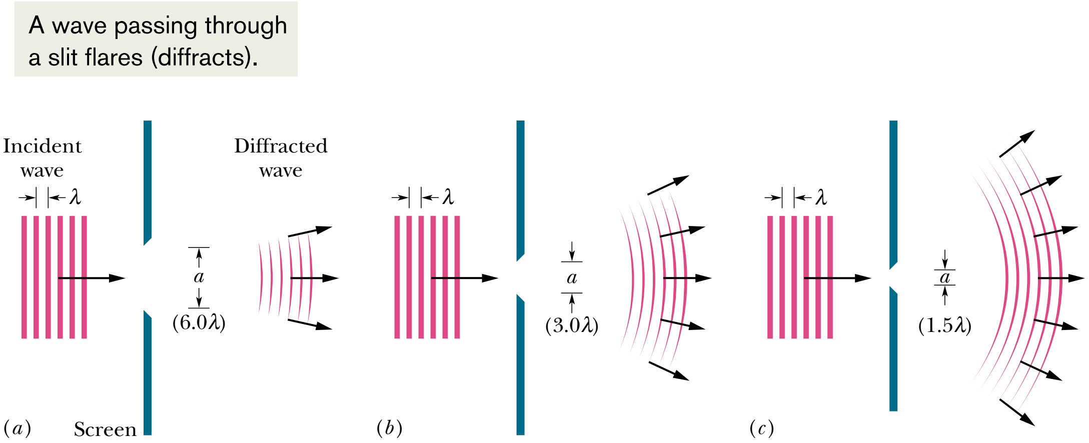


Figure 35-8 Diffraction represented schematically. For a given wavelength λ, the diffraction is more pronounced the smaller the slit width a. The figures show the cases for (*a*) slit width $a = 6.0\lambda$, (*b*) slit width $a = 3.0\lambda$, and (*c*) slit width $a = 1.5\lambda$. In all three cases, the screen and the length of the slit extend well into and out of the page, perpendicular to it.

to spread. Indeed, the narrower we make the slits (in the hope of producing a narrower beam), the greater the spreading is. Thus, geometrical optics holds only when slits or other apertures that might be located in the path of light do not have dimensions comparable to or smaller than the wavelength of the light.

35.4 | YOUNG'S INTERFERENCE EXPERIMENT

Key Concepts

- In Young's interference experiment, light passing through a single slit falls on two slits in a screen. The light leaving these slits flares out (by diffraction), and interference occurs in the region beyond the screen. A fringe pattern, due to the interference, forms on a viewing screen.
- The conditions for maximum and minimum intensity are

$$d \sin\theta = m\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima—bright fringes)},$$

$$d \sin\theta = \left(m + \frac{1}{2}\right)\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(minima—dark fringes)},$$

where θ is the angle the light path makes with a central axis and d is the slit separation.

- The distance between two adjacent maxima in given by

$$\Delta y = y_{m+1} - y_m = \frac{\lambda D}{d}.$$

In 1801, Thomas Young experimentally proved that light is a wave, contrary to what most other scientists then thought. He did so by demonstrating that light undergoes interference, as do water waves, sound waves, and waves of all other types. In addition, he was able to measure the average wavelength of sunlight; his value, 570 nm, is impressively close to the modern accepted value of 555 nm. We shall here examine Young's experiment as an example of the interference of light waves.

Figure 35-9 gives the basic arrangement of Young's experiment. Light from a distant monochromatic source illuminates slit S_0 in screen A. The emerging light then spreads via diffraction to illuminate two slits S_1 and S_2 in screen B. Diffraction of the light by these two slits sends overlapping circular waves into the region beyond screen B, where the waves from one slit interfere with the waves from the other slit.

The "snapshot" of Fig. 35-9 depicts the interference of the overlapping waves. However, we cannot see evidence for the interference except where a viewing screen C intercepts the light. Where it does so, points of interference maxima form visible bright rows—called *bright bands, bright fringes*, or (loosely speaking) *maxima*—that extend across the screen (into and out of the page in Fig. 35-9). Dark regions—called *dark bands, dark fringes*, or (loosely speaking) *minima*—result from fully destructive interference and are visible between adjacent pairs of bright fringes. (*Maxima* and *minima* more properly refer to the center of a band.) The pattern of bright and dark

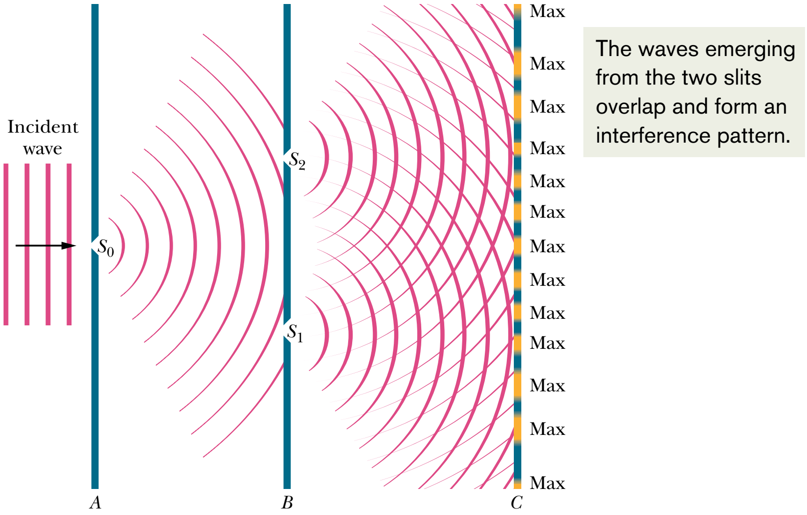

Incident wave
S_0
S_2
S_1
A
B
C
Max
Max
Max
Max
Max
Max
Max
Max
Max
Max
Max
Max
Max
The waves emerging from the two slits overlap and form an interference pattern.

Figure 35-9 In Young's interference experiment, incident monochromatic light is diffracted by slit S_0, which then acts as a point source of light that emits semicircular wavefronts. As that light reaches screen B, it is diffracted by slits S_1 and S_2, which then act as two point sources of light. The light waves traveling from slits S_1 and S_2 overlap and undergo interference, forming an interference pattern of maxima and minima on viewing screen C. This figure is a cross section; the screens, slits, and interference pattern extend into and out of the page. Between screens B and C, the semicircular wavefronts centered on S_2 depict the waves that would be there if only S_2 were open. Similarly, those centered on S_1 depict waves that would be there if only S_1 were open.

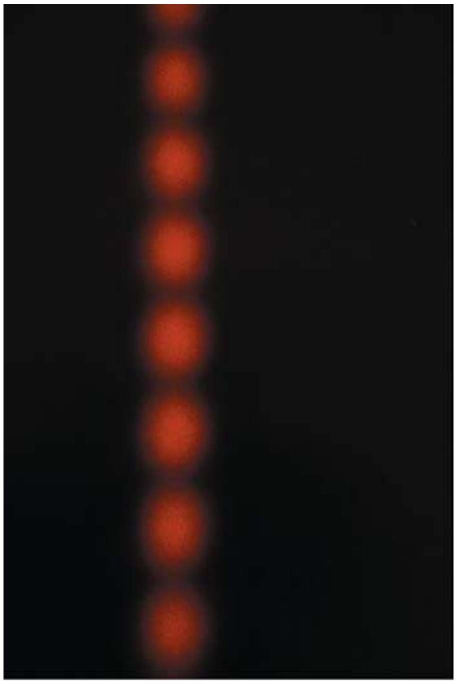
fringes on the screen is called an **interference pattern**. Figure 35-10 is a photograph of part of the interference pattern that would be seen by an observer standing to the left of screen C in the arrangement of Fig. 35-9.

Let us try to understand the phenomena of interference of waves. Any two waves reaching at a point may have a path difference between them, as we discussed in chapter on Longitudinal Waves, Vol. I. A path difference of 0.5 wavelength puts two waves exactly out of phase. If the waves had equal amplitudes and were to reach some common point, they would then undergo fully destructive interference, producing darkness at that point. With a path difference of 0.0 or 1.0 wavelength, they would, instead, undergo fully constructive interference, resulting in brightness at the common point. Our path difference of 0.6 wavelength is an intermediate situation but closer to fully destructive interference, and the waves would produce a dimly illuminated common point.

Locating the Fringes

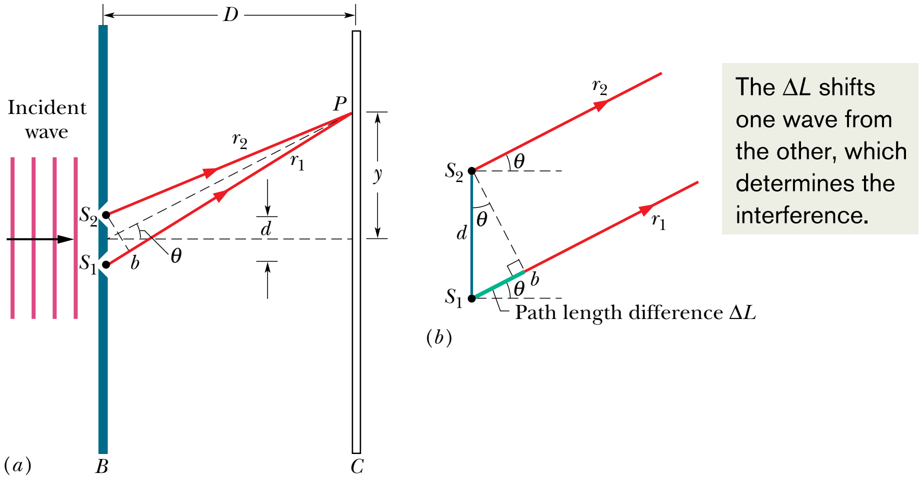
Light waves produce fringes in a *Young's double-slit interference experiment*, as it is called, but what exactly determines the locations of the fringes? To answer, we shall use the arrangement in Fig. 35-11*a*. There, a plane wave of monochromatic light is incident on two slits S_1 and S_2 in screen B; the light diffracts through the slits and produces an interference pattern on screen C. We draw a central axis from the point halfway between the slits to screen C as a reference. We then pick, for discussion, an arbitrary point P on the screen, at angle θ to the central axis. This point intercepts the wave of ray r_1 from the bottom slit and the wave of ray r_2 from the top slit.

Path Length Difference. These waves are in phase when they pass through the two slits because there they are just portions of the same



Courtesy Jearl Walker

Figure 35-10 A photograph of the interference pattern produced by the arrangement shown in Fig. 35-8, but with short slits. (The photograph is a front view of part of screen C.) The alternating maxima and minima are called *interference fringes* (because they resemble the decorative fringe sometimes used on clothing and rugs).



The ΔL shifts one wave from the other, which determines the interference.

Figure 35-11 (*a*) Waves from slits S_1 and S_2 (which extend into and out of the page) combine at *P*, an arbitrary point on screen *C* at distance *y* from the central axis. The angle θ serves as a convenient locator for *P*. (*b*) For $D \gg d$, we can approximate rays r_1 and r_2 as being parallel, at angle θ to the central axis.

incident wave. However, once they have passed the slits, the two waves must travel different distances to reach *P*. We saw a similar situation in Section 17.8 with sound waves and concluded that

> The phase difference between two waves can change if the waves travel paths of different lengths.

The change in phase difference is due to the *path length difference* ΔL in the paths taken by the waves. Consider two waves initially exactly in phase, traveling along paths with a path length difference ΔL, and then passing through some common point. When ΔL is zero or an integer number of wavelengths, the waves arrive at the common point exactly in phase and they interfere fully constructively there. If that is true for the waves of rays r_1 and r_2 in Fig. 35-11, then point *P* is part of a bright fringe. When, instead, ΔL is an odd multiple of half a wavelength, the waves arrive at the common point exactly out of phase and they interfere fully destructively there. If that is true for the waves of rays r_1 and r_2, then point *P* is part of a dark fringe. (And, of course, we can have intermediate situations of interference and thus intermediate illumination at *P*.) Thus,

> What appears at each point on the viewing screen in a Young's double-slit interference experiment is determined by the path length difference ΔL of the rays reaching that point.

Angle. We can specify where each bright fringe and each dark fringe is located on the screen by giving the angle θ from the central axis to that fringe. To find θ, we must relate it to ΔL. We start with Fig. 35-11*a* by finding a point *b* along ray r_1 such that the path length from *b* to *P* equals the path length from S_2 to *P*. Then the path length difference ΔL between the two rays is the distance from S_1 to *b*.

The relation between this S_1-to-*b* distance and θ is complicated, but we can simplify it considerably if we arrange for the distance *D* from the slits to the screen to be much greater than the slit separation *d*. Then we can approximate rays r_1 and r_2 as being parallel to each other and at angle θ to the central axis (Fig. 35-11*b*). We can also approximate the triangle formed by S_1, S_2, and *b* as being a right triangle, and approximate the angle inside that triangle at S_2 as being θ. Then, for that triangle, $\sin\theta = \Delta L/d$ and thus

$$\Delta L = d \sin\theta \quad \text{(path length difference).} \tag{35-12}$$

For a bright fringe, we saw that ΔL must be either zero or an integer number of wavelengths. Using Eq. 35-12, we can write this requirement as

$$\Delta L = d \sin \theta = (\text{integer})(\lambda), \tag{35-13}$$

or as

$$d \sin \theta = m\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad (\text{maxima—bright fringes}). \tag{35-14}$$

For a dark fringe, ΔL must be an odd multiple of half a wavelength. Again using Eq. 35-12, we can write this requirement as

$$\Delta L = d \sin \theta = (\text{odd number})\left(\frac{1}{2}\lambda\right), \tag{35-15}$$

or as

$$d \sin \theta = \left(m + \frac{1}{2}\right)\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad (\text{minima—dark fringes}). \tag{35-16}$$

With Eqs. 35-14 and 35-16, we can find the angle θ to any fringe and thus locate that fringe; further, we can use the values of m to label the fringes. For the value and label $m = 0$, Eq. 35-14 tells us that a bright fringe is at $\theta = 0$ and thus on the central axis. This *central maximum* is the point at which waves arriving from the two slits have a path length difference $\Delta L = 0$, hence zero phase difference.

For, say, $m = 2$, Eq. 35-14 tells us that *bright* fringes are at the angle

$$\theta = \sin^{-1}\left(\frac{2\lambda}{d}\right)$$

above and below the central axis. Waves from the two slits arrive at these two fringes with $\Delta L = 2\lambda$ and with a phase difference of two wavelengths. These fringes are said to be the *second-order bright fringes* (meaning $m = 2$) or the *second side maxima* (the second maxima to the side of the central maximum), or they are described as being the second bright fringes from the central maximum.

For $m = 1$, Eq. 35-16 tells us that *dark* fringes are at the angle

$$\theta = \sin^{-1}\left(\frac{1.5\lambda}{d}\right)$$

above and below the central axis. Waves from the two slits arrive at these two fringes with $\Delta L = 1.5\lambda$ and with a phase difference, in wavelengths, of 1.5. These fringes are called the *second-order dark fringes* or *second minima* because they are the second dark fringes to the side of the central axis. (The first dark fringes, or first minima, are at locations for which $m = 0$ in Eq. 35-16.)

Nearby Screen. We derived Eqs. 35-14 and 35-16 for the situation $D \gg d$. However, they also apply if we place a converging lens between the slits and the viewing screen and then move the viewing screen closer to the slits, to the focal point of the lens. (The screen is then said to be in the *focal plane* of the lens; that is, it is in the plane perpendicular to the central axis at the focal point.) One property of a converging lens is that it focuses all rays that are parallel to one another to the same point on its focal plane. Thus, the rays that now arrive at any point on the screen (in the focal plane) were exactly parallel (rather than approximately) when they left the slits. They are like the initially parallel rays in Fig. 34-34*a* that are directed to a point (the focal point) by a lens.

Let us now answer the basic question: What is the distance on screen C in Fig. 35-11*a* between adjacent maxima near the center of the interference pattern? Assume that θ in Fig. 35-11 is small enough to permit the use of the approximations $\sin \theta \approx \tan \theta \approx \theta$ in which θ is expressed in radian measure. This normally occurs in the case when we use visible light for performing Young's experiment. For this, first, let us pick a maximum with a low value of m to ensure that it is near the center of the pattern. Then, from the geometry of Fig. 35-11*a*, the maximum's vertical distance y_m from the center of the pattern is related to its angle θ from the central axis by

$$\tan \theta \approx \theta \approx \frac{y_m}{D}.$$

From Eq. 35-14, this angle θ for the mth maximum is given by

$$\sin\theta \approx \theta = \frac{m\lambda}{d}.$$

If we equate our two expressions for angle θ and then solve for y_m, we find

$$y_m = \frac{m\lambda D}{d}. \tag{35-17}$$

For the next maximum as we move away from the pattern center, we have

$$y_{m+1} = \frac{(m+1)\lambda D}{d}. \tag{35-18}$$

We find the distance between these adjacent maxima by subtracting Eq. 35-17 from Eq. 35-18:

$$\Delta y = y_{m+1} - y_m = \frac{\lambda D}{d}. \tag{35-19}$$

As long as d and λ in Fig. 35-11*a* are small, the separation of the interference fringes is independent of m, that is, the fringes are evenly spaced. In general, the equation is valid only if the wavelength of light is small in comparison to the slit separation d.

CHECKPOINT 3

In Fig. 35-11*a*, what are ΔL (as a multiple of the wavelength) and the phase difference (in wavelengths) for the two rays if point P is (a) a third side maximum and (b) a third minimum?

SAMPLE PROBLEM 35.02

Double-slit interference pattern

What is the distance on screen C in Fig. 35-11*a* between adjacent maxima near the center of the interference pattern? The wavelength λ of the light is 546 nm, the slit separation d is 0.12 mm, and the slit–screen separation D is 55 cm. Assume that θ in Fig. 35-11 is small enough to permit use of the approximations $\sin\theta \approx \tan\theta \approx \theta$, in which θ is expressed in radian measure.

KEY IDEAS

(1) First, let us pick a maximum with a low value of m to ensure that it is near the center of the pattern. Then, from the geometry of Fig. 35-11*a*, the maximum's vertical distance y_m from the center of the pattern is related to its angle θ from the central axis by

$$\tan\theta \approx \theta = \frac{y_m}{D}.$$

(2) From Eq. 35-14, this angle θ for the mth maximum is given by

$$\sin\theta \approx \theta = \frac{m\lambda}{d}.$$

Calculations: If we equate our two expressions for angle θ and then solve for y_m, we find

$$y_m = \frac{m\lambda D}{d}. \tag{35-20}$$

For the next maximum as we move away from the pattern's center, we have

$$y_{m+1} = \frac{(m+1)\lambda D}{d}. \tag{35-21}$$

We find the distance between these adjacent maxima by subtracting Eq. 35-20 from Eq. 35-21:

$$\begin{aligned}\Delta y = y_{m+1} - y_m &= \frac{\lambda D}{d}\\ &= \frac{(546\times10^{-9}\,\text{m})(55\times10^{-2}\,\text{m})}{0.12\times10^{-3}\,\text{m}}\\ &= 2.50\times10^{-3}\,\text{m} \approx 2.5\text{ mm}. \qquad \text{(Answer)}\end{aligned}$$

As long as d and θ in Fig. 35-11*a* are small, the separation of the interference fringes is independent of m; that is, the fringes are evenly spaced.

SAMPLE PROBLEM 35.03

Young's double-slit experiment performed in different mediums

Young's double-slit experiment is performed inside water ($\mu = 4/3$) with light of frequency 6×10^{14} Hz. If the slits are separated by 0.2 mm and the screen kept 1 m from the slits, find the fringe width. Using the same setup, what will be the fringe width if the experiment is performed in air?

KEY IDEA

Here the wavelength is much less than separation of the slits. Hence, we can use the formula for fringe width.

Calculations: The wavelength of given light in air can be obtained from the relation

$$c = f\lambda_{\text{air}}$$

$$\lambda_{\text{air}} = \frac{c}{f} = \frac{3\times10^{8}}{6\times10^{14}} = 5\times10^{-7}\ \text{m}.$$

The wavelength of light of same frequency (6×10^{14} Hz) in water will be different and

$$\lambda_{\text{water}} = \frac{\lambda_{\text{air}}}{\mu_{\text{water}}} = \frac{5\times10^{-7}}{4/3} = 3.75\times10^{-7}\,\text{m}.$$

As the experiment is performed in water, the fringe width is given by

$$\beta = \frac{\lambda_{\text{water}} D}{d}.$$

Here $D = 1$ m, $d = 0.2$ mm $= 0.2 \times 10^{-3}$ m; therefore

$$\beta = \frac{3.75\times10^{-7}\times1}{0.2\times10^{-3}}\,\text{m} = 1.87\ \text{mm}.$$

If the experiment is performed in air, the fringe width is given by

$$\beta = \frac{\lambda_{\text{air}} D}{d} = \frac{5\times10^{-7}\times1}{0.2\times10^{-3}}\,\text{m} = 2.5\ \text{mm}. \quad \text{(Answer)}$$

Learn: The fringe width in water reduces by the factor of refractive index because the wavelength reduces by the same factor.

SAMPLE PROBLEM 35.04

Young's double slit experiment (YDSE) with coherent parallel beam

A coherent parallel beam of microwaves of wavelength $\lambda = 0.5$ mm falls on a Young's double-slit apparatus. The separation between the slits is 1.0 mm. The intensity of microwaves is measured on a screen placed parallel to the plane of the slits at a distance of 1.0 m from it as shown in Fig. 35-12.

(a) If the incident beam falls normally on the double-slit apparatus, find the y coordinates of all the interference minima on the screen.

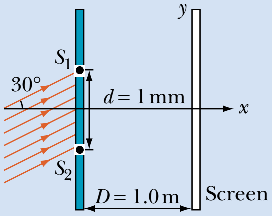


Figure 35-12 A coherent parallel beam of microwaves falling on an YDSE apparatus.

KEY IDEA

Refer to Fig. 35-13. When the incident beam falls normally on the slits S_1 and S_2, the path difference at the central point P_0 of the screen is zero. Hence, we have the central maximum at P_0. At the minima, the path difference between the waves is $\Delta = [m + (1/2)]\,\lambda$.

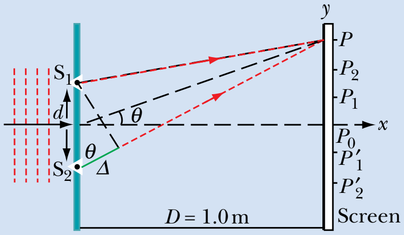


Figure 35-13 Interference pattern when the waves are incident normally.

This is a classic case when the wavelength of the light is comparable to the slit separation d. So we cannot apply the equation for fringe width as in Eq. 35-19.

Calculation: Let the minima appear along directions θ with respect to the incident direction. Coherent waves from S_1 and S_2 along this direction are brought to a focus at P. It is clear that the path difference between the waves from S_1 and S_2 on reaching P is

$$\Delta = d \sin\theta.$$

The interference minima will appear on the screen if

$$\Delta = \left(m + \frac{1}{2}\right)\lambda$$

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda,$$

where m is an integer. Thus, the direction of minima is given by

$$\sin\theta = \left(m + \frac{1}{2}\right) \times \left(\frac{\lambda}{d}\right).$$

It is given that $d = 1.0$ mm and $\lambda = 0.5$ mm. Therefore,

$$\sin\theta = \left(m + \frac{1}{2}\right) \times \left(\frac{0.5}{1.0}\right) = \frac{1}{2}\left(m + \frac{1}{2}\right). \qquad (35\text{-}22)$$

The allowed values of m are those integers for which sin θ is not more than +1 or less than −1. These values are $m = 1, 0, -1$, and -2. Hence, four minima will be observed. If follows from the figure that

$$\tan\theta = \frac{y}{D} \Rightarrow y = D\tan\theta$$

$$y = \frac{D\sin\theta}{(1 - \sin^2\theta)^{1/2}}. \qquad (35\text{-}23)$$

The first minima above the central maximum (P_0) corresponds to $m = 0$. Using Eq. 36-17, we have sin $\theta = 1/4$ for this minimum. Using this value in Eq. 35-22 the distance of the first minimum P_1 from P_0 is

$$y_1 = \frac{(1.0\text{ m}) \times (1/4)}{[1 - (1/16)]^{1/2}} = \frac{1}{\sqrt{15}}\text{ m.}$$

This is the y coordinate of the first minimum P_1 above P_0.

The second minima above P_0 corresponds to m = 1, for which, we have from Eq. 36-17, sin θ = 3/4. Using this value in Eq. 35-24, the y coordinate of the second minimum P_2 above P_0 is

$$y_2 = \frac{(1.0\text{ m}) \times (3/4)}{[1 - (9/16)]^{1/2}} = \frac{3}{\sqrt{7}}\text{ m.}$$

The first minimum P_1' below P_0 corresponds to $m = -1$, for which sin $\theta = -1/4$. The y coordinate of this minimum P_1' is

$$y_1' = -\frac{1}{\sqrt{15}}\text{ m.}$$

The second minimum P_2' below P_0 corresponds to $m = -2$ for which sin $\theta = -3/4$. The y coordinate of P_2' is (see Fig. 35-12)

$$P_2' = -\frac{3}{\sqrt{7}}\text{ m.} \qquad \text{(Answer)}$$

(b) If the incident beam makes an angle of 30º with the x axis (as shown in the figure), find the y coordinates of the first minima on either side of the central maximum.

KEY IDEA

When the incident wave makes an angle $\alpha = 30º$ with the x axis, the position of zero path difference, that is the central maximum, shifts from P_0 to P_0' (see Fig. 35-14).

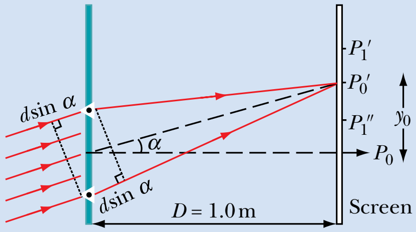


Figure 35-14 Interference pattern when the waves are incident at an angle of 30° to the normal.

Here we can easily see that the waves arriving at slits are having an initial path difference of d sin α. To compensate for this, the central maxima should shift at a point where the path of the wave coming through the lower slit is longer by the same amount. Clearly, this will occur at an angle α to the central line.

We have

$$y_0 = D\tan\alpha = (1.0\text{ m}) \times \tan 30° = \frac{1}{\sqrt{3}}\text{ m.}$$

Therefore, the y coordinate to the first minimum P_1' is the point where the path difference is $\lambda/2$. Therefore,

$$d\sin\theta - d\sin\alpha = -\frac{\lambda}{2}.$$

Solving, we get sin $\theta = 3/4$. Therefore,

$$y_1 = \frac{(1.0\text{ m}) \times (3/4)}{[1 - (9/16)]^{1/2}} = \frac{3}{\sqrt{7}}\text{ m.}$$

Similarly, the y coordinate of the first minimum P_1'' is where the path difference is $-\lambda/2$.

$$d\sin\theta - d\sin\alpha = -\frac{\lambda}{2}.$$

Solving, we get sin $\theta = 1/4$. Therefore,

$$y_1' = \frac{(1.0\text{ m}) \times (1/4)}{[1 - (1/16)]^{1/2}} = \frac{1}{\sqrt{15}}\text{ m.} \qquad \text{(Answer)}$$

Learn: How do we distinguish between the central maxima and other maxima? For this, we note that the central maximum is the point where the path difference is zero for all the wavelengths. The location of other maxima is at a path difference of $n\lambda$, which depends on the wavelength. So, if we use white light to obtain the interference pattern, the central maxima will be common to all the colors. Other fringes will be colored.

Geometrical Path and Optical Path

The fact that the wavelength of light depends on the index of refraction via Eq. 35-8 is important in certain situations involving the interference of light waves. For example, in Fig. 35-6, the *waves of the rays* (that is, the waves represented by the rays) have identical wavelengths and are initially in-phase in air ($n \approx 1$). One of the waves travels through medium 1 of index of refraction n_1 and length L. The other wave travels through medium 2 of index of refraction n_2 and the same length L. When the waves leave the two media, they will have the same wavelength λ: their wavelength λ in air. However, because their wavelengths differed in the two media, the two waves may no longer be in-phase.

The phase difference between two light waves can change if the waves travel through different materials having different indexes of refraction.

As we will discuss soon, this change in the phase difference can determine how the light waves will interfere if they reach some common point.

The actual distance travelled by light in a medium is called geometrical path (Δx). Consider a light wave given by the equation:

$$E = E_0 \sin(\omega t - kx + \phi).$$

If the light travels by Δx, its phase changes by $k\Delta x = (\omega/v)\Delta x$, where ω which is the frequency of light does not depend on the medium, but v which is the speed of light depends on the medium as $v = c/n$.

Consequently, change in phase is given by

$$\Delta\phi = \frac{\omega}{C} n\Delta x.$$

It is clear that a wave traveling a distance Δx in a medium of refractive index n suffers the same phase change as when it travels a distance $n\Delta x$ in vacuum. That is a path length of Δx in medium of refractive index n is equivalent to a path length of $n\Delta x$ in vacuum.

The quantity $n\Delta x$ is called the optical path length of light, Δx_{opt}. In terms of optical path length, phase difference would be given by

$$\Delta\phi = \frac{2\pi}{\lambda_0} \Delta x_{\text{opt}},$$

where λ_0 is the wavelength of light in vacuum. However, in terms of the geometrical path length Δx,

$$\Delta\phi = \frac{2\pi}{\lambda_0} n\Delta x = \frac{2\pi}{\lambda} \Delta x,$$

where λ is the wavelength of light in the medium ($\lambda = \lambda_0/n$). Figure 35-15 clearly shows that a thickness t of the medium has as many wavelengths as there are in a length nt of vacuum. Therefore, in terms of wavelengths, a thickness t in a medium of refractive index n is equivalent to a path length nt in vacuum. The quantity nt is called equivalent optical path length.

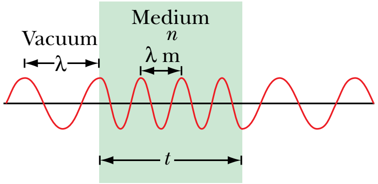


Figure 35-15 The shortening of wavelength in a medium.

Displacement of Fringe on Introduction of a Glass Slab in the Path of the Light Coming out of the Slits

In Young's double-slit experiment (YDSE), we are concerned with the phase difference of the waves that arrive at the screen. It can be easily seen that a thin film in front of the upper slit will introduce additional path in comparison to the imaginary film of thickness t in air in front of the lower slit (Fig. 35-16).

On introduction of the thin glass-slab of thickness t and refractive index n, the optical path of the ray S_1P increases by $t(n-1)$ in comparison to the similar film of air in front of the lower slit. Now the path difference between waves coming from S_1 and S_2 at any point P is

$$\begin{aligned}\Delta x &= S_2P - [S_1P + t(n-1)] \\ &= (S_2P - S_1P) - t(n-1).\end{aligned}$$

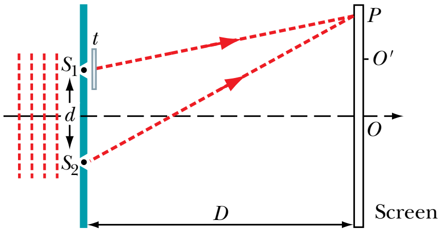


Figure 35-16 In a standard YDSE apparatus, a thin film is introduced in front of one of the slits. The effect is to shift the entire interference pattern in the direction of that slit to point P (central maxima).

This implies

$$\Delta x = d\sin\theta - t(n-1) \text{ if } d \ll D.$$

$$\Delta x = \frac{yd}{D} - t(n-1) \text{ if } y \ll D \text{ as well}.$$

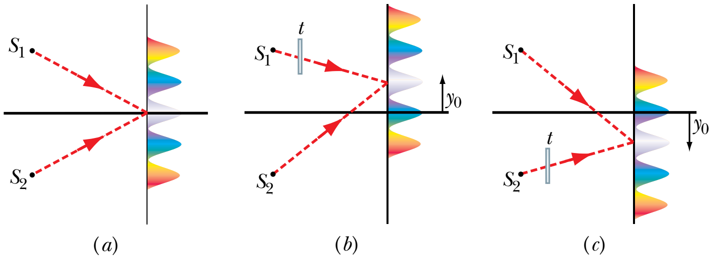
For central bright fringe,

$$\Delta\phi = 0 \Rightarrow \frac{yd}{D} = t(n-1).$$

The whole fringe pattern gets shifted by the same distance

$$y_0 = (n-1)t\frac{D}{d}.$$

The shift of the fringe pattern has been shown in Fig. 35-17.



(*a*) (*b*) (*c*)

Figure 35-17 Shift of the interference pattern in the presence of a thin glass slab. The intensity at all the maxima is the same, and to emphasize the central maxima, it has been shown white. Other fringes are colored. Notice that this shift is in the direction of the slit before which the glass slab is placed. If the glass slab is placed before the upper slit, the fringe pattern gets shifted upward and if the glass slab is placed before the lower slit the fringe pattern gets shifted downward.

SAMPLE PROBLEM 35.05

Measuring fringe width and thickness of thin glass plate

A double-slit apparatus is immersed in a transparent liquid of refractive index 1.33. The slit separation is 1 mm and the distance between the plate of slits and screen is 1.33 m. The slits are illuminated by a parallel beam of light whose wavelength in air is 6300 Å.

(a) Find the fringe width.

KEY IDEA

In this situation, the wavelength of the light in the medium changes. So, the fringe width will change. Also, the refractive index has to be taken relative to the medium in which interference has to be done.

Calculation: The wavelength in liquid is

$$\lambda_1 = \frac{\lambda_\alpha\mu_\alpha}{\mu_1} = \frac{6300\ \text{Å}\times 1}{1.33} = \frac{6.3\times10^{-7}}{1.33}\ \text{m}.$$

Therefore, fringe width in liquid is

$$\beta_1 = \frac{\lambda_1 D}{d} = \frac{6.3\times10^{-7}\times1.33}{1.33\times1\times10^{-3}} = 6.3\times10^{-4}\,\text{m} = 0.63\ \text{mm}.$$

(b) One of the slits of the apparatus is covered by a thin glass sheet of refractive index 1.53. Find the smallest thickness of the sheet to bring the adjacent minimum of the axis.

KEY IDEA

If t is the thickness of the glass plate, the fringes are displaced by an amount given by

$$\Delta y = \left(\frac{\mu_g}{\mu_l} - 1\right)\frac{tD}{d}. \tag{35-24a}$$

In order to bring the adjacent minimum on the axis (where we had the maximum before the sheet was introduced), the fringes must be displaced by half the fringe width, that is

$$\Delta y = \frac{\beta_1}{2}. \tag{35-24b}$$

Calculation: Equating Eqs. 35-24a and 35-24b we have

$$\frac{\beta_1}{2} = \left(\frac{\mu_g}{\mu_l} - 1\right)\frac{tD}{d}.$$

Substituting $\beta_1 = 6.3 \times 10^{-4}$ m, $\mu_g = 1.53$, $\mu_1 = 1.33$, $D = 1.33$ m and $d = 1 \times 10^{-3}$ m and solving we get

$$t = 1.575 \times 10^{-6} \text{ m}.$$

Learn: By measuring this fringe shift, we can measure the thickness of very thin plates.

35.5 | COHERENCE

Key Concept

- If two light waves that meet at a point are to interfere perceptibly, the phase difference between them must remain constant with time; that is, the waves must be coherent. When two coherent waves meet, the resulting intensity may be found by using phasors.

For the interference pattern to appear on viewing screen C in Fig. 35-9, the light waves reaching any point P on the screen must have a phase difference that does not vary in time. That is the case in Fig. 35-9 because the waves passing through slits S_1 and S_2 are portions of the single light wave that illuminates the slits. Because the phase difference remains constant, the light from slits S_1 and S_2 is said to be completely **coherent**.

Sunlight and Fingernails. Direct sunlight is partially coherent; that is, sunlight waves intercepted at two points have a constant phase difference only if the points are very close. If you look closely at your fingernail in bright sunlight, you can see a faint interference pattern called *speckle* that causes the nail to appear to be covered with specks. You see this effect because light waves scattering from very close points on the nail are sufficiently coherent to interfere with one another at your eye. The slits in a double-slit experiment, however, are not close enough, and in direct sunlight, the light at the slits would be **incoherent**. To get coherent light, we would have to send the sunlight through a single slit as in Fig. 35-9; because that single slit is small, light that passes through it is coherent. In addition, the smallness of the slit causes the coherent light to spread via diffraction to illuminate both slits in the double-slit experiment.

Incoherent Sources. If we replace the double slits with two similar but independent monochromatic light sources, such as two fine incandescent wires, the phase difference between the waves emitted by the sources varies rapidly and randomly. (This occurs because the light is emitted by vast numbers of atoms in the wires, acting randomly and independently for extremely short times—of the order of nanoseconds.) As a result, at any given point on the viewing screen, the interference between the waves from the two sources varies rapidly and randomly between fully constructive and fully destructive. The eye (and most common optical detectors) cannot follow such changes, and no interference pattern can be seen. The fringes disappear, and the screen is seen as being uniformly illuminated.

Coherent Source. A *laser* differs from common light sources in that its atoms emit light in a cooperative manner, thereby making the light coherent. Moreover, the light is almost monochromatic, is emitted in a thin beam with little spreading, and can be focused to a width that almost matches the wavelength of the light.

35.6 | INTENSITY IN DOUBLE-SLIT INTERFERENCE

Key Concept

- In Young's interference experiment, two waves, each with intensity I_0, yield a resultant wave of intensity I at the viewing screen, with

$$I = 4I_0 \cos^2 \frac{1}{2}\phi, \quad \text{where } \phi = \frac{2\pi d}{\lambda}\sin\theta.$$

Equations 35-14 and 35-16 tell us how to locate the maxima and minima of the double-slit interference pattern on screen C of Fig. 35-11 as a function of the angle θ in that figure. Here we wish to derive an expression for the intensity I of the fringes as a function of θ.

The light leaving the slits is in phase. However, let us assume that the light waves from the two slits are not in phase when they arrive at point P. Instead, the electric field components of those waves at point P are not in phase and vary with time as

$$E_1 = E_0 \sin \omega t \tag{35-25}$$

and

$$E_2 = E_0 \sin(\omega t + \phi), \tag{35-26}$$

where ω is the angular frequency of the waves and ϕ is the phase constant of wave E_2. Note that the two waves have the same amplitude E_0 and a phase difference of ϕ. Because that phase difference does not vary, the waves are coherent. We shall show that these two waves will combine at P to produce an intensity I given by

$$I = 4I_0 \cos^2 \frac{1}{2}\phi, \tag{35-27}$$

and that

$$\phi = \frac{2\pi d}{\lambda}\sin\theta. \tag{35-28}$$

In Eq. 35-27, I_0 is the intensity of the light that arrives on the screen from one slit when the other slit is temporarily covered. We assume that the slits are so narrow in comparison to the wavelength that this single-slit intensity is essentially uniform over the region of the screen in which we wish to examine the fringes.

Equations 35-27 and 35-28, which together tell us how the intensity I of the fringe pattern varies with the angle θ in Fig. 35-11, necessarily contain information about the location of the maxima and minima. Let us see if we can extract that information to find equations about those locations.

Maxima. Study of Eq. 35-27 shows that intensity maxima will occur when

$$\frac{1}{2}\phi = m\pi, \quad \text{for } m = 0, 1, 2, \ldots. \tag{35-29}$$

If we put this result into Eq. 35-28, we find

$$2m\pi = \frac{2\pi d}{\lambda}\sin\theta, \quad \text{for } m = 0, 1, 2, \ldots$$

or

$$d \sin\theta = m\lambda, \quad \text{for } m = 0, 1, 2, \ldots \text{ (maxima)}, \tag{35-30}$$

which is exactly Eq. 35-14, the expression that we derived earlier for the locations of the maxima.

Minima. The minima in the fringe pattern occur when

$$\frac{1}{2}\phi = \left(m + \frac{1}{2}\right)\pi, \quad \text{for } m = 0, 1, 2, \ldots. \tag{35-31}$$

If we combine this relation with Eq. 35-28, we are led at once to

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(minima)}, \tag{35-32}$$

which is just Eq. 35-16, the expression we derived earlier for the locations of the fringe minima.

Figure 35-18, which is a plot of Eq. 35-27, shows the intensity of double-slit interference patterns as a function of the phase difference ϕ between the waves at the screen. The horizontal solid line is I_0, the (uniform) intensity on the screen when one of the slits is covered up. Note in Eq. 35-27 and the graph that the intensity I varies from zero at the fringe minima to $4I_0$ at the fringe maxima.

If the waves from the two sources (slits) were *incoherent*, so that no enduring phase relation existed between them, there would be no fringe pattern and the intensity would have the uniform value $2I_0$ for all points on the screen; the horizontal dashed line in Fig. 35-18 shows this uniform value.

Interference cannot create or destroy energy but merely redistributes it over the screen. Thus, the *average* intensity on the screen must be the same $2I_0$ regardless of whether the sources are coherent. This follows at once from Eq. 35-27; if we substitute 1/2, the average value of the cosine-squared function, this equation reduces to $I_{\text{avg}} = 2I_0$.

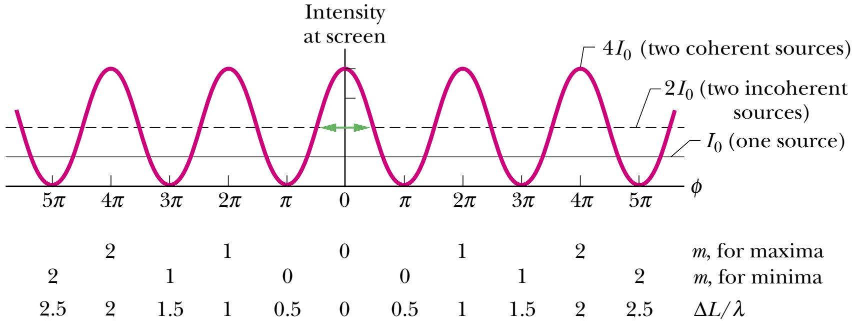


Figure 35-18 A plot of Eq. 35-27, showing the intensity of a double-slit interference pattern as a function of the phase difference between the waves when they arrive from the two slits. I_0 is the (uniform) intensity that would appear on the screen if one slit were covered. The average intensity of the fringe pattern is $2I_0$, and the *maximum* intensity (for coherent light) is $4I_0$.

Proof of Eqs. 35-27 and 35-28

We shall combine the electric field components E_1 and E_2, given by Eqs. 35-25 and 35-26, respectively, by the method of phasors as is discussed in Section 16.9. In Fig. 35-19*a*, the waves with components E_1 and E_2 are represented by phasors of magnitude E_0 that rotate around the origin at angular speed ω. The values of E_1 and E_2 at any time are the projections of the corresponding phasors on the vertical axis. Figure 35-19*a* shows the phasors and their projections at an arbitrary time t. Consistent with Eqs. 35-25 and 35-26, the phasor for E_1 has a rotation angle ωt and the phasor for E_2 has a rotation angle $\omega t + \phi$ (it is phase-shifted ahead of E_1). As each phasor rotates, its projection on the vertical axis varies with time in the same way that the sinusoidal functions of Eqs. 35-25 and 35-26 vary with time.

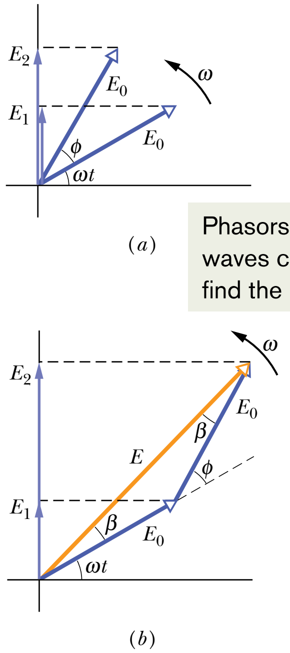
To combine the field components E_1 and E_2 at any point P in Fig. 35-11, we add their phasors vectorially, as shown in Fig. 35-19*b*. The magnitude of the vector sum is the amplitude E of the resultant wave at point P, and that wave has a certain phase constant β. To find the amplitude E in Fig. 35-19*b*, we first note that the two angles marked β are equal because they are opposite equal-length sides of a triangle. From the theorem (for triangles) that an exterior angle (here ϕ, as shown in Fig. 35-19*b*) is equal to the sum of the two opposite interior angles (here that sum is $\beta + \beta$), we see that $\beta = 1/2\,\phi$. Thus, we have

$$E = 2(E_0 \cos \beta)$$

$$= 2E_0 \cos \frac{1}{2}\phi. \qquad (35\text{-}33)$$

If we square each side of this relation, we obtain

$$E^2 = 4E_0^2 \cos^2 \frac{1}{2}\phi. \qquad (35\text{-}34)$$



Phasors that represent waves can be added to find the net wave.

Figure 35-19 (*a*) Phasors representing, at time t, the electric field components given by Eqs. 35-20 and 35-21. Both phasors have magnitude E_0 and rotate with angular speed ω. Their phase difference is ϕ. (*b*) Vector addition of the two phasors gives the phasor representing the resultant wave, with amplitude E and phase constant β.

Intensity. Now, from Eq. 33-29, we know that the intensity of an electromagnetic wave is proportional to the square of its amplitude. Therefore, the waves we are combining in Fig. 35-19*b*, whose amplitudes are E_0, each has an intensity I_0 that is proportional to E_0^2, and the resultant wave, with amplitude E, has an intensity I that is proportional to E^2. Thus,

$$\frac{I}{I_0} = \frac{E^2}{E_0^2}.$$

Substituting Eq. 35-34 into this equation and rearranging then yield

$$I = 4I_0 \cos^2 \frac{1}{2}\phi,$$

which is Eq. 35-27, which we set out to prove.

We still must prove Eq. 35-28, which relates the phase difference ϕ between the waves arriving at any point P on the screen of Fig. 35-11 to the angle θ that serves as a locator of that point.

The phase difference ϕ in Eq. 35-26 is associated with the path length difference S_1b in Fig. 35-11*b*. If S_1b is $1/2\lambda$, then ϕ is π; if S_1b is λ, then ϕ is 2π, and so on. This suggests

$$\begin{pmatrix}\text{phase}\\ \text{difference}\end{pmatrix} = \frac{2\pi}{\lambda}\begin{pmatrix}\text{path length}\\ \text{difference}\end{pmatrix}. \qquad (35\text{-}35)$$

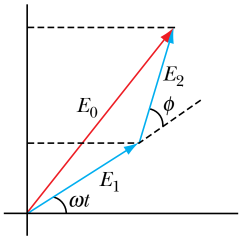


Figure 35-20 Phasors representing, at time t, the electric field components given by Eqs. 35-30 and 35-31. Phasors have magnitude E_1 and E_2 and rotate with angular speed ω. Their phase difference is ϕ. Vector addition of the two phasors gives the phasor representing the resultant wave, with amplitude E_0.

The path length difference S_1b in Fig. 35-11*b* is $d \sin\theta$ (a leg of the right triangle); so Eq. 35-35 for the phase difference between the two waves arriving at point P on the screen becomes

$$\phi = \frac{2\pi d}{\lambda}\sin\theta,$$

which is Eq. 35-28, the other equation that we set out to prove to relate ϕ to the angle θ that locates P.

Here we assumed that the intensity due to both the sources is the same. But in many of the cases, especially when the slits are of unequal width, the intensity due to the two silts on the screen is not the same. In such a case, we can find the resultant electric field amplitude by adding the electric field amplitudes vectorially as can be seen in Fig. 35-20.

If two waves of amplitudes E_1 and E_2, differing in phase by ϕ, superpose, the resultant amplitude A is given by,

$$E_0^2 = E_1^2 + E_2^2 + 2E_1E_2\cos\phi.$$

Since intensity of a wave is proportional to the square of its amplitude, if two waves of intensities I_1 and I_2 superpose, the resultant intensity I is given by

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi. \qquad (35\text{-}36)$$

Combining More Than Two Waves

In a more general case, we might want to find the resultant of more than two sinusoidally varying waves at a point. Whatever the number of waves is, our general procedure is this:

1. Construct a series of phasors representing the waves to be combined. Draw them end to end, maintaining the proper phase relations between adjacent phasors.
2. Construct the vector sum of this array. The length of this vector sum gives the amplitude of the resultant phasor. The angle between the vector sum and the first phasor is the phase of the resultant with respect to this first phasor. The projection of this vector-sum phasor on the vertical axis gives the time variation of the resultant wave.

Constructive Interference

The resultant intensity I is maximum if $\cos\phi = +1$, that is

$$\phi = 2m\pi, \qquad (35\text{-}37)$$

where $m = 0, 1, 2, 3, \ldots$, etc. is an integer,

or

$$\frac{2\pi\Delta}{\lambda} = 2m\pi \quad \text{or} \quad \Delta = m\lambda, \qquad (35\text{-}38)$$

where Δ is the path difference between the interfering waves. In this case

$$I_{\text{max}} = I_1 + I_2 + 2\sqrt{I_1 I_2},$$
$$I_{\text{min}} = I_1 + I_2 - 2\sqrt{I_1 I_2}. \qquad (35\text{-}39)$$

CHECKPOINT 4

Each of four pairs of light waves arrives at a certain point on a screen. The waves have the same wavelength. At the arrival point, their amplitudes and phase differences are (a) $2E_0$, $6E_0$, and π rad; (b) $3E_0$, $5E_0$, and π rad; (c) $9E_0$, $7E_0$, and 3π rad; (d) $2E_0$, $2E_0$, and 0 rad. Rank the four pairs according to the intensity of the light at the arrival point, greatest first. (*Hint:* Draw phasors.)

SAMPLE PROBLEM 35.06

Combining three light waves by using phasors

Three light waves combine at a certain point where their electric field components are

$$E_1 = E_0 \sin \omega t,$$
$$E_2 = E_0 \sin(\omega t + 60°),$$
$$E_3 = E_0 \sin(\omega t - 30°).$$

Find their resultant component $E(t)$ at that point.

KEY IDEA

The resultant wave is

$$E(t) = E_1(t) + E_2(t) + E_3(t).$$

We can use the method of phasors to find this sum, and we are free to evaluate the phasors at any time t.

Calculations: To simplify the solution, we choose $t = 0$, for which the phasors representing the three waves are shown in Fig. 35-21. We can add these three phasors either directly on a vector-capable calculator or by components. For the component approach, we first write the sum of their horizontal components as

$$\sum E_h = E_0 \cos 0 + E_0 \cos 60° + E_0 \cos(-30°) = 2.37E_0.$$

The sum of their vertical components, which is the value of E at $t = 0$, is

$$\sum E_v = E_0 \sin 0 + E_0 \sin 60° + E_0 \sin(-30°) = 0.366E_0.$$

The resultant wave $E(t)$ thus has an amplitude E_R of

$$E_R = \sqrt{(2.37E_0)^2 + (0.366E_0)^2} = 2.4E_0,$$

and a phase angle β relative to the phasor representing E_1 of

$$\beta = \tan^{-1}\left(\frac{0.366E_0}{2.37E_0}\right) = 8.8°.$$

We can now write, for the resultant wave $E(t)$,

$$E = E_R \sin(\omega t + \beta)$$
$$= 2.4E_0 \sin(\omega t + 8.8°). \qquad \text{(Answer)}$$

Be careful to interpret the angle β correctly in Fig. 35-21: It is the constant angle between E_R and the phasor representing E_1 as the four phasors rotate as a single unit around the origin. The angle between E_R and the horizontal axis in Fig. 35-21 does not remain equal to β.

Phasors that represent waves can be added to find the net wave.

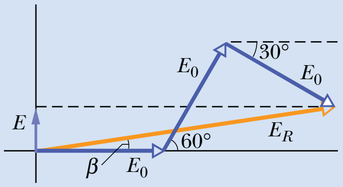


Figure 35-21 Three phasors, representing waves with equal amplitudes E_0 and with phase constants 0°, 60°, and –30°, shown at time $t = 0$. The phasors combine to give a resultant phasor with magnitude E_R, at angle β.

SAMPLE PROBLEM 35.07

Interference pattern with two coherent light sources

Interference pattern is obtained with two coherent light sources of intensity ratio n. Show that in the interference pattern,

$$\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}=\frac{2\sqrt{n}}{n+1}.$$

KEY IDEA

Here we can apply Eq. 35-39.

Calculation: Given $I_1/I_2 = n$; therefore, the amplitude ratio is

$$\frac{A_1}{A_2}=\sqrt{n}.$$

Also, $I_{\max} = A_1 + A_2$ and $I_{\min} = A_1 + A_2$. Therefore,

$$\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}=\frac{(A_1+A_2)^2-(A_1-A_2)^2}{(A_1+A_2)^2+(A_1-A_2)^2}$$

$$=\frac{2A_1A_2}{A_1^2+A_2^2}$$

$$=\frac{2A_1/A_2}{(A_1^2/A_2^2)+1}=\frac{2\sqrt{n}}{n+1}.$$

SAMPLE PROBLEM 35.08

Young's double-slit experiment

In Young's double-slit experiment, the upper slit is covered by a thin glass plate of refractive index 1.4 while the lower slit is covered by another glass plate having the same thickness as the first but having refractive index 1.7. Interference pattern is observed using light of wavelength 5400 Å. It is found that the point P on the screen where the central maximum ($n = 0$) fell before the glass plates were introduced now has three-fourth (3/4) of the original intensity. It is further observed that what used to be the fifth maximum earlier lies below the point P, while the sixth minimum lies above P. Calculate the thickness of the glass plate. (Absorption of light by glass plate may be neglected.)

KEY IDEA

Refer to Fig. 35-22. In Young's double-slit experiment, slits S_1 and S_2 are equidistant from the source S and the slits have equal widths. Hence, light emerging from S_1 and S_2 has the same intensity, say I_0. Before the plates are introduced, the path difference (and hence the phase difference ϕ) at the central point P between the interfering waves is zero.

Calculation: The intensity at point P is

$$I = I_0 + I_0 + 2\sqrt{I_0I_0}\cos 0° = 4I_0.$$

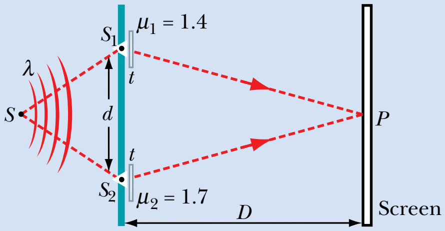


Figure 35-22 The interference pattern when both the slits are covered by thin slabs.

When the two plates are introduced, the path difference at the central point P between the interfering waves becomes

$$\Delta = (\mu_2 - 1)t - (\mu_1 - 1)t = (\mu_2 - \mu_1)t.$$

Therefore, the phase difference is given by

$$\phi = \frac{2\pi\Delta}{\lambda} = \frac{2\pi}{\lambda}(\mu_2 - \mu_1)t.$$

Hence, the intensity at point P becomes

$$I' = I_0 + I_0 + 2\sqrt{I_0I_0}\cos\phi = 2I_0 + 2I_0\cos\phi.$$

It is given that

$$I' = \frac{3I}{4} = \frac{3}{4}\times 4I_0 = 3I_0.$$

Thus

$$3I_0 = 2I_0 + 2I_0 \cos\phi.$$

Now

$$\cos\phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \ldots$$

$$\phi = \frac{\pi}{3}(6m \pm 1);\ m = 0, 1, 2, 3, \ldots \qquad (35\text{-}40)$$

where the negative value $\phi = -\pi/3$ corresponding to $m = 0$ is not admissible since ϕ is positive as $\mu_2 > \mu_1$. The upward shift of the interference pattern due to the upper plate is

$$(\Delta y)_1 = (\mu_1 - 1)\frac{tD}{d}.$$

The downward shift due to the lower plate is

$$(\Delta y)_2 = (\mu_2 - 1)\frac{tD}{d}.$$

Therefore, the new downward shift is

$$\Delta y = (\Delta y)_2 - (\Delta y)_1 = (\mu_2 - \mu_1)\frac{tD}{d}$$

$$= (1.7 - 1.4)\frac{tD}{d} = \frac{0.3tD}{d}.$$

Now, the distance of the fifth maximum from P is $5\lambda D/d$, and the distance of sixth minimum from P is $[6 - (1/2)]\,\lambda D/d = 5.5\,\lambda D/d$. It is given that the net shift lies between $5\lambda D/d$ and $5.5\,\lambda D/d$, that is

$$\frac{5\lambda D}{d} < \frac{0.3tD}{d} < \frac{5.5\lambda D}{d}$$

$$\frac{5\lambda}{0.3} < t < \frac{5.5\lambda}{0.3}$$

$$\frac{5 \times 5.4 \times 10^{-7}}{0.3}\,\text{m} < t < \frac{5.5 \times 5.4 \times 10^{-7}}{0.3}\,\text{m}$$

$$9.0 \times 10^{-6}\,\text{m} < t < 9.9 \times 10^{-6}\,\text{m}. \qquad (35\text{-}41)$$

We have seen that the phase difference,

$$\phi = \frac{2\pi}{\lambda}(\mu_2 - \mu_1)t$$

$$= \frac{2\pi}{\lambda}(1.7 - 1.4)t$$

$$= \frac{2\pi}{\lambda}(0.3)t. \qquad (35\text{-}42)$$

Using Eqs. 35-41 and 35-42, we have

$$\frac{\pi}{3}(6m \pm 1) = \frac{2\pi}{\lambda}(0.3t)$$

$$6m \pm 1 = \frac{6}{\lambda}(0.3)t = \frac{1.8t}{5.4 \times 10^{-7}}$$

$$t = 3(6m \pm 1) \times 10^{-7}\,\text{m}. \qquad (35\text{-}43)$$

It is clear that Eq. 35-43 will yield a value of t which satisfies the requirement (Eq. 35-41) if $m = 5$. Hence,

$$t = 3(6 \times 5 \pm 1) \times 10^{-7}$$

$$= 3(30 \pm 1) \times 10^{-7}$$

$$= 9.3 \times 10^{-6}\,\text{m} \ \text{ or }\ 8.7 \times 10^{-6}\,\text{m}.$$

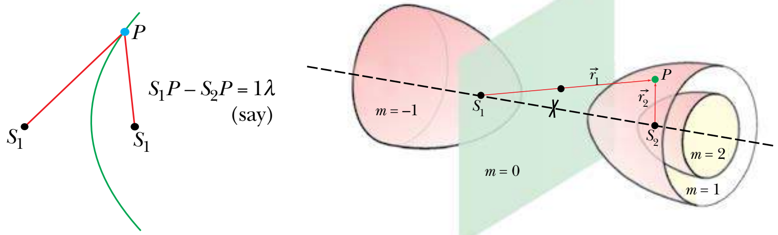
The value of $t = 8.7 \times 10^{-6}$ m does not satisfy the requirement in Eq. 35-40. Hence, $t = 9.3 \times 10^{-6}$ m.

Shape of Fringes

As for a given fringe, n is constant, each fringe will be the locus of the point P for which

$$S_1P - S_2P = \text{constant}.$$

But from coordinate geometry we know that this locus is a hyperbola with S_1 and S_2 as foci, as seen in Fig. 35-23*a*. If there are two point sources, we can say that they will produce such hyperbola in all the directions. To get this directional distribution, we rotate a hyperbola about the line S_1S_2, as seen in Fig. 35-23*b*. We thus get hyperboloids of revolution, *that is, the interference fringes for two coherent point sources are hyperboloids with axis S_1S_2*. The fringes seen on a screen are sections of these hyperboloids so:

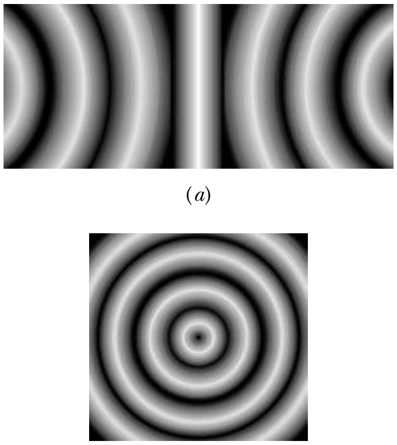


(a) (b)

Figure 35-23 (a) The locus of points in space where the path difference is constant is a hyperbola. (b) A combination of all such hyperbolas makes a hyperboloid in space. Please note that the figure is greatly exaggerated for the purpose of clarity. In reality, these hyperboloids are so flat that they almost look like a plane.

1. If the screen is parallel to line S_1S_2 which is usually the case, the fringes will be hyperbolic with straight-central bright fringe (Fig. 35-24a).
2. If the screen is perpendicular to line joining the sources, the fringes will be circular concentric circles with their centers on the point intersection of screen with line S_1S_2 (Fig. 35-24b). In this situation central fringe will be bright if $S_1S_2 = n\lambda$ and dark if $S_1S_2 = (2n - 1)\ \lambda/2$.

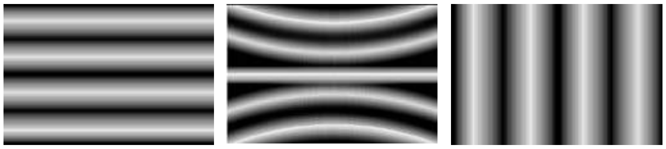
If instead of pinholes the sources are slits, we can visualize that in space fringes will be obtained on a hyperbolic plane. This hyperbolic plane intersects with the screen to produce an interference pattern. For a screen perpendicular to the slits, the shape of fringes will be hyperbolic with straight-central bright fringe (Fig. 35-25b) while for screen parallel to the slits, fringes will be straight lines parallel to the slits, as shown in Figs. 35-25a and c. For the screen parallel to the slits as well as parallel to the plane of the slits, central fringe will always be bright. For the screen parallel to the slits but perpendicular to the plane of the slits, central fringe (the fringe in the plane of the slits) will be bright if $S_1S_2 = n\lambda$ and dark if $S_1S_2 = (2n - 1)\ \lambda/2$. For the screen perpendicular to the slits as well as the plane of the slits, central fringe will always be bright. From all this it is clear that *shape of fringes depends on nature of sources* (pinholes or slits) *and direction of observation,* that is, the position of screen.



(a)

(b)

Figure 35-24 Shape of interference pattern on a screen kept (a) parallel to the line S_1S_2, (b) perpendicular to line S_1S_2.



(a) (b) (c)

Figure 35-25 Shape of interference pattern on a screen kept (a) parallel to the slits, (b) perpendicular to the slits, and (c) parallel to the slits but perpendicular the plane of the slits.

35.7 | MODIFICATIONS OF YOUNG'S DOUBLE-SLIT EXPERIMENT

There are many other arrangements that can be considered for the formation of an interference pattern on the screen. In all these situations, note how the arrangement produces two coherent sources whose waves interfere to produce an interference pattern. Here we are going to consider a few popular examples of the same.

SAMPLE PROBLEM 35.09

Interference pattern of two point sources oscillating in the same phase

Two point sources oscillating in the same phase are on a straight line perpendicular to a screen. The nearest source is at a distance of $D \gg \lambda$ from the screen. What shape will the interference fringes have on the screen? What is the distance on the screen from the perpendicular to the nearest bright fringe if the distance between the sources is $l = n\lambda$ (n is an integer)? Note that $\lambda \ll l$.

KEY IDEA

Here we first try to see why an interference pattern will be obtained. The interference pattern is observed because as seen on the screen, the paths of the waves coming from two sources are not equal.

Calculation: The intensity on the screen will be maximum when $d_2 - d_1 = k\lambda$. The locus of points on the screen reached by rays from both sources with this difference is a circle with its center at point A (Fig. 35-26). For this reason, the interference fringes will have the form of concentric circle.

When $l = n\lambda$ the intensity will be maximum at point A (an interference maximum of the nth order). The nearest bright interference fringe circle of the $(n-1)$th order is at a distance from point A determined from the equation,

$$d_2 - d_1 = \sqrt{(n\lambda + D)^2 + h_{n-1}^2} - \sqrt{D^2 + h_{n-1}^2} = (n-1)\lambda.$$

Bearing in mind the conditions of the problem that $h \ll D$ and $\lambda \ll l$, we get

$$h_{n-1} \cong \sqrt{\frac{2D(D+n\lambda)}{n}} = \sqrt{2D\lambda\left(\frac{D}{l}+1\right)}.$$

Learn: One interesting aspect of this problem is that the order of maxima at any point other than the center will be less than n. Recall your mathematics classes. In any triangle, the third side is always longer than the difference of the other two sides. Since in the triangle formed by the two sources and point on the screen, the distance between two sources is $n\lambda$, the path difference at any arbitrary point should be less than that.

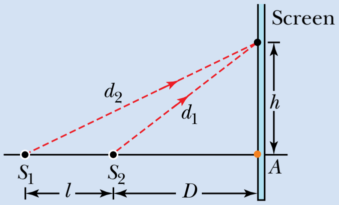


Figure 35-26 Interference pattern when two point sources S_1 and S_2 are kept on a line perpendicular to the screen.

SAMPLE PROBLEM 35.10

Interference pattern of light incident on a Fresnel biprism

Light from source S is incident on the Fresnel biprism shown in Fig. 35-27. The light beams refracted by the different faces of the prism partly overlap and produce an interference pattern on a screen on its section AB.

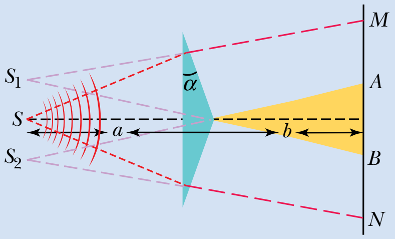


Figure 35-27 Fresnel biprism used to form interference pattern on screen MN. Here the light refracted from the upper portion and the light refracted from the lower portion interfere to produce the interference pattern.

(a) Find the distance between adjacent interference fringes if the distance from the source to the prism is $a = 1$ m and that from the prism to the screen is $b = 4$ m. The angle of refraction of the prism is $a = 2 \times 10^{-3}$ rad.

The glass of prism has a refractive index of $n = 1.5$. The wavelength of the light is $\lambda = 6000$ Å.

KEY IDEA

To find the distance d it is first necessary to calculate the distance l between the virtual source S_1 and S_2 located at the point of intersection to the continuation of the rays refracted by the prism faces. For this purpose, it will be the simplest to consider the path of the ray normally incident on the face of the prism (Fig. 35-28).

Calculation: No such ray actually exists, but it can be plotted by mentally extending the upper prism downward.

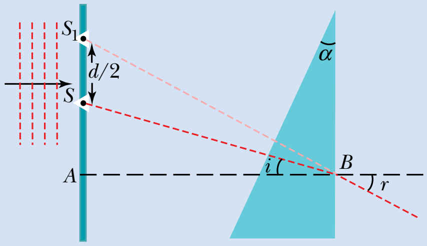


Figure 35-28 The path of an imaginary ray incident from S on upper part of biprism. Since a is small, all rays undergo the same deviation. The refracted rays can be extended to meet at S_1, which acts as virtual source.

All the beams from point source refracted by a prism can be assumed to converge at a point. Since the refracting angle of the prism is small (the prism is thin), the virtual images S_1 and S_2 of the source can be regarded as lying at the same distance from the prism as source S. It can be seen from Fig. 35-28 that $i = \alpha$ and $SA = a\alpha$. According to the law of refraction, $r \cong n\alpha$. Considering the triangle AS_1B, we can write

$$\frac{d}{2} + a\alpha \cong a\alpha n.$$

Hence,

$$d = 2a\alpha(n-1).$$

We find that

$$\beta = \frac{\lambda D}{d} = \frac{\lambda(a+b)}{2a\alpha(n-1)} = 0.15\,\text{cm}.$$

(b) How many interference fringes can be observed on a screen?

KEY IDEA

$N = L/\beta$, where L is the width of the interference pattern.

Calculation: It can be seen from Fig. 35-27 that $AB = (b/a)d$ where d is the distance between the virtual images S_1 and S_2 of the source. So, we get

$$N = \frac{4ab\alpha^2(n-1)^2}{(a+b)\lambda} \cong 5.$$

SAMPLE PROBLEM 35.11

Interference pattern of light incident on two flat mirrors (Fresnels' mirror method)

Two flat mirrors form an angle close to 180º (Fig. 35-29). A source of light S is placed at an equal distance b from the mirrors. Find the distance between adjacent interference fringes on screen MN at a distance $OA = \alpha$, from the point of intersection of the mirrors. The wavelength of the light is equal to λ. Shield C does not allow the light to pass directly from the source to the screen.

KEY IDEA

As you might recall, the object and image distances from the mirror are equal. This would imply that the image is located behind the mirror at an angle $\alpha/2$ behind the line OA. Similarly, for the second mirror. The image should also be located at a distance b from the point of intersection of the mirrors. This can be seen in Fig. 35-30.

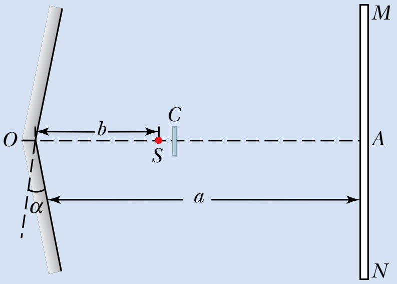


Figure 35-29 Fresnel's mirror method to produce the interference pattern on a screen.

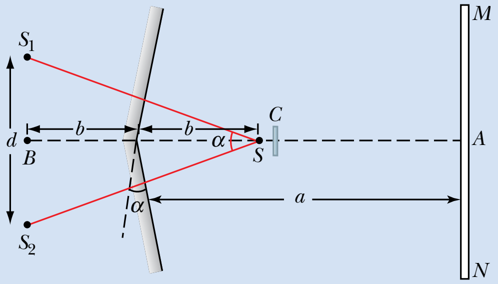


Figure 35-30 In Fresnel's mirrors, the images of source S act as two coherent sources of light.

This is again a situation where the two mirrors are producing the two images. These two images act as coherent sources.

Calculation: The distance between interference bands is $\beta = \lambda D/d$. In our case $D = AB \cong a + b$, and $d = S_1S_2$ is the distance between the images S_1 and S_2 of the source in the flat mirrors (Fig. 35-30). The value of d can be found from triangle S_1SB:

$$\frac{d}{2} = 2b\frac{\alpha}{2}$$

$$d = 2b\alpha.$$

Hence,

$$\beta = \frac{\lambda(a+b)}{2b\alpha}.$$

Learn: The light from source S, after reflection from M_1 and M_2, splits into two beams which illuminate different regions of the screen. The formation of interference fringes can be interpreted as due to two wavefronts coming from virtual sources S_1 and S_2 which are images of S in mirrors M_1 and M_2, respectively. The linear distance between S_1 and S_2 is, therefore, equivalent to separation d between the slits.

SAMPLE PROBLEM 35.12

Interference pattern of a light from monochromatic point source incident on thin lens (Billet split lens method)

In Fig. 35-31, S is a monochromatic point source emitting light of wavelength l = 500 nm. A thin lens of circular shape and of focal length 0.10 m is cut into two identical halves L_1 and L_2 by a plane passing through a diameter. The two halves are placed symmetrically about the central axis SO with a gap of 0.5 mm. The distance along the axis from S to L_1 and L_2 is 0.15 m, while that from L_1 and L_2 to O is 1.30 m. The screen at O is normal to SO. If the third intensity maximum occurs at point A on the screen, find the distance OA.

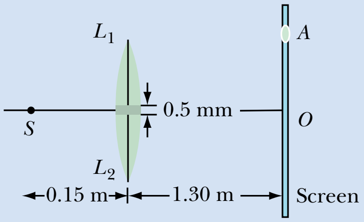


Figure 35-31 Billet split lens method to produce interference pattern on the screen.

KEY IDEA

The two pieces L_1 and L_2 act as independent lenses. S_1 is the image of source S due to lens L_1 and S_2 that due to lens L_2. S_1 and S_2 serve as the two coherent light sources which produce interference pattern on the screen (Fig. 35-32).

Calculation: Given f = 0.10 m and u = – 0.15 m. The image distance v is obtained by using the lens formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

which gives

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u} = \frac{1}{0.10} + \frac{1}{-0.15} = \frac{1}{0.30}$$

or v = 0.30 m. Rays from S that pass through the optical center O_1 of lens L_1 and through O_2 of lens L_2 go through undeviated. Therefore, triangles SO_1O_2 and SS_1S_2 are similar. Hence,

$$\frac{S_1S_2}{O_1O_2} = \frac{|u|+|v|}{|u|}. \qquad (35\text{-}44)$$

However, O_1O_2 = 0.5 mm (given). Therefore,

$$d = S_1S_2 = \frac{(0.15+0.30)\ \text{m}}{(0.15)\ \text{m}} \times 0.5\ \text{mm} = 1.5\ \text{mm}.$$

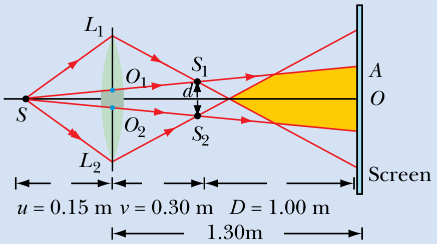


Figure 35-32 L_1 and L_2 produce two images S_1 and S_2 which act as coherent sources of light producing interference pattern on the screen shown by yellowish color.

The distance of the vertical plane containing S_1 and S_2 from the screen is

$$D = 1.30 - 0.30 = 1.00 \text{ m}.$$

The distance of the nth maximum from the central point O is given by

$$y_n = \frac{n\lambda D}{d}.$$

Since the third-order (n = 3) maximum occurs at A, we have

$$y_3 = OA = \frac{3\lambda D}{d} = \frac{3\times 500\times 10^{-9}\times 1.00}{1.5\times 10^{-3}}$$
$$= 1.0\times 10^{-3} \text{ m} = 1.0 \text{ mm}.$$

Note: What happens if the gap O_1O_2 is reduced? If the gap O_1O_2 is reduced, it follows from Eq. 35-44 that $S_1S_2 = d$ is also reduced as u and v remain unchanged. Hence, OA increases.

35.8 | INTERFERENCE FROM THIN FILMS

Key Concepts

- When light is incident on a thin transparent film, the light waves reflected from the front and back surfaces interfere. For near-normal incidence, the wavelength conditions for maximum and minimum intensity of the light reflected from a *film in air* are

$$2L = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima—bright film in air)},$$

$$2L = m\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(minima—dark film in air)},$$

where n_2 is the index of refraction of the film, L is its thickness, and λ is the wavelength of the light in air.

- If a film is sandwiched between media other than air, these equations for bright and dark films may be interchanged, depending on the relative indexes of refraction.
- If the light incident at an interface between media with different indexes of refraction is in the medium with the smaller index of refraction, the reflection causes a phase change of π rad, or half a wavelength, in the reflected wave. Otherwise, there is no phase change due to the reflection. Refraction causes no phase shift.

The colors on a sunlit soap bubble or an oil slick are caused by the interference of light waves reflected from the front and back surfaces of a thin transparent film. The thickness of the soap or oil film is typically of the order of magnitude of the wavelength of the (visible) light involved. (Greater thicknesses spoil the coherence of the light needed to produce the colors due to interference.)

Figure 35-33 shows a thin transparent film of uniform thickness L and index of refraction n_2, illuminated by bright light of wavelength λ from a distant point source. For now, we assume that air lies on both sides of the film and thus that $n_1 = n_3$ in Fig. 35-33. For simplicity, we also assume that the light rays are almost perpendicular to the film ($\theta \approx 0$). We are interested in whether the film is bright or dark to an observer viewing it almost perpendicularly. (Since the film is brightly illuminated, how could it possibly be dark? You will see.)

The incident light, represented by ray i, intercepts the front (left) surface of the film at point a and undergoes

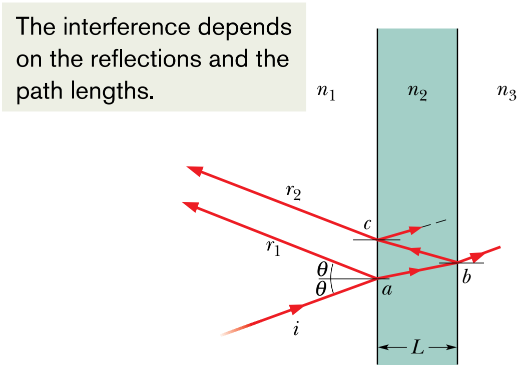

The interference depends on the reflections and the path lengths.

Figure 35-33 Light waves, represented with ray i, are incident on a thin film of thickness L and index of refraction n_2. Rays r_1 and r_2 represent light waves that have been reflected by the front and back surfaces of the film, respectively. (All three rays are actually nearly perpendicular to the film.) The interference of the waves of r_1 and r_2 with each other depends on their phase difference. The index of refraction n_1 of the medium at the left can differ from the index of refraction r_3 of the medium at the right, but for now we assume that both media are air, with $n_1 = n_3 = 1.0$, which is less than n_2.

both reflection and refraction there. The reflected ray r_1 is intercepted by the observer's eye. The refracted light crosses the film to point b on the back surface, where it undergoes both reflection and refraction. The light reflected at b crosses back through the film to point c, where it undergoes both reflection and refraction. The light refracted at c, represented by ray r_2, is intercepted by the observer's eye.

If the light waves of rays r_1 and r_2 are exactly in phase at the eye, they produce an interference maximum and region ac on the film is bright to the observer. If they are exactly out of phase, they produce an interference minimum and region ac is dark to the observer, *even though it is illuminated*. If there is some intermediate phase difference, there are intermediate interference and brightness.

The Key. Thus, the key to what the observer sees is the phase difference between the waves of rays r_1 and r_2. Both rays are derived from the same ray i, but the path involved in producing r_2 involves light traveling twice across the film (a to b, and then b to c), whereas the path involved in producing r_1 involves no travel through the film. Because θ is about zero, we approximate the path length difference between the waves of r_1 and r_2 as $2L$. However, to find the phase difference between the waves, we cannot just find the number of wavelengths λ that is equivalent to a path length difference of $2L$. This simple approach is impossible for two reasons: (1) the path length difference occurs in a medium other than air, and (2) reflections are involved, which can change the phase.

The phase difference between two waves can change if one or both are reflected.

Let's next discuss changes in phase that are caused by reflections.

Reflection Phase Shifts

Refraction at an interface never causes a phase change—but reflection can, depending on the indexes of refraction on the two sides of the interface. Figure 35-34 shows what happens when reflection causes a phase change, using as an example pulses on a denser string (along which pulse travel is relatively slow) and a lighter string (along which pulse travel is relatively fast).

When a pulse traveling relatively slowly along the denser string in Fig. 35-34a reaches the interface with the lighter string, the pulse is partially transmitted and partially reflected, with no change in orientation. For light, this situation corresponds to the incident wave traveling in the medium of greater index of refraction n (recall that greater n means slower speed). In that case, the wave that is reflected at the interface does not undergo a change in phase; that is, its *reflection phase shift* is zero.

When a pulse traveling more quickly along the lighter string in Fig. 35-34b reaches the interface with the denser string, the pulse is again partially transmitted and partially reflected. The transmitted pulse again has the same orientation as the incident pulse, but now the reflected pulse is inverted. For a sinusoidal wave, such an inversion involves a phase change of π rad, or half a wavelength. For light, this situation corresponds to the incident wave traveling in the medium of lesser index of refraction (with greater speed). In that case, the wave that is reflected at the interface undergoes a phase shift of π rad, or half a wavelength.

We can summarize these results for light in terms of the index of refraction of the medium off which (or from which) the light reflects:

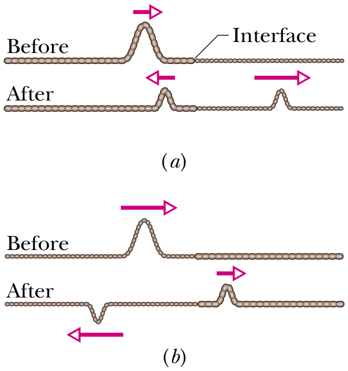


Figure 35-34 Phase changes when a pulse is reflected at the interface between two stretched strings of different linear densities. The wave speed is greater in the lighter string. (a) The incident pulse is in the denser string. (b) The incident pulse is in the lighter string. Only here is there a phase change, and only in the reflected wave.

Reflection	Reflection phase shift
Off lower index	0
Off higher index	0.5 wavelength

This might be remembered as "higher means half."

Equations for Thin-Film Interference

In this chapter we have now seen three ways in which the phase difference between two waves can change:

1. by reflection
2. by the waves traveling along paths of different lengths
3. by the waves traveling through media of different indexes of refraction

When light reflects from a thin film, producing the waves of rays r_1 and r_2 shown in Fig. 35-33, all three ways are involved. Let us consider them one by one.

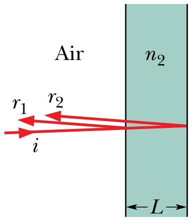


Figure 35-35 Reflections from a thin film in air.

Reflection Shift. We first reexamine the two reflections in Fig. 35-33. At point a on the front interface, the incident wave (in air) reflects from the medium having the higher of the two indexes of refraction; so the wave of reflected ray r_1 has its phase shifted by 0.5 wavelength. At point b on the back interface, the incident wave reflects from the medium (air) having the lower of the two indexes of refraction; so the wave reflected there is not shifted in phase by the reflection, and thus neither is the portion of it that exits the film as ray r_2. We can organize this information with the first line in Table 35-1, which refers to the simplified drawing in Fig. 35-35 for a thin film in air. So far, as a result of the reflection phase shifts, the waves of r_1 and r_2 have a phase difference of 0.5 wavelength and thus are exactly out of phase.

Table 35-1 An Organizing Table for Thin-Film Interference in Air (Fig. 35-35)[a]

Reflection	r_1	r_2
phase shifts	0.5 wavelength	0
Path length difference	$2L$	
Index in which path length difference occurs	n_2	
In phase[a]:	$2L = \frac{\text{odd number}}{2} \times \frac{\lambda}{n_2}$,	
Out of phase[a]:	$2L = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_2}$.	

[a] Valid for $n_2 > n_1$ and $n_2 > n_3$.

Path Length Difference. Now we must consider the path length difference $2L$ that occurs because the wave of ray r_2 crosses the film twice. (This difference $2L$ is shown on the second line in Table 35-1.) If the waves of r_1 and r_2 are to be exactly in phase so that they produce fully constructive interference, the path length $2L$ must cause an additional phase difference of 0.5, 1.5, 2.5,... wavelengths. Only then will the net phase difference be an integer number of wavelengths. Thus, for a bright film, we must have

$$2L = \frac{\text{odd number}}{2} \times \text{wavelength} \quad \text{(in-phase waves).} \tag{35-45}$$

The wavelength we need here is the wavelength λ_{n2} of the light in the medium containing path length $2L$—that is, in the medium with index of refraction n_2. Thus, we can rewrite Eq. 35-45 as

$$2L = \frac{\text{odd number}}{2} \times \lambda_{n2} \quad \text{(in-phase waves).} \tag{35-46}$$

If, instead, the waves are to be exactly out of phase so that there is fully destructive interference, the path length $2L$ must cause either no additional phase difference or a phase difference of 1, 2, 3,... wavelengths. Only then will the net phase difference be an odd number of half-wavelengths. For a dark film, we must have

$$2L = \text{integer} \times \text{wavelength} \quad \text{(out-of-phase waves).} \tag{35-47}$$

where, again, the wavelength is the wavelength λ_{n2} in the medium containing $2L$. Thus, this time we have

$$2L = \text{integer} \times \lambda_{n2} \quad \text{(out-of-phase waves).} \tag{35-48}$$

Now we can use Eq. 35-6 ($\lambda_n = \lambda/n$) to write the wavelength of the wave of ray r_2 inside the film as

$$\lambda_{n2} = \frac{\lambda}{n_2}, \tag{35-49}$$

where λ is the wavelength of the incident light in vacuum (and approximately also in air). Substituting Eq. 35-49 into Eq. 35-46 and replacing "odd number/2" with $(m + 1/2)$ give us

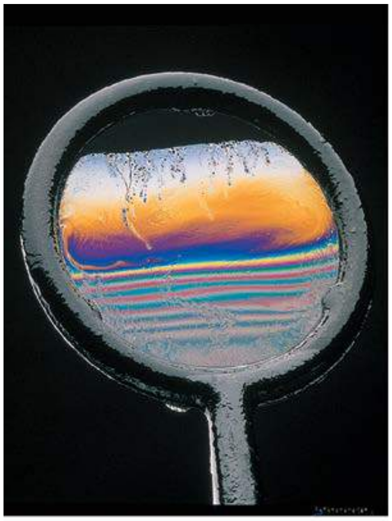
$$2L = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima—bright film in air).} \tag{35-50}$$

Similarly, with m replacing "integer," Eq. 35-48 yields

$$2L = m\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(minima—dark film in air).} \qquad (35\text{-}51)$$

For a given film thickness L, Eqs. 35-50 and 35-51 tell us the wavelengths of light for which the film appears bright and dark, respectively, one wavelength for each value of m. Intermediate wavelengths give intermediate brightnesses. For a given wavelength λ, Eqs. 35-50 and 35-51 tell us the thicknesses of the films that appear bright and dark in that light, respectively, one thickness for each value of m. Intermediate thicknesses give intermediate brightnesses.

Heads Up. (1) For a thin film surrounded by air, Eq. 35-50 corresponds to bright reflections and Eq. 35-51 corresponds to no reflections. For transmissions, the roles of the equations are reversed (after all, if the light is brightly reflected, then it is not transmitted, and vice versa). (2) If we have a different set of values of the indexes of refraction, the roles of the equations may be reversed. For any given set of indexes, you must go through the thought process behind Table 35-1 and, in particular, determine the reflection shifts to see which equation applies to bright reflections and which applies to no reflections. (3) The index of refraction in the equations is that of the thin film, where the path length difference occurs.



Richard Megna/Fundamental Photographs

Figure 35-36 The reflection of light from a soapy water film spanning a vertical loop. The top portion is so thin (due to gravitational slumping) that the light reflected there undergoes destructive interference, making that portion dark. Colored interference fringes, or bands, decorate the rest of the film but are marred by circulation of liquid within the film as the liquid is gradually pulled downward by gravitation.

Film Thickness Much Less Than λ

A special situation arises when a film is so thin that L is much less than λ, say, $L < 0.1\lambda$. Then the path length difference $2L$ can be neglected, and the phase difference between r_1 and r_2 is due *only* to reflection phase shifts. If the film of Fig. 35-35, where the reflections cause a phase difference of 0.5 wavelength, has thickness $L < 0.1\lambda$, then r_1 and r_2 are exactly out of phase, and thus the film is dark, regardless of the wavelength and intensity of the light. This special situation corresponds to $m = 0$ in Eq. 35-51. We shall count *any* thickness $L < 0.1\lambda$ as being the least thickness specified by Eq. 35-49 to make the film of Fig. 35-35 dark. (Every such thickness will correspond to $m = 0$.) The next greater thickness that will make the film dark is that corresponding to $m = 1$.

In Fig. 35-36, bright white light illuminates a vertical soap film whose thickness increases from top to bottom. However, the top portion is so thin that it is dark. In the (somewhat thicker) middle we see fringes, or bands, whose color depends primarily on the wavelength at which reflected light undergoes fully constructive interference for a particular thickness. Toward the (thickest) bottom the fringes become progressively narrower and the colors begin to overlap and fade.

Why is Interference Observed Only in Thin Films?

Interference effects can be ignored with a thick film because it thickness is large. In this case the alternate bright and dark regions will be so close to each other that these will appear to merge into one another, and interference pattern will not be visible.

Further, when we see such a film in white light, the various complementary colors will be so close to each other that these merge into one another and make the appearance of the thick film white. To understand this let us consider film of water (refractive index, $\mu = 4/3$) 1 cm thick on top of a glass surface (RI = 3/2). Assume that it is illuminated from above (Fig. 35-37). Then

$$\Delta x = \text{path difference between 1 and 2} = 2\mu_\omega t.$$

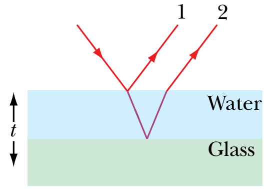


Figure 35-37 A light wave undergoing successive reflections.

For constructive interference in the reflected waves

$$2\mu_\omega t = n\lambda.$$

Because $t = 1$ cm is very large in comparison to the wavelength of visible light, the values of n in this equation will be large. For instance, if wavelength corresponding to orange-red light ($\lambda = 667$ nm) is to be strongly reflected then

$$n = \frac{2\mu_\omega t}{\lambda} = \frac{(2)(1.33)(1\times10^{-2})}{6.67\times10^{-7}} = 40\ 000.$$

We may feel that this thick film would appear strongly orange-red; however, there will also be strong reflection of a slightly longer wavelength corresponding to n = 39,999 and of slightly shorter wavelength corresponding to n = 40,001. These wavelengths differ from each other by only one part in 40,000 or 667 nm/40,000 = 0.17 nm. This is a much smaller wavelength difference than the eye can detect. Hence, no one wavelength appears to be reinforced more than any other in the light which is reflected from a thick film. If the film is illuminated by a white light, the reflected light appears white. For thin films, the integer n will be small, the difference between adjacent strongly reflected wavelengths will be substantial, and the preferential reflection of certain wavelengths will be easily observed by the eye.

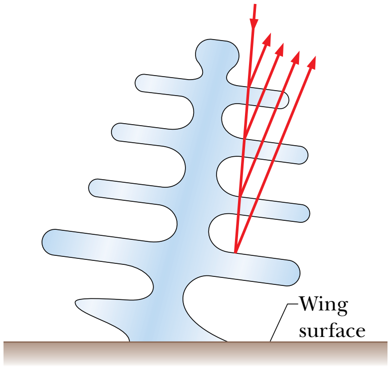


Figure 35-38 Reflecting structure extending up from a Morpho butterfly wing. Reflections from the top surfaces of the transparent "terraces" give an interference color to the wing.

Color Shifting by Butterflies, Inks, and Paints

A surface that displays colors due to thin-film interference is said to be *iridescent* because the tints of the colors change as you change your view of the surface. The iridescence of the top surface of a *Morpho* butterfly wing is due to thin-film interference of light reflected by thin terraces of transparent cuticle-like material on the wing (Fig. 35-38). These terraces are arranged like wide, flat branches on a tree-like structure that extends perpendicular to the wing.

Suppose you look directly down on these terraces as white light shines directly down on the wing. Then the light reflected back up to you from the terraces undergoes fully constructive interference in the blue-green region of the visible spectrum. Light in the yellow and red regions, at the opposite end of the spectrum, is weaker because it undergoes only intermediate interference. Thus, the top surface of the wing looks blue-green to you.

If you intercept light that reflects from the wing in some other direction, the light has traveled along a slanted path through the terraces. Then the wavelength at which there is fully constructive interference is somewhat different from that for light reflected directly upward. Thus, if the wing moves in your view so that the angle at which you view it changes, the color at which the wing is brightest changes somewhat, producing the iridescence of the wing.

The color-shifting inks and paints used on paper currencies, cars, guitars, and other objects function in almost the same way as the color-shifting wing on a *Morpho* butterfly. Figure 35-39*a* shows a cross section of the ink layer used on some currencies. The color shifting is due to thin multilayered flakes suspended in regular ink. Figure 35-39*b* shows a cross section through one of the flakes. Light penetrating the regular ink above the flake travels through thin layers of chromium (Cr), magnesium fluoride (MgF_2), and aluminum (Al). The Cr layers function as weak mirrors, the Al layer functions as a better mirror, and the MgF_2 layers function like soap films. The result is that light reflected upward from each boundary between layers passes back through the regular ink and then undergoes interference at an observer's eye. Which color undergoes fully constructive interference depends on the thickness L of the MgF_2 layers. In Indian currency printed with color-shifting inks, the value of L is designed to give fully constructive interference for green or greenish yellow

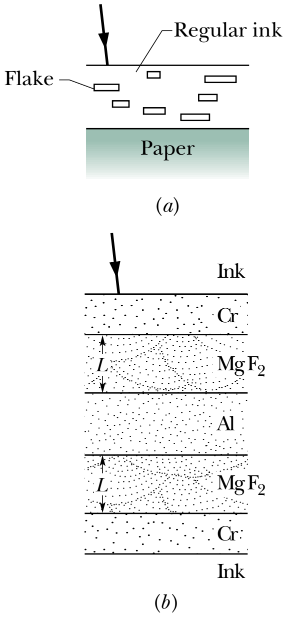


Figure 35-39 (*a*) Color-shifting ink on a paper currency consists of multilayered thin-film flakes suspended in regular ink. (*b*) Cross section of one of the flakes. Light penetrates the five layers, reflecting from each boundary. The color that results from the interference of these reflected light waves is determined by the thickness L of the magnesium fluoride layers.

light when the observer looks directly down on the currency. When the observer tilts the currency and thus each flake, the light reaching the observer from the flakes undergoes constructive interference for blue light. Thus, by changing the angle of view, the observer can shift the color. Other countries use other designs of thin-film flakes to achieve different shifts in the colors on their currencies.

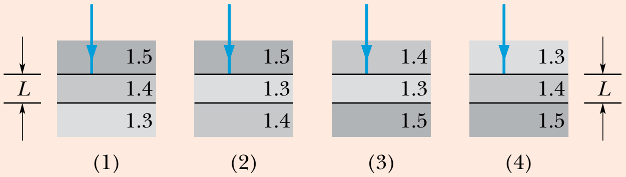
PROBLEM-SOLVING TACTIC

Tactic 1: Thin-Film Equations Some students believe that Eq. 35-50 gives the maxima and Eq. 35-51 gives the minima for *all* thin-film situations. This is not true. These relations were derived only for the situation in which $n_2 > n_1$ and $n_2 > n_3$ in Fig. 35-33 and for the situation of Fig. 35-35, in which a thin film lies in air.

The appropriate equations for other relative values of the indexes of refraction can be derived by following the reasoning of this section and constructing new versions of Table 35-1. In each case you will end up with Eqs. 35-50 and 35-51, but sometimes Eq. 35-50 will give the minima and Eq. 35-51 will give the maxima–the opposite of what we found here. Which equation gives which depends on whether the reflections at the two interfaces give the same reflection phase shift.

CHECKPOINT 5

The figure shows four situations in which light reflects perpendicularly from a thin film of thickness L, with indexes of refraction as given. (a) For which situations does reflection at the film interfaces cause a zero phase difference for the two reflected rays? (b) For which situations will the film be dark if the path length difference $2L$ causes a phase difference of 0.5 wavelength?



SAMPLE PROBLEM 35.13

Thin-film interference of a water film in air

White light, with a uniform intensity across the visible wavelength range of 400 to 690 nm, is perpendicularly incident on a water film, of index of refraction $n_2 = 1.33$ and thickness $L = 320$ nm, that is suspended in air. At what wavelength λ is the light reflected by the film brightest to an observer?

KEY IDEA

The reflected light from the film is brightest at the wavelengths λ for which the reflected rays are in phase with one another. The equation relating these wavelengths λ to the given film thickness L and film index of refraction n_2 is either Eq. 35-50 or Eq. 35-51, depending on the reflection phase shifts for this particular film.

Calculations: To determine which equation is needed, we should fill out an organizing table like Table 35-1. However, because there is air on both sides of the water film, the situation here is exactly like that in Fig. 35-35, and thus the table would be exactly like Table 35-1. Then from Table 35-1, we see that the reflected rays are in phase (and thus the film is brightest) when

$$2L = \frac{\text{odd number}}{2} \times \frac{\lambda}{n_2},$$

which leads to Eq. 35-50:

$$2L = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_2}.$$

Solving for λ and substituting for L and n_2, we find

$$\lambda = \frac{2n_2L}{m+\frac{1}{2}} = \frac{(2)(1.33)(320\text{ nm})}{m+\frac{1}{2}} = \frac{851\text{ nm}}{m+\frac{1}{2}}.$$

For $m = 0$, this gives us $\lambda = 1700$ nm, which is in the infrared region. For $m = 1$, we find $\lambda = 567$ nm, which is yellow-green light, near the middle of the visible spectrum. For $m = 2$, $\lambda = 340$ nm, which is in the ultraviolet region. Thus, the wavelength at which the light seen by the observer is brightest is

$$\lambda = 567\text{ nm}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 35.14

Thin-film interference of a coating on a glass lens

In Fig. 35-40, a glass lens is coated on one side with a thin film of magnesium fluoride (MgF_2) to reduce reflection from the lens surface. The index of refraction of MgF_2 is 1.38; that of the glass is 1.50. What is the least coating thickness that eliminates (via interference) the reflections at the middle of the visible spectrum ($\lambda = 550$ nm)? Assume that the light is approximately perpendicular to the lens surface.

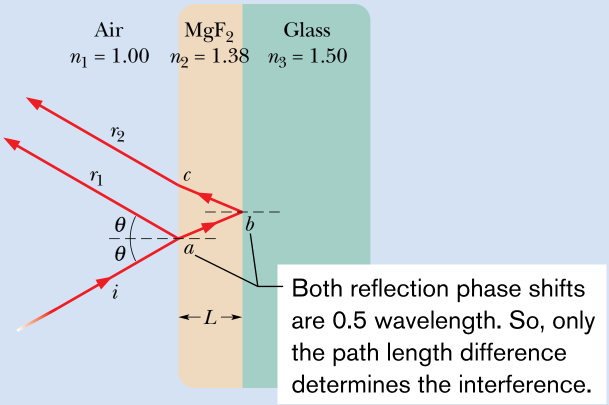

Air
$n_1 = 1.00$
MgF_2
$n_2 = 1.38$
Glass
$n_3 = 1.50$
r_2
r_1
c
b
a
θ
θ
i
L
Both reflection phase shifts are 0.5 wavelength. So, only the path length difference determines the interference.

Figure 35-40 Unwanted reflections from glass can be suppressed (at a chosen wavelength) by coating the glass with a thin transparent film of magnesium fluoride of the properly chosen thickness.

KEY IDEA

Reflection is eliminated if the film thickness L is such that light waves reflected from the two film interfaces are exactly out of phase. The equation relating L to the given wavelength λ and the index of refraction n_2 of the thin film is either Eq. 35-50 or Eq. 35-51, depending on the reflection phase shifts at the interfaces.

Calculations: To determine which equation is needed, we fill out an organizing table like Table 35-1. At the first interface, the incident light is in air, which has a lesser index of refraction than the MgF_2 (the thin film). Thus, we fill in 0.5 wavelength under r_1 in our organizing table (meaning that the waves of ray r_1 are shifted by 0.5λ at the first interface). At the second interface, the incident light is in the MgF_2, which has a lesser index of refraction than the glass on the other side of the interface. Thus, we fill in 0.5 wavelength under r_2 in our table.

Because both reflections cause the same phase shift, they tend to put the waves of r_1 and r_2 in phase. Since we want those waves to be *out of phase*, their path length difference $2L$ must be an odd number of half-wavelengths:

$$2L = \frac{\text{odd number}}{2} \times \frac{\lambda}{n_2}.$$

This leads to Eq. 35-50 (for a bright film sandwiched in air but for a dark film in the arrangement here). Solving that equation for L then gives us the film thicknesses that will eliminate reflection from the lens and coating:

$$L = \left(m + \frac{1}{2}\right)\frac{\lambda}{2n_2}, \quad \text{for } m = 0, 1, 2, \ldots. \qquad (35\text{-}52)$$

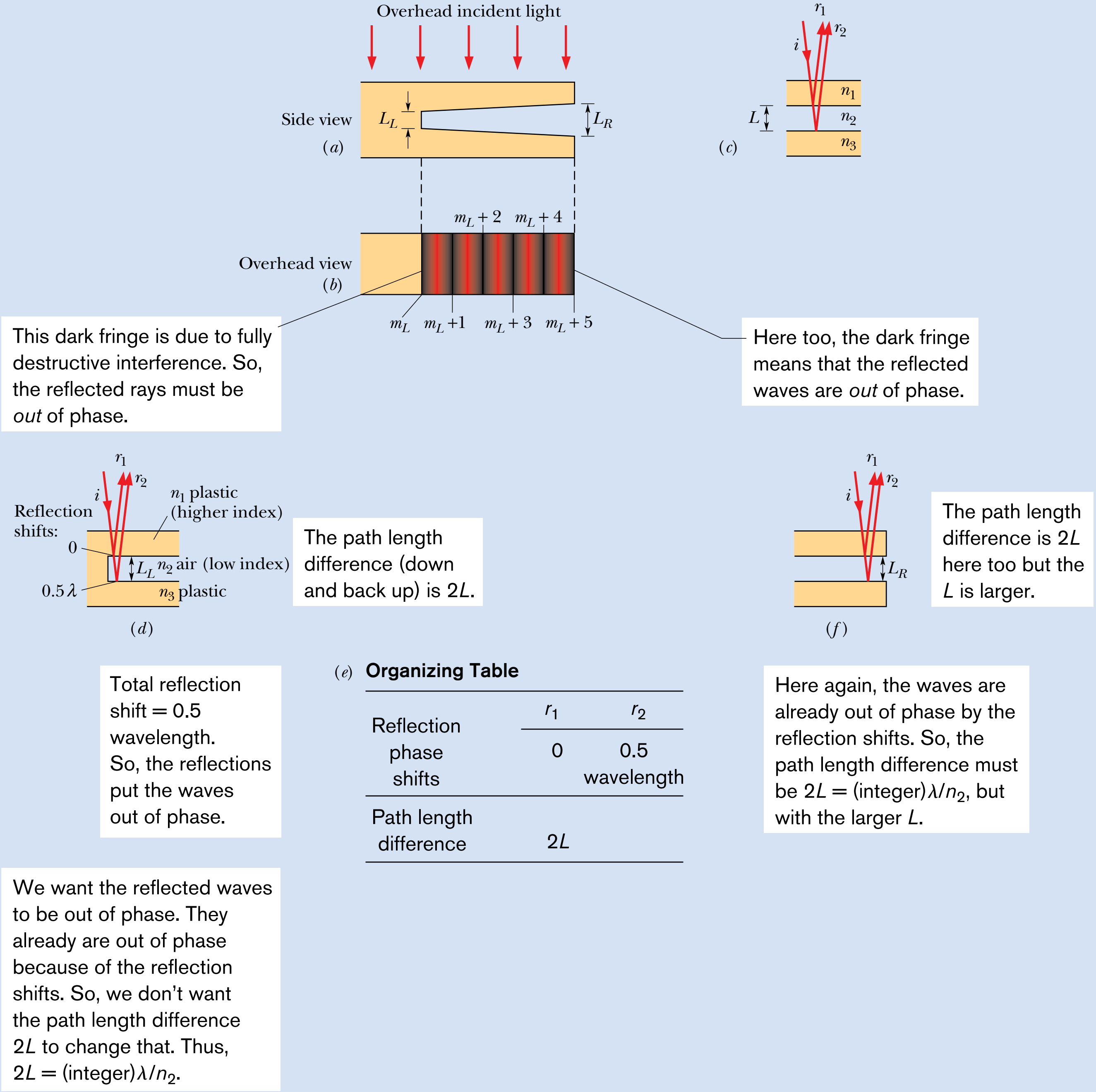
We want the least thickness for the coating—that is, the smallest value of L. Thus, we choose $m = 0$, the smallest possible value of m. Substituting it and the given data in Eq. 35-52, we obtain

$$L = \frac{\lambda}{4n_2} = \frac{550\,\text{nm}}{(4)(1.38)} = 99.6\text{ nm}. \qquad \text{(Answer)}$$

SAMPLE PROBLEM 35.15

Thin-film interference of a transparent wedge

Figure 35-41*a* shows a transparent plastic block with a thin wedge of air at the right. (The wedge thickness is exaggerated in the figure.) A broad beam of red light, with wavelength $\lambda = 632.8$ nm, is directed downward through the top of the block (at an incidence angle of 0°). Some of the light that passes into the plastic is reflected back up from the top and bottom surfaces of the wedge, which acts as a thin film (of air) with a thickness that varies uniformly and gradually from L_L at the left-hand end to L_R at the righthand end. (The plastic layers above and below the wedge of air are too thick to act as thin films.) An observer looking down on the block sees an interference pattern consisting of six dark fringes and five bright red fringes along the wedge. What is the change in thickness ΔL ($= L_R - L_L$) along the wedge?



This dark fringe is due to fully destructive interference. So, the reflected rays must be *out* of phase.

Here too, the dark fringe means that the reflected waves are *out* of phase.

The path length difference (down and back up) is 2*L*.

The path length difference is 2*L* here too but the *L* is larger.

Total reflection shift = 0.5 wavelength. So, the reflections put the waves out of phase.

(*e*) **Organizing Table**

	r_1	r_2
Reflection phase shifts	0	0.5 wavelength
Path length difference	2*L*	

Here again, the waves are already out of phase by the reflection shifts. So, the path length difference must be $2L = (\text{integer})\lambda/n_2$, but with the larger *L*.

We want the reflected waves to be out of phase. They already are out of phase because of the reflection shifts. So, we don't want the path length difference 2*L* to change that. Thus, $2L = (\text{integer})\lambda/n_2$.

Figure 35-41 (*a*) Red light is incident on a thin, air-filled wedge in the side of a transparent plastic block. The thickness of the wedge is L_L at the left end and L_R at the right end. (*b*) The view from above the block: an interference pattern of six dark fringes and five bright red fringes lies over the region of the wedge. (*c*) A representation of the incident ray *i*, reflected rays r_1 and r_2, and thickness *L* of the wedge anywhere along the length of the wedge. The reflection rays at the (*d*) left and (*f*) right ends of the wedge and (*e*) their organizing table.

KEY IDEAS

(1) The brightness at any point along the left–right length of the air wedge is due to the interference of the waves reflected at the top and bottom interfaces of the wedge.

(2) The variation of brightness in the pattern of bright and dark fringes is due to the variation in the thickness of the wedge. In some regions, the thickness puts the reflected waves in phase and thus produces a bright reflection

(a bright red fringe). In other regions, the thickness puts the reflected waves out of phase and thus produces no reflection (a dark fringe).

Organizing the reflections: Because the observer sees more dark fringes than bright fringes, we can assume that a dark fringe is produced at both the left and right ends of the wedge. Thus, the interference pattern is that shown in Fig. 35-41*b*.

We can represent the reflection of light at the top and bottom interfaces of the wedge, at any point along its length, with Fig. 35-41*c*, in which L is the wedge thickness at that point. Let us apply this figure to the left end of the wedge, where the reflections give a dark fringe.

We know that, for a dark fringe, the waves of rays r_1 and r_2 in Fig. 35-41*d* must be out of phase. We also know that the equation relating the film thickness L to the light's wavelength λ and the film's index of refraction n_2 is either Eq. 35-50 or Eq. 35-51, depending on the reflection phase shifts. To determine which equation gives a dark fringe at the left end of the wedge, we should fill out an organizing table like Table 35-1, as shown in Fig. 35-41*e*.

At the top interface of the wedge, the incident light is in the plastic, which has a greater n than the air beneath that interface. So, we fill in 0 under r_1 in our organizing table. At the bottom interface of the wedge, the incident light is in air, which has a lesser n than the plastic beneath that interface. So we fill in 0.5 wavelength under r_2. So, the phase difference due to the reflection shifts is 0.5 wavelength. Thus the reflections alone tend to put the waves of r_1 and r_2 out of phase.

Reflections at left end (Fig. 35-41d): Because we see a dark fringe at the left end of the wedge, which the reflection phase shifts alone would produce, we do not want the path length difference to alter that condition. So, the path length difference $2L$ at the left end must be given by

$$2L = \text{integer} \times \frac{\lambda}{n_2},$$

which leads to Eq. 35-49:

$$2L = m\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots. \tag{35-53}$$

Reflections at right end (Fig. 35-41f): Equation 35-53 holds not only for the left end of the wedge but also for any point along the wedge where a dark fringe is observed, including the right end, with a different integer value of m for each fringe. The least value of m is associated with the least thickness of the wedge where a dark fringe is observed. Progressively greater values of m are associated with progressively greater thicknesses of the wedge where a dark fringe is observed. Let m_L be the value at the left end. Then the value at the right end must be $m_L + 5$ because, from Fig. 35-41*b*, the right end is located at the fifth dark fringe from the left end.

Thickness difference: To find ΔL, we first solve Eq. 35-53 twice—once for the thickness L_L at the left end and once for the thickness L_R at the right end:

$$L_L = (m_L)\frac{\lambda}{2n_2}, \quad L_R = (m_L + 5)\frac{\lambda}{2n_2}. \tag{35-54}$$

We can now subtract L_L from L_R and substitute $n_2 = 1.00$ for the air within the wedge and $\lambda = 632.8 \times 10^{-9}$ m:

$$\Delta L = L_R - L_L = \frac{(m_L + 5)\lambda}{2n_2} - \frac{m_L\lambda}{2n_2} = \frac{5}{2}\frac{\lambda}{n_2}$$
$$= 1.58 \times 10^{-6}\,\text{m}. \qquad \text{(Answer)}$$

35.9 | DIFFRACTION AND THE WAVE THEORY OF LIGHT

Key Concept

- When waves encounter an edge, an obstacle, or an aperture the size of which is comparable to the wavelength of the waves, those waves spread out as they travel and, as a result, undergo interference. This type of interference is called diffraction.

We defined diffraction rather loosely as the flaring of light as it emerges from a narrow slit. More than just flaring occurs, however, because the light produces an interference pattern called a **diffraction pattern**. For example, when monochromatic light from a distant source (or a laser) passes through a narrow slit and is then intercepted by a viewing screen, the light produces on the screen a diffraction pattern like that in Fig. 35-42. This pattern consists of a broad and intense (very bright) central maximum plus a number of narrower and less intense maxima (called

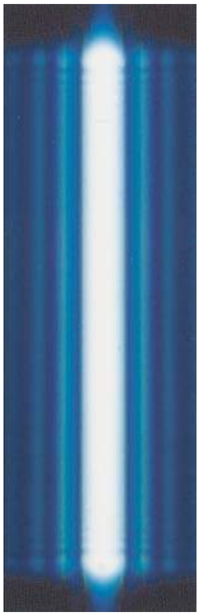
secondary or **side** maxima) to both sides. In between the maxima are minima. Light flares into those dark regions, but the light waves cancel out one another.

Such a pattern would be totally unexpected in geometrical optics: If light traveled in straight lines as rays, then the slit would allow some of those rays through to form a sharp rendition of the slit on the viewing screen instead of a pattern of bright and dark bands as we see in Fig. 35-42. As in Chapter 35, we must conclude that geometrical optics is only an approximation.

Edges. Diffraction is not limited to situations in which light passes through a narrow opening (such as a slit or pinhole). It also occurs when light passes an edge, such as the edges of the razor blade whose diffraction pattern is shown in Fig. 35-43. Note the lines of maxima and minima that run approximately parallel to the edges, at both the inside edges of the blade and the outside edges. As the light passes, say, the vertical edge at the left, it flares left and right and undergoes interference, producing the pattern along the left edge. The rightmost portion of that pattern actually lies behind the blade, within what would be the blade's shadow if geometrical optics prevailed.

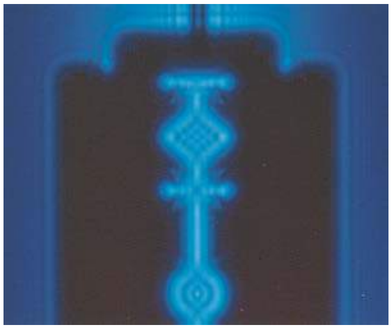
Floaters. You encounter a common example of diffraction when you look at a clear blue sky and see tiny specks and hairlike structures floating in your view. These *floaters*, as they are called, are produced when light passes the edges of tiny deposits in the vitreous humor, the transparent material filling most of the eyeball. What you are seeing when a floater is in your field of vision is the diffraction pattern produced on the retina by one of these deposits. If you sight through a pinhole in a piece of cardboard so as to make the light entering your eye approximately a plane wave, you can distinguish individual maxima and minima in the patterns.

Cheerleaders. Diffraction is a wave effect. That is, it occurs because light is a wave and it occurs with other types of waves as well. For example, you have probably seen diffraction in action at football games. When a cheerleader near the playing field yells up at several thousand noisy fans, the yell can hardly be heard because the sound waves diffract when they pass through the narrow opening of the cheerleader's mouth. This flaring leaves little of the waves traveling toward the fans in front of the cheerleader. To offset the diffraction, the cheerleader can yell through a megaphone. The sound waves then emerge from the much wider opening at the end of the megaphone. The flaring is thus reduced, and much more of the sound reaches the fans in front of the cheerleader.



Ken Kay/Fundamental Photographs

Figure 35-42 This diffraction pattern appeared on a viewing screen when light that had passed through a narrow vertical slit reached the screen. Diffraction caused the light to flare out perpendicular to the long sides of the slit. That flaring produced an interference pattern consisting of a broad central maximum plus less intense and narrower secondary (or side) maxima, with minima between them.



Ken Kay/Fundamental Photographs

Figure 35-43 The diffraction pattern produced by a razor blade in monochromatic light. Note the lines of alternating maximum and minimum intensity.

35.10 | DIFFRACTION BY A SINGLE SLIT: LOCATING THE MINIMA

Key Concepts

- Waves passing through a long narrow slit of width a produce, on a viewing screen, a single-slit diffraction pattern that includes a central maximum (bright fringe) and other maxima. They are separated by minima that are located relative to the central axis by angles θ:

$$a \sin \theta = m\lambda, \quad \text{for } m = 1, 2, 3, \ldots \quad \text{(minima)}.$$

- The maxima are located approximately halfway between minima.

Let us now examine the diffraction pattern of plane waves of light of wavelength λ that are diffracted by a single long, narrow slit of width a in an otherwise opaque screen B, as shown in cross section in Fig. 35-44. (In that figure, the slit's length extends into and out of the page, and the incoming wavefronts are parallel to screen B.) When the diffracted light reaches viewing screen C, waves from different points within the slit undergo interference and produce a diffraction pattern of bright and dark fringes (interference maxima and minima) on the screen. To locate the fringes, we shall use a procedure somewhat similar to the one we used to locate the fringes in a two-slit interference pattern. However, diffraction is more mathematically challenging, and here we shall be able to find equations for only the dark fringes.

Before we do that, however, we can justify the central bright fringe seen in Fig. 35-42 by noting that the Huygens wavelets from all points in the slit travel about the same distance to reach the center of the pattern and thus are in phase there. As for the other bright fringes, we can say only that they are approximately halfway between adjacent dark fringes.

Pairings. To find the dark fringes, we shall use a clever (and simplifying) strategy that involves pairing up all the rays coming through the slit and then finding what conditions cause the wavelets of the rays in each pair to cancel each other. We apply this strategy in Fig. 35-44 to locate the first dark fringe, at point P_1. First, we mentally divide the slit into two *zones* of equal widths $a/2$. Then we extend to P_1 a light ray r_1 from the top point of the top zone and a light ray r_2 from the top point of the bottom zone. We want the wavelets along these two rays to cancel each other when they arrive at P_1. Then any similar pairing of rays from the two zones will give cancellation. A central axis is drawn from the center of the slit to screen C, and P_1 is located at an angle θ to that axis.

This pair of rays cancel each other at P_1. So do all such pairings.

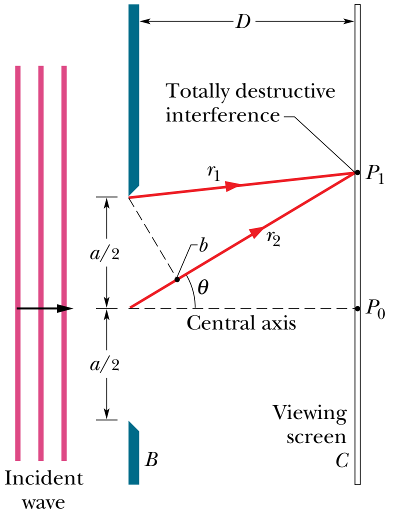
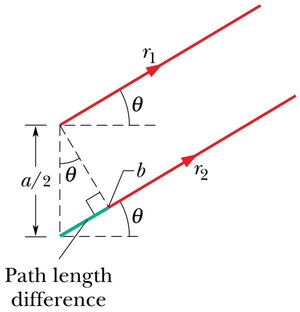


Figure 35-44 Waves from the top points of two zones of width $a/2$ undergo fully destructive interference at point P_1 on viewing screen C.

Path Length Difference. The wavelets of the pair of rays r_1 and r_2 are in phase within the slit because they originate from the same wavefront passing through the slit, along the width of the slit. However, to produce the first dark fringe they must be out of phase by $\lambda/2$ when they reach P_1; this phase difference is due to their path length difference, with the path traveled by the wavelet of r_2 to reach P_1 being longer than the path traveled by the wavelet of r_1. To display this path length difference, we find a point b on ray r_2 such that the path length from b to P_1 matches the path length of ray r_1. Then the path length difference between the two rays is the distance from the center of the slit to b.

When viewing screen C is near screen B, as in Fig. 35-44, the diffraction pattern on C is difficult to describe mathematically. However, we can simplify the mathematics considerably if we arrange for the screen separation D to be much larger than the slit width a. Then, as in Fig. 35-45, we can approximate rays r_1 and r_2 as being parallel, at angle θ to the central axis. We can also approximate the triangle formed by point b, the top point of the slit, and the center point of the slit as being a right triangle, and one of the angles inside that triangle as being θ. The path length difference between rays r_1 and r_2 (which is still the distance from the center of the slit to point b) is then equal to $(a/2)\sin\theta$.

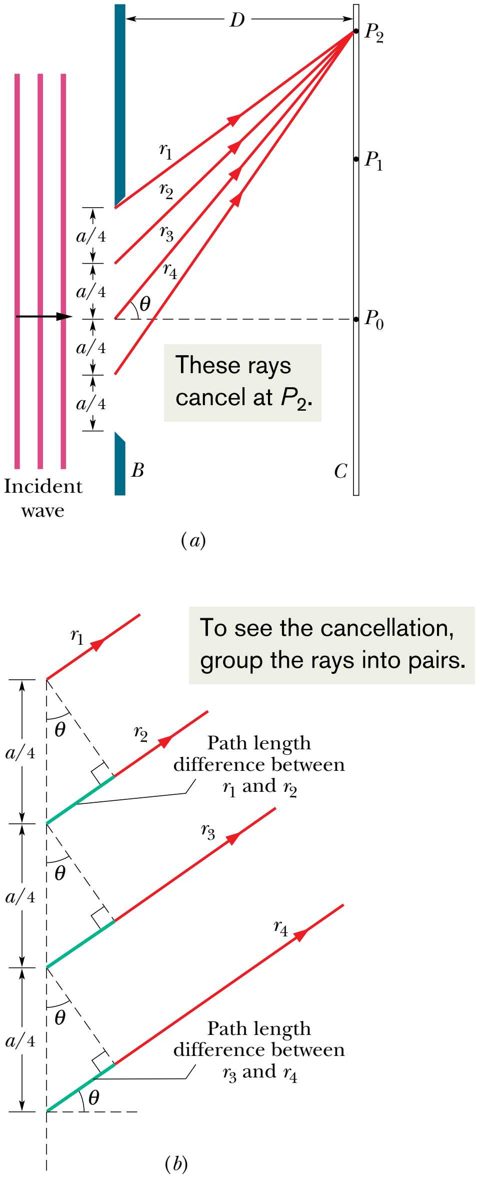


This path length difference shifts one wave from the other, which determines the interference.

Figure 35-45 For $D \gg a$, we can approximate rays r_1 and r_2 as being parallel, at angle θ to the central axis.

First Minimum. We can repeat this analysis for any other pair of rays originating at corresponding points in the two zones (say, at the midpoints of the zones) and extending to point P_1. Each such pair of rays has the same path length difference $(a/2)\sin\theta$. Setting this common path length difference equal to $\lambda/2$ (our condition for the first dark fringe), we have

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2},$$



These rays cancel at P_2.

To see the cancellation, group the rays into pairs.

Figure 35-46 (*a*) Waves from the top points of four zones of width $a/4$ undergo fully destructive interference at point P_2. (*b*) For $D \gg a$, we can approximate rays r_1, r_2, r_3, and r_4 as being parallel, at angle θ to the central axis.

which gives us

$$a \sin \theta = \lambda \quad \text{(first minimum).} \tag{35-55}$$

Given slit width a and wavelength λ, Eq. 35-53 tells us the angle θ of the first dark fringe above and (by symmetry) below the central axis.

Narrowing the Slit. Note that if we begin with $a > \lambda$ and then narrow the slit while holding the wavelength constant, we increase the angle at which the first dark fringes appear; that is, the extent of the diffraction (the extent of the flaring and the width of the pattern) is *greater* for a *narrower* slit. When we have reduced the slit width to the wavelength (that is, $a = \lambda$), the angle of the first dark fringes is 90°. Since the first dark fringes mark the two edges of the central bright fringe, that bright fringe must then cover the entire viewing screen.

Second Minimum. We find the second dark fringes above and below the central axis as we found the first dark fringes, except that we now divide the slit into *four* zones of equal widths $a/4$, as shown in Fig. 35-46*a*. We then extend rays r_1, r_2, r_3, and r_4 from the top points of the zones to point P_2, the location of the second dark fringe above the central axis. To produce that fringe, the path length difference between r_1 and r_2, that between r_2 and r_3, and that between r_3 and r_4 must all be equal to $\lambda/2$.

For $D \gg a$, we can approximate these four rays as being parallel, at angle θ to the central axis. To display their path length differences, we extend a perpendicular line through each adjacent pair of rays, as shown in Fig. 35-46*b*, to form a series of right triangles, each of which has a path length difference as one side. We see from the top triangle that the path length difference between r_1 and r_2 is $(a/4) \sin \theta$. Similarly, from the bottom triangle, the path length difference between r_3 and r_4 is also $(a/4) \sin \theta$. In fact, the path length difference for any two rays that originate at corresponding points in two adjacent zones is $(a/4) \sin \theta$. Since in each such case the path length difference is equal to $\lambda/2$, we have

$$\frac{a}{4} \sin \theta = \frac{\lambda}{2},$$

which gives us

$$a \sin \theta = 2\lambda \quad \text{(second minimum).} \tag{35-56}$$

All Minima. We could now continue to locate dark fringes in the diffraction pattern by splitting up the slit into more zones of equal width. We would always choose an even number of zones so that the zones (and their waves) could be paired as we have been doing. We would find that the dark fringes above and below the central axis can be located with the general equation

$$a \sin \theta = m\lambda, \quad \text{for } m = 1, 2, 3, \ldots \quad \text{(minima—dark fringes).} \tag{35-57}$$

You can remember this result in the following way. Draw a triangle like the one in Fig. 35-45, but for the full slit width a, and note that the path length difference between the top and bottom rays equals $a \sin \theta$. Thus, Eq. 35-57 says:

In a single-slit diffraction experiment, dark fringes are produced where the path length differences ($a \sin \theta$) between the top and bottom rays are equal to $\lambda, 2\lambda, 3\lambda, \ldots$.

This may seem to be wrong because the waves of those two particular rays will be exactly in phase with each other when their path length difference is an integer number of wavelengths. However, they each will still be part of a pair of waves that are exactly out of phase with each other; thus, *each* wave will be canceled by some other wave, resulting in darkness. (Two light waves that are exactly out of phase will always cancel each other, giving a net wave of zero, even if they happen to be exactly in phase with other light waves.)

Using a Lens. Equations 35-53, 35-54, and 35-57 are derived for the case of $D \gg a$. However, they also apply if we place a converging lens between the slit and the viewing screen and then move the screen in so that it coincides with the focal plane of the lens. The lens ensures that rays which now reach any point on the screen are *exactly* parallel (rather than approximately) back at the slit. They are like the initially parallel rays of Fig. 34-34*a* that are directed to the focal point by a converging lens.

☑ CHECKPOINT 6

We produce a diffraction pattern on a viewing screen by means of a long narrow slit illuminated by blue light. Does the pattern expand away from the bright center (the maxima and minima shift away from the center) or contract toward it if we (a) switch to yellow light or (b) decrease the slit width?

SAMPLE PROBLEM 35.16

Single-slit diffraction pattern with white light

A slit of width a is illuminated by white light.

(a) For what value of a will the first minimum for red light of wavelength $\lambda = 650$ nm appear at $\theta = 15°$?

KEY IDEA

Diffraction occurs separately for each wavelength in the range of wavelengths passing through the slit, with the locations of the minima for each wavelength given by Eq. 35-57 ($a \sin\theta = m\lambda$).

Calculation: When we set $m = 1$ (for the first minimum) and substitute the given values of θ and λ, Eq. 35-54 yields

$$a = \frac{m\lambda}{\sin\theta} = \frac{(1)(650\,\text{nm})}{\sin 15°} = 2511\text{ nm} \approx 2.5\,\mu\text{m}. \qquad \text{(Answer)}$$

For the incident light to flare out that much (±15° to the first minima) the slit has to be very fine indeed—in this case, a mere four times the wavelength. For comparison, note that a fine human hair may be about 100 μm in diameter.

(b) What is the wavelength λ' of the light whose first side diffraction maximum is at 15°, thus coinciding with the first minimum for the red light?

KEY IDEA

The first side maximum for any wavelength is about halfway between the first and second minima for that wavelength.

Calculations: Those first and second minima can be located with Eq. 35-57 by setting $m = 1$ and $m = 2$, respectively. Thus, the first side maximum can be located *approximately* by setting $m = 1.5$. Then Eq. 35-57 becomes

$$a \sin\theta = 1.5\lambda'.$$

Solving for λ' and substituting known data yield

$$\lambda' = \frac{a \sin\theta}{1.5} = \frac{(2511\text{ nm})(\sin 15°)}{1.5} = 430\text{ nm}. \qquad \text{(Answer)}$$

Light of this wavelength is violet (far blue, near the short-wavelength limit of the human range of visible light). From the two equations we used, can you see that the first side maximum for light of wavelength 430 nm will always coincide with the first minimum for light of wavelength 650 nm, no matter what the slit width is? However, the angle θ at which this overlap occurs does depend on slit width. If the slit is relatively narrow, the angle will be relatively large, and conversely.

35.11 | INTENSITY IN SINGLE-SLIT DIFFRACTION, QUANTITATIVELY

Key Concept

- The intensity of the diffraction pattern at any given angle θ is

$$I(\theta) = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2,$$

where I_m is the intensity at the center of the pattern and

$$\alpha = \frac{\pi a}{\lambda}\sin\theta.$$

Equation 35-57 tells us how to locate the minima of the single-slit diffraction pattern on screen C of Fig. 35-44 as a function of the angle θ in that figure. Here we wish to derive an expression for the intensity $I(\theta)$ of the pattern as a function of θ. We state, and shall prove below, that the intensity is given by

$$I(\theta) = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2, \tag{35-58}$$

where

$$\alpha = \frac{1}{2}\phi = \frac{\pi a}{\lambda}\sin\theta. \tag{35-59}$$

The symbol α is just a convenient connection between the angle θ that locates a point on the viewing screen and the light intensity $I(\theta)$ at that point. The intensity I_m is the greatest value of the intensities $I(\theta)$ in the pattern and occurs at the central maximum (where $\theta = 0$), and ϕ is the phase difference (in radians) between the top and bottom rays from the slit of width a.

Study of Eq. 35-56 shows that intensity minima will occur where

$$\alpha = m\pi, \quad \text{for } m = 1, 2, 3, \ldots. \tag{35-60}$$

If we put this result into Eq. 35-57, we find

$$m\pi = \frac{\pi a}{\lambda}\sin\theta, \quad \text{for } m = 1, 2, 3, \ldots,$$

or

$$a\sin\theta = m\lambda, \quad \text{for } m = 1, 2, 3, \ldots \quad \text{(minima—dark fringes)}, \tag{35-61}$$

which is exactly Eq. 35-57, the expression that we derived earlier for the location of the minima.

Plots. Figure 35-47 shows plots of the intensity of a single-slit diffraction pattern, calculated with Eqs. 35-58 and 35-59 for three slit widths: $a = \lambda$, $a = 5\lambda$, and $a = 10\lambda$. Note that as the slit width increases (relative to the wavelength), the width of the *central diffraction maximum* (the central hill-like region of the graphs) decreases; that is, the light undergoes less flaring by the slit. The secondary maxima also decrease in width (and become weaker). In the limit of slit width a being much greater than wavelength λ, the secondary maxima due to the slit disappear; we then no longer have single-slit diffraction (but we still have diffraction due to the edges of the wide slit, like that produced by the edges of the razor blade in Fig. 35-43).

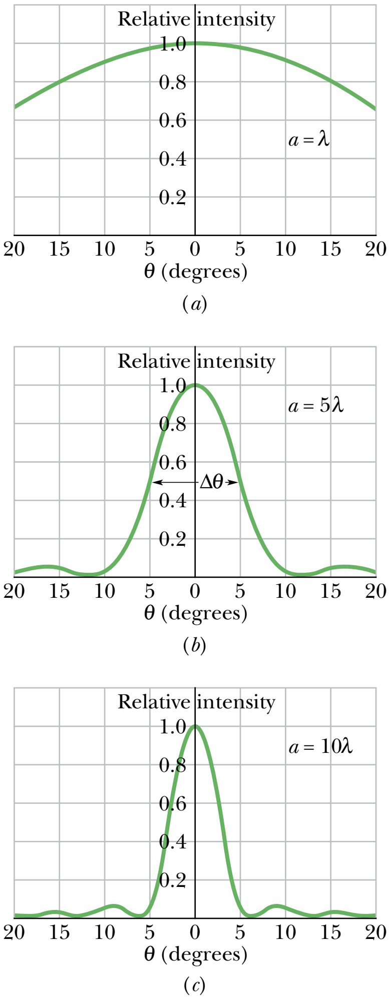


Figure 35-47 The relative intensity in single-slit diffraction for three values of the ratio a/λ. The wider the slit is, the narrower is the central diffraction maximum.

SAMPLE PROBLEM 35.17

Intensities of the maxima in a single-slit interference pattern

Find the intensities of the first three secondary maxima (side maxima) in the single-slit diffraction pattern of Fig. 35-42, measured as a percentage of the intensity of the central maximum.

KEY IDEAS

The secondary maxima lie approximately halfway between the minima, whose angular locations are given by Eq. 35-60 ($\alpha = m\pi$). The locations of the secondary maxima are then given (approximately) by

$$a = \left(m + \frac{1}{2}\right)\pi, \quad \text{for } m = 1, 2, 3, \ldots,$$

with α in radian measure. We can relate the intensity I at any point in the diffraction pattern to the intensity I_m of the central maximum via Eq. 35-58.

Calculations: Substituting the approximate values of α for the secondary maxima into Eq. 35-58 to obtain the relative intensities at those maxima, we get

$$\frac{I}{I_m} = \left(\frac{\sin\alpha}{\alpha}\right)^2 = \left(\frac{\sin\left(m + \frac{1}{2}\right)\pi}{\left(m + \frac{1}{2}\right)\pi}\right)^2, \quad \text{for } m = 1, 2, 3, \ldots.$$

The first of the secondary maxima occurs for $m = 1$, and its relative intensity is

$$\frac{I_1}{I_m} = \left(\frac{\sin\left(1 + \frac{1}{2}\right)\pi}{\left(1 + \frac{1}{2}\right)\pi}\right)^2 = \left(\frac{\sin 1.5\pi}{1.5\pi}\right)^2$$

$$= 4.50 \times 10^{-2} \approx 4.5\%. \qquad \text{(Answer)}$$

For $m = 2$ and $m = 3$, we find that

$$\frac{I_2}{I_m} = 1.6\% \quad \text{and} \quad \frac{I_3}{I_m} = 0.83\%. \qquad \text{(Answer)}$$

As you can see from these results, successive secondary maxima decrease rapidly in intensity. Figure 35-42 was deliberately overexposed to reveal them.

35.12 | DIFFRACTION BY A CIRCULAR APERTURE

Key Concepts

- Diffraction by a circular aperture or a lens with diameter d produces a central maximum and concentric maxima and minima, with the first minimum at an angle θ given by

$$\sin\theta = 1.22\frac{\lambda}{d} \quad \text{(first minimum—circular aperture).}$$

- Rayleigh's criterion suggests that two objects are on the verge of resolvability if the central diffraction maximum of one is at the first minimum of the other. Their angular separation can then be no less than

$$\theta_R = 1.22\frac{\lambda}{d} \quad \text{(Rayleigh's criterion),}$$

in which d is the diameter of the aperture through which the light passes.

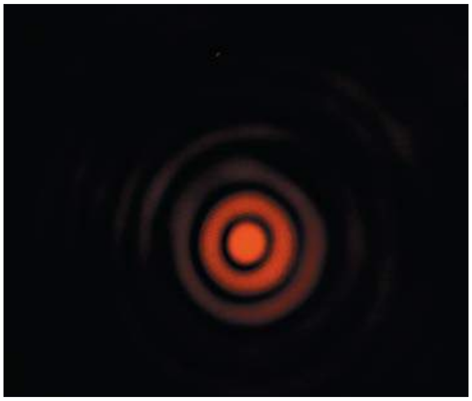
Here we consider diffraction by a circular aperture—that is, a circular opening, such as a circular lens, through which light can pass. Figure 35-48 shows the image formed by light from a laser that was directed onto a circular aperture with a very small diameter. This image is not a point, as geometrical optics would suggest, but a circular disk surrounded by several progressively fainter secondary rings. Comparison with Fig. 35-42 leaves little doubt that we are dealing with a diffraction phenomenon. Here, however, the aperture is a circle of diameter d rather than a rectangular slit.

The (complex) analysis of such patterns shows that the first minimum for the diffraction pattern of a circular aperture of diameter d is located by

$$\sin\theta = 1.22\frac{\lambda}{d} \quad \text{(first minimum—circular aperture).} \qquad (35\text{-}62)$$

The angle θ here is the angle from the central axis to any point on that (circular) minimum. Compare this with Eq. 35-55,

$$\sin\theta = \frac{\lambda}{a} \quad \text{(first minimum—single slit),} \qquad (35\text{-}63)$$



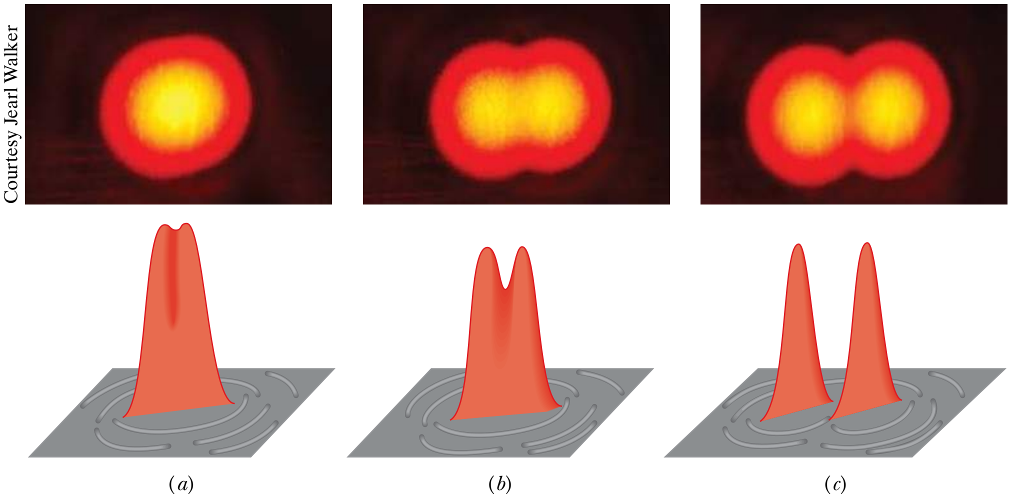
Courtesy Jearl Walker

Figure 35-48 The diffraction pattern of a circular aperture. Note the central maximum and the circular secondary maxima. The figure has been overexposed to bring out these secondary maxima, which are much less intense than the central maximum.

which locates the first minimum for a long narrow slit of width a. The main difference is the factor 1.22, which enters because of the circular shape of the aperture.

Resolvability

The fact that lens images are diffraction patterns is important when we wish to *resolve* (distinguish) two distant point objects whose angular separation is small. Figure 35-49 shows, in three different cases, the visual appearance and corresponding intensity pattern for two distant point objects (stars, say) with small angular separation. In Figure 35-49*a*, the objects are not resolved because of diffraction; that is, their diffraction patterns (mainly their central maxima) overlap so much that the two objects cannot be distinguished from a single point object. In Fig. 35-49*b* the objects are barely resolved, and in Fig. 35-49*c* they are fully resolved.


Courtesy Jearl Walker

(*a*) (*b*) (*c*)

Figure 35-49 At the top, the images of two point sources (stars) formed by a converging lens. At the bottom, representations of the image intensities. In (*a*) the angular separation of the sources is too small for them to be distinguished, in (*b*) they can be marginally distinguished, and in (*c*) they are clearly distinguished. Rayleigh's criterion is satisfied in (*b*), with the central maximum of one diffraction pattern coinciding with the first minimum of the other.

In Fig. 35-49*b* the angular separation of the two point sources is such that the central maximum of the diffraction pattern of one source is centered on the first minimum of the diffraction pattern of the other, a condition called **Rayleigh's criterion** for resolvability. From Eq. 35-62, two objects that are barely resolvable by this criterion must have an angular separation θ_R of

$$\theta_R \sin^{-1}\frac{1.22\lambda}{d}.$$

Since the angles are small, we can replace $\sin\theta_R$ with θ_R expressed in radians:

$$\theta_R = 1.22\frac{\lambda}{d} \quad \text{(Rayleigh's criterion).} \tag{35-64}$$

Human Vision. Applying Rayleigh's criterion for resolvability to human vision is only an approximation because visual resolvability depends on many factors, such as the relative brightness of the sources and their surroundings, turbulence in the air between the sources and the observer, and the functioning of the observer's visual system. Experimental results show that the least angular separation that can actually be resolved by a person is generally somewhat greater than the value given by Eq. 35-64. However, for calculations here, we shall take Eq. 35-64 as being a precise criterion: If the angular separation θ between the sources is greater than θ_R, we can visually resolve the sources; if it is less, we cannot.

When we wish to use a lens instead of our visual system to resolve objects of small angular separation, it is desirable to make the diffraction pattern as small as possible. According to Eq. 35-64, this can be done either by increasing the lens diameter or by using light of a shorter wavelength. For this reason ultraviolet light is often used with microscopes because its wavelength is shorter than a visible light wavelength.

CHECKPOINT 7

Suppose that you can barely resolve two red dots because of diffraction by the pupil of your eye. If we increase the general illumination around you so that the pupil decreases in diameter, does the resolvability of the dots improve or diminish? Consider only diffraction. (You might experiment to check your answer.)

SAMPLE PROBLEM 35.18

Rayleigh's criterion for resolving two distant objects

A circular converging lens, with diameter d = 32 mm and focal length $f = 24$ cm, forms images of distant point objects in the focal plane of the lens. The wavelength is $\lambda = 550$ nm.

(a) Considering diffraction by the lens, what angular separation must two distant point objects have to satisfy Rayleigh's criterion?

KEY IDEA

Figure 35-50 shows two distant point objects P_1 and P_2, the lens, and a viewing screen in the focal plane of the lens. It also shows, on the right, plots of light intensity I versus position on the screen for the central maxima of the images formed by the lens. Note that the angular separation θ_o of the objects equals the angular separation θ_i of the images. Thus, if the images are to satisfy Rayleigh's criterion, these separations must be given by Eq. 35-64 (for small angles).

Calculations: From Eq. 35-62, we obtain

$$\theta_o = \theta_i = \theta_R = 1.22\frac{\lambda}{d}$$

$$= \frac{(1.22)(550\times10^{-9}\,\text{m})}{32\times10^{-3}\,\text{m}} = 2.1\times10^{-5}\,\text{rad.} \quad \text{(Answer)}$$

Each central maximum in the two intensity curves of Fig. 35-50 is centered on the first minimum of the other curve.

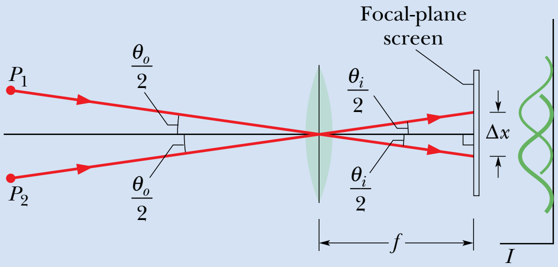


Figure 35-50 Light from two distant point objects P_1 and P_2 passes through a converging lens and forms images on a viewing screen in the focal plane of the lens. Only one representative ray from each object is shown. The images are not points but diffraction patterns, with intensities approximately as plotted at the right.

(b) What is the separation Δx of the centers of the *images* in the focal plane? (That is, what is the separation of the *central* peaks in the two intensity-versus-position curves?)

Calculations: From either triangle between the lens and the screen in Fig. 35-50, we see that $\tan \theta_i/2 = \Delta x/2f$. Rearranging this equation and making the approximation $\tan \theta \approx \theta$, we find

$$\Delta x = f\theta_i, \tag{35-65}$$

where θ_i is in radian measure. We then find

$$\Delta x = (0.24\text{ m})(2.1\times10^{-5}\text{ rad}) = 5.0\ \mu\text{m}. \quad \text{(Answer)}$$

35.13 | DIFFRACTION BY A DOUBLE SLIT

Key Concepts

- Waves passing through two slits produce a combination of double-slit interference and diffraction by each slit.
- For identical slits with width a and center-to-center separation d, the intensity in the pattern varies with the angle θ from the central axis as

$$I(\theta) = I_m(\cos^2\beta)\left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \text{(double slit)},$$

where I_m is the intensity at the center of the pattern,

$$\beta = \left(\frac{\pi d}{\lambda}\right)\sin\theta,$$

and

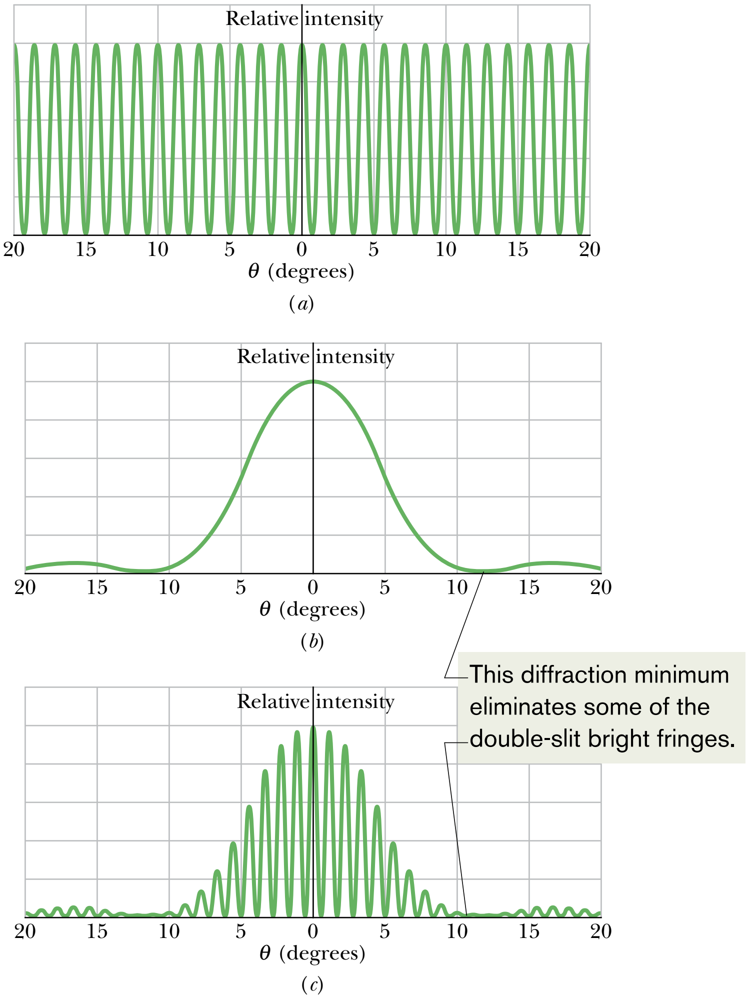
$$\alpha = \left(\frac{\pi a}{\lambda}\right)\sin\theta.$$

In the double-slit experiments earlier in this chapter, we implicitly assumed that the slits were much narrower than the wavelength of the light illuminating them; that is, $a \ll \lambda$. For such narrow slits, the central maximum of the diffraction pattern of either slit covers the entire viewing screen. Moreover, the interference of light from the two slits produces bright fringes with approximately the same intensity (Fig. 35-18).

In practice with visible light, however, the condition $a \ll \lambda$ is often not met. For relatively wide slits, the interference of light from two slits produces bright fringes that do not all have the same intensity. That is, the intensities of the fringes produced by double-slit interference (as discussed in Section 35.6) are modified by diffraction of the light passing through each slit (as discussed in this chapter).

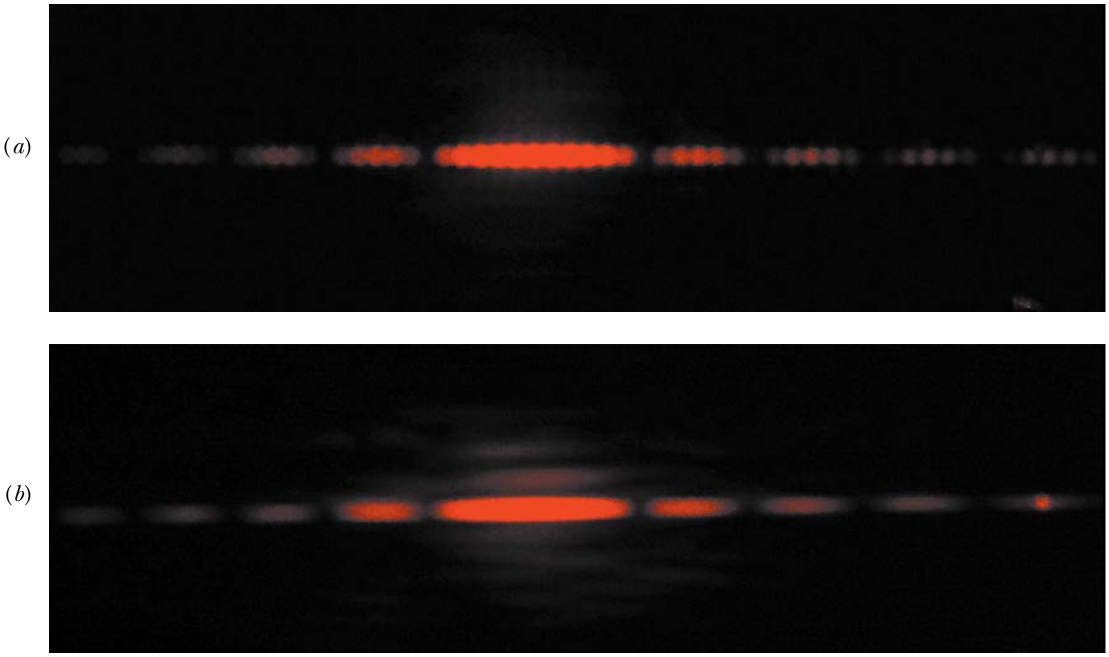
Plots. As an example, the intensity plot of Fig. 35-51*a* suggests the doubleslit interference pattern that would occur if the slits were infinitely narrow (and thus $a \ll \lambda$); all the bright interference fringes would have the same intensity. The intensity plot of Fig. 35-51*b* is that for diffraction by a single actual slit; the diffraction pattern has a broad central maximum and weaker secondary maxima at ±17°. The plot of Fig. 35-51*c* suggests the interference pattern for two actual slits. That plot was constructed by using the curve of Fig. 35-51*b* as an *envelope* on the intensity plot in Fig. 35-51*a*. The positions of the fringes are not changed; only the intensities are affected.

Photos. Figure 35-52*a* shows an actual pattern in which both double-slit interference and diffraction are evident. If one slit is covered, the single-slit diffraction pattern of Fig. 35-52*b* results. Note the correspondence between Figs. 35-52*a* and 35-51*c*, and between Figs. 35-52*b* and 35-51*b*. In comparing these figures, bear in mind that Fig. 35-52 has been deliberately overexposed to bring out the faint secondary maxima and that several secondary maxima (rather than one) are shown.



This diffraction minimum eliminates some of the double-slit bright fringes.

Figure 35-51 (*a*) The intensity plot to be expected in a double-slit interference experiment with vanishingly narrow slits. (*b*) The intensity plot for diffraction by a typical slit of width *a* (not vanishingly narrow). (*c*) The intensity plot to be expected for two slits of width *a*. The curve of (*b*) acts as an envelope, limiting the intensity of the double-slit fringes in (*a*). Note that the first minima of the diffraction pattern of (*b*) eliminate the double-slit fringes that would occur near 12° in (*c*).



Courtesy Jearl Walker

Figure 35-52 (*a*) Interference fringes for an actual double-slit system; compare with Fig. 35-51*c*. (*b*) The diffraction pattern of a single slit; compare with Fig. 35-51*b*.

Intensity. With diffraction effects taken into account, the intensity of a doubleslit interference pattern is given by

$$I(\theta) = I_m(\cos^2\beta) = \left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \text{(double slit)}, \tag{35-66}$$

in which

$$\beta = \frac{\pi d}{\lambda}\sin\theta \tag{35-67}$$

and

$$\alpha = \frac{\pi a}{\lambda}\sin\theta. \tag{35-68}$$

Here d is the distance between the centers of the slits and a is the slit width. Note carefully that the right side of Eq. 35-66 is the product of I_m and two factors. (1) The *interference factor* $\cos^2\beta$ is due to the interference between two slits with slit separation d (as given by Eqs. 35-27 and 35-28). (2) The *diffraction factor* $[(\sin\alpha)/\alpha]^2$ is due to diffraction by a single slit of width a (as given by Eqs. 35-58 and 35-59).

Let us check these factors. If we let $a \to 0$ in Eq. 35-68, for example, then $\alpha \to 0$ and $(\sin\alpha)/\alpha \to 1$. Equation 35-65 then reduces, as it must, to an equation describing the interference pattern for a pair of vanishingly narrow slits with slit separation d. Similarly, putting $d = 0$ in Eq. 35-67 is equivalent physically to causing the two slits to merge into a single slit of width a. Then Eq. 35-67 yields $\beta = 0$ and $\cos^2\beta = 1$. In this case Eq. 35-66 reduces, as it must, to an equation describing the diffraction pattern for a single slit of width a.

Language. The double-slit pattern described by Eq. 35-66 and displayed in Fig. 35-52*a* combines interference and diffraction in an intimate way. Both are superposition effects, in that they result from the combining of waves with different phases at a given point. If the combining waves originate from a small number of elementary coherent sources—as in a double-slit experiment with $a \ll \lambda$—we call the process *interference*. If the combining waves originate in a single wavefront—as in a single-slit experiment—we call the process *diffraction*. This distinction between interference and diffraction (which is somewhat arbitrary and not always adhered to) is a convenient one, but we should not forget that both are superposition effects and usually both are present simultaneously (as in Fig. 35-52*a*).

SAMPLE PROBLEM 35.19

Double-slit experiment with diffraction of each slit included

In a double-slit experiment, the wavelength λ of the light source is 405 nm, the slit separation d is 19.44 μm, and the slit width a is 4.050 μm. Consider the interference of the light from the two slits and also the diffraction of the light through each slit.

(a) How many bright interference fringes are within the central peak of the diffraction envelope?

KEY IDEAS

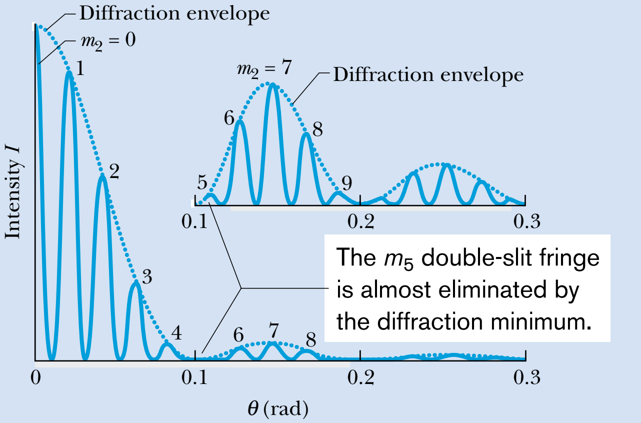
We first analyze the two basic mechanisms responsible for the optical pattern produced in the experiment:

1. *Single-slit diffraction:* The limits of the central peak are the first minima in the diffraction pattern due to either slit individually. (See Fig. 35-51.) The angular locations of those minima are given by Eq. 35-56 ($a \sin \theta = m\lambda$). Here let us rewrite this equation as $a \sin \theta = m_1\lambda$, with the subscript 1 referring to the one-slit diffraction. For the first minima in the diffraction pattern, we substitute $m_1 = 1$, obtaining
$$a \sin \theta = \lambda. \qquad (35\text{-}69)$$
2. *Double-slit interference:* The angular locations of the bright fringes of the double-slit interference pattern are given by Eq. 35-14, which we can write as
$$d \sin \theta = m_2\lambda, \quad \text{for } m_2 = 0, 1, 2, \ldots. \qquad (35\text{-}70)$$
Here the subscript 2 refers to the double-slit interference.

Calculations: We can locate the first diffraction minimum within the double-slit fringe pattern by dividing Eq. 35-68 by Eq. 35-67 and solving for m_2. By doing so and then substituting the given data, we obtain

$$m_2 = \frac{d}{a} = \frac{19.44\ \mu\text{m}}{4.050\ \mu\text{m}} = 4.8.$$

This tells us that the bright interference fringe for $m_2 = 4$ fits into the central peak of the one-slit diffraction pattern, but the fringe for $m_2 = 5$ does not fit. Within the central diffraction peak we have the central bright fringe ($m_2 = 0$), and four bright fringes (up to $m_2 = 4$) on each side of it. Thus, a total of nine bright fringes of the double-slit interference pattern are within the central peak of the diffraction envelope. The bright fringes to one side of the central bright fringe are shown in Fig. 35-53.



The m_5 double-slit fringe is almost eliminated by the diffraction minimum.

Figure 35-53 One side of the intensity plot for a two-slit interference experiment. The inset shows (vertically expanded) the plot within the first and second side peaks of the diffraction envelope.

(b) How many bright fringes are within either of the first side peaks of the diffraction envelope?

KEY IDEA

The outer limits of the first side diffraction peaks are the second diffraction minima, each of which is at the angle θ given by $a \sin \theta = m_1\lambda$ with $m_1 = 2$:

$$a \sin \theta = 2\lambda. \qquad (35\text{-}71)$$

Calculation: Dividing Eq. 35-68 by Eq. 35-69, we find

$$m_2 = \frac{2d}{a} = \frac{(2)(19.44\ \mu\text{m})}{4.050\ \mu\text{m}} = 9.6.$$

This tells us that the second diffraction minimum occurs just before the bright interference fringe for $m_2 = 10$ in Eq. 35-68. Within either first side diffraction peak we have the fringes from $m_2 = 5$ to $m_2 = 9$, for a total of five bright fringes of the double-slit interference pattern (shown in the inset of Fig. 35-53). However, if the $m_2 = 5$ bright fringe, which is almost eliminated by the first diffraction minimum, is considered too dim to count, then only four bright fringes are in the first side diffraction peak.

REVIEW AND SUMMARY

Huygens' Principle The three-dimensional transmission of waves, including light, may often be predicted by *Huygens' principle*, which states that all points on a wavefront serve as point sources of spherical secondary wavelets. After a time t, the new position of the wavefront will be that of a surface tangent to these secondary wavelets.

The law of refraction can be derived from Huygens' principle by assuming that the index of refraction of any medium is $n = c/v$, in which v is the speed of light in the medium and c is the speed of light in vacuum.

Wavelength and Index of Refraction The wavelength λ_n of light in a medium depends on the index of refraction n of the medium:

$$\lambda_n = \frac{\lambda}{n}, \tag{35-6}$$

in which λ is the wavelength in vacuum. Because of this dependency, the phase difference between two waves can change if they pass through different materials with different indexes of refraction.

Young's Experiment In Young's interference experiment, light passing through a single slit falls on two slits in a screen. The light leaving these slits flares out (by diffraction), and interference occurs in the region beyond the screen. A fringe pattern, due to the interference, forms on a viewing screen.

The light intensity at any point on the viewing screen depends in part on the difference in the path lengths from the slits to that point. If this difference is an integer number of wavelengths, the waves interfere constructively and an intensity maximum results. If it is an odd number of half-wavelengths, there is destructive interference and an intensity minimum occurs. The conditions for maximum and minimum intensity are

$$d \sin\theta = m\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima—bright fringes)}, \tag{35-14}$$

$$d \sin\theta = \left(m + \frac{1}{2}\right)\lambda, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(minima—dark fringes)}, \tag{35-16}$$

where θ is the angle the light path makes with a central axis and d is the slit separation.

Coherence If two light waves that meet at a point are to interfere perceptibly, the phase difference between them must remain constant with time; that is, the waves must be **coherent**. When two coherent waves meet, the resulting intensity may be found by using phasors.

Intensity in Two-Slit Interference In Young's interference experiment, two waves, each with intensity I_0, yield a resultant wave of intensity I at the viewing screen, with

$$I = 4I_0 \cos^2 \frac{1}{2}\phi, \quad \text{where } \phi = \frac{2\pi d}{\lambda}\sin\theta. \tag{35-27, 35-28}$$

Equations 35-14 and 35-16, which identify the positions of the fringe maxima and minima, are contained within this relation.

If two waves of intensities I_1 and I_2 superpose, the resultant intensity I is given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \tag{35-36}$$

Thin-Film Interference When light is incident on a thin transparent film, the light waves reflected from the front and back surfaces interfere. For near-normal incidence, the wavelength conditions for maximum and minimum intensity of the light reflected from a *film in air* are

$$2L = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(maxima—bright film in air)}, \tag{35-50}$$

$$2L = m\frac{\lambda}{n_2}, \quad \text{for } m = 0, 1, 2, \ldots \quad \text{(minima—dark film in air)}, \tag{35-51}$$

where n_2 is the index of refraction of the film, L is its thickness, and λ is the wavelength of the light in air.

If the light incident at an interface between media with different indexes of refraction is in the medium with the smaller index of refraction, the reflection causes a phase change of π rad, or half a wavelength, in the reflected wave. Otherwise, there is no phase change due to the reflection. Refraction causes no phase shift.

Diffraction When waves encounter an edge, an obstacle, or an aperture the size of which is comparable to the wavelength of the waves, those waves spread out as they travel and, as a result, undergo interference. This is called **diffraction**.

Single-Slit Diffraction Waves passing through a long narrow slit of width a produce, on a viewing screen, a **single-slit diffraction pattern** that includes a central maximum and other maxima, separated by minima located at angles θ to the central axis that satisfy

$$a \sin\theta = m\lambda, \quad \text{for } m = 1, 2, 3, \ldots \quad \text{(minima)}. \tag{35-57}$$

The intensity of the diffraction pattern at any given angle θ is

$$I(\theta) = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2, \quad \text{where } \alpha = \frac{\pi a}{\lambda}\sin\theta \tag{35-58, 35-59}$$

and I_m is the intensity at the center of the pattern.

Circular-Aperture Diffraction Diffraction by a circular aperture or a lens with diameter d produces a central maximum and concentric maxima and minima, with the first minimum at an angle θ given by

$$\sin\theta = 1.22\frac{\lambda}{d} \quad \text{(first minimum—circular aperture)}. \tag{35-62}$$

Rayleigh's Criterion *Rayleigh's criterion* suggests that two objects are on the verge of resolvability if the central diffraction maximum of one is at the first minimum of the other. Their angular separation can then be no less than

$$\theta_R = 1.22\frac{\lambda}{d} \quad \text{(Rayleigh's criterion)}, \qquad (35\text{-}64)$$

in which d is the diameter of the aperture through which the light passes.

Double-Slit Diffraction Waves passing through two slits, each of width a, whose centers are a distance d apart, display diffraction patterns whose intensity I at angle θ is

$$I(\theta) = I_m(\cos^2 \beta)\left(\frac{\sin \alpha}{\alpha}\right)^2 \quad \text{(double slit)}, \qquad (35\text{-}66)$$

with $\beta = (\pi d/\lambda) \sin \theta$ and α as for single-slit diffraction.

PROBLEMS

1. In Fig. 35-54, a broad beam of light of wavelength 630 nm is sent directly downward through the top plate of a pair of glass plates touching at the left end. The air between the plates acts as a thin film, and an interference pattern can be seen from above the plates. Initially, a dark fringe lies at the left end, a bright fringe lies at the right end, and nine dark fringes lie between those two end fringes. The plates are then very gradually squeezed together at a constant rate to decrease the angle between them. As a result, the fringe at the right side changes between being bright to being dark every 15.0 s. (a) At what rate is the spacing between the plates at the right end being changed? (b) By how much has the spacing there changed when both left and right ends have a dark fringe and there are five dark fringes between them?

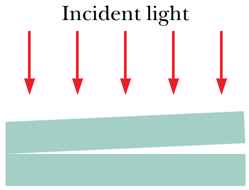


Figure 35-54 Problems 1 through 5.

2. In Fig. 35-54, two microscope slides touch at one end and are separated at the other end. When light of wavelength 420 nm shines vertically down on the slides, an overhead observer sees an interference pattern on the slides with the dark fringes separated by 1.9 mm. What is the angle between the slides?

3. In Fig. 35-54, a broad beam of monochromatic light is directed perpendicularly through two glass plates that are held together at one end to create a wedge of air between them. An observer intercepting light reflected from the wedge of air, which acts as a thin film, sees 4001 dark fringes along the length of the wedge. When the air between the plates is evacuated, only 4000 dark fringes are seen. Calculate to six significant figures the index of refraction of air from these data.

4. In Fig. 35-54, a broad beam of light of wavelength 683 nm is sent directly downward through the top plate of a pair of glass plates. The plates are 240 mm long, touch at the left end, and are separated by 51.4 μm at the right end. The air between the plates acts as a thin film. How many bright fringes will be seen by an observer looking down through the top plate?

5. Two rectangular glass plates ($n = 1.60$) are in contact along one edge and are separated along the opposite edge (Fig. 35-54). Light with a wavelength of 700 nm is incident perpendicularly onto the top plate. The air between the plates acts as a thin film. Nine dark fringes and eight bright fringes are observed from above the top plate. If the distance between the two plates along the separated edges is increased by 700 nm, how many dark fringes will there then be across the top plate?

6. Two waves of light in air, of wavelength λ = 600.0 nm, are initially in phase. They then both travel through a layer of plastic as shown in Fig. 35-55, with $L_1 = 4.00$ μm, L_2 = 3.50 μm, n_1 = 1.42, and n_2 = 1.60. (a) What multiple of λ gives their phase difference after they both have emerged from the layers? (b) If the waves later arrive at some common point with the same amplitude, is their interference fully constructive, fully destructive, intermediate but closer to fully constructive, or intermediate but closer to fully destructive?

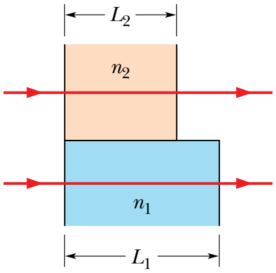


Figure 35-55 Problem 6.

7. The wavelength of yellow sodium light in air is 589 nm. (a) What is its frequency? (b) What is its wavelength in glass whose index of refraction is 1.52? (c) From the results of (a) and (b), find its speed in this glass.

8. Two waves of the same frequency have amplitudes 1.60 and 2.20. They interfere at a point where their phase difference is 60.0°. What is the resultant amplitude?

9. A double-slit arrangement produces interference fringes for sodium light (λ = 589 nm) that have an angular separation of 3.50×10^{-3} rad. For what wavelength would the angular separation be 6.0% greater?

10. Suppose that Young's experiment is performed with blue-green light of wavelength 410 nm. The slits are 1.20 mm apart, and the viewing screen is 5.40 m from the slits. How far apart are the bright fringes near the center of the interference pattern?

11. White light is sent downward onto a horizontal thin film that is sandwiched between two materials. The indexes of refraction are 1.80 for the top material, 1.65 for the thin film, and 1.50 for the bottom material. The film thickness is 5.00×10^{-7} m. Of the visible wavelengths (400 nm to 700 nm) that result in fully constructive interference at an observer above the film, which is the (a) longer and (b) shorter wavelength? The materials and film are then heated so that the film thickness increases. (c) Does the light resulting in fully constructive interference shift toward longer or shorter wavelengths?

12. In a double-slit experiment, the distance between slits is 5.0 mm and the slits are 1.8 m from the screen. Two interference patterns can be seen on the screen: one due to light of wavelength 480 nm, and the other due to light of wavelength 600 nm. What is the separation on the screen between the third-order ($m = 3$) bright fringes of the two interference patterns?

13. In Fig. 35-56, a light ray is incident at angle $\theta_1 = 55°$ on a series of five transparent layers with parallel boundaries. For layers 1 and 3, $L_1 = 20\ \mu\text{m}$, $L_3 = 25\ \mu\text{m}$, $n_1 = 1.6$, and $n_3 = 1.45$. (a) At what angle does the light emerge back into air at the right? (b) How much time does the light take to travel through layer 3?

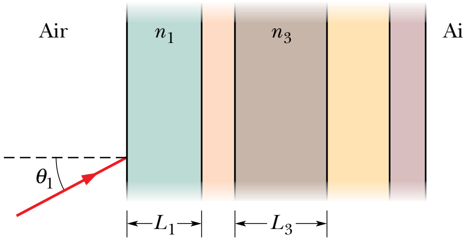


Figure 35-56 Problem 13.

14. In Fig. 35-4, assume that two waves of light in air, of wavelength 585 nm, are initially in phase. One travels through a glass layer of index of refraction $n_1 = 1.60$ and thickness L. The other travels through an equally thick plastic layer of index of refraction $n_2 = 1.50$. (a) What is the smallest value L should have if the waves are to end up with a phase difference of 5.65 rad? (b) If the waves arrive at some common point with the same amplitude, is their interference fully constructive, fully destructive, intermediate but closer to fully constructive, or intermediate but closer to fully destructive?

15. The rhinestones in costume jewelry are glass with index of refraction 1.50. To make them more reflective, they are often coated with a layer of silicon monoxide of index of refraction 2.00. What is the minimum coating thickness needed to ensure that light of wavelength 500 nm and of perpendicular incidence will be reflected from the two surfaces of the coating with fully constructive interference?

16. A disabled tanker leaks kerosene ($n = 1.20$) into the Persian Gulf, creating a large slick on top of the water ($n = 1.30$). (a) If you are looking straight down from an airplane, while the Sun is overhead, at a region of the slick where its thickness is 380 nm, for which wavelength(s) of visible light is the reflection brightest because of constructive interference? (b) If you are scuba diving directly under this same region of the slick, for which wavelength(s) of visible light is the transmitted intensity strongest?

17. The speed of yellow light (from a sodium lamp) in a certain liquid is measured to be 1.81×10^8 m/s. What is the index of refraction of this liquid for the light?

18. A thin film ($n = 1.25$) coats a thick glass plate ($n = 1.50$). White light is incident normal to the film. In the reflections, fully destructive interference occurs at 600 nm and fully constructive interference at 700 nm. Calculate the thickness of the film.

19. In Fig. 35-57, a light wave along ray r_1 reflects once from a mirror and a light wave along ray r_2 reflects twice from that same mirror and once from a tiny mirror at distance L from the bigger mirror. (Neglect the slight tilt of the rays.) The waves have wavelength 350 nm and are initially in phase. (a)What is the smallest value of L that puts the final light waves exactly out of phase? (b)With the tiny mirror initially at that value of L, how far must it be moved away from the bigger mirror to again put the final waves out of phase?

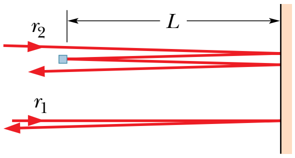


Figure 35-57 Problems 19 and 20.

20. In Fig. 35-57, a light wave along ray r_1 reflects once from a mirror and a light wave along ray r_2 reflects twice from that same mirror and once from a tiny mirror at distance L from the bigger mirror. (Neglect the slight tilt of the rays.) The waves have wavelength λ and are initially exactly out of phase. What are the (a) smallest, (b) second smallest, and (c) fourth smallest values of L/λ that result in the final waves being exactly in phase?

21. In the double-slit experiment of Fig. 35-11, the viewing screen is at distance $D = 4.00$ m, point P lies at distance $y = 20.5$ cm from the center of the pattern, the slit separation d is 4.50 μm, and the wavelength λ is 650 nm. (a) Determine where point P is in the interference pattern by giving the maximum or minimum on which it lies, or the maximum and minimum between which it lies. (b) What is the ratio of the intensity I_P at point P to the intensity I_{cen} at the center of the pattern?

22. In Fig. 35-58, a broad beam of light of wavelength 420 nm is incident at 90° on a thin, wedge-shaped film with index of refraction 1.70. Transmission gives 10 bright and 9 dark fringes along the film's length. What is the left-to-right change in film thickness?

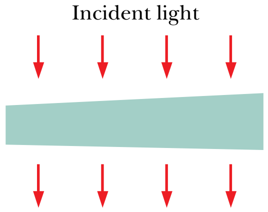


Figure 35-58 Problem 22.

23. In Fig. 35-59, two light rays go through different paths by reflecting from the various flat surfaces shown. The light waves have a wavelength of 411.0 nm and are initially in phase. What are the (a) smallest and (b) second smallest value of distance L that will put the waves exactly out of phase as they emerge from the region?

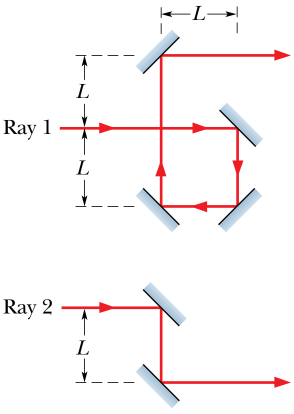


Figure 35-59 Problem 23.

24. We wish to coat flat glass ($n = 1.50$) with a transparent material ($n = 1.45$) so that reflection of light at wavelength 500 nm is eliminated by interference. What minimum thickness can the coating have to do this?

25. A plane wave of monochromatic light is incident normally on a uniform thin film of oil that covers a glass plate. The wavelength of the source can be varied continuously. Fully destructive interference of the reflected light is observed for wavelengths of 500 nm and 700 nm and for no wavelengths in between. If the index of refraction of the oil is 1.26 and that of the glass is 1.50, find the thickness of the oil film.

26. In Fig. 35-6, assume that the two light waves, of wavelength 515 nm in air, are initially out of phase by π rad. The indexes of refraction of the media are $n_1 = 1.45$ and $n_2 = 1.75$. What are the (a) smallest and (b) second smallest value of L that will put the waves exactly in phase once they pass through the two media?

27. In a double-slit experiment, the fourth-order maximum for a wavelength of 450 nm occurs at an angle of $\theta = 90°$. (a) What range of wavelengths in the visible range (400 nm to 700 nm) are not present in the third-order maxima? To eliminate all visible light in the fourth-order maximum, (b) should the slit separation be increased or decreased and (c) what least change is needed?

28. Light of wavelength 424 nm is incident perpendicularly on a soap film (n = 1.33) suspended in air. What are the (a) least and (b) second least thicknesses of the film for which the reflections from the film undergo fully constructive interference?

29. In the double-slit experiment of Fig. 35-11, the electric fields of the waves arriving at point P are given by

$$E_1 = (2.00\ \mu\text{V/m})\sin[(1.26 \times 10^{15})t]$$
$$E_2 = (2.00\ \mu\text{V/m})\sin[(1.26 = 10^{15})t + 39.6\text{ rad}],$$

where time t is in seconds. (a) What is the amplitude of the resultant electric field at point P? (b) What is the ratio of the intensity I_P at point P to the intensity I_{cen} at the center of the interference pattern? (c) Describe where point P is in the interference pattern by giving the maximum or minimum on which it lies, or the maximum and minimum between which it lies. In a phasor diagram of the electric fields, (d) at what rate would the phasors rotate around the origin and (e) what is the angle between the phasors?

30. Three electromagnetic waves travel through a certain point P along an x axis. They are polarized parallel to a y axis, with the following variations in their amplitudes. Find their resultant at P.

$$E_1 = (8.00\ \mu\text{V/m})\sin[(2.0 \times 10^{14}\text{ rad/s})t]$$
$$E_2 = (5.00\ \mu\text{V/m})\sin[(2.0 \times 10^{14}\text{ rad/s})t + 55.0°]$$
$$E_3 = (5.00\ \mu\text{V/m})\sin[(2.0 \times 10^{14}\text{ rad/s})t - 55.0°]$$

31. In Fig. 35-60, sources A and B emit long-range radio waves of wavelength 400 m, with the phase of the emission from A ahead of that from source B by 90.0°. The distance r_A from A to detector D is greater than the corresponding distance r_B by 150 m. What is the phase difference of the waves at D?

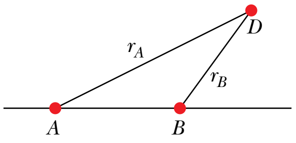


Figure 35-60 Problem 31.

32. In Fig. 35-61, two isotropic point sources S_1 and S_2 emit light in phase at wavelength λ and at the same amplitude. The sources are separated by distance $2d = 6.00\lambda$. They lie on an axis that is parallel to an x axis, which runs along a viewing screen at distance $D = 20.0\lambda$. The origin lies on the perpendicular bisector between the sources. The figure shows two rays reaching point P on the screen, at position x_P. (a) At what value of x_P do the rays have the minimum possible phase difference? (b) What multiple of λ gives that minimum phase difference? (c) At what value of x_P do the rays have the maximum possible phase difference? What multiple of λ gives (d) that maximum phase difference and (e) the phase difference when $x_P = 6.00\lambda$? (f) When $x_P = 6.00\lambda$, is the resulting intensity at point P maximum, minimum, intermediate but closer to maximum, or intermediate but closer to minimum?

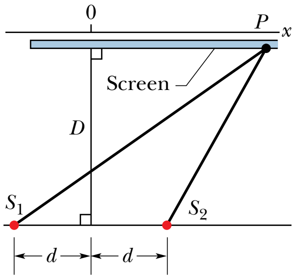


Figure 35-61 Problem 32.

33. Suppose that the two waves in Fig. 35-6 have wavelength λ = 610 nm in air. What multiple of λ gives their phase difference when they emerge if (a) $n_1 = 1.50$, $n_2 = 1.60$, and $L = 7.00\ \mu$m; (b) $n_1 = 1.62$, $n_2 = 1.72$, and $L = 7.00\ \mu$m; and (c) $n_1 = 1.83$, $n_2 = 1.59$, and $L = 2.16\ \mu$m? (d) Suppose that in each of these three situations the waves arrive at a common point (with the same amplitude) after emerging. Rank the situations according to the brightness the waves produce at the common point.

34. Monochromatic green light, of wavelength 550 nm, illuminates two parallel narrow slits 5.60 μm apart. Calculate the angular deviation (θ in Fig. 35-11) of the third-order ($m = 3$) bright fringe (a) in radians and (b) in degrees.

35. A thin flake of mica ($n = 1.58$) is used to cover one slit of a double-slit interference arrangement. The central point on the viewing screen is now occupied by what had been the fifth bright side fringe ($m = 5$). If $\lambda = 550$ nm, what is the thickness of the mica?

36. In a double-slit arrangement the slits are separated by a distance equal to 100 times the wavelength of the light passing through the slits. (a) What is the angular separation in radians between the central maximum and an adjacent maximum? (b) What is the distance between these maxima on a screen 90.0 cm from the slits?

37. The reflection of perpendicularly incident white light by a soap film in air has an interference maximum at 478.8 nm and a minimum at 598.5 nm, with no minimum in between. If $n = 1.33$ for the film, what is the film thickness, assumed uniform?

38. A double-slit arrangement produces interference fringes for sodium light ($\lambda = 589$ nm) that are 0.25° apart. What is the angular separation if the arrangement is immersed in water ($n = 1.33$)?

39. In Fig. 35-62, two isotropic point sources of light (S_1 and S_2) are separated by distance 2.70 μm along a y axis and emit in phase at wavelength 900 nm and at the same amplitude. A light detector is located at point P at coordinate x_P on the x axis. What are (a) the greatest value of x_P and (b) the second greatest value at which the detected light is minimum due to destructive interference?

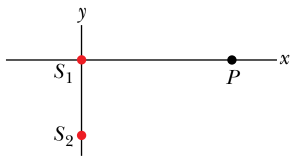


Figure 35-62 Problems 39 and 40.

40. Figure 35-62 shows two isotropic point sources of light (S_1 and S_2) that emit in phase at wavelength 400 nm and at the same amplitude. A detection point P is shown on an x axis that extends through source S_1. The phase difference ϕ between the light arriving at point P from the two sources is to be measured as P is moved along the x axis from $x = 0$ out to $x = +\infty$. The results out to $x_s = 10 \times 10^{-7}$ m are given in Fig. 35-63. On the way out to $+\infty$, what is the greatest value of x at which the light arriving at P from S_1 is exactly out of phase with the light arriving at P from S_2?

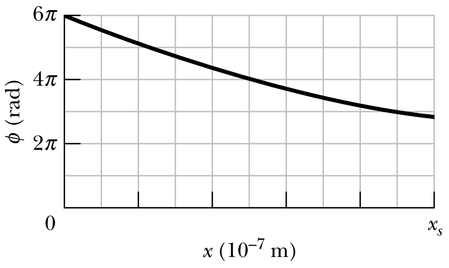


Figure 35-63 Problem 40.

41. In Fig. 35-64, two radio-frequency point sources S_1 and S_2, separated by distance $d = 1.8$ m, are radiating in phase with $\lambda = 0.50$ m. A detector moves in a large circular path around the two sources in a plane containing them. How many (a) maxima and (b) minima does it detect?

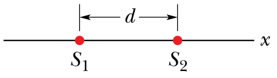


Figure 35-64 Problems 41 and 42.

42. In Fig. 35-64, two isotropic point sources S_1 and S_2 emit identical light waves in phase at wavelength λ. The sources lie at separation d on an x axis, and a light detector is moved in a circle of large radius around the midpoint between them. It detects 30 points of zero intensity, including two on the x axis, one of them to the left of the sources and the other to the right of the sources. What is the value of d/λ?

43. The ratio of intensities due to upper and lower slits on the screen of a Young's double-slit experiment is 16:9. Find the ratio of maximum to average intensity on screen.

44. In a Young's double-slit experiment apparatus, two identical slits are separated by 1 mm and distance between slits and screen is 1 m. The wavelength of light used is 6000 Å. What is the minimum distance between two points on the screen having 75% intensity of the maximum intensity?

45. Two coherent point sources S_1 and S_2 are located at a distance $4l$ from each other. Perpendicular to the straight line joining S_1 and S_2, there is a large screen at a distance of $1000\,\lambda$ from S_1 as shown in Fig. 35-65. Find the radius of the third maxima starting from the point O.

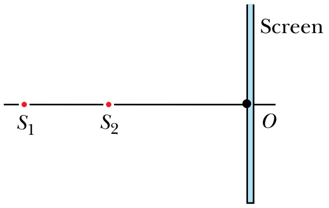


Figure 35-65 Problem 45.

46. In Lloyd's mirror experiment, the source slit S lies in a plane 20 cm to the left from the left edge of the mirror (Fig. 35-66). The mirror is 30 cm long and a screen is placed touching the right edge of the mirror. The distance of S from plane of mirror is 2 mm and wavelength of light used = 6400 Å. Find the number of maxima formed.

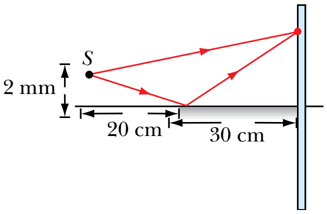


Figure 35-66 Problem 46.

47. The distance between a slit and a biprism of acute angle 2° is 10 cm. Find (a) the fringe width and (b) the width of the entire fringe pattern when observation is made on screen at a distance of 90 cm from the biprism. The refractive index of the biprism is 1.5 and $l = 5890$ Å.

48. Suppose that the central diffraction envelope of a double-slit diffraction pattern contains 13 bright fringes and the first diffraction minima eliminate (are coincident with) bright fringes. How many bright fringes lie between the second and third minima of the diffraction envelope?

49. Estimate the linear separation of two objects on Mars that can just be resolved under ideal conditions by an observer on Earth (a) using the naked eye and (b) using the 200 in. (= 5.1 m) Mount Palomar telescope. Use the following data: distance to Mars = 8.0×10^7 km, diameter of pupil = 5.0 mm, wavelength of light = 550 nm.

50. Light of wavelength 620 nm passes through a double slit, yielding a diffraction pattern whose graph of intensity I versus angular position θ is shown in Fig. 35-67. Calculate (a) the slit width and (b) the slit separation. (c) Verify the displayed intensities of the $m = 1$ and $m = 2$ interference fringes.

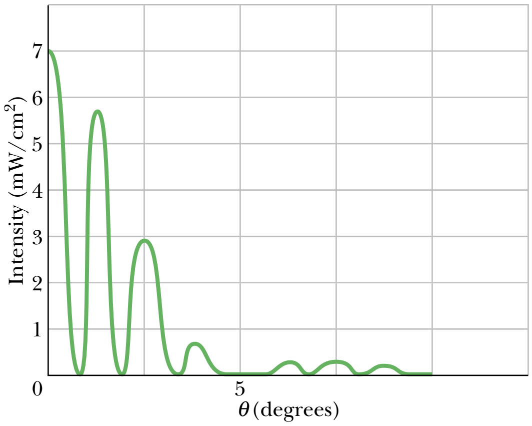


Figure 35-67 Problem 50.

51. Figure 35-68 gives α versus the sine of the angle θ in a single-slit diffraction experiment using light of wavelength 610 nm. The vertical axis scale is set by α_s = 12 rad. What are (a) the slit width, (b) the total number of diffraction minima in the pattern (count them on both sides of the center of the diffraction pattern), (c) the least angle for a minimum, and (d) the greatest angle for a minimum?

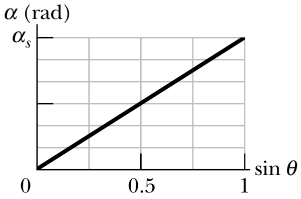


Figure 35-68 Problem 51.

52. In a double-slit experiment, the slit separation d is 5.00 times the slit width w. How many bright interference fringes are in the central diffraction envelope?

53. Sound waves with frequency 2500 Hz and speed 343 m/s diffract through the rectangular opening of a speaker cabinet and into a large auditorium of length d = 100 m. The opening, which has a horizontal width of 30.0 cm, faces a wall 100 m away (Fig. 35-69). Along that wall, how far from the central axis will a listener be at the first diffraction minimum and thus have difficulty hearing the sound? (Neglect reflections.)

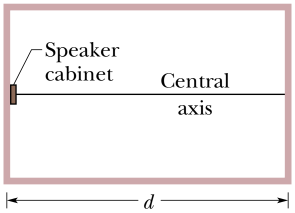


Figure 35-69 Problem 53.

54. (a) How many bright fringes appear between the first diffraction-envelope minima to either side of the central maximum in a double-slit pattern if λ = 550 nm, d = 0.180 mm, and a = 30.0 μm? (b) What is the ratio of the intensity of the third bright fringe to the intensity of the central fringe?

55. A single slit is illuminated by light of wavelengths λ_a and λ_b, chosen so that the first diffraction minimum of the λ_a component coincides with the second minimum of the λ_b component. (a) If λ_b = 420 nm, what is λ_a? For what order number m_b (if any) does a minimum of the λ_b component coincide with the minimum of the λ_a component in the order number (b) m_a = 2 and (c) m_a = 3?

56. A slit 1.00 mm wide is illuminated by light of wavelength 650 nm. We see a diffraction pattern on a screen 3.00 m away. What is the distance between the first and third diffraction minima on the same side of the central diffraction maximum?

57. The two headlights of an approaching automobile are 1.4 m apart. At what (a) angular separation and (b) maximum distance will the eye resolve them? Assume that the pupil diameter is 4.5 mm, and use a wavelength of 550 nm for the light. Also assume that diffraction effects alone limit the resolution so that Rayleigh's criterion can be applied.

58. What must be the ratio of the slit width to the wavelength for a single slit to have the first diffraction minimum at $\theta = 30°$?

59. A 0.20 mm wide slit is illuminated by light of wavelength 420 nm. Consider a point P on a viewing screen on which the diffraction pattern of the slit is viewed; the point is at 30° from the central axis of the slit. What is the phase difference between the Huygens wavelets arriving at point P from the top and midpoint of the slit? (*Hint:* See Eq. 35-55.)

60. In conventional television, signals are broadcast from towers to home receivers. Even when a receiver is not in direct view of a tower because of a hill or building, it can still intercept a signal if the signal diffracts enough around the obstacle, into the obstacle's "shadow region." Previously, television signals had a wavelength of about 50 cm, but digital television signals that are transmitted from towers have a wavelength of about 10 mm. (a) Did this change in wavelength increase or decrease the diffraction of the signals into the shadow regions of obstacles? Assume that a signal passes through an opening of 6.0 m width between two adjacent buildings. What is the angular spread of the central diffraction maximum (out to the first minima) for wavelengths of (b) 50 cm and (c) 10 mm?

61. *Entoptic halos.* If someone looks at a bright outdoor lamp in otherwise dark surroundings, the lamp appears to be surrounded by bright and dark rings (hence *halos*) that are actually a circular diffraction pattern as in Fig. 35-11, with the central maximum overlapping the direct light from the lamp. The diffraction is produced by structures within the cornea or lens of the eye (hence *entoptic*). If the lamp is monochromatic at wavelength 550 nm and the first dark ring subtends angular diameter 2.0° in the observer's view, what is the (linear) diameter of the structure producing the diffraction?

62. In the single-slit diffraction experiment of Fig. 35-44, let the wavelength of the light be 500 nm, the slit width be 6.00 μm, and the viewing screen be at distance D = 4.00 m. Let a y axis extend upward along the viewing screen, with its origin at the center of the diffraction pattern. Also let I_P represent the intensity of the diffracted light at point P at y = 15.0 cm. (a) What is the ratio of I_P to the intensity I_m at the center of the pattern? (b) Determine where point P is in the diffraction pattern by giving the maximum and minimum between which it lies, or the two minima between which it lies.

63. A plane wave of wavelength 420 nm is incident on a slit with a width of $a = 0.60$ mm. A thin converging lens of focal length +70 cm is placed between the slit and a viewing screen and focuses the light on the screen. (a) How far is the screen from the lens? (b) What is the distance on the screen from the center of the diffraction pattern to the first minimum?

64. Assume that Rayleigh's criterion gives the limit of resolution of an astronaut's eye looking down on Earth's surface from a typical space shuttle altitude of 420 km. (a) Under that idealized assumption, estimate the smallest linear width on Earth's surface that the astronaut can resolve. Take the astronaut's pupil diameter to be 5 mm and the wavelength of visible light to be 550 nm. (b) Can the astronaut resolve the Great Wall of China (Fig. 35-70), which is more than 3000 km long, 5 to 10 m thick at its base, 4 m thick at its top, and 8 m in height? (c) Would the astronaut be able to resolve any unmistakable sign of intelligent life on Earth's surface?

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Figure 35-70 Problem 64. The Great Wall of China.

65. (a) Show that the values of α at which intensity maxima for single-slit diffraction occur can be found exactly by differentiating Eq. 35-56 with respect to α and equating the result to zero, obtaining the condition $\tan \alpha = \alpha$. To find values of α satisfying this relation, plot the curve $y = \tan \alpha$ and the straight line $y = \alpha$ and then find their intersections, or use a calculator to find an appropriate value of α by trial and error. Next, from $\alpha = (m + 1/2)\pi$, determine the values of m associated with the maxima in the single-slit pattern. (These m values are *not* integers because secondary maxima do not lie exactly halfway between minima.) What are the (b) smallest α and (c) associated m, the (d) second smallest α and (e) associated m, and the (f) third smallest α and (g) associated m?

66. The telescopes on some commercial surveillance satellites can resolve objects on the ground as small as 85 cm across (see Google Earth), and the telescopes on military surveillance satellites reportedly can resolve objects as small as 10 cm across. Assume first that object resolution is determined entirely by Rayleigh's criterion and is not degraded by turbulence in the atmosphere. Also assume that the satellites are at a typical altitude of 420 km and that the wavelength of visible light is 550 nm. What would be the required diameter of the telescope aperture for (a) 85 cm resolution and (b) 10 cm resolution? (c) Now, considering that turbulence is certain to degrade resolution and that the aperture diameter of the Hubble Space Telescope is 2.4 m, what can you say about the answer to (b) and about how the military surveillance resolutions are accomplished?

67. Light of wavelength 420 nm is incident on a narrow slit. The angle between the first diffraction minimum on one side of the central maximum and the first minimum on the other side is 2.00°. What is the width of the slit?

68. (a) A circular diaphragm 50 cm in diameter oscillates at a frequency of 25 kHz as an underwater source of sound used for submarine detection. Far from the source, the sound intensity is distributed as the diffraction pattern of a circular hole whose diameter equals that of the diaphragm. Take the speed of sound in water to be 1450 m/s and find the angle between the normal to the diaphragm and a line from the diaphragm to the first minimum. (b) Is there such a minimum for a source having an (audible) frequency of 1.0 kHz?

69. Monochromatic light of wavelength 441 nm is incident on a narrow slit. On a screen 2.00 m away, the distance between the second diffraction minimum and the central maximum is 1.80 cm. (a) Calculate the angle of diffraction θ of the second minimum. (b) Find the width of the slit.

70. Monochromatic light with wavelength 420 nm is incident on a slit with width 0.050 mm. The distance from the slit to a screen is 3.5 m. Consider a point on the screen 2.2 cm from the central maximum. Calculate (a) θ for that point, (b) α, and (c) the ratio of the intensity at that point to the intensity at the central maximum.

71. Two slits of width a and separation d are illuminated by a coherent beam of light of wavelength λ. What is the linear separation of the bright interference fringes observed on a screen that is at a distance D away?

72. (a) In a double-slit experiment, what ratio of d to a causes diffraction to eliminate the fourth bright side fringe? (b) what other bright fringes are also eliminated?

PRACTICE QUESTIONS

Single Correct Choice Type

1. The intensity ratio of the two interfering beams of light is m. What is the value of $I_{\max} - I_{\min}/I_{\max} + I_{\min}$?

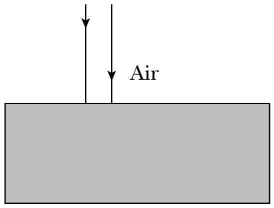
(a) $2\sqrt{m}$

(b) $\dfrac{2\sqrt{m}}{1+m}$

(c) $\dfrac{2}{1+m}$

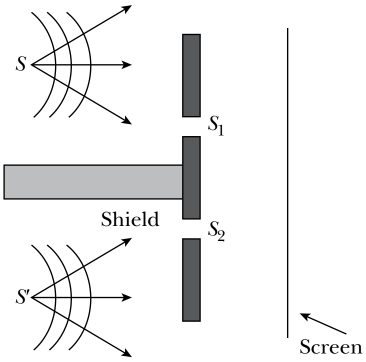
(d) $\dfrac{1+m}{2\sqrt{m}}$

2. A parallel beam of light of intensity I_0 is incident on a glass plate, 25% of light is reflected by upper surface and 50% of light is reflected from lower surface (see figure). The ratio of maximum to minimum intensity in interference region of reflected rays is:



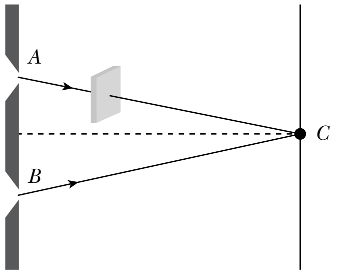
(a) $\left[\dfrac{(1/2)+(\sqrt{3}/\sqrt{8})}{(1/2)-(\sqrt{3}/\sqrt{8})}\right]^2$
(b) $\left[\dfrac{(1/4)+(\sqrt{3}/\sqrt{8})}{(1/2)-(\sqrt{3}/\sqrt{8})}\right]^2$
(c) $\dfrac{5}{8}$
(d) $\dfrac{8}{5}$

3. What happens to the interference pattern if the two slits in a Young's experiment are illuminated by two independent sources such as two sodium lamps S and S' as shown in the below figure?



(a) Two sets of interference fringes overlap
(b) No fringes are observed
(c) The intensity of the bright fringes is doubled
(d) The intensity of the bright fringes becomes four times

4. In Young's experiment (see figure), monochromatic light is used to illuminate the two slits A and B. Interference fringes are observed on a screen placed in front of the slits. Now, if a thin glass plate is placed normally in the path of the beam coming from the slit, then



(a) The fringes will disappear
(b) The fringe width will increase
(c) The fringe width will decrease
(d) There will be no change in the fringe width, but the pattern shifts

5. In Young's double-slit experiment, the intensity is I at a point where the path difference is $\lambda/6$, where λ is wavelength of light used. If I_0 denotes the maximum intensity, then I/I_0 is equal to
(a) 3/4
(b) $1/\sqrt{2}$
(c) $\sqrt{3}/2$
(d) 1/2

6. The angle of polarization for any medium is 60°, what will be critical angle for this
(a) $\sin^{-1}\sqrt{3}$
(b) $\tan^{-1}\sqrt{3}$
(c) $\cos^{-1}\sqrt{3}$
(d) $\sin^{-1}\dfrac{1}{\sqrt{3}}$

7. White light is used to illuminate the two slits in Young's experiment. The separation between the slits is d and the screen is at a distance $D(\gg d)$ from the slits. At a point directly in front of one of the slits, certain wavelength is missing. The missing wavelength
(a) $\lambda = d^2/D$
(b) $\lambda = d^2/5D$
(c) $\lambda = d^2/3D$
(d) All of these

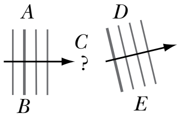
8. Light passes successively through two polarimeter tubes each of length 0.29 m. The first tube contains dextro-rotatory solution of concentration 60 kg/m^3 and specific rotation 0.01 rad m^2/kg. The second tube contains levo-rotatory solution of concentration 30 kg/m^3 and specific rotation 0.02 rad m^2/kg. The net rotation produced is
(a) 15°
(b) 0°
(c) 20°
(d) 10°

9. Two Nicols are oriented with their principal planes making an angle of 60°. The percentage of incident unpolarized light which passes through the system is
(a) 50%
(b) 100%
(c) 12.5%
(d) 37.5%

10. In the visible region of the spectrum the rotation of the plane of polarization is given by $\theta = a + (B/\lambda^2)$. The optical rotation produced by a particular material is found to be 30° per mm at $\lambda = 5000$ Å and 50° per mm at $\lambda = 4000$ Å. The value of constant a will be
(a) $+\dfrac{50^\circ}{9}$ per mm
(b) $-\dfrac{50^\circ}{9}$ per mm
(c) $+\dfrac{9^\circ}{50}$ per mm
(d) $-\dfrac{9^\circ}{50}$ per mm

11. When a ray of light of frequency 6×10^{14} Hz travels from water of refractive index 4/3 to glass of refractive index 8/5, its
(a) Frequency becomes five-sixth (5/6) of its initial value
(b) Speed becomes five-sixth (5/6) of its initial value
(c) Wavelength becomes six-fifth (6/5) of its initial value
(d) Speed becomes six-fifth (6/5) of its initial value

12. A wavefront AB passing through a system C emerges as DE (As shown in the following figure). The system C could be



(a) A slit
(b) A biprism
(c) A prism
(d) A glass slab

13. The wavefront of a light beam is given by the equation $x + 2y + 3z = c$, (where c is arbitary constant) then the angle made by the direction of light with the y axis is

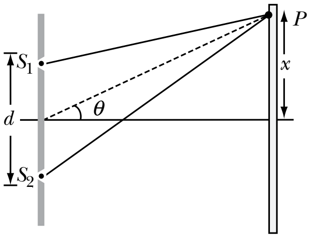
(a) $\cos^{-1}\frac{1}{\sqrt{14}}$ **(b)** $\sin^{-1}\frac{2}{\sqrt{14}}$

(c) $\cos^{-1}\frac{2}{\sqrt{14}}$ **(d)** $\sin^{-1}\frac{3}{\sqrt{14}}$

14. In Young's double-slit interference experiment a first screen with a single narrow slit is used in addition to the double-slit screen. An interference pattern is observed on the screen as shown in the Fig. 35-9. What happens if the first screen is removed and light from an extended but monochromatic source, e.g., yellow light from large sodium vapor lamp, is allowed to illuminate the double-slit screen directly?

(a) The interference pattern disappears
(b) The intensity of light on the viewing screen is reduced
(c) The width of the dark fringes reduces noticeably
(d) The width of the bright fringes reduces noticeably

15. Two sources, in phase and a distance d apart, each emit a wave of wavelength l (as shown in the following figure). Which of the choices for the path difference $\Delta L = \Delta P = S_2P - S_1P$ $\Delta L = L_1 - L_2$ will always produce constructive interference at point P?



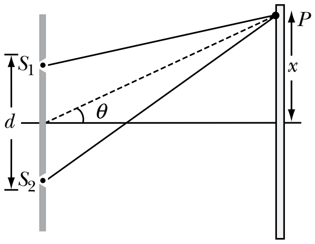
(a) $d\sin\theta$ **(b)** λ

(c) $(x/\lambda)d$ **(d)** $\lambda/2$

16. In a double-slit interference pattern, the first maxima for infrared light would be

(a) At the same place as the first maxima for green light
(b) Closer to the center than the first maxima for green light
(c) Farther from the center than the first maxima for green light
(d) Infrared light does not produce an interference pattern

17. Light shining through two very narrow slits produces an interference maximum at point P when the entire apparatus is in air as shown in the following figure. For the interference maximum represented, light through the bottom slit travels one wavelength further than light through the top slit before reaching point P. If the entire apparatus is immersed in water, the angle θ to the interference maximum



(a) Is unchanged
(b) Decreases because the frequency decreases
(c) Decreases because the wavelength decreases
(d) Increases because the frequency increases

18. In a Young's double-slit experiment the slit separation is 0.5 mm and the screen is 0.5 m from the slit. For a monochromatic light of wavelength 500 nm the distance of third maxima from the second minima on the other side is

(a) 2.75 mm **(b)** 2.5 mm

(c) 22.5 mm **(d)** 2.25 mm

19. Two different color beams (yellow and blue) from a point source are incident on both slits of a Young's double-slit experiment setup

(a) There will be no interference pattern seen on the screen
(b) There will be interference pattern and where there is constructive interference the spot will be green
(c) There will be alternate bands of yellow, blue, and dark regions with equal separation between them
(d) There will be yellow and blue light interference patterns which are slightly separated from each other because the angles at which the maxima and minima occur for the two wavelengths are different

20. Two wavelength of light λ_1 and λ_2 are sent through Young's double-slit apparatus simultaneously. What must be true about λ_1 and λ_2 if the third-order bright fringe of λ_1 coincides with fifth-order dark fringe of λ_2

(a) $3\lambda_1 = 2\lambda_2$ **(b)** $2\lambda_1 = 3\lambda_2$

(c) $3\lambda_1 = 5\lambda_2$ **(d)** $5\lambda_1 = 3\lambda_2$

21. In a Young's double-slit experiment, if the incident light consists of two wavelengths λ_1 and λ_2, the slit separation is d, and the distance between the slit and the screen is D, the maxima due to each wavelength will coincide at a distance from the central maxima, given by

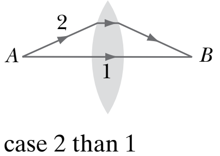
(a) $\frac{\lambda_1 + \lambda_2}{2Dd}$ **(b)** LCM of $\frac{\lambda_1 D}{d}$ and $\frac{\lambda_2 D}{d}$

(c) $(\lambda_1 - \lambda_2)\frac{2D}{d}$ **(d)** HCF of $\frac{\lambda_1 D}{d}$ and $\frac{\lambda_2 D}{d}$

22. Two plane monochromatic coherent waves produce interference pattern (of alternately bright and dark bands) on a screen. When the angle θ between the two beams is decreased the fringe width

(a) Increases
(b) Is unaffected
(c) Decreases
(d) Increases and then decreases

23. Path followed by two rays through a thin lens in air is shown in the following figure. (Assume value of μ of lens same for both rays.) Optical path followed for two rays from A to B is



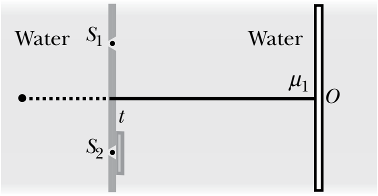
(a) Greater in case 2 than 1
(b) Greater in case 1 than 2
(c) Equal in both the cases
(d) Value of μ is needed to compare

24. An interference pattern is formed on a screen by shining a planar wave on a double-slit arrangement. If we cover one slit with a glass plate, the phases of the two emerging waves will be different because the wavelength is shorter in glass than in air. If the phase difference is 180°, how is the interference pattern altered?
(a) The pattern vanishes
(b) The bright spots lie closer together
(c) The bright spots are farther apart
(d) The pattern gets shifted in such a way that center of the screen is a dark band

25. A glass slab of thickness 4 cm contains the same number of waves as 5 cm of water when both are traversed by the same monochromatic light. If the refractive index of water is 4/3, what is the refractive index of glass?
(a) $\frac{5}{3}$ **(b)** $\frac{5}{4}$
(c) $\frac{16}{15}$ **(d)** $\frac{3}{2}$

26. On introducing a thin sheet of mica (thickness 12×10^{-5} cm) in path of one of the interfering beams ($\lambda_{\text{vacuum}} = 600$ nm) in Young's double-slit experiment, the central fringe is shifted through a distance equal to the spacing between successive bright fringes. Calculate the refractive index of mica.
(a) 1.33 **(b)** 1.4
(c) 1.5 **(d)** 2.5

27. A Young's double-slit experiment is conducted in water (n_1) as shown in the following figure, and a glass plate of thickness t and refractive index n_2 is placed in the path of S_2. Wavelength of light in water is λ. Find the magnitude of the phase difference between waves coming from S_1 and S_2 at O.



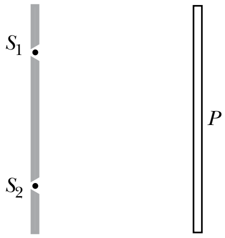
(a) $\left|\left(\frac{n_2}{n_1}-1\right)t\right|\frac{2\pi}{\lambda}$ **(b)** $\left|\left(\frac{n_2}{n_1}-1\right)t\right|\frac{2\pi}{\lambda}$

(c) $\left|(n_2-n_1)t\right|\frac{2\pi}{\lambda}$ **(d)** $\left|(n_2-1)t\right|\frac{2\pi}{\lambda}$

28. In a Young's double-slit experiment, green light is incident on the two slits. The interference pattern is observed on a screen. Which one of the following changes would cause the observed fringes to be more closely spaced?
(a) Reducing the separation between the slits
(b) Using blue light instead of green light
(c) Using red light instead of green light
(d) Moving the light source further away from the slits

29. Why isn't the interference pattern like the one from Young's double-slit experiment produced from the two headlights of a car?
(a) The lights are too far apart
(b) The lights are not bright enough
(c) The lights are not rectangular slit in shape
(d) The light from one lamp is not coherent with the other

30. Coherent light is incident on two fine parallel slits S_1 and S_2 as shown in the following figure. A dark fringe occurs at P when, m being an integer, the phase difference between the wave trains from S_1 and S_2 is



(a) $m\pi$ rad **(b)** $(m + 1/2)\pi$ rad
(c) $(2m - 1)\pi$ rad **(d)** $(2m + 1/2)\pi$ rad

31. Three coherent, equal intensity light rays arrive at a point P on a screen to produce an interference minimum of zero intensity. If any two of the rays are blocked, the intensity of the light at P is I_1. What is the intensity of the light at P if only one of the rays is blocked?
(a) 0 **(b)** $I_1/2$
(c) I_1 **(d)** $2I_1$

32. In a Young's double slit experiment, if the slits are of unequal width,
(a) Fringes will not be formed
(b) The positions of minimum intensity will not be completely dark
(c) Bright fringe will not be formed at the center of the screen
(d) Distance between two consecutive bright fringes will not be equal to the distance between two consecutive dark fringes

33. Intensities of light due to the two slits of Young's double-slit experiment are I and $4I$. How far from the central maxima will the intensity be equal to the average intensity on the screen? (β is the fringe width.)
(a) $\frac{\beta}{2}$ **(b)** $\frac{\beta}{3}$
(c) $\frac{\beta}{6}$ **(d)** $\frac{\beta}{4}$

34. In a Young's double-slit interference pattern, the intensity of the central fringe at $P = I_0$. When one slit width is reduced to half, the intensity at P will be
(a) $I_0/2$ **(b)** $I_0/4$
(c) $\frac{I_0}{8}(3+2\sqrt{2})$ **(d)** $\frac{9}{4}I_0$

35. In a Young's double-slit experiment, the central bright fringe can be identified
(a) As it has greater intensity than the other bright fringes
(b) As it is wider than the other bright fringes
(c) As it is narrower than the other bright fringes
(d) By using white light instead of monochromatic light

36. As shown in Fig. 35-32 of billet lens, a point light source is placed at distance $2f$ from a lens with focus length f and a screen is placed at $4f$ from the lens. The lens is then cut at the middle into two equal portions: upper half and lower half. The upper half is moved upward by a small distance *d* comparable to the light wavelength and the lower half is moved downward by the same distance *d*. What is the light pattern on the screen?

(a) Bright and dark strips similar to the pattern seen in Young's double-slit experiment
(b) Bright and dark concentric rings
(c) Two large, bright, and partly overlapping patches
(d) Two separate bright spots

37. A Fresnel biprism is used to form the interference fringes. The distance between the source and the biprism is 20 cm and that between the biprism and the screen is 80 cm. If $\lambda = 6563$Å and the separation between the virtual sources is 3.6 mm, then the fringe width is

(a) 1.82 cm
(b) 0.182 cm
(c) 0.0182 cm
(d) 0.00182 cm

38. In a biprism experiment, the biprism is made of glass ($n = 1.5$). When the setup is shifted from air to the inside of a still, clear lake water ($n = 4/3$) then

(a) The fringe pattern gets enlarged
(b) The fringe pattern gets shrunk
(c) The fringe pattern gets shifted
(d) The fringe pattern remains unchanged

39. Two radio stations that are 250 m apart emit radio waves of wavelength 100 m. Point *A* is 400 m from both stations and point *B* is 450 m from both stations. Point *C* is 400 m from one station and 450 m from the other. The radio stations emit in phase. Which of the following statements is true?

(a) There will be constructive interference at *A* and *B* and destructive interference at *C*
(b) There will be destructive interference at *A* and *B* and constructive interference at *C*
(c) There will be constructive interference at *A* and destructive interference at *B* and *C*
(d) There will be constructive interference at *B* and *C* and destructive interference at *A*

40. A soap film of thickness *t* is surrounded by air. It is illuminated at near normal incidence by monochromatic light which has wavelength λ in the film. What will be the film thickness that will produce maximum brightness of the reflected light?

(a) $(1/4)\lambda$
(b) $(1/2)\lambda$
(c) λ
(d) 2λ

41. A very thin oil film of refractive index 1.2 floats on the surface of water ($n = 4/3$). When its thickness is 10^{-7} m, then it appears dark when seen normally. The minimum change in thickness for which it will appear bright in normally reflected light of the same color is

(a) 10^{-7} m
(b) 2×10^{-7} m
(c) 3×10^{-7} m
(d) 5×10^{-7} m

42. White light is normally incident on a soap bubble in air, and the reflected light is viewed. As the wall thickness of the bubble approaches zero (just before it breaks) the reflected light

(a) Becomes dim for all wavelengths
(b) Becomes unusually bright for all wavelengths
(c) Becomes bright only for certain wavelengths, dependent on the index of refraction of the bubble
(d) None of the above occurs

43. Light is reflecting off a wedge-shaped thin piece of glass producing bright and dark interference fringes. If a certain location has a bright fringe, a nearby point will have a dark fringe if the thickness of the glass increases by

(a) 1/8 of a wavelength of the light in glass
(b) 1/4 of a wavelength of the light in glass
(c) 1/2 of a wavelength of the light in glass
(d) 1 wavelength of the light in glass

44. Consider two identical microscope slides in air illuminated with monochromatic light. The bottom slide is rotated (counterclockwise about the point of contact in the side view) so that the wedge angle gets a bit smaller. What happens to the fringes?

(a) They are spaced farther apart
(b) They are spaced closer together
(c) They do not change
(d) Bright and dark spots are interchanged

45. Radio waves are readily diffracted around buildings whereas light waves are negligibly diffracted around buildings. This is because radio waves

(a) Are plane polarized
(b) Have much longer wavelengths than light waves
(c) Have much shorter wavelengths than light waves
(d) Are nearly monochromatic (single frequency)

46. The shimmering or wavy lines that can often be seen near the ground on a hot day are due to

(a) Brownian movement
(b) Reflection
(c) Refraction
(d) Diffraction

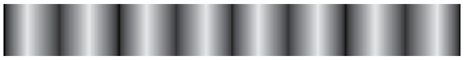
47. If the Huygen's principle applies to any point anywhere in a beam path, why does not a laser beam without any slit spread out in all directions?

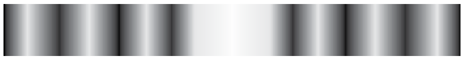
(a) Because all waves that spread interfere destructively
(b) It does spread, but the spread is so small that we normally do not notice it
(c) We cannot apply the Huygen's principle anywhere but in slits and apertures

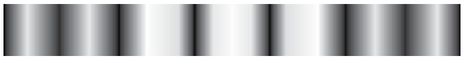
48. No fringes are seen in a single-slit diffraction pattern if

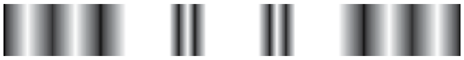
(a) The screen is far away
(b) The wavelength is less than the slit width
(c) The wavelength is greater than the slit width
(d) The distance to the screen is greater than the slit width

49. Monochromatic plane waves of light are incident normally on a single slit. Which option correctly shows the diffraction pattern observed on a distant screen?

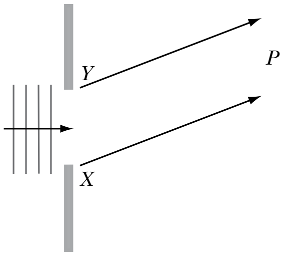
(a)


(b)


(c)


(d)


50. A single slit with the direction to a point *P* on a distant screen as shown in the following figure. At *P*, the pattern has its second minimum (from its central maximum). If X and Y are the edge of the slit, what is the path length difference $(PX) - (PY)$?



(a) $\lambda/2$ **(b)** λ
(c) $3\lambda/2$ **(d)** 2λ

51. At the first minimum adjacent to the central maximum of a single-slit diffraction pattern the phase difference between the Huygen's wavelet from the top of the slit and the wavelet from the midpoint of the slit is

(a) $\pi/8$ rad **(b)** $\pi/4$ rad
(c) $\pi/2$ rad **(d)** π rad

52. A diffraction-limited laser of length l and aperture diameter d generates light of wavelength λ. If the beam is directed at the surface of the Moon a distance D away, the radius of the illuminated area on the Moon is approximately

(a) dD/l **(b)** dD/λ
(c) $D\lambda/l$ **(d)** D

53. Two stars that are close together are photographed through a telescope. The black and white film is equally sensitive to all colors. Which situation would result in the most clearly separated images of the stars?

(a) Small lens, red stars **(b)** Small lens, blue stars
(c) Large lens, red stars **(d)** Large lens, blue stars

54. The resolving power of a telescope can be increased by

(a) Increasing the objective focal length and decreasing the eyepiece focal length
(b) Increasing the lens diameters
(c) Decreasing the lens diameters
(d) Inserting a correction lens between objective and eyepiece

55. In a double-slit diffraction experiment, the number of interference fringes within the central diffraction maximum can be increased by

(a) Increasing the wavelength
(b) Decreasing the wavelength
(c) Decreasing the slit separation
(d) Decreasing the slit width

56. A double slit is illuminated with monochromatic light of wavelength 6.00×10^2 nm. The $m = 0$ and $m = 1$ bright fringes are separated by 3.0 cm on a screen which is located 4.0 m from the slits. What is the separation between the slits?

(a) 4.0×10^{-5} m **(b)** 1.2×10^{-4} m
(c) 8.0×10^{-5} m **(d)** 1.6×10^{-4} m

57. Two slits are separated by 2.00×10^{-5} m. They are illuminated by light of wavelength 5.60×10^{-7} m. If the distance from the slits to the screen is 6.00 m, what is the separation between the central bright fringe and the third dark fringe?

(a) 0.421 m **(b)** 0.168 m
(c) 0.224 m **(d)** 0.084 m

58. A 4.0×10^2 nm thick film of kerosene ($n = 1.2$) is floating on water. White light is normally incident on the film. What is the visible wavelength *in air* that has a maximum intensity after the light is reflected?

Note: The visible wavelength range is 380 nm to 750 nm.

(a) 380 nm **(b)** 480 nm
(c) 430 nm **(d)** 530 nm

59. A portion of a soap bubble appears green ($\lambda = 500.0$ nm in vacuum) when viewed at normal incidence in white light. Determine the two smallest, non-zero thicknesses for the soap film if its index of refraction is 1.40.

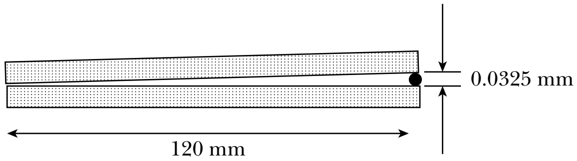
(a) 89 nm and 179 nm **(b)** 125 nm and 250 nm
(c) 89 nm and 268 nm **(d)** 125 nm and 375 nm.

60. Light of wavelength λ in vacuum strikes a lens that is made of glass with index of refraction 1.6. The lens has been coated with a film of thickness t and index of refraction 1.3. For which one of the following conditions will there be no reflection?

(a) $2t = \dfrac{\lambda}{2}$ **(b)** $2t = \dfrac{1}{2}\left(\dfrac{\lambda}{1.3}\right)$

(c) $2t = \dfrac{\lambda}{1.33}$ **(d)** $2t = \dfrac{1}{2}\left(\dfrac{\lambda}{1.6}\right)$

61. Light of wavelength 650 nm is incident normally upon a glass plate. The glass plate rests on top of a second plate so that they touch at one end and are separated by 0.0325 mm at the other end as shown in the figure.

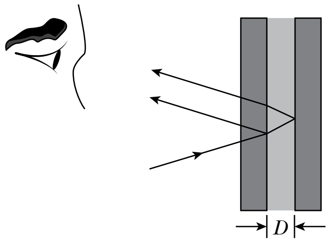


Which range of values contains the horizontal separation between adjacent bright fringes?

(a) 1.1 mm to 1.4 mm **(b)** 2.8 mm to 4.2 mm
(c) 1.4 mm to 2.8 mm **(d)** 4.2 mm to 5.6 mm

62. Two glass plates, each with an index of refraction of 1.55, are separated by a small distance D. The space between the plates is filled with water ($n = 1.33$) as shown. For which one of the following conditions will the reflected light appear green?

Note: The wavelength of green light is 460 nm in vacuum.



(a) $D = \left(\dfrac{460\text{ nm}}{2}\right)$ **(b)** $2D = \dfrac{1}{2}\left(\dfrac{460\text{ nm}}{1.33}\right)$

(c) $2D = \left(\dfrac{460\text{ nm}}{1.33}\right)$ **(d)** $2D = \dfrac{1}{2}\left(\dfrac{460\text{ nm}}{1.55}\right)$

63. Light of 600.0 nm is incident upon a single slit. The resulting diffraction pattern is observed on a screen that is 0.50 m from the slit. The distance between the first and third minima of the diffraction pattern is 0.80 mm. Which range of values listed below contains the width of the slit?

(a) 0.1 mm to 0.4 mm **(b)** 0.8 mm to 1.2 mm

(c) 0.4 mm to 0.8 mm **(d)** 1.2 mm to 1.6 mm

64. A spy satellite is in orbit at a distance of 1.0×10^6 m above the ground. It carries a telescope that can resolve the two rails of a railroad track that are 1.4 m apart using light of wavelength 600 nm. Which one of the following statements best describes the diameter of the lens in the telescope?

(a) It is less than 0.14 m.

(b) It is greater than 0.14 m and less than 0.23 m.

(c) It is greater than 0.23 m and less than 0.35 m.

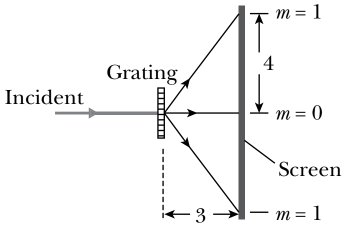
(d) It is greater than 0.52 m.

65. Light from two sources, $\lambda_1 = 623$ nm and $\lambda_2 = 488$ nm, is incident on a diffraction grating that has 5550 lines/cm. What is the angular separation, $\lambda_1 - \lambda_2$, of the second order maxima of the two waves?

(a) 11.0° **(b)** 25.0°

(c) 15.0° **(d)** 32.8°

66. Light from a laser ($\lambda = 640$ nm) passes through a diffraction grating and spreads out into three beams as shown in the figure. Determine the spacing between the slits of the grating.



(a) 240 nm **(b)** 680 nm

(c) 410 nm **(d)** 800 nm

More than One Correct Choice Type

67. If white light is used in a Young's double-slit experiment, the

(a) Bright white fringe is formed at the center of the screen

(b) Fringes of different colors are observed clearly only in the first order

(c) The first-order violet fringes are closer to the center of the screen than the first-order red fringes

(d) The first-order red fringes are closer to the center of the screen than the first-order violet fringes

68. In a Young's double-slit experiment, let A and B be the two slits. A thin film of thickness t and refractive index n is placed in front of A. Let β be the fringe width. The central maximum will shift

(a) Toward A **(b)** Toward B

(c) By $t(n-1)\dfrac{\beta}{\lambda}$ **(d)** By $nt\dfrac{\beta}{\lambda}$

69. In Young's double-slit experiment, the ratio of intensities of bright and dark fringes is 9. This means that

(a) The intensities of individual sources are 5 units and 4 units, respectively

(b) The intensities of individual sources are 4 units and 1 unit, respectively

(c) The ratio of their amplitudes is 3

(d) The ratio of their amplitudes is 2

70. In a Young's double-slit experiment, the separation between the two slits is d and the wavelength of the light is λ. The intensity of light falling on slit 1 is four times the intensity of light falling on slit 2. Choose the correct choice(s):

(a) If $d = \lambda$, the screen will contain only one maximum.

(b) If $\lambda < d < 2\lambda$, at least one more maximum (besides the central maximum) will be observed on the screen.

(c) If the intensity of light failing on slit 1 is reduced so that it becomes equal to that of slit 2, the intensities of the observed dark and bright fringes will increase.

(d) If the intensity of light falling on slit 2 is increased so that it becomes equal to that of slit 1, the intensities of the observed dark and bright fringes will increase.

71. White light is used to illuminate the two slits in a Young's double-slit experiment. The separation between the slits is d and the screen is at a distance D ($\gg d$) from the slits. At a point on the screen directly in front of one of the slits, certain wavelengths are missing. Some of the missing wavelengths are

(a) $\lambda = \dfrac{d^2}{D}$ **(b)** $\lambda = \dfrac{2d^2}{D}$

(c) $\lambda = \dfrac{d^2}{3D}$ **(d)** $\lambda = \dfrac{2d^2}{3D}$

72. In Young's double-slit experiment, white light is used. The separation between the slits is d. The screen is at a distance D ($D \gg d$) from the slits. Some wavelengths are missing exactly in front of one slit. These wavelengths are

(a) $\lambda = \dfrac{d^2}{D}$ **(b)** $\lambda = \dfrac{2d^2}{D}$

(c) $\lambda = \dfrac{d^2}{3D}$ **(d)** $\lambda = \dfrac{2d^2}{3D}$

73. If the first minima in a Young's double-slit experiment occurs directly in front of one of the slits (distance between slit 1 and screen is $D = 12$ cm and distance between slits is $d = 5$ cm), then the wavelength of the radiation used can be

(a) 2 cm **(b)** 4 cm

(c) 2/3 cm **(d)** 4/3 cm

74. An interference pattern is formed on the screen, when light from two different monochromatic sources are allowed to interfere. Then, it is true that

(a) Frequencies of light from the two sources are equal to each other

(b) The sources are coherent

(c) The sources should be located in the same medium

(d) The path difference should either be an even or an odd multiple of $\lambda/2$, where λ is the wavelength of light

75. The intensity of a plane progressive wave in loss free medium is

(a) Directly proportional to the square of amplitude of the wave
(b) Directly proportional to the velocity of the wave
(c) Directly proportional to the square of frequency of the wave
(d) Inversely proportional to the density of the medium

76. If one of the slits of a standard Young's double-slit experiment is covered by a thin parallel sided glass slab so that it transmits only one-half the light intensity of the other, then

(a) The fringe pattern will get shifted toward the covered slit
(b) The fringe pattern will get shifted away from the covered slit
(c) The bright fringes will become less bright and the dark ones will become more bright
(d) The fringe width will remain unchanged

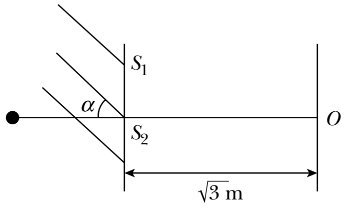
77. If instead of monochromatic light, white light is used in Young's double-slit experiment, then

(a) A bright white fringe is formed at the center of the screen
(b) Fringes of different colors are clearly observed only in the first-order place
(c) The first-order red fringes are relatively close to the screen than the first-order violet fringes
(d) The first-order violet fringes are relatively close to the center of the screen than the first-order red fringes

78. Huygens' principle of secondary wavelets may be used to

(a) Find the velocity of light in vacuum
(b) Explain the particle behavior of light
(c) Find the new position of a wavefront
(d) Explain Snell's law

79. A parallel beam of light (λ = 5000 Å) is incident at an angle $\alpha = 30°$ with the normal to the slit plane in a Young's double-slit experiment. Assume that the intensity due to each slit at any point on the screen is I_0. Point O is equidistant from S_1 and S_2 (see the following figure). The distance between slits is 1 mm



(a) The intensity at O is $4I_0$
(b) The intensity at O is zero
(c) The intensity at a point on the screen 1 m below O is $4I_0$
(d) The intensity at a point on the screen 1 m below O is zero

80. Which of the following can give sustained interference?

(a) Two independent laser sources
(b) Two independent light bulbs
(c) Two independent sound sources
(d) Two independent microwave sources

81. A Young's double-slit experiment is performed with white light. Then

(a) The central fringe will be white
(b) There will not be a completely dark fringe
(c) The fringe next to the central will be red
(d) The fringe next to the central will be violet

82. Fraunhoffer diffraction can be observed with

(a) One wide slit
(b) One narrow slit
(c) Two narrow slits
(d) Large number of narrow slits

83. A person views the interference pattern, produced by two slits illuminated by white light, on placing a green filter in front of his eyes. Then

(a) He will see sharply distinguishable dark and bright fringes
(b) The fringes will not be sharp but are differentiable
(c) On replacing the green filter by a blue filter, the fringes will be again sharp but closer than those by the green filter
(d) On using both the filters simultaneously, the central bands will be maximum bright

84. In a Young's double-slit experiment apparatus, we use white light. Then

(a) The fringe next to the central will be red
(b) The central fringe will be white
(c) The fringe next to the central will be violet
(d) There will not be a completely dark fringe

85. Two sources are called coherent if they produce waves

(a) Of equal wavelength
(b) Of equal velocity
(c) Having same shape of wavefront
(d) Having a constant phase difference

86. To observe a stationary interference pattern formed by two light waves, it is not necessary that they must have

(a) The same frequency
(b) Same amplitude
(c) A constant phase difference
(d) The same intensity

87. If one of the slits of a standard Young's double-slit experiment apparatus is covered by a thin parallel sided glass slab so that it transmits only one-half of the light intensity of the other, then

(a) The fringe pattern will get shifted toward the covered slit
(b) The fringe pattern will get shifted away from the covered slit
(c) The bright fringes will be less bright, and the dark ones will be more bright
(d) The fringe width will remain unchanged

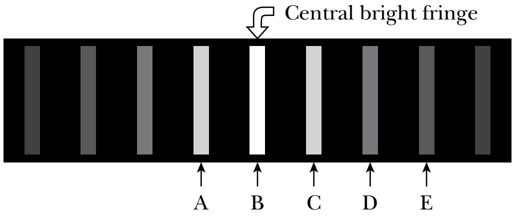
88. Mark the correct statements(s):

(a) Direction of wave propagation is along the normal to wavefront
(b) For a point source of light, the shape of wavefronts can be considered to be plane at very large distance from the source
(c) A point source of light is placed at the focus of a thin spherical lens, then the shape of the wavefront for emerged light can be plane
(d) The shape of the wavefront for the light incident on a thin spherical lens is plane, the shape of the wavefront corresponding to emergent light would always be spherical

89. The fringes produced on a screen by blue monochromatic light falling upon a double-slit arrangement were found to be too close together for measurements to be taken on them. The fringe separation could be increased by
- **(a)** Increasing the distance between the slits and the screen
- **(b)** Replacing the light with monochromatic red light
- **(c)** Increasing the separation of the slits
- **(d)** Decreasing the wavelength of the incident light

Linked Comprehension

Paragraph for Questions 90–93: The figure shows the interference pattern obtained in a double-slit experiment using light of wavelength 600 nm.



90. Which fringe is the same distance from both slits?
- **(a)** A
- **(b)** C
- **(c)** B
- **(d)** D

91. Which fringe is the third order maximum?
- **(a)** A
- **(b)** E
- **(c)** B
- **(d)** D

92. Which fringe is 300 nm closer to one slit than to the other?
- **(a)** A
- **(b)** C
- **(c)** B
- **(d)** D

93. Which fringe results from a phase difference of 4π?
- **(a)** A
- **(b)** C
- **(c)** B
- **(d)** D

Paragraph for Questions 94 and 95**:** Angular width of central maxima in single-slit diffraction pattern for light of wavelength 600 nm is measured. For light of wavelength λ, the angular width decreases by 30%. The same decrease in angular width is measured when entire apparatus is immersed in a liquid of refractive index n.

94. Value of λ is
- **(a)** 4200 nm
- **(b)** 420 nm
- **(c)** 800 nm
- **(d)** 320 nm

95. Value of μ is
- **(a)** $\frac{4}{3}$
- **(b)** 1.5
- **(c)** 1.43
- **(d)** 1.86

Paragraph for Questions 96–98**:** An initially parallel cylindrical beam travels in a medium of refractive index $n(I) = n_1 + n_2 I$, where, n_1 and n_2 are positive constants and I is the intensity of the light beam. The intensity of the light beam is decreasing with increasing radius.

96. The initial shape of the wavefront is
- **(a)** Convex
- **(b)** Concave
- **(c)** Convex near the axis and concave near the periphery
- **(d)** Planar

97. The speed of the light in the medium is
- **(a)** Maximum on the axis of the beam
- **(b)** Minimum on the axis of the beam
- **(c)** The same everywhere in the beam
- **(d)** Directly proportional to the intensity I

98. As the beam enters the medium, it will
- **(a)** Diverge
- **(b)** Converge
- **(c)** Diverge near the axis and converge near the periphery
- **(d)** Travel as a cylindrical beam

Paragraph for Questions 99–103**:** When waves from two coherent sources, having amplitudes a and b superimpose, the amplitude R of the resultant wave is given by $R = \sqrt{a^2 + b^2 + 2ab\cos\phi}$, where ϕ is the constant phase angle between the two waves. The resultant intensity I is directly proportional to the square of the amplitude of the resultant wave, that is, $I \propto R^2$, that is, $I \propto (a^2 + b^2\ 2ab\cos\phi)$. For constructive interference, $\phi = 2n\pi$ and $I_{\max} = (a + b)^2$. For destructive interference, $\phi = (2n - 1)\ \phi$ and $I_{\min} = (a - b)^2$. If I_1, I_2 are intensities of light from two slits of widths ω_1 and ω_2, $I_1/I_2 = \omega_1/\omega_2 = a^2/b^2$. Light waves from two coherent sources of intensity ratio 81 : 1 produce interference.

99. The ratio of amplitudes of two sources is
- **(a)** 9 : 1
- **(b)** 81 : 1
- **(c)** 1 : 9
- **(d)** 1 : 81

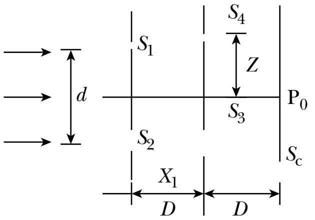
100. The ratio of slit widths of the two sources is
- **(a)** 9 : 1
- **(b)** 81 : 1
- **(c)** 1 : 9
- **(d)** 1 : 81

101. The ratio of maxima and minima in the interference pattern is
- **(a)** 9 : 1
- **(b)** 81 : 1
- **(c)** 25 : 16
- **(d)** 16 : 25

102. If two slits in Young's experiment have width ratio 1 : 4, the ratio of maximum and minimum intensity in the interference pattern would be
- **(a)** 1 : 4
- **(b)** 1 : 16
- **(c)** 9 : 1
- **(d)** 9 : 16

103. The ratio of amplitudes of light waves from two sources is
- **(a)** 1 : 4
- **(b)** 4 : 1
- **(c)** 2 : 1
- **(d)** 1 : 2

Paragraph for Questions 104–106**:** Consider the situation shown in the below figure. The two slits S_1 and S_2 placed symmetrically around the central line are illuminated by monochromatic light of wavelength, λ. The separation between the slits is d. The light transmitted by the slits falls on a screen S_0 placed at a distance D from the slits. The slit S_3 is at the central line and the slit S_4 is at a distance Z from S_3. Another screen S_c is placed a further distance D away from S_c. Find the ratio of the maximum to minimum intensity observed on S_c.



104. If $Z = \lambda D/2d$

(a) 1 **(b)** 1/2

(c) 3/2 **(d)** 2

105. If $Z = \lambda D/d$

(a) 4 **(b)** 2

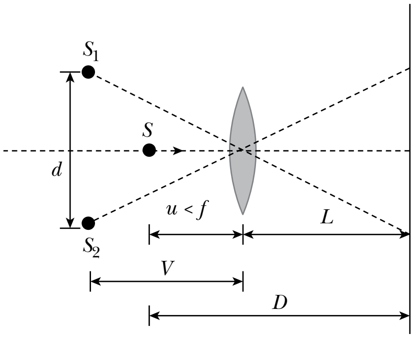
(c) ∞ **(d)** 1

106. If $Z = \lambda D/4d$

(a) $[3-2\sqrt{2}]^2$ **(b)** $[3+2\sqrt{2}]^2$

(c) $[3-\sqrt{2}]^2$ **(d)** $[3+2\sqrt{2}]^2$

***Paragraph for Questions 107 and 108*:** A lens of focal length $f = 40$ cm is cut along the diameter into two identical halves. In this process, a layer of the lens $t = 1$ mm in thickness is lost, then the halves are put together to form a composite lens. In between the focal plane and the composite lens, a narrow slit is placed, near the focal plane. The slit is emitting monochromatic light with wavelength $\lambda = 0.6$ mm. Behind the lens, a screen is located at a distance $L = 0.5$ m from it (see the following figure).



107. Find the fringe width for the pattern obtained under given arrangement on the screen.

(a) 0.19 m **(b)** 0.24 m

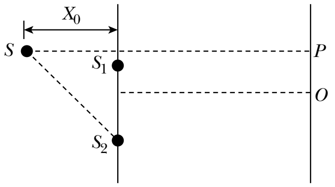
(c) 0.3 m **(d)** 0.43 m

108. The expression for the number of visible maxima which is obtained through above said arrangement will turn out to be

(a) $\dfrac{Lt^2}{\lambda f^2}$ **(b)** $\dfrac{2Lt^2}{\lambda f^2}$

(c) $\dfrac{Lt}{2\lambda f^2}$ **(d)** $\dfrac{Lt^2}{2\lambda f^2}$

***Paragraph for Questions 109 and 110*:** In a modified Young's double-slit experiment, source S is kept in front of slit S_1 as shown in the following figure. Find the phase difference at a point O that is equidistant from slits S_1 and S_2, and point P that is in front of slit S_1 in the following situations:



109. If a liquid of refractive index n is filled between the screen and slits, what is the phase difference?

(a) $\dfrac{2\pi}{\lambda}\left[\left[\sqrt{d^2+x_0^2}+x_0\right]+\dfrac{nd^2}{2D}\right]$

(b) $\dfrac{2\pi}{\lambda}\left[\left[\sqrt{d^2+x_0^2}-x_0\right]+\dfrac{nd^2}{2D}\right]$

(c) $\dfrac{2\pi}{\lambda}\left[\left[\sqrt{d^2-x_0^2}+x_0\right]+\dfrac{nd^2}{2D}\right]$

(d) $\dfrac{2\pi}{\lambda}\left[\left[\sqrt{d^2-x_0^2}-x_0\right]+\dfrac{nd^2}{2D}\right]$

110. If a liquid is filled between the slit and the source S, what is the phase difference?

(a) $\dfrac{2\pi}{\lambda}\left[n\sqrt{d^2+x_0^2}-x_0-\dfrac{d^2}{2D}\right]$

(b) $\dfrac{2\pi}{\lambda}\left[n\sqrt{d^2+x_0^2}-x_0+\dfrac{d^2}{2D}\right]$

(c) $\dfrac{2\pi}{\lambda}\left[n\sqrt{d^2-x_0^2}+x_0+\dfrac{d^2}{2D}\right]$

(d) $\dfrac{2\pi}{\lambda}\left[n\sqrt{d^2-x_0^2}-x_0-\dfrac{d^2}{2D}\right]$

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

111. Match the property with the concept it is based on.

Column I	Column II
(a) Interference	**(p)** Corpuscular theory
(b) Compton effect	**(q)** Longitudinal waves
(c) Diffraction	**(r)** Transverse waves
(d) Polarization	**(s)** Dual theory

112. Match the observation given in Column I with the experiments given in Column II.

Column I	Column II
(a) Central fringes is bright and all fringes are equally spaced	**(p)** Interference in wedge-shaped thin
(b) Central fringes is dark and all fringes are equally spaced	**(q)** Young's double-slit experiment
(c) Central fringes is bright and fringes are not equally spaced	**(r)** Fraunhoffer single-slit diffraction experiment
(d) All fringes are equally spaced	**(s)** Lloyd's mirror experiment

Directions for Questions 113 and 114: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

113. Interference is an important property of light. In the given table, Column I shows the properties of interfering waves, Column II shows the phase difference, and Column III shows the path difference between two interfering waves.

Column I	Column II	Column III
(I) Produced by systematic fluctuations.	**(i)** Phase difference $= 2n\pi$	**(J)** Path difference will vary continuously
(II) Points of maximum intensity	**(ii)** Phase difference $= (2n - 1)\pi$	**(K)** Path difference $= n\lambda$
(III) Not very short-lived (produced by random fluctuations)	**(iii)** Phase difference $= 2\pi/\lambda(xd/D)$	**(L)** Path difference $= xd/D$
(IV) Points of minimum intensity	**(iv)** Phase difference will vary continuously	**(M)** Path difference $= (2n-1)\lambda/2$

(1) What are the conditions for constructive interference?
(a) (I) (iv) (M) **(b)** (IV) (ii) (L)
(c) (II) (i) (K) **(d)** (II) (i) (K)

(2) What are the conditions for destructive interference?
(a) (I) (ii) (J) **(b)** (IV) (ii) (M)
(c) (II) (iii) (L) **(d)** (I) (i) (M)

(3) What are the conditions for sustained interference?
(a) (II) (ii) (J) **(b)** (III) (iii) (L)
(c) (IV) (i) (L) **(d)** (II) (iii) (M)

114. The amplitudes of two coherent waves are vector quantities. In the given table, Column I shows the phase difference between two coherent waves, Column II shows the path difference between them and Column III shows their resultant amplitudes.

Column I	Column II	Column III
(I) $\phi = 0$	**(i)** $\delta = 0$	**(J)** $E_R = 2E_0$
(II) $\phi = 120$	**(ii)** $\delta = \lambda/6$	**(K)** $E_R = 0$
(III) $\phi = 60(2\pi)$	**(iii)** $\delta = \lambda/3$	**(L)** $E_R = 3E_0$
(IV) $\phi = 180$	**(iv)** $\delta = \lambda/2$	**(M)** $E_R = E_0$

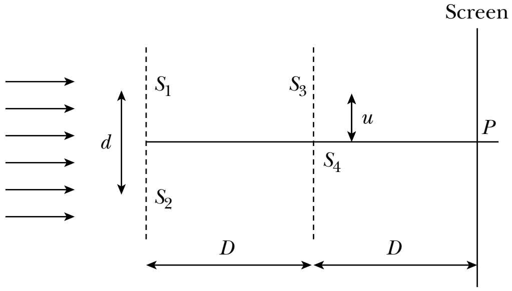
(1) When is the phasor diagram at primary maxima?
(a) (I) (iii) (M) **(b)** (III) (ii) (K)
(c) (I) (i) (L) **(d)** (I) (iii) (J)

(2) When is E_R maximum?
(a) (I) (ii) (M) **(b)** (III) (ii) (J)
(c) (II) (ii) (K) **(d)** (II) (iv) (M)

(3) When is the phasor diagram at secondary maxima?
(a) (III) (i) (L) **(b)** (IV) (i) (J)
(c) (III) (iv) (J) **(d)** (IV) (iv) (M)

Integer Type

115. In the arrangement shown in the following figure, the wavelength of light used is λ. The distance between slits S_1 and S_2 is d ($\ll D$). The distance between S_3 and S_4 is $u = \lambda D/3d$. If the ratio of maximum to minimum intensity observed on screen is k. Find k.

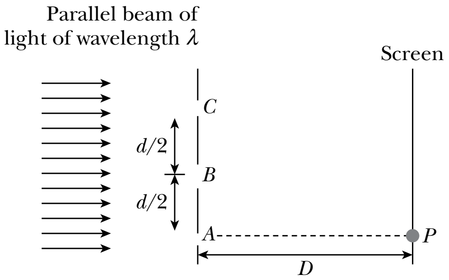


116. Light used in a Young's double-slit experiment consists of two wavelengths 450 nm and 720 nm. In the interference pattern, eighth maximum of the first coincides with the mth maximum of the second. What is m?

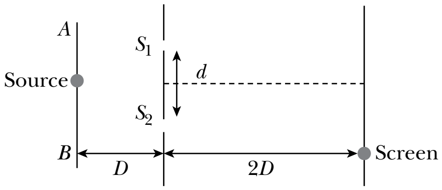
117. In Young's double-slit experiment, $d = 2$, $D = 2$ m, and $\lambda = 500$ nm. If intensity of two slits are I_0 and $9I_0$, then the intensity at $y = (1/6)$ mm is KI_0. Find K.

118. Two identical coherent sources of wavelength λ are placed at $(100\lambda, 0)$ and $(-50\lambda, 0)$, respectively. A detector moves slowly from origin to $(2\lambda, 0)$ along x-axis. What is the number of maxima detected [including origin and $(2\lambda, 0)$]?

119. For the situation depicted in the following figure, $BP - AP = \lambda/3$ and $D \gg d$. The slits are of equal widths, having intensity I_0. If the intensity of P is kI_0, find k.



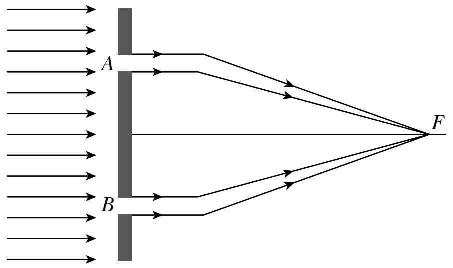
120. In a Young's double-slit experiment, the point source of light is placed as shown in the following figure. The source starts moving at $t = 0$ from the shown position in upward direction along AB with constant velocity 3 m/s the wavelength is λ and $D \gg d$ and $d \gg \lambda$. Find the velocity of central maxima (in m/s).



121. A plane monochromatic light wave falls normally on a single slit. At a distance $D = 4$ m from it, there is a screen showing a certain diffraction pattern. The slit width is decreased $n = 2$ times. Calculate the new distance D' at which the screen should be positioned to obtain the diffraction pattern similar to previous one but diminished n times.

122. One slit of a Young's experiment is covered by a glass plate ($n_1 = 1.4$) and the other by another glass plate ($n_2 = 1.7$) of the same thickness. The point of central maxima on the screen, before the plates were introduced is now occupied by the third bright fringe. Find the thickness of the plate, the wavelength of light used is 4000 Å.

123. In a modified Young's double-slit experiment, a monochromatic uniform and parallel beam of light of wavelength 6000 Å and intensity ($10/\pi$) W/m^2 is incident normally on two apertures A and B of radii 0.001 m and 0.002 m, respectively. A perfectly transparent film of thickness 2000 Å and refractive index 1.5 for the wavelength of 6000 Å is placed in front of aperture A (see the following figure). Calculate the power (in W) received at the focal spot F of the lens. The lens is symmetrically placed with respect to the apertures. Assume that 10% of the power received by each aperture goes in the original direction and is brought to the focal spot.



ANSWER KEY

Checkpoints

1. b (least n), c, a

2. (a) top; (b) bright intermediate illumination (phase difference is 2.1 wavelengths)

3. (a) 3λ, 3; (b) 2.5λ, 2.5

4. a and d tie (amplitude of resultant wave is $4E_0$), then b and c tie (amplitude of resultant wave is $2E_0$)

5. (a) 1 and 4; (b) 1 and 4

6. (a) expand; (b) expand

7. diminsh

Problems

1. (a) 10.3 nm/s; (b) 1.10 μm

2. 0.0063°

3. 1.00025

4. 150

5. 11

6. (a) 0.700; (b) intermediate closer to fully destructive

7. (a) 5.09×10^{14} Hz; (b) 387.5 nm; (c) 1.97×10^8 m/s

8. 3.30

9. 624 nm

10. 1.85 mm

11. (a) 550 nm; (b) 413 nm; (c) longer wavelength

12. 0.13 mm

13. (a) 55° (same as incident angle); (b) 1.4×10^{-13} s

14. (a) 5.26 mm; (b) intermediate closer to fully constructive

15. 6.25×10^{-8} m

16. (a) 456 nm; (b) 608 nm

17. 1.66

18. 840 nm

19. (a) 87.5 nm; (b) 175 nm

20. (a) 0.25; (b) 0.75; (c) 1.25

21. (a) between the central maximum (zero wavelength difference) and the first minimum (1/2 wavelength difference); (b) 0.196

22. 1.11 μm

23. (a) 51.38 nm; (b) 154.1 nm

24. 86.2 nm

25. 694 nm

26. (a) 0.858 μm; (b) 2.58 μm

27. (a) 600 nm, any wavelength greater than this will not be seen. Thus, 600 nm $< \theta \leq$ 700 nm are absent; (b) the slit separation d needs to be decreased; (c) 0.20 μm

28. (a) 79.7 nm; (b) 239 nm

29. (a) 2.33 μV/m; (b) 0.338; (c) point P is between the sixth side maximum (at which $\phi = 6$ wavelengths) and the seventh minimum (at which $\phi = 6\frac{1}{2}$ wavelengths); (d) 1.26×10^{15} rad/s; (e) 2270°

30. (13.7 μV/m) sin[(2.0×10^{14} rad/s)t

31. 0.785 rad

32. (a) 0; (b) 0; (c) ∞; (d) $12p$; (e) $3.6p$; (f) "intermediate" but closer to a minimum than to a maximum

33. (a) 1.15; (b) 1.15; (c) 0.85; (d) all tie

34. (a) 0.299 rad; (b) 17.1°

35. 4.74×10^{-6} m

36. (a) 0.010 rad; (b) 9.0 mm

37. 450 nm

38. 0.19°

39. (a) 7.88 μm; (b) 2.03 μm
40. 3.5 μm
41. (a) 14; (b) 16
42. 7.5
43. 49/25
44. 0.20 mm
45. $\sqrt{3} \times 1000\lambda$
46. 38
47. (a) 0.168 mm; (b) 31.5 mm
48. 6
49. (a) 1.1×10^7 m = 1.1×10^4 km; (b) 1.1×10^4 m = 11 km
50. (a) 7.1 μm; (b) 28 μm; (c) for m = 1, I = 5.7 mW/cm^2; for m = 2, I = 2.9 mW/cm^2
51. (a) 2.33 μm; (b) 3.82; (c) 15.2°; (d) 51.8°
52. 9
53. 51.4 m
54. (a) 11; (b) 0.405
55. (a) 840 nm; ((b) 4; (c) 6
56. 3.90 mm
57. (a) 1.5×10^{-4} rad; (b) 9.39×10^4 m ≈ 9.4 km
58. 2.00
59. 0.09 π rad = 17°
60. (a) diffraction effects in general would decrease; (b) 9.6; (c) 0.010 m.
61. 38 μm
62. (a) 0.487; (b) 0.60
63. (a) 70 cm; (b) 0.49 mm
64. (a) ≈ 56 m; (b) no; (c) no
65. (b) 0; (c) – 0.500; (d) 4.493 rad; (e) 0.930; (f) 7.725 rad; (g) 1.96
66. (a) 0.33 m; (b) 2.8 m; (c) the military satellites do not use Hubble Telescope-sized apertures
67. 2.41×10^{-5} m
68. (a) 8.1; (b) since sin θ cannot exceed 1 there is no minimum
69. (a) 0.516°; (b) 0.098 mm
70. (a) 0.36°; (b) 2.35 rad ≈ 2.4 rad; (c) 0.091
71. $\dfrac{D\lambda}{d}$
72. (a) 4*a*; (b) every fourth fringe is missing

Practice Questions

Single Correct Choice Type

1. (b)
2. (a)
3. (b)
4. (d)
5. (a)
6. (d)
7. (d)
8. (b)
9. (c)
10. (b)
11. (b)
12. (c)
13. (c)
14. (a)
15. (b)
16. (c)
17. (c)
18. (b)
19. (d)
20. (b)
21. (b)
22. (a)
23. (c)
24. (d)
25. (a)
26. (c)
27. (a)
28. (b)
29. (d)
30. (c)
31. (c)
32. (b)
33. (d)
34. (d)
35. (d)
36. (a)
37. (c)
38. (a)
39. (a)
40. (a)
41. (a)
42. (a)
43. (b)
44. (a)
45. (b)
46. (c)
47. (a)
48. (c)
49. (b)
50. (d)
51. (d)
52. (d)
53. (d)
54. (b)
55. (d)
56. (c)
57. (a)
58. (b)
59. (c)
60. (b)
61. (a)
62. (b)
63. (c)
64. (d)
65. (a)
66. (d)

More than One Correct Choice Type

67. (a), (b), (c)
68. (a), (c)
69. (b), (d)
70. (a), (b)
71. (a), (c)
72. (a), (c)
73. (a), (c)
74. (a), (b)
75. (a), (b), (c)
76. (a), (c), (d)
77. (a), (b), (d)
78. (c), (d)
79. (a), (c)
80. (a), (c), (d)
81. (a), (b), (d)
82. (b), (c), (d)
83. (a), (c)
84. (b), (c), (d)
85. (a), (b), (c), (d)
86. (b), (d)
87. (a), (c), (d)
88. (a), (b), (c), (d)
89. (a), (b)

Linked Comprehension

90. (c)
91. (b)
92. (a)
93. (d)
94. (b)
95. (c)
96. (d)
97. (b)
98. (b)
99. (a)
100. (b)
101. (c)
102. (a)
103. (d)
104. (a)
105. (c)
106. (d)
107. (c)
108. (c)
109. (b)
110. (a)

Matrix-Match

111. (a) → (r), (s); (b) → (p), (s); (c) → (q), (r), (s); (d) → (r), (s)

112. (a) → (q); (b) → (s); (c) → (r); (d) → (p)

113. (1) → (d); (2) → (b); (3) → (b)

114. (1) → (c); (2) → (b); (3) → (d)

Integer Type

115. 9

116. 5

117. 7

118. 3

119. 3

120. 6

121. 6

122. 4

123. 7

CHAPTER

36

Relativity

36.1 | WHAT IS PHYSICS?

One principal subject of physics is **relativity**, the field of study that measures events (things that happen): where and when they happen, and by how much any two events are separated in space and in time. In addition, relativity has to do with transforming such measurements (and also measurements of energy and momentum) between reference frames that move relative to each other. (Hence the name *relativity.*)

Transformations and moving reference frames, such as those we discussed in Chapter 4, Vol. 1, were well understood and quite routine to physicists in 1905. Then Albert Einstein (Fig. 36-1) published his **special theory of relativity**. The adjective *special* means that the theory deals only with **inertial reference frames**, which are frames in which Newton's laws are valid. (Einstein's *general theory of relativity* treats the more challenging situation in which reference frames can undergo gravitational acceleration; in this chapter the term *relativity* implies only inertial reference frames.)

Starting with two deceivingly simple postulates, Einstein stunned the scientific world by showing that the old ideas about relativity were wrong, even though everyone was so accustomed to them that they seemed to be unquestionable common sense. This supposed common sense, however, was derived only from experience with things that move rather slowly. Einstein's relativity, which turns out to be correct for all physically possible speeds, predicted many effects that were, at first study, bizarre because no one had ever experienced them.

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Figure 36-1 Einstein posing for a photograph as fame began to accumulate.

Contents

36.1 What is Physics?
36.2 The Postulates
36.3 Measuring an Event
36.4 The Relativity of Simultaneity
36.5 The Relativity of Time
36.6 The Relativity of Length
36.7 The Lorentz Transformation
36.8 Electricity and Magnetism
36.9 The Relativity of Velocities
36.10 Doppler Effect for Light
36.11 A New Look at Momentum
36.12 A New Look at Energy

Entangled. In particular, Einstein demonstrated that space and time are entangled; that is, the time between two events depends on how far apart they occur, and vice versa. Also, the entanglement is different for observers who move relative to each other. One result is that time does not pass at a fixed rate, as if it were ticked off with mechanical regularity on some master grandfather clock that controls the universe. Rather, that rate is adjustable: Relative motion can change the rate at which time passes. Prior to 1905, no one but a few daydreamers would have thought that. Now, engineers and scientists take it for granted because their experience with special relativity has reshaped their common sense. For example, any engineer involved with the Global Positioning System of the NAVSTAR satellites must routinely use relativity (both special relativity and general relativity) to determine the rate at which time passes on the satellites because that rate differs from the rate on Earth's surface. If the engineers failed to take relativity into account, GPS would become almost useless in less than one day.

Special relativity has the reputation of being difficult. It is not difficult mathematically, at least not here. However, it is difficult in that we must be very careful about *who* measures *what* about an event and just *how* that measurement is made—and it can be difficult because it can contradict routine experience.

36.2 | THE POSTULATES

Key Concept

- Einstein's special theory of relativity is based on two postulates: (1) The laws of physics are the same for observers in all inertial reference frames. (2) The speed of light in vacuum has the same value c in all directions and in all inertial reference frames.

We now examine the two postulates of relativity, on which Einstein's theory is based:

1. The Relativity Postulate: The laws of physics are the same for observers in all inertial reference frames. No one frame is preferred over any other.

Galileo assumed that the laws of *mechanics* were the same in all inertial reference frames. Einstein extended that idea to include *all* the laws of physics, especially those of electromagnetism and optics. This postulate does *not* say that the measured values of all physical quantities are the same for all inertial observers; most are not the same. It is the *laws of physics*, which relate these measurements to one another, that are the same.

2. The Speed of Light Postulate: The speed of light in vacuum has the same value c in all directions and in all inertial reference frames.

We can also phrase this postulate to say that there is in nature an *ultimate speed* c, the same in all directions and in all inertial reference frames. Light happens to travel at this ultimate speed. However, no entity that carries energy or information can exceed this limit. Moreover, no particle that has mass can actually reach speed c, no matter how much or for how long that particle is accelerated. (Alas, the faster-than-light warp drive used in many science fiction stories appears to be impossible.)

Both postulates have been exhaustively tested, and no exceptions have ever been found.

The Ultimate Speed

The existence of a limit to the speed of accelerated electrons was shown in a 1964 experiment by W. Bertozzi, who accelerated electrons to various measured speeds and—by an independent method—measured their kinetic energies. He found that as the force on a very fast electron is increased, the electron's measured kinetic energy increases toward very large values but its speed does not increase appreciably (Fig. 36-2). Electrons have been accelerated in laboratories to at least 0.999 999 999 95 times the speed of light but—close though it may be—that speed is still less than the ultimate speed c.

This ultimate speed has been defined to be exactly

$$c = 299\ 792\ 458\ \text{m/s}. \tag{36-1}$$

Caution: So far in this book we have (appropriately) approximated c as 3.0×10^8 m/s, but in this chapter we shall often use the exact value. You might want to store the exact value in your calculator's memory (if it is not there already), to be called up when needed.

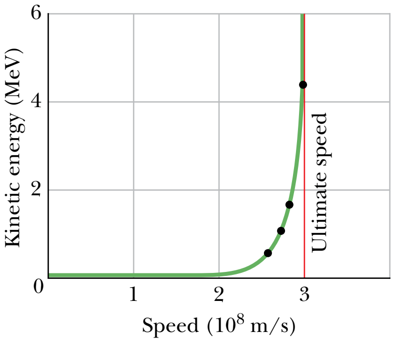


Figure 36-2 The dots show measured values of the kinetic energy of an electron plotted against its measured speed. No matter how much energy is given to an electron (or to any other particle having mass), its speed can never equal or exceed the ultimate limiting speed c. (The plotted curve through the dots shows the predictions of Einstein's special theory of relativity.)

Testing the Speed of Light Postulate

If the speed of light is the same in all inertial reference frames, then the speed of light emitted by a source moving relative to, say, a laboratory should be the same as the speed of light that is emitted by a source at rest in the laboratory. This claim has been tested directly, in an experiment of high precision. The "light source" was the *neutral pion* (symbol π^0), an unstable, short-lived particle that can be produced by collisions in a particle accelerator. It decays (transforms) into two gamma rays by the process

$$\pi^0 \rightarrow \gamma + \gamma. \tag{36-2}$$

Gamma rays are part of the electromagnetic spectrum (at very high frequencies) and so obey the speed of light postulate, just as visible light does. (In this chapter we shall use the term light for any type of electromagnetic wave, visible or not.)

In 1964, physicists at CERN, the European particle-physics laboratory near Geneva, generated a beam of pions moving at a speed of $0.999\ 75c$ with respect to the laboratory. The experimenters then measured the speed of the gamma rays emitted from these very rapidly moving sources. They found that the speed of the light emitted by the pions was the same as it would be if the pions were at rest in the laboratory, namely c.

36.3 | MEASURING AN EVENT

Key Concept

- Three space coordinates and one time coordinate specify an event. One task of special relativity is to relate these coordinates as assigned by two observers who are in uniform motion with respect to each other.

An **event** is something that happens, and every event can be assigned three space coordinates and one time coordinate. Among many possible events are (1) the turning on or off of a tiny lightbulb, (2) the collision of two particles, (3) the passage of a pulse of light through a specified point, (4) an explosion, and (5) the sweeping of the hand of a clock past a marker on the rim of the clock. A certain observer, fixed in a certain inertial reference frame, might, for example, assign to an event A the coordinates given in Table 36-1. Because space and time are entangled with each other in relativity, we can describe these coordinates collectively as *spacetime* coordinates. The coordinate system itself is part of the reference frame of the observer.

Table 36-1 Record of Event *A*

Coordinate	Value
x	3.58 m
y	1.29 m
z	0 m
t	34.5 s

A given event may be recorded by any number of observers, each in a different inertial reference frame. In general, different observers will assign different spacetime coordinates to the same event. Note that an event does not "belong" to any particular inertial reference frame. An event is just something that happens, and anyone in any reference frame may detect it and assign spacetime coordinates to it.

Travel Times. Making such an assignment can be complicated by a practical problem. For example, suppose a balloon bursts 1 km to your right while a firecracker pops 2 km to your left, both at 9:00 A.M. However, you do not detect either event precisely at 9:00 A.M. because at that instant light from the events has not yet reached you.

Because light from the firecracker pop has farther to go, it arrives at your eyes later than does light from the balloon burst, and thus the pop will seem to have occurred later than the burst. To sort out the actual times and to assign 9:00 A.M. as the happening time for both events, you must calculate the travel times of the light and then subtract these times from the arrival times.

This procedure can be very messy in more challenging situations, and we need an easier procedure that automatically eliminates any concern about the travel time from an event to an observer. To set up such a procedure, we shall construct an imaginary array of measuring rods and clocks throughout the observer's inertial frame (the array moves rigidly with the observer). This construction may seem contrived, but it spares us much confusion and calculation and allows us to find the coordinates, as follows.

1. **The Space Coordinates.** We imagine the observer's coordinate system fitted with a close-packed, three-dimensional array of measuring rods, one set of rods parallel to each of the three coordinate axes. These rods provide a way to determine coordinates along the axes. Thus, if the event is, say, the turning on of a small lightbulb, the observer, in order to locate the position of the event, need only read the three space coordinates at the bulb's location.
2. **The Time Coordinate.** For the time coordinate, we imagine that every point of intersection in the array of measuring rods includes a tiny clock, which the observer can read because the clock is illuminated by the light generated by the event. Figure 36-3 suggests one plane in the "jungle gym" of clocks and measuring rods we have described.

   The array of clocks must be synchronized properly. It is not enough to assemble a set of identical clocks, set them all to the same time, and then move them to their assigned positions. We do not know, for example, whether moving the clocks will change their rates. (Actually, it will.) We must put the clocks in place and *then* synchronize them.

   If we had a method of transmitting signals at infinite speed, synchronization would be a simple matter. However, no known signal has this property. We therefore choose light (any part of the electromagnetic spectrum) to send out our synchronizing signals because, in vacuum, light travels at the greatest possible speed, the limiting speed c.

   Here is one of many ways in which an observer might synchronize an array of clocks using light signals: The observer enlists the help of a great number of temporary helpers, one for each clock. The observer then stands at a point selected as the origin and sends out a pulse of light when the origin clock reads $t = 0$. When the light pulse reaches the location of a helper, that helper sets the clock there to read $t = r/c$, where r is the distance between the helper and the origin. The clocks are then synchronized.
3. **The Spacetime Coordinates.** The observer can now assign spacetime coordinates to an event by simply recording the time on the clock nearest the event and the position as measured on the nearest measuring rods. If there are two events, the observer computes their separation in time as the difference in the times on clocks near each and their separation in space from the differences in coordinates on rods near each. We thus avoid the practical problem of calculating the travel times of the signals to the observer from the events.

We use this array to assign spacetime coordinates.

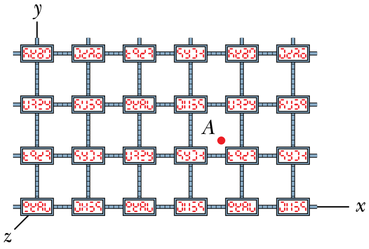


Figure 36-3 One section of a three-dimensional array of clocks and measuring rods by which an observer can assign spacetime coordinates to an event, such as a flash of light at point A. The event's space coordinates are approximately $x = 3.6$ rod lengths, $y = 1.3$ rod lengths, and $z = 0$. The time coordinate is whatever time appears on the clock closest to A at the instant of the flash.

36.4 | THE RELATIVITY OF SIMULTANEITY

Key Concepts

- If two observers are in relative motion, they generally will not agree as to whether two events are simultaneous.
- If two successive events occur at the same place in an inertial reference frame, the time interval Δt_0 between them, measured on a single clock where they occur, is the proper time between them. Observers in frames moving relative to that frame will always measure a *larger* value Δt_0 for the time interval, an effect known as time dilation.

Suppose that one observer (Sam) notes that two independent events (event Red and event Blue) occur at the same time. Suppose also that another observer (Sally), who is moving at a constant velocity $\vec{v}$ with respect to Sam, also records these same two events. Will Sally also find that they occur at the same time?

The answer is that in general she will not:

> If two observers are in relative motion, they will not, in general, agree as to whether two events are simultaneous. If one observer finds them to be simultaneous, the other generally will not.

We cannot say that one observer is right and the other wrong. Their observations are equally valid, and there is no reason to favor one over the other.

The realization that two contradictory statements about the same natural events can be correct is a seemingly strange outcome of Einstein's theory. However, in Chapter 17 we saw another way in which motion can affect measurement without balking at the contradictory results: In the Doppler effect, the frequency an observer measures for a sound wave depends on the relative motion of observer and source. Thus, two observers moving relative to each other can measure different frequencies for the same wave, and both measurements are correct.

We conclude the following:

> Simultaneity is not an absolute concept but rather a relative one, depending on the motion of the observer.

If the relative speed of the observers is very much less than the speed of light, then measured departures from simultaneity are so small that they are not noticeable. Such is the case for all our experiences of daily living; that is why the relativity of simultaneity is unfamiliar.

A Closer Look at Simultaneity

Let us clarify the relativity of simultaneity with an example based on the postulates of relativity, no clocks or measuring rods being directly involved. Figure 36-4 shows two long spaceships (the SS *Sally* and the SS *Sam*), which can serve as inertial reference frames for observers Sally and Sam. The two observers are stationed at the midpoints of their ships. The ships are separating along a common x axis, the relative velocity of *Sally* with respect to *Sam* being $\vec{v}$. Figure 36-4a shows the ships with the two observer stations momentarily aligned opposite each other.

Two large meteorites strike the ships, one setting off a red flare (event Red) and the other a blue flare (event Blue), not necessarily simultaneously. Each event leaves a permanent mark on each ship, at positions RR' and BB'.

Let us suppose that the expanding wavefronts from the two events happen to reach Sam at the same time, as Fig. 36-4b shows. Let us further suppose that, after the episode, Sam finds, by measuring the marks on his spaceship, that he was indeed stationed exactly halfway between the markers B and R on his ship when the two events occurred. He will say:

Sam: Light from event Red and light from event Blue reached me at the same time. From the marks on my spaceship, I find that I was standing halfway between the two sources. Therefore, event Red and event Blue were simultaneous events.

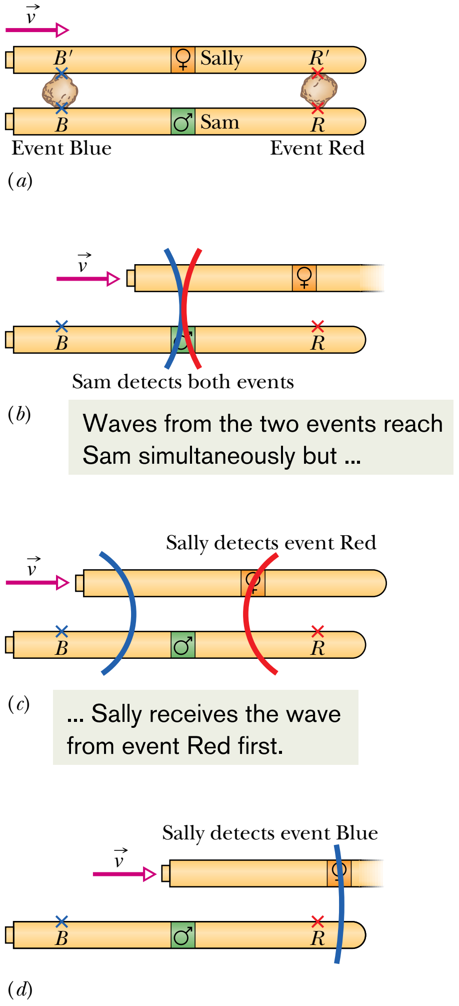


Figure 36-4 The spaceships of Sally and Sam and the occurrences of events from Sam's view. Sally's ship moves rightward with velocity $\vec{v}$. (a) Event Red occurs at positions RR' and event Blue occurs at positions BB'; each event sends out a wave of light. (b) Sam simultaneously detects the waves from event Red and event Blue. (c) Sally detects the wave from event Red. (d) Sally detects the wave from event Blue.

As study of Fig. 36-4 shows, Sally and the expanding wavefront from event Red are moving *toward* each other, while she and the expanding wavefront from event Blue are moving in the *same direction*. Thus, the wavefront from event Red will reach Sally *before* the wavefront from event Blue does. She will say:

Sally: Light from event Red reached me before light from event Blue did. From the marks on my spaceship, I found that I too was standing halfway between the two sources. Therefore, the events were not simultaneous; event Red occurred first, followed by event Blue.

These reports do not agree. Nevertheless, *both* observers are correct.

Note carefully that there is only one wavefront expanding from the site of each event and that *this wavefront travels with the same speed c in both reference frames*, exactly as the speed of light postulate requires.

It *might* have happened that the meteorites struck the ships in such a way that the two hits appeared to Sally to be simultaneous. If that had been the case, then Sam would have declared them not to be simultaneous.

Let us study this concept more closely. What we are going to do is show that relativity of simultaneity implies that a moving observer sees that the reading in two clocks may be different, even though they may have been synchronized in their own frame. For this, let us consider two clocks positioned at the ends of a train of length L (as measured in its own frame). They are synchronized in the train frame. We will show here that the clocks are not synchronized in the observer's frame. (Observer is standing on the ground.)

Let us put a light source on the train, and position it so that the light hits the clocks at the ends of the train at the same time *in the observer's frame*. The relative speeds of the wavefront of light and the clocks are $c + v$ and $c - v$ (as viewed in the observer's frame). To make the time of travel for the wavefront equal, we need to divide the train into lengths in this ratio, in the observer's frame. So, in the train frame, one can quickly show that two numbers that are in this ratio, and that add up to L, are $L(c + v)/2c$ and $L(c - v)/2c$.

The source of light is to be positioned such that the light hits the rear end and front end simultaneously. This can happen only if the light source is closer to the front end. After all, the light has the same velocity in all the directions. The front end is running away from the wavefront and the rear end is running toward it.

The situation in the train frame therefore looks like that in Fig. 36-5. The light must travel an extra distance of $L(c + v)/2c - L(c - v)/2c = Lv/c$ to reach the rear clock. The extra time is therefore Lv/c^2. Hence, the rear clock reads Lv/c^2 more when it is hit by the backward wavefront, compared to what the front clock reads when it is hit by the forward wavefront. Note that our result does *not* say that you see the rear clock ticking at a faster rate than the front clock. They run at the same rate. The back clock is simply a fixed time ahead of the front clock, as seen by you. If we were to make an array of clocks, the clocks stationary with respect to you would look like in Fig. 36-6*a* and the clocks moving with respect to you would look like in Fig. 36-6*b*.

What is the moral of the story? The clocks that are synchronized in one frame are not necessarily synchronized in another frame. This follows from the relativity of simultaneity: What is simultaneous in the one frame is not simultaneous in the other frame.

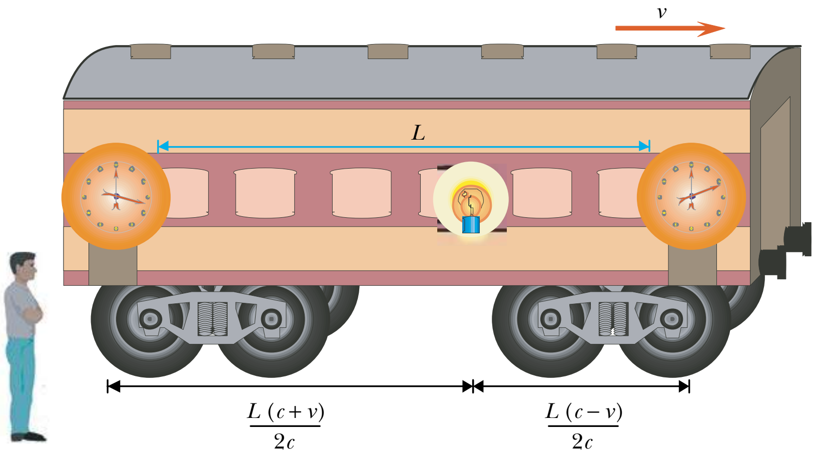


Figure 36-5 A light source is positioned in a train so that its light hits the rear end and the front end simultaneously as seen by an observer on ground.

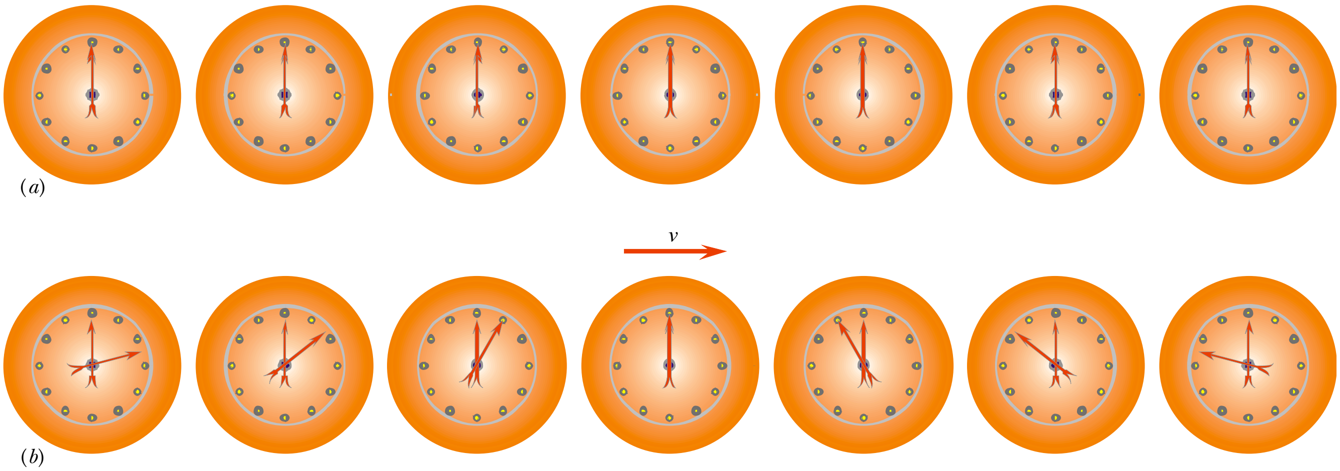


Figure 36-6 (*a*) The time shown by the synchronized clocks which are stationary in the observer's frame. (*b*) The time shown by the clocks are moving with respect to the observer.

36.5 | THE RELATIVITY OF TIME

Key Concept

- If the relative speed between the two frames is v, then

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \frac{\Delta t_0}{\sqrt{1-\beta^2}} = \gamma \Delta t_0,$$

where $\beta = v/c$ is the speed parameter and $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor.

If observers who move relative to each other measure the time interval (or *temporal separation*) between two events, they generally will find different results. Why? Because the spatial separation of the events can affect the time intervals measured by the observers.

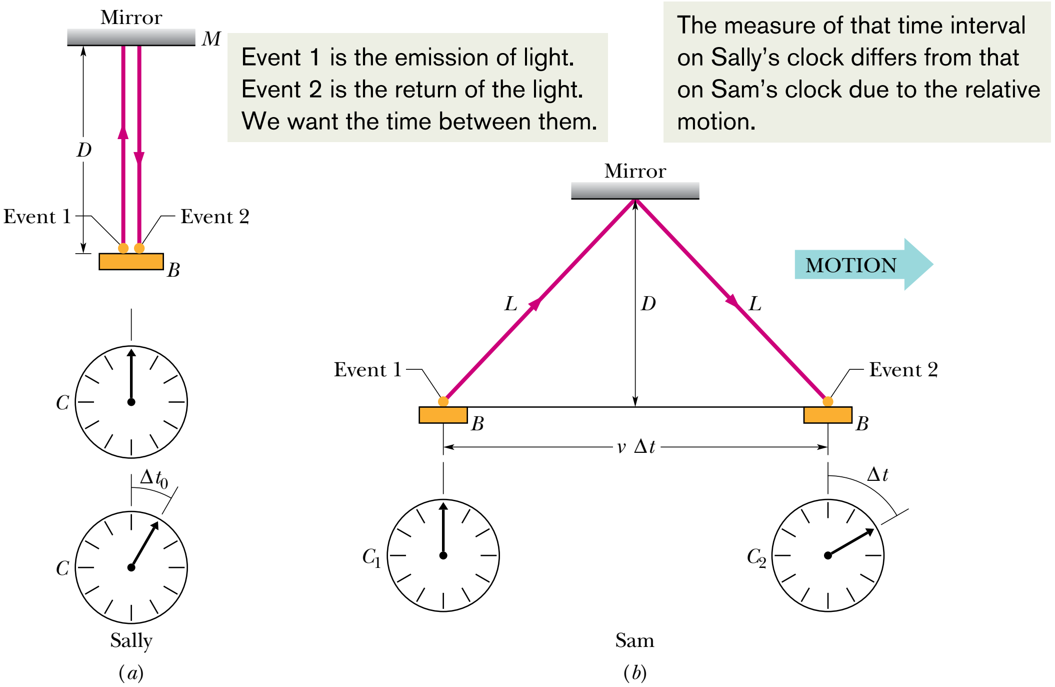
The time interval between two events depends on how far apart they occur in both space and time; that is, their spatial and temporal separations are entangled.

In this section we discuss this entanglement by means of an example; however, the example is restricted in a crucial way: *To one of two observers, the two events occur at the same location*. We shall not get to more general examples until Section 36.7.

Figure 36-7*a* shows the basics of an experiment Sally conducts while she and her equipment—a light source, a mirror, and a clock—ride in a train moving with constant velocity $\vec{v}$ relative to a station. A pulse of light leaves the light source B (event 1), travels vertically upward, is reflected vertically downward by the mirror, and then is detected back at the source (event 2). Sally measures a certain time interval Δt_0 between the two events, related to the distance D from source to mirror by

$$\Delta t_0 = \frac{2D}{c} \quad \text{(Sally)}. \tag{36-3}$$

The two events occur at the same location in Sally's reference frame, and she needs only one clock C at that location to measure the time interval. Clock C is shown twice in Fig. 36-7*a*, at the beginning and end of the interval.


Event 1 is the emission of light.
Event 2 is the return of the light.
We want the time between them.

The measure of that time interval on Sally's clock differs from that on Sam's clock due to the relative motion.

Figure 36-7 (*a*) Sally, on the train, measures the time interval Δt_0 between events 1 and 2 using a single clock C on the train. That clock is shown twice: first for event 1 and then for event 2. (*b*) Sam, watching from the station as the events occur, requires two synchronized clocks, C_1 at event 1 and C_2 at event 2, to measure the time interval between the two events; his measured time interval is Δt.

Consider now how these same two events are measured by Sam, who is standing on the station platform as the train passes. Because the equipment moves with the train during the travel time of the light, Sam sees the path of the light as shown in Fig. 36-7*b*. For him, the two events occur at different places in his reference frame, and so to measure the time interval between events, Sam must use *two* synchronized clocks, C_1 and C_2, one at each event. According to Einstein's speed of light postulate, the light travels at the same speed c for Sam as for Sally. Now, however, the light travels distance $2L$ between events 1 and 2. The time interval measured by Sam between the two events is

$$\Delta t = \frac{2L}{c} \quad \text{(Sam)}, \tag{36-4}$$

in which

$$L = \sqrt{\left(\frac{1}{2}v\,\Delta t\right)^2 + D^2}. \tag{36-5}$$

From Eq. 36-3, we can write this as

$$L = \sqrt{\left(\frac{1}{2}v\,\Delta t\right)^2 + \left(\frac{1}{2}c\,\Delta t_0\right)^2}. \tag{36-6}$$

If we eliminate L between Eqs. 36-4 and 36-6 and solve for Δt, we find

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}}. \tag{36-7}$$

Equation 36-7 tells us how Sam's measured interval Δt between the events compares with Sally's interval Δt_0. Because v must be less than c, the denominator in Eq. 36-7 must be less than unity. Thus, Δt must be greater than Δt_0: Sam measures a *greater* time interval between the two events than does Sally. Sam and Sally have measured the time interval between the *same* two events, but the relative motion between Sam and Sally made their measurements *different*. We conclude that relative motion can change the *rate* at which time passes between two events; the key to this effect is the fact that the speed of light is the same for both observers.

We distinguish between the measurements of Sam and Sally in this way:

When two events occur at the same location in an inertial reference frame, the time interval between them, measured in that frame, is called the **proper time interval** or the **proper time**. Measurements of the same time interval from any other inertial reference frame are always greater.

Thus, Sally measures a proper time interval, and Sam measures a greater time interval. (The term *proper* is unfortunate in that it implies that any other measurement is improper or nonreal. That is just not so.) The amount by which a measured time interval is greater than the corresponding proper time interval is called **time dilation**. (To dilate is to expand or stretch; here the time interval is expanded or stretched.)

Often the dimensionless ratio v/c in Eq. 36-7 is replaced with β, called the **speed parameter**, and the dimensionless inverse square root in Eq. 36-7 is often replaced with γ, called the **Lorentz factor**:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(v/c)^2}}. \qquad (36\text{-}8)$$

With these replacements, we can rewrite Eq. 36-7 as

$$\Delta t = \gamma \Delta t_0 \quad \text{(time dilation)}. \qquad (36\text{-}9)$$

The speed parameter β is always less than unity, and, provided v is not zero, γ is always greater than unity. However, the difference between γ and 1 is not significant unless $v > 0.1c$. Thus, in general, "old relativity" works well enough for $v < 0.1c$, but we must use special relativity for greater values of v. As shown in Fig. 36-8, γ increases rapidly in magnitude as β approaches 1 (as v approaches c). Therefore, the greater the relative speed between Sally and Sam is, the greater will be the time interval measured by Sam, until at a great enough speed, the interval takes "forever."

You might wonder what Sally says about Sam's having measured a greater time interval than she did. His measurement comes as no surprise to her, because to her, he failed to synchronize his clocks C_1 and C_2 in spite of his insistence that he did. Recall that observers in relative motion generally do not agree about simultaneity. Here, Sam insists that his two clocks simultaneously read the same time when event 1 occurred. To Sally, however, Sam's clock C_2 was erroneously set ahead during the synchronization process. Thus, when Sam read the time of event 2 on it, to Sally he was reading off a time that was too large, and that is why the time interval he measured between the two events was greater than the interval she measured.

As the speed parameter goes to 1.0 (as the speed approaches c), the Lorentz factor approaches infinity.

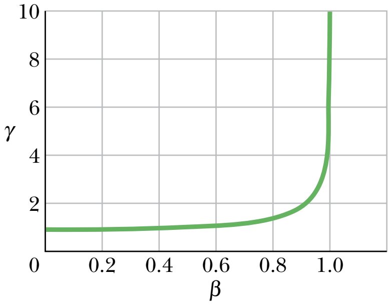


Figure 36-8 A plot of the Lorentz factor γ as a function of the speed parameter β $(= v/c)$.

Two Tests of Time Dilation

1. **Microscopic Clocks.** Subatomic particles called *muons* are unstable; that is, when a muon is produced, it lasts for only a short time before it *decays* (transforms into particles of other types). The *lifetime* of a muon is the time interval between its production (event 1) and its decay (event 2). When muons are stationary and their lifetimes are measured with stationary clocks (say, in a laboratory), their average lifetime is 2.200 μs. This is a proper time interval because, for each muon, events 1 and 2 occur at the same location in the reference frame of the muon—namely, at the muon itself. We can represent this proper time interval with Δt_0; moreover, we can call the reference frame in which it is measured the *rest frame* of the muon.

   If, instead, the muons are moving, say, through a laboratory, then measurements of their lifetimes made with the laboratory clocks should yield a greater average lifetime (a dilated average lifetime). To check this conclusion, measurements were made of the average lifetime of muons moving with a speed of 0.9994c relative to laboratory clocks. From Eq. 36-8, with $\beta = 0.9994$, the Lorentz factor for this speed is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(0.9994)^2}} = 28.87.$$

Equation 36-9 then yields, for the average dilated lifetime,

$$\Delta t = \gamma \Delta t_0 = (28.87)(2.200\ \mu\text{s}) = 63.51\ \mu\text{s}.$$

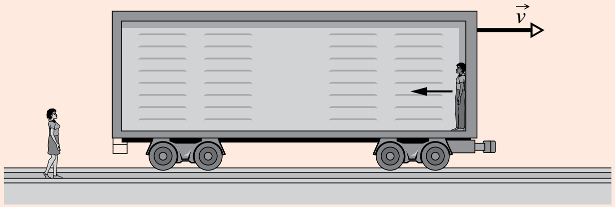
The actual measured value matched this result within experimental error.

2. **Macroscopic Clocks.** In October 1971, Joseph Hafele and Richard Keating carried out what must have been a grueling experiment. They flew four portable atomic clocks twice around the world on commercial airlines, once in each direction. Their purpose was "to test Einstein's theory of relativity with macroscopic clocks." As we have just seen, the time dilation predictions of Einstein's theory have been confirmed on a microscopic scale, but there is great comfort in seeing a confirmation made with an actual clock. Such macroscopic measurements became possible only because of the very high precision of modern atomic clocks. Hafele and Keating verified the predictions of the theory to within 10%. (Einstein's *general* theory of relativity, which predicts that the rate at which time passes on a clock is influenced by the gravitational force on the clock, also plays a role in this experiment.)

   A few years later, physicists at the University of Maryland flew an atomic clock round and round over Chesapeake Bay for flights lasting 15 h and succeeded in checking the time dilation prediction to better than 1%. Today, when atomic clocks are transported from one place to another for calibration or other purposes, the time dilation caused by their motion is always taken into account.

CHECKPOINT 1

Standing beside railroad tracks, we are suddenly startled by a relativistic boxcar traveling past us as shown in the figure. Inside, a well-equipped hobo fires a laser pulse from the front of the boxcar to its rear. (a) Is our measurement of the speed of the pulse greater than, less than, or the same as that measured by the hobo? (b) Is his measurement of the flight time of the pulse a proper time? (c) Are his measurement and our measurement of the flight time related by Eq. 36-9?



SAMPLE PROBLEM 36.01

Time dilation for a space traveler who returns to Earth

Your starship passes Earth with a relative speed of 0.9990c. After traveling 10.0 y (your time), you stop at lookout post LP13, turn, and then travel back to Earth with the same relative speed. The trip back takes another 10.0 y (your time). How long does the round trip take according to measurements made on Earth? (Neglect any effects due to the accelerations involved with stopping, turning, and getting back up to speed.)

KEY IDEAS

We begin by analyzing the outward trip:

1. This problem involves measurements made from two (inertial) reference frames, one attached to Earth and the other (your reference frame) attached to your ship.
2. The outward trip involves two events: the start of the trip at Earth and the end of the trip at LP13.

3. Your measurement of 10.0 y for the outward trip is the proper time Δt_0 between those two events, because the events occur at the same location in your reference frame—namely, on your ship.
4. The Earth-frame measurement of the time interval Δt for the outward trip must be greater than Δt_0, according to Eq. 36-9 ($\Delta t = \gamma \Delta t_0$) for time dilation.

Calculations: Using Eq. 36-8 to substitute for γ in Eq. 36-9, we find

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \frac{10.0 \text{ y}}{\sqrt{1-(0.9990c/c)^2}}$$
$$= (22.37)(10.0 \text{ y}) = 224 \text{ y}.$$

On the return trip, we have the same situation and the same data. Thus, the round trip requires 20 y of your time but

$$\Delta t_{\text{total}} = (2)(224 \text{ y}) = 448 \text{ y} \qquad \text{(Answer)}$$

of Earth time. In other words, you have aged 20 y while the Earth has aged 448 y. Although you cannot travel into the past (as far as we know), you can travel into the future of, say, Earth, by using high-speed relative motion to adjust the rate at which time passes.

SAMPLE PROBLEM 36.02

Time dilation and travel distance for a relativistic particle

The elementary particle known as the *positive kaon* (K^+) is unstable in that it can *decay* (transform) into other particles. Although the decay occurs randomly, we find that, on average, a positive kaon has a lifetime of 0.1237 μs when stationary—that is, when the lifetime is measured in the rest frame of the kaon. If a positive kaon has a speed of 0.990c relative to a laboratory reference frame when the kaon is produced, how far can it travel in that frame during its lifetime according to *classical physics* (which is a reasonable approximation for speeds much less than c) and according to special relativity (which is correct for all physically possible speeds)?

KEY IDEAS

1. We have two (inertial) reference frames, one attached to the kaon and the other attached to the laboratory.
2. This problem also involves two events: the start of the kaon's travel (when the kaon is produced) and the end of that travel (at the end of the kaon's lifetime).
3. The distance traveled by the kaon between those two events is related to its speed v and the time interval for the travel by

$$v = \frac{\text{distance}}{\text{time interval}}. \qquad (36\text{-}10)$$

With these ideas in mind, let us solve for the distance first with classical physics and then with special relativity.

Classical physics: In classical physics we would find the same distance and time interval (in Eq. 36-10) whether we measured them from the kaon frame or from the laboratory frame. Thus, we need not be careful about the frame in which the measurements are made. To find the kaon's travel distance d_{cp} according to classical physics, we first rewrite Eq. 36-10 as

$$d_{cp} = v\,\Delta t, \qquad (36\text{-}11)$$

where Δt is the time interval between the two events in either frame. Then, substituting 0.990c for v and 0.1237 μs for Δt in Eq. 36-11, we find

$$\begin{aligned} d_{cp} &= (0.990c)\,\Delta t \\ &= (0.990)(299\,792\,458 \text{ m/s})(0.1237 \times 10^{-6} \text{ s}) \\ &= 36.7 \text{ m}. \qquad \text{(Answer)} \end{aligned}$$

This is how far the kaon would travel if classical physics were correct at speeds close to c.

Special relativity: In special relativity we must be very careful that both the distance and the time interval in Eq. 36-10 are measured in the *same* reference frame—especially when the speed is close to c, as here. Thus, to find the actual travel distance d_{sr} of the kaon *as measured from the laboratory frame* and according to special relativity, we rewrite Eq. 36-10 as

$$d_{sr} = v\,\Delta t, \qquad (36\text{-}12)$$

where Δt is the time interval between the two events *as measured from the laboratory frame.*

Before we can evaluate d_{sr} in Eq. 36-12, we must find Δt. The 0.1237 μs time interval is a proper time because the two events occur at the same location in the kaon frame—namely, at the kaon itself. Therefore, let Δt_0 represent this proper time interval. Then we can use Eq. 36-9 ($\Delta t = \gamma \Delta t_0$) for time dilation to find the time interval Δt as measured from the laboratory frame. Using Eq. 36-8 to substitute for γ in Eq. 36-9 leads to

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \frac{0.1237 \times 10^{-6}\,\text{s}}{\sqrt{1-(0.990c/c)^2}} = 8.769 \times 10^{-7}\,\text{s}.$$

This is about seven times longer than the kaon's proper lifetime. That is, the kaon's lifetime is about seven times longer in the laboratory frame than in its own frame—the kaon's lifetime is dilated. We can now evaluate Eq. 36-12 for the travel distance d_{sr} in the laboratory frame as

$$\begin{aligned} d_{sr} &= v\,\Delta t = (0.990c)\,\Delta t \\ &= (0.990)(299\,792\,458\text{ m/s})(8.769 \times 10^{-7}\text{ s}) \\ &= 260\text{ m.} \qquad \text{(Answer)} \end{aligned}$$

This is about seven times d_{cp}. Experiments like the one outlined here, which verify special relativity, became routine in physics laboratories decades ago. The engineering design and the construction of any scientific or medical facility that employs highspeed particles must take relativity into account.

36.6 | THE RELATIVITY OF LENGTH

Key Concepts

- The length L_0 of an object measured by an observer in an inertial reference frame in which the object is at rest is called its proper length. Observers in frames moving relative to that frame and parallel to that length will always measure a shorter length, an effect known as length contraction.
- If the relative speed between frames is v, the contracted length L and the proper length L_0 are related by
$$L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma},$$
where $\beta = v/c$ is the speed parameter and $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor.

If you want to measure the length of a rod that is at rest with respect to you, you can—at your leisure—note the positions of its end points on a long stationary scale and subtract one reading from the other. If the rod is moving, however, you must note the positions of the end points *simultaneously* (in your reference frame) or your measurement cannot be called a length. Figure 36-9 suggests the difficulty of trying to measure the length of a moving penguin by locating its front and back at different times. Because simultaneity is relative and it enters into length measurements, length should also be a relative quantity. It is.

Let L_0 be the length of a rod that you measure when the rod is stationary (meaning you and it are in the same reference frame, the rod's rest frame). If, instead, there is relative motion at speed v between you and the rod *along the length of the rod*, then with simultaneous measurements you obtain a length L given by

$$L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma} \quad \text{(length contraction).} \qquad (36\text{-}13)$$

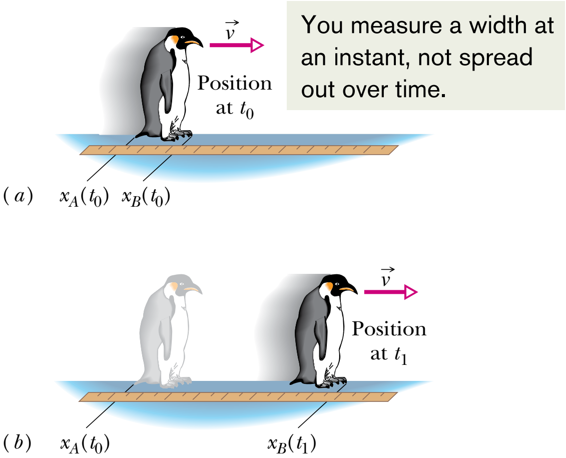


Figure 36-9 If you want to measure the front-to-back length of a penguin while it is moving, you must mark the positions of its front and back simultaneously (in your reference frame), as in (*a*), rather than at different times, as in (*b*).

Because the Lorentz factor γ is always greater than unity if there is relative motion, L is less than L_0. The relative motion causes a *length contraction*, and L is called a *contracted length*. A greater speed v results in a greater contraction.

The length L_0 of an object measured in the rest frame of the object is its **proper length** or **rest length**. Measurements of the length from any reference frame that is in relative motion *parallel* to that length are always less than the proper length.

Be careful: Length contraction occurs only along the direction of relative motion. Also, the length that is measured does not have to be that of an object like a rod or a circle. Instead, it can be the length (or distance) between two objects in the same rest frame—for example, the Sun and a nearby star (which are, at least approximately, at rest relative to each other).

Does a moving object *really* shrink? Reality is based on observations and measurements; if the results are always consistent and if no error can be determined, then what is observed and measured is real. In that sense, the object really does shrink. However, a more precise statement is that the object *is really measured* to shrink—motion affects that measurement and thus reality.

When you measure a contracted length for, say, a rod, what does an observer moving with the rod say of your measurement? To that observer, you did not locate the two ends of the rod simultaneously. (Recall that observers in motion relative to each other do not agree about simultaneity.) To the observer, you first located the rod's front end and then, slightly later, its rear end, and that is why you measured a length that is less than the proper length.

Proof of Eq. 36-13

Length contraction is a direct consequence of time dilation. Consider once more our two observers. This time, both Sally, seated on a train moving through a station, and Sam, again on the station platform, want to measure the length of the platform. Sam, using a tape measure, finds the length to be L_0, a proper length because the platform is at rest with respect to him. Sam also notes that Sally, on the train, moves through this length in a time $\Delta t = L_0/v$, where v is the speed of the train; that is,

$$L_0 = v\,\Delta t \quad \text{(Sam)}. \tag{36-14}$$

This time interval Δt is not a proper time interval because the two events that define it (Sally passes the back of the platform and Sally passes the front of the platform) occur at two different places, and therefore Sam must use two synchronized clocks to measure the time interval Δt.

For Sally, however, the platform is moving past her. She finds that the two events measured by Sam occur *at the same place* in her reference frame. She can time them with a single stationary clock, and so the interval Δt_0 that she measures is a proper time interval. To her, the length L of the platform is given by

$$L = v\,\Delta t_0 \quad \text{(Sally)}. \tag{36-15}$$

If we divide Eq. 36-15 by Eq. 36-14 and apply Eq. 36-9, the time dilation equation, we have

$$\frac{L}{L_0} = \frac{v\,\Delta t_0}{v\,\Delta t} = \frac{1}{\gamma},$$

or

$$L = \frac{L_0}{\gamma}, \tag{36-16}$$

which is Eq. 36-13, the length contraction equation.

SAMPLE PROBLEM 36.03

Time dilation and length contraction as seen from each frame

In Fig. 36-10, Sally (at point A) and Sam's spaceship (of proper length $L_0 = 230$ m) pass each other with constant relative speed v. Sally measures a time interval of 3.57 μs for the ship to pass her (from the passage of point B in Fig. 36-10*a* to the passage of point C in Fig. 36-10*b*). In terms of c, what is the relative speed v between Sally and the ship?

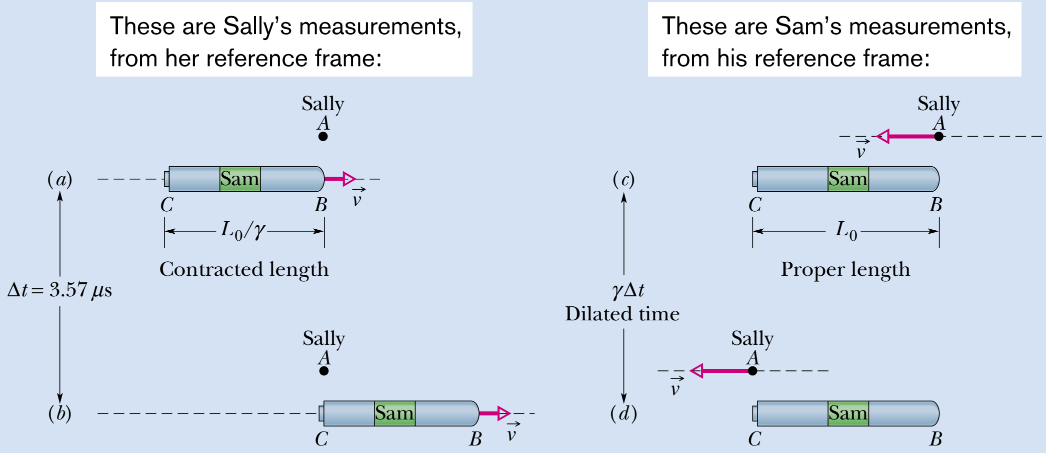


Figure 36-10 (*a*)–(*b*) Event 1 occurs when point *B* passes Sally (at point *A*) and event 2 occurs when point *C* passes her. (*c*)–(*d*) Event 1 occurs when Sally passes point *B* and event 2 occurs when she passes point *C*.

KEY IDEAS

Let's assume that speed v is near c. Then:

1. This problem involves measurements made from two (inertial) reference frames, one attached to Sally and the other attached to Sam and his spaceship.
2. This problem also involves two events: the first is the passage of point *B* past Sally (Fig. 36-10*a*) and the second is the passage of point *C* past her (Fig. 36-10*b*).
3. From either reference frame, the other reference frame passes at speed v and moves a certain distance in the time interval between the two events:

$$v = \frac{\text{distance}}{\text{time interval}}. \tag{36-17}$$

Because speed v is assumed to be near the speed of light, we must be careful that the distance and the time interval in Eq. 36-17 are measured in the *same* reference frame.

Calculations: We are free to use either frame for the measurements. Because we know that the time interval Δt between the two events measured from Sally's frame is 3.57 μs, let us also use the distance L between the two events measured from her frame. Equation 36-17 then becomes

$$v = \frac{L}{\Delta t}. \tag{36-18}$$

We do not know L, but we can relate it to the given L_0: The distance between the two events as measured from Sam's frame is the ship's proper length L_0. Thus, the distance L measured from Sally's frame must be less than L_0, as given by Eq. 36-13 ($L = L_0/\gamma$) for length contraction. Substituting L_0/γ for L in Eq. 36-18 and then substituting Eq. 36-8 for γ, we find

$$v = \frac{L_0/\gamma}{\Delta t} = \frac{L_0\sqrt{(1-(v/c)^2}}{\Delta t}.$$

Solving this equation for v (notice that it is on the left and also buried in the Lorentz factor) leads us to

$$\begin{aligned} v &= \frac{L_0 c}{\sqrt{(c\,\Delta t)^2 + L_0^2}} \\ &= \frac{(230\text{ m})c}{\sqrt{(299\,792\,458\text{ m/s})^2(3.57\times10^{-6}\text{ s})^2 + (230\text{ m})^2}} \\ &= 0.210c. \qquad \text{(Answer)} \end{aligned}$$

Note that only the relative motion of Sally and Sam matters here; whether either is stationary relative to, say, a space station is irrelevant. In Figs. 36-10*a* and *b* we took Sally to be stationary, but we could instead have taken the ship to be stationary, with Sally moving to the left past it. Event 1 is again when Sally and point *B* are aligned (Fig. 36-10*c*), and event 2 is again when Sally and point *C* are aligned (Fig. 36-10*d*). However, we are now using Sam's measurements. So the length between the two events in *his* frame is the proper length L_0 of the ship and the time interval between them is not Sally's measurement Δt but a dilated time interval $\gamma\Delta t$.

Substituting Sam's measurements into Eq. 36-17, we have

$$v = \frac{L_0}{\gamma\Delta t},$$

which is exactly what we found using Sally's measurements. Thus, we get the same result of $v = 0.210c$ with either set of measurements, *but we must be careful not to mix the measurements from the two frames*.

SAMPLE PROBLEM 36.04

Time dilation and length contraction in outrunning a supernova

Caught by surprise near a supernova, you race away from the explosion in your spaceship, hoping to outrun the high-speed material ejected toward you. Your Lorentz factor γ relative to the inertial reference frame of the local stars is 22.4.

(a) To reach a safe distance, you figure you need to cover 9.00×10^{16} m as measured in the reference frame of the local stars. How long will the flight take, as measured in that frame?

KEY IDEAS

From Chapter 2, for constant speed, we know that

$$\text{speed} = \frac{\text{distance}}{\text{time interval}}. \tag{36-19}$$

From Fig. 36-8, we see that because your Lorentz factor γ relative to the stars is 22.4 (large), your relative speed v is almost c—so close that we can approximate it as c. Then for speed $v \approx c$, we must be careful that the distance and the time interval in Eq. 36-19 are measured in the *same* reference frame.

Calculations: The given distance (9.00×10^{16} m) for the length of your travel path is measured in the reference frame of the stars, and the requested time interval Δt is to be measured in that same frame. Thus, we can write

$$\left(\begin{array}{c}\text{time interval}\\ \text{relative to stars}\end{array}\right) = \frac{\text{distance relative to stars}}{c}.$$

Then substituting the given distance, we find that

$$\left(\begin{array}{c}\text{time interval}\\ \text{relative to stars}\end{array}\right) = \frac{9.00\times10^{16}\ \text{m}}{299\ 792\ 458\ \text{m/s}}$$
$$= 3.00\times10^{8}\ \text{s} = 9.51\ \text{y}. \qquad \text{(Answer)}$$

(b) How long does that trip take according to you (in your reference frame)?

KEY IDEAS

1. We now want the time interval measured in a different reference frame—namely, yours. Thus, we need to transform the data given in the reference frame of the stars to your frame.
2. The given path length of 9.00×10^{16} m, measured in the reference frame of the stars, is a proper length L_0, because the two ends of the path are at rest in that frame. As observed from your reference frame, the stars' reference frame and those two ends of the path race past you at a relative speed of $v \approx c$.
3. You measure a contracted length L_0/γ for the path, not the proper length L_0.

Calculations: We can now rewrite Eq. 36-19 as

$$\left(\begin{array}{c}\text{time interval}\\ \text{relative to you}\end{array}\right) = \frac{\text{distance relative to you}}{c} = \frac{L_0/\gamma}{c}.$$

Substituting known data, we find

$$\left(\begin{array}{c}\text{time interval}\\ \text{relative to you}\end{array}\right) = \frac{(9.00\times10^{16}\ \text{m})/22.4}{299\ 792\ 458\ \text{m/s}}$$
$$= 1.340\times10^{7}\ \text{s} = 0.425\ \text{y}. \qquad \text{(Answer)}$$

In part (a) we found that the flight takes 9.51 y in the reference frame of the stars. However, here we find that it takes only 0.425 y in your frame, due to the relative motion and the resulting contracted length of the path.

36.7 | THE LORENTZ TRANSFORMATION

Key Concept

- The Lorentz transformation equations relate the spacetime coordinates of a single event as seen by observers in two inertial frames, S and S', where S' is moving relative to S with velocity v in the positive x and x' direction. The four coordinates are related by

$$x' = \gamma(x - vt),$$
$$y' = y,$$
$$z' = z,$$
$$t' = \gamma(t - vx/c^2).$$

Figure 36-11 shows inertial reference frame S' moving with speed v relative to frame S, in the common positive direction of their horizontal axes (marked x and x'). An observer in S reports spacetime coordinates x, y, z, t for an event, and an observer in S' reports x', y', z', t' for the same event. How are these sets of numbers related? We claim at once (although it requires proof) that the y and z coordinates, which are perpendicular to the motion, are not affected by the motion; that is, $y = y'$ and $z = z'$. Our interest then reduces to the relation between x and x' and that between t and t'.

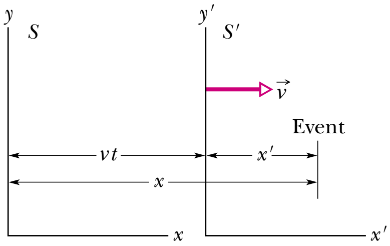


Figure 36-11 Two inertial reference frames: frame S' has velocity $\vec{v}$ relative to frame S.

The Galilean Transformation Equations

Prior to Einstein's publication of his special theory of relativity, the four coordinates of interest were assumed to be related by the *Galilean transformation equations:*

$$\begin{aligned} x' &= x - vt \quad &&\text{(Galilean transformation equations;} \\ t' &= t, &&\text{approximately valid at low speeds).} \end{aligned} \tag{36-20}$$

(These equations are written with the assumption that $t = t' = 0$ when the origins of S and S' coincide.) You can verify the first equation with Fig. 36-11. The second equation effectively claims that time passes at the same rate for observers in both reference frames. That would have been so obviously true to a scientist prior to Einstein that it would not even have been mentioned. When speed v is small compared to c, Eqs. 36-20 generally work well.

The Lorentz Transformation Equations

Equations 36-20 work well when speed v is small compared to c, but they are actually incorrect for any speed and are very wrong when v is greater than about $0.10c$. The equations that are correct for any physically possible speed are called the **Lorentz transformation equations*** (or simply the Lorentz transformations). We can derive them from the postulates of relativity, but here we shall instead first examine them and then justify them by showing them to be consistent with our results for simultaneity, time dilation, and length contraction. Assuming that $t = t' = 0$ when the origins of S and S' coincide in Fig. 36-11 (event 1), then the spatial and temporal coordinates of any other event are given by

$$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - vx/c^2) \end{aligned} \qquad \begin{array}{l}\text{(Lorentz transformation equations;} \\ \text{valid at all physically possible speeds).}\end{array} \tag{36-21}$$

Note that the spatial values x and the temporal values t are bound together in the first and last equations. This entanglement of space and time was a prime message of Einstein's theory, a message that was long rejected by many of his contemporaries.

It is a formal requirement of relativistic equations that they should reduce to familiar classical equations if we let c approach infinity. That is, if the speed of light were infinitely great, *all* finite speeds would be "low" and classical equations would never fail. If we let $c \to \infty$ in Eqs. 36-21, $\gamma \to 1$ and these equations reduce—as we expect—to the Galilean equations (Eqs. 36-20). You should check this.

Equations 36-21 are written in a form that is useful if we are given x and t and wish to find x' and t'. We may wish to go the other way, however. In that case we simply solve Eqs. 36-21 for x and t, obtaining

$$x = \gamma(x' + vt') \quad \text{and} \quad t = \gamma(t' + vx'/c^2). \tag{36-22}$$

Comparison shows that, starting from either Eqs. 36-21 or Eqs. 36-22, you can find the other set by interchanging primed and unprimed quantities and reversing the sign of the relative velocity v. (For example, if the S' frame has

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*You may wonder why we do not call these the *Einstein transformation equations* (and why not the *Einstein factor* for γ). H. A. Lorentz actually derived these equations before Einstein did, but as the great Dutch physicist graciously conceded, he did not take the further bold step of interpreting these equations as describing the true nature of space and time. It is this interpretation, first made by Einstein, that is at the heart of relativity.

a positive velocity relative to an observer in the S frame as in Fig. 36-11, then the S frame has a *negative* velocity relative to an observer in the S' frame.)

Equations 36-21 relate the coordinates of a second event when the first event is the passing of the origins of S and S' at $t = t' = 0$. However, in general we do not want to restrict the first event to being such a passage. So, let's rewrite the Lorentz transformations in terms of any pair of events 1 and 2, with spatial and temporal separations

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta t = t_2 - t_1,$$

as measured by an observer in S, and

$$\Delta x' = x_2' - x_1' \quad \text{and} \quad \Delta t' = t_2' - t_1',$$

as measured by an observer in S'. Table 36-2 displays the Lorentz equations in difference form, suitable for analyzing pairs of events. The equations in the table were derived by simply substituting differences (such as Δx and $\Delta x'$) for the four variables in Eqs. 36-21 and 36-22.

Be careful: When substituting values for these differences, you must be consistent and not mix the values for the first event with those for the second event. Also, if, say, Δx is a negative quantity, you must be certain to include the minus sign in a substitution.

Table 36-2 The Lorentz Transformation Equations for Pairs of Events

1. $\Delta x = \gamma(\Delta x' + v\,\Delta t')$	1′. $\Delta x' = \gamma(\Delta x - v\,\Delta t)$
2. $\Delta t = \gamma(\Delta t' + v\,\Delta x'/c^2)$	2′. $\Delta t' = \gamma(\Delta t - v\,\Delta x/c^2)$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

Frame S' moves at velocity v relative to frame S.

CHECKPOINT 2

In Fig. 36-11, frame S' has velocity $0.90c$ relative to frame S. An observer in frame S' measures two events as occurring at the following spacetime coordinates: event Yellow at (5.0 m, 20 ns) and event Green at (–2.0 m, 45 ns). An observer in frame S wants to find the temporal separation $\Delta t_{GY} = t_G - t_Y$ between the events. (a) Which equation in Table 36-2 should be used? (b) Should $+0.90c$ or $-0.90c$ be substituted for v in the parentheses on the equation's right side and in the Lorentz factor γ? What value should be substituted into the (c) first and (d) second term in the parentheses?

Some Consequences of the Lorentz Equations

Here we use the equations of Table 36-2 to affirm some of the conclusions that we reached earlier by arguments based directly on the postulates.

Simultaneity

Consider Eq. 2 of Table 36-2,

$$\Delta t = \gamma\left(\Delta t' + \frac{v\,\Delta x'}{c^2}\right). \tag{36-23}$$

If two events occur at different places in reference frame S' of Fig. 36-11, then $\Delta x'$ in this equation is not zero. It follows that even if the events are simultaneous in S' (thus $\Delta t' = 0$), they will not be simultaneous in frame S. (This is in accord with our conclusion in Section 36.4.) The time interval between the events in S will be

$$\Delta t = \gamma\,\frac{v\,\Delta x'}{c^2} \quad \text{(simultaneous events in } S'\text{)}.$$

Thus, the spatial separation $\Delta t'$ guarantees a temporal separation Δt.

Time Dilation

Suppose now that two events occur at the same place in S' (thus $\Delta x' = 0$) but at different times (thus $\Delta t' \neq 0$). Equation 36-23 then reduces to

$$\Delta t = \gamma\,\Delta t' \quad \text{(events in same place in } S'\text{)}. \tag{36-24}$$

This confirms time dilation between frames S and S'. Moreover, because the two events occur at the same place in S', the time interval $\Delta t'$ between them can be measured with a single clock, located at that place. Under these conditions, the measured interval is a proper time interval, and we can label it Δt_0 as we have previously labeled proper times. Thus, with that label Eq. 36-24 becomes

$$\Delta t = \gamma\, \Delta t_0 \quad \text{(time dilation)},$$

which is exactly Eq. 36-9, the time dilation equation. Thus, time dilation is a special case of the more general Lorentz equations.

Length Contraction

Consider Eq. 1′ of Table 36-2,

$$\Delta x' = \gamma(\Delta x - v\,\Delta t). \tag{36-25}$$

If a rod lies parallel to the x and x' axes of Fig. 36-11 and is at rest in reference frame S', an observer in S' can measure its length at leisure. One way to do so is by subtracting the coordinates of the end points of the rod. The value of $\Delta x'$ that is obtained will be the proper length L_0 of the rod because the measurements are made in a frame where the rod is at rest.

Suppose the rod is moving in frame S. This means that Δx can be identified as the length L of the rod in frame S only if the coordinates of the rod's end points are measured *simultaneously*—that is, if $\Delta t = 0$. If we put $\Delta x' = L_0$, $\Delta x = L$, and $\Delta t = 0$ in Eq. 36-25, we find

$$L = \frac{L_0}{\gamma} \quad \text{(length contraction)}, \tag{36-26}$$

which is exactly Eq. 36-13, the length contraction equation. Thus, length contraction is a special case of the more general Lorentz equations.

SAMPLE PROBLEM 36.05

Lorentz transformations and reversing the sequence of events

An Earth starship has been sent to check an Earth outpost on the planet P1407, whose moon houses a battle group of the often hostile Reptulians. As the ship follows a straight-line course first past the planet and then past the moon, it detects a high-energy microwave burst at the Reptulian moon base and then, 1.10 s later, an explosion at the Earth outpost, which is 4.00×10^8 m from the Reptulian base as measured from the ship's reference frame. The Reptulians have obviously attacked the Earth outpost, and so the starship begins to prepare for a confrontation with them.

(a) The speed of the ship relative to the planet and its moon is $0.980c$. What are the distance and time interval between the burst and the explosion as measured in the planet–moon frame (and thus according to the occupants of the stations)?

KEY IDEAS

1. This problem involves measurements made from two reference frames, the planet–moon frame and the starship frame.
2. We have two events: the burst and the explosion.
3. We need to transform the given data as measured in the starship frame to the corresponding data as measured in the planet–moon frame.

Starship frame: Before we get to the transformation, we need to carefully choose our notation. We begin with a sketch of the situation as shown in Fig. 36-12. There, we have chosen the ship's frame S to be stationary and the planet–moon frame S' to be moving with positive velocity (rightward). (This is an arbitrary choice; we could, instead, have chosen the planet–moon frame to be stationary. Then we would redraw $\vec{v}$ in Fig. 36-12 as being attached to the S frame and indicating leftward motion; v would then be a negative quantity. The results would be the same.) Let subscripts e and b represent the explosion and burst, respectively. Then the given data, all in the unprimed (starship) reference frame, are

$$\Delta x = x_e - x_b = +4.00 \times 10^8 \text{ m}$$

and

$$\Delta t = t_e - t_b = +1.10 \text{ s}.$$

The relative motion alters the time intervals between events and maybe even their sequence.

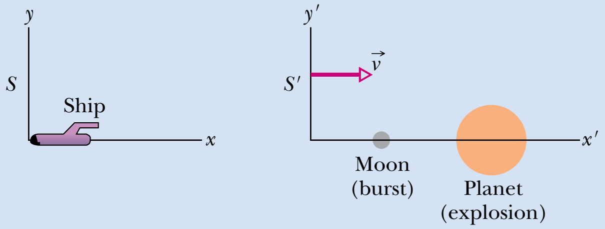


Figure 36-12 A planet and its moon in reference frame S' move rightward with speed v relative to a starship in reference frame S.

Here, Δx is a positive quantity because in Fig. 36-12, the coordinate x_e for the explosion is greater than the coordinate x_b for the burst; Δt is also a positive quantity because the time t_e of the explosion is greater (later) than the time t_b of the burst.

Planet–moon frame: We seek $\Delta x'$ and $\Delta t'$, which we shall get by transforming the given S-frame data to the planet–moon frame S'. Because we are considering a pair of events, we choose transformation equations from Table 36-2—namely, Eqs. 1′ and 2′:

$$\Delta x' = \gamma(\Delta x - v\,\Delta t) \tag{36-27}$$

and

$$\Delta t' = \gamma\left(\Delta t - \frac{v\,\Delta x}{c^2}\right). \tag{36-28}$$

Here, $v = +0.980c$ and the Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(+0.980c/c)^2}} = 5.0252.$$

Equation 36-27 then becomes

$$\begin{aligned}\Delta x' &= (5.0252)[4.00 \times 10^8\ \text{m} - (+0.980\,c)(1.10\ \text{s})]\\ &= 3.86 \times 10^8\ \text{m}, \qquad \text{(Answer)}\end{aligned}$$

and Eq. 36-28 becomes

$$\begin{aligned}\Delta t' &= (5.0252)\left[(1.10\,\text{s}) - \frac{(+0.980c)(4.00\times 10^8\,\text{m})}{c^2}\right]\\ &= -1.04\ \text{s}. \qquad \text{(Answer)}\end{aligned}$$

(b) What is the meaning of the minus sign in the value for $\Delta t'$?

Reasoning: We must be consistent with the notation we set up in part (a). Recall how we originally defined the time interval between burst and explosion: $\Delta t = t_e - t_b = +1.10$ s. To be consistent with that choice of notation, our definition of $\Delta t'$ must be $t'_e - t'_b$; thus, we have found that

$$\Delta t' = t'_e - t'_b = -1.04\ \text{s}.$$

The minus sign here tells us that $t'_b > t'_e$; that is, in the planet–moon reference frame, the burst occurred 1.04 s *after* the explosion, not 1.10 s *before* the explosion as detected in the ship frame.

(c) Did the burst cause the explosion, or vice versa?

KEY IDEA

The sequence of events measured in the planet–moon reference frame is the reverse of that measured in the ship frame. In either situation, if there is a causal relationship between the two events, information must travel from the location of one event to the location of the other to cause it.

Checking the speed: Let us check the required speed of the information. In the ship frame, this speed is

$$v_{\text{info}} = \frac{\Delta x}{\Delta t} = \frac{4.00\times 10^8\,\text{m}}{1.10\ \text{s}} = 3.64\times 10^8\,\text{m/s},$$

but that speed is impossible because it exceeds c. In the planet–moon frame, the speed comes out to be 3.70×10^8 m/s, also impossible. Therefore, neither event could possibly have caused the other event; that is, they are *unrelated* events. Thus, the starship should stand down and not confront the Reptulians.

36.8 | ELECTRICITY AND MAGNETISM

The theoretical implications of the relation between electricity and magnetism led Einstein to discover special relativity. The relation between electric and magnetic field can be explained taking into account the theory of relativity.

Though the full explanation of how relativity links electricity and magnetism is mathematically complex, some aspects of it are quite easy to understand. Let us consider an example: the origin of magnetic force between two parallel currents. Note that that the electric charge is relativistically invariant which implies that a charge whose magnitude is Q in one frame of reference is also Q in all other frames.

Consider two ideal conductors (Fig. 36-13*a*) having equally spaced equal numbers of positive and negative charges at rest. As the conductors are electrically neutral, there is no force between them.

Figure 36-13 explains how two parallel current carrying wires attract each other. We assume that the current flows because of the flow of positive as well as negative charges in the opposite direction with same velocity v. In Fig. 36-13*a*, since the charges are at rest and being equal in number do not exert any net force on each other. In Fig. 36-13*b*, since the charges are moving they exert magnetic force on each other. Figure 36-13*c* explains the reason why. As there is a negative charge in wire I, the negative charges in the other wire are at rest; however, the positive charges move with a velocity of $2v$. This results in Lorentz contraction between them as well as a net attractive force, because to the electron the number of positive charges in the other wire seems more. The same phenomena from the perspective of a proton in the wire is explained in Fig. 36-13*c*. Thus, the magnetic force in one frame reduces to electric force in the other frame.

Figure 36-13*b* illustrates the same conductors when they carry currents i_{I} and i_{II} in the same direction. The positive and negative charges move to the right and the left, respectively, both at the same speed v as seen from the laboratory frame of reference. (Actual currents in metal consist of flows of negative electrons only, of course, but the electrically equivalent model here is easier to analyze and the results are the same.) Because the charges are moving, their spacing is smaller than before by the factor $\sqrt{1-v^2/c^2}$. Since n is the same for both sets of charges, their spacings shrink by the same amounts, and both conductors remain neutral to an observer in the laboratory. However, the conductors now attract each other. Let us see why.

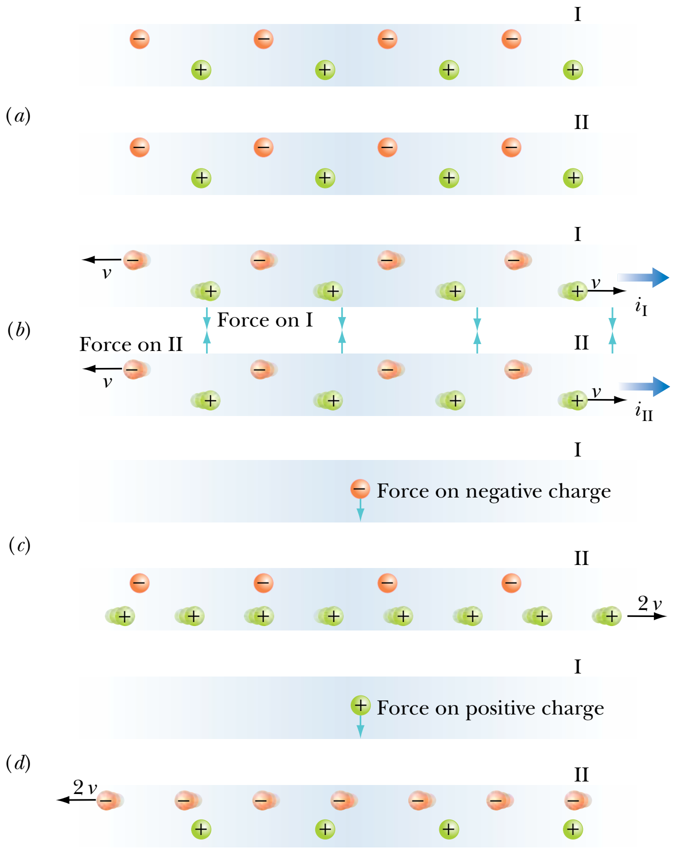


Figure 36-13 Magnetic attraction between two parallel current-carrying wires. (a) Two neutral conductors do not exert any force on each other. (b) If the conductors carry current, the distance between the moving charges undergoes contraction as seen from lab frame of reference. (c) If the currents in both the wires are the same, the electrons have the same velocity. As seen by electrons of the first wire, the electrons of the other wire are at rest, but the protons are moving. This relative motion produces a contraction. So, this electron will see protons more closely spaced, and thus feel a net attractive force. (d) Similar phenomenon occurs as seen from the frame of a positive charge. The distances are exaggerated for clarity.

Consider conductor II from the frame of reference of one of the negative charges in conductor I. As can be seen in Fig. 36-13*c* because the negative charges in conductor II appear at rest in this frame, their spacing is not contracted. On the other hand, the positive charges in II have a velocity $2n$, and their spacing is accordingly contracted to a greater extent than they are in the laboratory form. Conductor II, therefore, appears to have a net positive charge, and an attractive force acts on the negative charge in conductor I.

Look at conductor II from the frame of reference of one of the positive charges in conductor I. The positive charges in conductor II are now at rest, and the negative charges there move to the left with speed $2n$. Hence, the negative charges are closer together than the positive ones, as in Fig. 36-13*d*, and the entire conductor appears negatively charged. An attractive force, therefore, acts on the positive charges in conductor I.

To each charge, the force on it is an “ordinary” electric force that arises because the charges of opposite sign in the other conductor are closer together than the charges of the same sign, in which case the other conductor appears to have a net charge. From the laboratory frame, the situation is less straightforward. Both conductors are electrically neutral in this frame, and it is natural to explain their mutual attraction by attributing it to a special “magnetic” interaction between the currents.

Although it is convenient to think of magnetic forces as being different from electric ones, they both result from a single electromagnetic interaction that occurs between charged particles.

Clearly a current-carrying conductor that is electrically neutral in one frame of reference might not be neutral in another frame. How can this observation be reconciled with charge invariance? Answer: We must consider the entire circuit of which the conductor is a part. As the circuit must be closed for a current to run in it, for every current element in one direction that a moving observer finds to have, say a positive charge, there must be another current element in the opposite direction which the same observer finds to have, here a negative charge. Thus, the circuit as a whole appears electrically neutral to all observers.

The discussion here considered only a particular magnetic effect. All other magnetic phenomena can also be interpreted on the basis of Coulomb's law, charge invariance, and special relativity, although the analysis is usually much more complicated.

36.9 | THE RELATIVITY OF VELOCITIES

Key Concept

- When a particle is moving with speed u' in the positive x' direction in an inertial reference frame S' that itself is moving with speed v parallel to the x direction of a second inertial frame S, the speed u of the particle as measured in S is

$$u = \frac{u'+v}{1+u'v/c^2} \quad \text{(relativistic velocity).}$$

Here we wish to use the Lorentz transformation equations to compare the velocities that two observers in different inertial reference frames S and S' would measure for the same moving particle. Let S' move with velocity v relative to S.

Suppose that the particle, moving with constant velocity parallel to the x and x' axes in Fig. 36-14, sends out two signals as it moves. Each observer measures the space interval and the time interval between these two events. These four measurements are related by Eqs. 1 and 2 of Table 36-2,

$$\Delta x = \gamma(\Delta x' + v\,\Delta t')$$

and

$$\Delta t = \gamma\left(\Delta t' + \frac{v\,\Delta x'}{c^2}\right).$$

The speed of the moving particle depends on the frame.

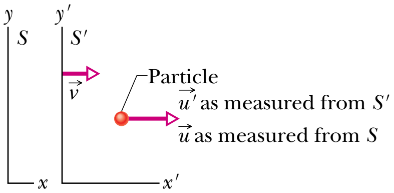

y
S
y'
S'
$\vec{v}$
Particle
$\vec{u}'$ as measured from S'
$\vec{u}$ as measured from S
x
x'

Figure 36-14 Reference frame S' moves with velocity $\vec{v}$ relative to frame S. A particle has velocity $\vec{u}'$ relative to reference frame S' and velocity $\vec{u}$ relative to reference frame S.

If we divide the first of these equations by the second, we find

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\,\Delta t'}{\Delta t' + v\,\Delta x'/c^2}.$$

Dividing the numerator and denominator of the right side by $\Delta t'$, we find

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x'/\Delta t' + v}{1 + v(\Delta x'/\Delta t')/c^2}.$$

However, in the differential limit, $\Delta x/\Delta t$ is u, the velocity of the particle as measured in S, and $\Delta x'/\Delta t'$ is u', the velocity of the particle as measured in S'. Then we have, finally,

$$u = \frac{u'+v}{1+u'v/c^2} \quad \text{(relativistic velocity transformation)} \qquad (36\text{-}29)$$

as the relativistic velocity transformation equation. (*Caution:* Be careful to substitute the correct signs for the velocities.) Equation 36-29 reduces to the classical, or Galilean, velocity transformation equation,

$$u = u' + v \quad \text{(classical velocity transformation)}, \tag{36-30}$$

when we apply the formal test of letting $c \to \infty$. In other words, Eq. 36-29 is correct for all physically possible speeds, but Eq. 36-30 is approximately correct for speeds much less than c.

36.10 | DOPPLER EFFECT FOR LIGHT

Key Concepts

- When a light source and a light detector move relative to each other, the wavelength of the light as measured in the rest frame of the source is the proper wavelength λ_0. The detected wavelength λ is either longer (a red shift) or shorter (a blue shift) depending on whether the source–detector separation is increasing or decreasing.
- When the separation is increasing, the wavelengths are related by

  $$\lambda = \lambda_0\sqrt{\frac{1+\beta}{1-\beta}} \quad \text{(source and detector separating)},$$

  where $\beta = v/c$ and v is the relative radial speed (along a line through the source and detector). If the separation is decreasing, the signs in front of the β symbols are reversed.
- For speeds much less than c, the magnitude of the Doppler wavelength shift $\Delta\lambda = \lambda - \lambda_0$ is approximately related to v by

  $$v = \frac{|\Delta\lambda|}{\lambda_0}c \quad (v \ll c).$$
- If the relative motion of the light source is perpendicular to a line through the source and detector, the detected frequency f is related to the proper frequency f_0 by

  $$f = f_0\sqrt{1-\beta^2}.$$

  This transverse Doppler effect is due to time dilation.

In Chapter 17 we discussed the Doppler effect (a shift in detected frequency) for sound waves, finding that the effect depends on the source and detector velocities relative to the air. That is not the situation with light waves, which require no medium (they can even travel through vacuum). The Doppler effect for light waves depends on only the relative velocity $\vec{v}$ between source and detector, as measured from the reference frame of either. Let f_0 represent the **proper frequency** of the source—that is, the frequency that is measured by an observer in the rest frame of the source. Let f represent the frequency detected by an observer moving with velocity $\vec{v}$ relative to that rest frame. Then, when the direction of $\vec{v}$ is directly away from the source,

$$f = f_0\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{(source and detector separating)}, \tag{36-31}$$

where $\beta = v/c$.

Because measurements involving light are usually done in wavelengths rather than frequencies, let's rewrite Eq. 36-31 by replacing f with c/λ and f_0 with c/λ_0, where λ is the measured wavelength and λ_0 is the **proper wavelength** (the wavelength associated with f_0). After canceling c from both sides, we then have

$$\lambda = \lambda_0\sqrt{\frac{1+\beta}{1-\beta}} \quad \text{(source and detector separating)}. \tag{36-32}$$

When the direction of $\vec{v}$ is directly toward the source, we must change the signs in front of the β symbols in Eqs. 36-31 and 36-32.

For an increasing separation, we can see from Eq. 36-32 (with an addition in the numerator and a subtraction in the denominator) that the measured wavelength is greater than the proper wavelength. Such a Doppler shift is described as being a *red shift*, where *red* does not mean the measured wavelength is red or even visible. The term merely serves as a memory device because red is at the *long*-wavelength end of the visible spectrum. Thus λ is longer than λ_0. Similarly, for a decreasing separation, λ is shorter than λ_0, and the Doppler shift is described as being a *blue shift*.

Low-Speed Doppler Effect

For low speeds ($\beta \ll 1$), Eq. 36-31 can be expanded in a power series in β and approximated as

$$f = f_0\left(1 - \beta + \frac{1}{2}\beta^2\right) \quad \text{(source and detector separating, } \beta \ll 1\text{).} \tag{36-33}$$

The corresponding low-speed equation for the Doppler effect with sound waves (or any waves except light waves) has the same first two terms but a different coefficient in the third term. Thus, the relativistic effect for low-speed light sources and detectors shows up only with the β^2 term.

A police radar unit employs the Doppler effect with microwaves to measure the speed v of a car. A source in the radar unit emits a microwave beam at a certain (proper) frequency f_0 along the road. A car that is moving toward the unit intercepts that beam but at a frequency that is shifted upward by the Doppler effect due to the car's motion toward the radar unit. The car reflects the beam back toward the radar unit. Because the car is moving toward the radar unit, the detector in the unit intercepts a reflected beam that is further shifted up in frequency. The unit compares that detected frequency with f_0 and computes the speed v of the car.

Astronomical Doppler Effect

In astronomical observations of stars, galaxies, and other sources of light, we can determine how fast the sources are moving, either directly away from us or directly toward us, by measuring the *Doppler shift* of the light that reaches us. If a certain star were at rest relative to us, we would detect light from it with a certain proper frequency f_0. However, if the star is moving either directly away from us or directly toward us, the light we detect has a frequency f that is shifted from f_0 by the Doppler effect. This Doppler shift is due only to the *radial* motion of the star (its motion directly toward us or away from us), and the speed we can determine by measuring this Doppler shift is only the *radial speed* v of the star—that is, only the radial component of the star's velocity relative to us.

Suppose a star (or any other light source) moves away from us with a radial speed v that is low enough (β is small enough) for us to neglect the β term in Eq. 36-33. Then we have

$$f = f_0(1 - \beta). \tag{36-34}$$

Because astronomical measurements involving light are usually done in wavelengths rather than frequencies, let's rewrite Eq. 36-34 as

$$\frac{c}{\lambda} = \frac{c}{\lambda_0}(1 - \beta),$$

or

$$\lambda = \lambda_0(1 - \beta)^{-1}.$$

Because we assume β is small, we can expand $(1 - \beta)^{-1}$ in a power series. Doing so and retaining only the first power of β, we have

$$\lambda = \lambda_0(1 + \beta),$$

or

$$\beta = \frac{\lambda - \lambda_0}{\lambda_0}. \tag{36-35}$$

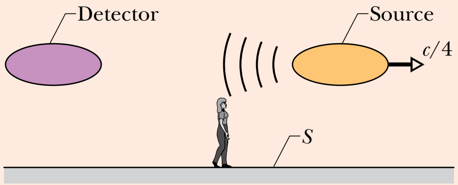
Replacing β with v/c and $\lambda - \lambda_0$ with $|\Delta\lambda|$ leads to

$$v = \frac{|\Delta\lambda|}{\lambda_0}c \quad \text{(radial speed of light source, } v \ll c\text{).} \tag{36-36}$$

The difference $\Delta\lambda$ is the *wavelength Doppler shift* of the light source. We enclose it with an absolute sign so that we always have a magnitude of the shift. Equation 36-36 is an approximation that can be applied whether the light source is moving toward or away from us but only when $v \ll c$.

CHECKPOINT 3

The figure shows a source that emits light of proper frequency f_0 while moving directly toward the right with speed $c/4$ as measured from reference frame S. The figure also shows a light detector, which measures a frequency $f > f_0$ for the emitted light. (a) Is the detector moving toward the left or the right? (b) Is the speed of the detector as measured from reference frame S more than $c/4$, less than $c/4$, or equal to $c/4$?



Transverse Doppler Effect

So far, we have discussed the Doppler effect, here and in Chapter 17, only for situations in which the source and the detector move either directly toward or directly away from each other. Figure 36-15 shows a different arrangement, in which a source S moves past a detector D. When S reaches point P, the velocity of S is perpendicular to the line joining P and D, and at that instant S is moving neither toward nor away from D. If the source is emitting sound waves of frequency f_0, D detects that frequency (with no Doppler effect) when it intercepts the waves that were emitted at point P. However, if the source is emitting light waves, there is still a Doppler effect, called the **transverse Doppler effect**. In this situation, the detected frequency of the light emitted when the source is at point P is

$$f = f_0\sqrt{1-\beta^2} \quad \text{(transverse Doppler effect).} \tag{36-37}$$

For low speeds ($\beta \ll 1$), Eq. 36-37 can be expanded in a power series in β and approximated as

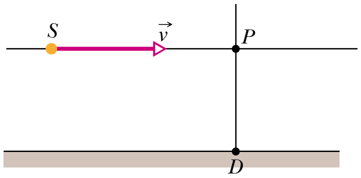


Figure 36-15 A light source S travels with Velocity $\vec{v}$ past a detector at D. The special theory of relativity predicts a transverse Doppler effect as the source passes through point P, where the direction of travel is perpendicular to the line extending through D. Classical theory predicts no such effect.

$$f = f_0\left(1-\frac{1}{2}\beta^2\right) \quad \text{(low speeds).} \tag{36-38}$$

Here the first term is what we would expect for sound waves, and again the relativistic effect for low-speed light sources and detectors appears with the β^2 term.

In principle, a police radar unit can determine the speed of a car even when the path of the radar beam is perpendicular (transverse) to the path of the car. However, Eq. 36-38 tells us that because β is small even for a fast car, the relativistic term $\beta^2/2$ in the transverse Doppler effect is extremely small. Thus, $f \approx f_0$ and the radar unit computes a speed of zero.

The transverse Doppler effect is really another test of time dilation. If we rewrite Eq. 36-37 in terms of the period T of oscillation of the emitted light wave instead of the frequency, we have, because $T = 1/f$,

$$T = \frac{T_0}{\sqrt{1-\beta^2}} = \gamma T_0, \tag{36-39}$$

in which T_0 ($= 1/f_0$) is the **proper period** of the source. As comparison with Eq. 36-9 shows, Eq. 36-39 is simply the time dilation formula because a period is a time interval.

36.11 | A NEW LOOK AT MOMENTUM

Key Concept

- The following definition of linear momentum $\vec{p}$, for a particle of mass m is valid at any physically possible speed:

$$\vec{p} = \gamma m\vec{v}.$$

Here λ is the Lorentz factor for the particle's motion.

Suppose that a number of observers, each in a different inertial reference frame, watch an isolated collision between two particles. In classical mechanics, we have seen that—even though the observers measure different velocities for the colliding particles—they all find that the law of conservation of momentum holds. That is, they find that the total momentum of the system of particles after the collision is the same as it was before the collision.

How is this situation affected by relativity? We find that if we continue to define the momentum $\vec{p}$ of a particle as $m\vec{v}$, the product of its mass and its velocity, total momentum is *not* conserved for the observers in different inertial frames. So, we need to redefine momentum in order to save that conservation law.

Consider a particle moving with constant speed v in the positive direction of an x axis. Classically, its momentum has magnitude

$$p = mv = m\frac{\Delta x}{\Delta t} \quad \text{(classical momentum)}, \tag{36-40}$$

in which Δx is the distance it travels in time Δx. To find a relativistic expression for momentum, we start with the new definition

$$p = m\frac{\Delta x}{\Delta t_0}.$$

Here, as before, Δx is the distance traveled by a moving particle as viewed by an observer watching that particle. However, Δt_0 is the time required to travel that distance, measured not by the observer watching the moving particle but by an observer moving with the particle. The particle is at rest with respect to this second observer; thus that measured time is a proper time.

Using the time dilation formula, $\Delta t = \gamma \Delta t_0$ (Eq. 36-9), we can then write

$$p = m\frac{\Delta x}{\Delta t_0} = m\frac{\Delta x}{\Delta t}\frac{\Delta x}{\Delta t_0} = m\frac{\Delta x}{\Delta t}\gamma.$$

However, since $\Delta x/\Delta t$ is just the particle velocity v, we have

$$\mathrm{p} = \gamma mv \quad \text{(momentum)}. \tag{36-41}$$

Note that this differs from the classical definition of Eq. 36-40 only by the Lorentz factor γ. However, that difference is important: Unlike classical momentum, relativistic momentum approaches an infinite value as v approaches c.

We can generalize the definition of Eq. 36-41 to vector form as

$$\vec{p} = \gamma m\vec{v} \quad \text{(momentum)}. \tag{36-42}$$

This equation gives the correct definition of momentum for all physically possible speeds. For a speed much less than c, it reduces to the classical definition of momentum ($\vec{p} = m\vec{v}$).

36.12 | A NEW LOOK AT ENERGY

Key Concepts

- The following definitions of kinetic energy K, and total energy E for a particle of mass m are valid at any physically possible speed:

$$E = mc^2 + K = \gamma mc^2 \quad \text{(total energy)},$$

$$K = mc^2(\gamma - 1) \quad \text{(kinetic energy)}.$$

Here γ is the Lorentz factor for the particle's motion, and mc^2 is the *mass energy*, or *rest energy*, associated with the mass of the particle.

- These equations lead to the relationships

$$(pc)^2 = K^2 + 2Kmc^2$$

and $$E^2 = (pc)^2 + (mc^2)^2.$$

- When a system of particles undergoes a chemical or nuclear reaction, the Q of the reaction is the negative of the change in the system's total mass energy:

$$Q = M_ic^2 - M_fc^2 = -\Delta M\, c^2,$$

where M_i is the system's total mass before the reaction and M_f is its total mass after the reaction.

Mass Energy

The science of chemistry was initially developed with the assumption that in chemical reactions, energy and mass are conserved separately. In 1905, Einstein showed that as a consequence of his theory of special relativity, mass can be considered to be another form of energy. Thus, the law of conservation of energy is really the law of conservation of mass–energy.

In a *chemical reaction* (a process in which atoms or molecules interact), the amount of mass that is transferred into other forms of energy (or vice versa) is such a tiny fraction of the total mass involved that there is no hope of measuring the mass change with even the best laboratory balances. Mass and energy truly *seem* to be separately conserved. However, in a *nuclear reaction* (in which nuclei or fundamental particles interact), the energy released is often about a million times greater than in a chemical reaction, and the change in mass can easily be measured.

An object's mass m and the equivalent energy E_0 are related by

$$E_0 = mc^2, \tag{36-43}$$

which, without the subscript 0, is the best-known science equation of all time. This energy that is associated with the mass of an object is called **mass energy** or **rest energy**. The second name suggests that E_0 is an energy that the object has even when it is at rest, simply because it has mass. (If you continue your study of physics beyond this book, you will see more refined discussions of the relation between mass and energy. You might even encounter disagreements about just what that relation is and means.)

Table 36-3 shows the (approximate) mass energy, or rest energy, of a few objects. The mass energy of, say, a U.S. penny is enormous; the equivalent amount of electrical energy would cost well over a million dollars. On the other hand, the entire annual U.S. electrical energy production corresponds to a mass of only a few hundred kilograms of matter (stones, burritos, or anything else).

In practice, SI units are rarely used with Eq. 36-43 because they are too large to be convenient. Masses are usually measured in atomic mass units, where

Table 36-3 The Energy Equivalents of a Few Objects

Object	Mass (kg)	Energy Equivalent	
Electron	$\approx 9.11 \times 10^{-31}$	$\approx 8.19 \times 10^{-14}$ J	(≈ 511 keV)
Proton	$\approx 1.67 \times 10^{-27}$	$\approx 1.50 \times 10^{-10}$ J	(≈ 938 MeV)
Uranium atom	$\approx 3.95 \times 10^{-25}$	$\approx 3.55 \times 10^{-8}$ J	(≈ 225 GeV)
Dust particle	$\approx 1 \times 10^{-13}$	$\approx 1 \times 10^{4}$ J	(≈ 2 kcal)
U.S. penny	$\approx 3.1 \times 10^{-3}$	$\approx 2.8 \times 10^{14}$ J	(≈ 78 GW·h)

$$1 \text{ u} = 1.660\,538\,86 \times 10^{-27} \text{ kg}, \tag{36-44}$$

and energies are usually measured in electron-volts or multiples of it, where

$$1 \text{ eV} = 1.602\,176\,462 \times 10^{-19} \text{ J}. \tag{36-45}$$

In the units of Eqs. 36-44 and 36-45, the multiplying constant c^2 has the values

$$\begin{aligned} c^2 &= 9.314\,940\,13 \times 10^{8} \text{ eV/u} = 9.314\,940\,13 \times 10^{5} \text{ keV/u} \\ &= 931.494\,013 \text{ MeV/u}. \end{aligned} \tag{36-46}$$

Total Energy

Equation 36-43 gives, for any object, the mass energy E_0 that is associated with the object's mass m, regardless of whether the object is at rest or moving. If the object is moving, it has additional energy in the form of kinetic energy K. If we assume that the object's potential energy is zero, then its total energy E is the sum of its mass energy and its kinetic energy:

$$E = E_0 + K = mc^2 + K. \tag{36-47}$$

Although we shall not prove it, the total energy E can also be written as

$$E = \gamma mc^2, \tag{36-48}$$

where γ is the Lorentz factor for the object's motion.

Since Chapter 8, we have discussed many examples involving changes in the total energy of a particle or a system of particles. However, we did not include mass energy in the discussions because the changes in mass energy were either zero or small enough to be neglected. The law of conservation of total energy still applies when changes in mass energy are significant. Thus, regardless of what happens to the mass energy, the following statement from Chapter 8 is still true:

The total energy E of an *isolated system* cannot change.

For example, if the total mass energy of two interacting particles in an isolated system decreases, some other type of energy in the system must increase because the total energy cannot change.

***Q* Value.** In a system undergoing a chemical or nuclear reaction, a change in the total mass energy of the system due to the reaction is often given as a Q value. The Q value for a reaction is obtained from the relation

$$\begin{pmatrix}\text{system's initial}\\ \text{total mass energy}\end{pmatrix} = \begin{pmatrix}\text{system's final}\\ \text{total mass energy}\end{pmatrix} + Q$$

or

$$E_{0i} = E_{0f} + Q. \tag{36-49}$$

Using Eq. 36-43 ($E_0 = mc^2$), we can rewrite this in terms of the initial *total* mass M_i and the final *total* mass M_f as

$$M_i c^2 = M_f c^2 + Q$$

or

$$Q = M_i c^2 - M_f c^2 = -\Delta M\, c^2, \tag{36-50}$$

where the change in mass due to the reaction is $\Delta M = M_f - M_i$.

If a reaction results in the transfer of energy from mass energy to, say, kinetic energy of the reaction products, the system's total mass energy E_0 (and total mass M) decreases and Q is positive. If, instead, a reaction requires that energy be transferred to mass energy, the system's total mass energy E_0 (and its total mass M) increases and Q is negative.

For example, suppose two hydrogen nuclei undergo a *fusion reaction* in which they join together to form a single nucleus and release two particles:

$$^1\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^2\mathrm{H} + \mathrm{e}^+ + \nu,$$

where ^{2}H is another type of hydrogen nucleus (with a neutron in addition to the proton), e^+ is a positron, and ν is a neutrino. The total mass energy (and total mass) of the resultant single nucleus and two released particles is less than the total mass energy (and total mass) of the initial hydrogen nuclei. Thus, the Q of the fusion reaction is positive, and energy is said to be *released* (transferred from mass energy) by the reaction. This release is important to you because the fusion of hydrogen nuclei in the Sun is one part of the process that results in sunshine on Earth and makes life here possible.

Kinetic Energy

In Chapter 8 we defined the kinetic energy K of an object of mass m moving at speed v well below c to be

$$K = \frac{1}{2}mv^2. \tag{36-51}$$

However, this classical equation is only an approximation that is good enough when the speed is well below the speed of light.

Let us now find an expression for kinetic energy that is correct for *all* physically possible speeds, including speeds close to c. Solving Eq. 36-47 for K and then substituting for E from Eq. 36-48 lead to

$$\begin{aligned} K &= E - mc^2 = \gamma mc^2 - mc^2 \\ &= mc^2(\gamma - 1) \quad \text{(kinetic energy)}, \end{aligned} \tag{36-52}$$

where $\gamma (= 1/\sqrt{1-(v/c)^2})$ is the Lorentz factor for the object's motion.

Figure 36-16 shows plots of the kinetic energy of an electron as calculated with the correct definition (Eq. 36-52) and the classical approximation (Eq. 36-51), both as functions of v/c. Note that on the left side of the graph the two plots coincide; this is the part of the graph—at lower speeds—where we have calculated kinetic energies so far in this book. This part of the graph tells us that we have been justified in calculating kinetic energy with the classical expression of Eq. 36-51. However, on the right side of the graph—at speeds near c—the two plots differ significantly. As v/c approaches 1.0, the plot for the classical definition of kinetic energy increases only moderately while the plot for the correct definition of kinetic energy increases dramatically, approaching an infinite value as v/c approaches 1.0. Thus, when an object's speed v is near c, we *must* use Eq. 36-52 to calculate its kinetic energy.

Work. Figure 36-16 also tells us something about the work we must do on an object to increase its speed by, say, 1%. The required work W is equal to the resulting change ΔK in the object's kinetic energy. If the change is to occur on the low-speed, left side of Fig. 36-16, the required work might be modest. However, if the change is to occur on the high-speed, right side of Fig. 36-16, the required work could be enormous because the kinetic energy K increases so rapidly there with an increase in speed v. To increase an object's speed to c would require, in principle, an infinite amount of energy; thus, doing so is impossible.

The kinetic energies of electrons, protons, and other particles are often stated with the unit electron-volt or one of its multiples used as an adjective. For example, an electron with a kinetic energy of 20 MeV may be described as a 20 MeV electron.

As v/c approaches 1.0, the actual kinetic energy approaches infinity.

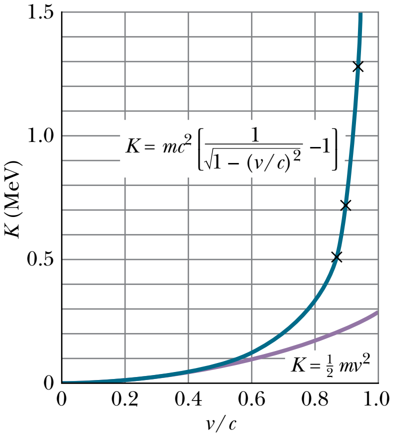


Figure 36-16 The relativistic (Eq. 36-52) and classical (Eq. 36-51) equations for the kinetic energy of an electron, plotted as a function of v/c, where v is the speed of the electron and c is the speed of light. Note that the two curves blend together at low speeds and diverge widely at high speeds. Experimental data (at the × marks) show that at high speeds the relativistic curve agrees with experiment but the classical curve does not.

Momentum and Kinetic Energy

In classical mechanics, the momentum p of a particle is mv and its kinetic energy K is $(1/2)mv^2$. If we eliminate v between these two expressions, we find a direct relation between momentum and kinetic energy:

$$p^2 = 2Km \quad \text{(classical)}. \tag{36-53}$$

We can find a similar connection in relativity by eliminating v between the relativistic definition of momentum (Eq. 36-41) and the relativistic definition of kinetic energy (Eq. 36-52). Doing so leads, after some algebra, to

$$(pc)^2 = K^2 + 2Kmc^2. \tag{36-54}$$

With the aid of Eq. 36-47, we can transform Eq. 36-54 into a relation between the momentum p and the total energy E of a particle:

$$E^2 = (pc)^2 + (mc^2)^2. \tag{36-55}$$

The right triangle of Fig. 36-17 can help you keep these useful relations in mind. You can also show that, in that triangle,

$$\sin\theta = \beta \quad \text{and} \quad \cos\theta = 1/\gamma. \tag{36-56}$$

This might help you to remember the relations.

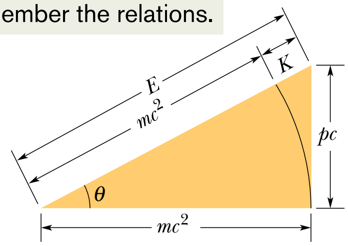


Figure 36-17 A useful memory diagram for the relativistic relations among the total energy E, the rest energy or mass energy mc^2, the kinetic energy K, and the momentum magnitude p.

With Eq. 36-55 we can see that the product pc must have the same unit as energy E; thus, we can express the unit of momentum p as an energy unit divided by c, usually as MeV/c or GeV/c in fundamental particle physics.

CHECKPOINT 4

Are (a) the kinetic energy and (b) the total energy of a 1 GeV electron more than, less than, or equal to those of a 1 GeV proton?

SAMPLE PROBLEM 36.06

Energy and momentum of a relativistic electron

(a) What is the total energy E of a 2.53 MeV electron?

KEY IDEA

From Eq. 36-47, the total energy E is the sum of the electron's mass energy (or rest energy) mc^2 and its kinetic energy:

$$E = mc^2 + K. \tag{36-57}$$

Calculations: The adjective "2.53 MeV" in the problem statement means that the electron's kinetic energy is 2.53 MeV. To evaluate the electron's mass energy mc^2, we substitute the electron's mass m from Appendix B, obtaining

$$Mc^2 = (9.109 \times 10^{-31}\ \text{kg})(299\,792\,458\ \text{m/s})^2$$
$$= 8.187 \times 10^{-14}\ \text{J}.$$

Then dividing this result by 1.602×10^{-13} J/MeV gives us 0.511 MeV as the electron's mass energy (confirming the value in Table 36-3). Equation 36-57 then yields

$$E = 0.511\ \text{MeV} + 2.53\ \text{MeV} = 3.04\ \text{MeV}. \quad \text{(Answer)}$$

(b) What is the magnitude p of the electron's momentum, in the unit MeV/c? (Note that c is the symbol for the speed of light and not itself a unit.)

KEY IDEA

We can find p from the total energy E and the mass energy mc^2 via Eq. 36-55,

$$E^2 = (pc)^2 + (mc^2)^2.$$

Calculations: Solving for pc gives us

$$pc = \sqrt{E^2 - (mc^2)^2}$$
$$= \sqrt{(3.04\ \text{MeV})^2 - (0.511\ \text{MeV})^2} = 3.00\ \text{MeV}.$$

Finally, dividing both sides by c we find

$$p = 3.00\ \text{MeV}/c. \quad \text{(Answer)}$$

SAMPLE PROBLEM 36.07

Energy and an astounding discrepancy in travel time

The most energetic proton ever detected in the cosmic rays coming to Earth from space had an astounding kinetic energy of 3.0×10^{20} eV (enough energy to warm a teaspoon of water by a few degrees).

(a) What were the proton's Lorentz factor γ and speed v (both relative to the ground-based detector)?

KEY IDEAS

(1) The proton's Lorentz factor γ relates its total energy E to its mass energy mc^2 via Eq. 36-48 ($E = \gamma mc^2$). (2) The proton's total energy is the sum of its mass energy mc^2 and its (given) kinetic energy K.

Calculations: Putting these ideas together we have

$$\gamma = \frac{E}{mc^2} = \frac{mc^2 + K}{mc^2} = 1 + \frac{K}{mc^2}. \tag{36-58}$$

From Table 36-3, the proton's mass energy mc^2 is 938 MeV. Substituting this and the given kinetic energy into Eq. 36-58, we obtain

$$\gamma = 1 + \frac{3.0 \times 10^{20}\ \text{eV}}{938 \times 10^{6}\ \text{eV}}$$
$$= 3.198 \times 10^{11} \approx 3.2 \times 10^{11}. \quad \text{(Answer)}$$

This computed value for γ is so large that we cannot use the definition of γ (Eq. 36-8) to find v. Try it; your calculator will tell you that β is effectively equal to 1 and thus that v is effectively equal to c. Actually, v is almost c, but we want a more accurate answer, which we can obtain by first solving Eq. 36-8 for $1 - \beta$. To begin we write

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1-\beta)(1+\beta)}} \approx \frac{1}{\sqrt{2(1-\beta)}},$$

where we have used the fact that β is so close to unity that $1 + \beta$ is very close to 2. (We can round off the sum of two very close numbers but not their difference.) The velocity we seek is contained in the $1 - \beta$ term. Solving for $1 - \beta$ then yields

$$1-\beta = \frac{1}{2\gamma^2} = \frac{1}{(2)(3.198\times10^{11})^2}$$
$$= 4.9\times10^{-24} \approx 5\times10^{-24}.$$

Thus, $$\beta = 1 - 5 \times 10^{-24}$$

and, since $v = \beta c$,

$$v \approx 0.999\,999\,999\,999\,999\,999\,999\,995c. \qquad \text{(Answer)}$$

(b) Suppose that the proton travels along a diameter of the Milky Way galaxy (9.8×10^4 ly). Approximately how long does the proton take to travel that diameter as measured from the common reference frame of Earth and the Galaxy?

Reasoning: We just saw that this *ultrarelativistic* proton is traveling at a speed barely less than c. By the definition of light-year, light takes 1 y to travel a distance of 1 ly, and so light should take 9.8×10^4 y to travel 9.8×10^4 ly, and this proton should take almost the same time. Thus, from our Earth–Milky Way reference frame, the proton's trip takes

$$\Delta t = 9.8 \times 10^4 \text{ y}. \qquad \text{(Answer)}$$

(c) How long does the trip take as measured in the reference frame of the proton?

KEY IDEAS

1. This problem involves measurements made from two (inertial) reference frames: one is the Earth–Milky Way frame and the other is attached to the proton.
2. This problem also involves two events: the first is when the proton passes one end of the diameter along the Galaxy, and the second is when it passes the opposite end.
3. The time interval between those two events as measured in the proton's reference frame is the proper time interval Δt_0 because the events occur at the same location in that frame—namely, at the proton itself.
4. We can find the proper time interval Δt_0 from the time interval Δt measured in the Earth–Milky Way frame by using Eq. 36-9 ($\Delta t = \gamma\,\Delta t_0$) for time dilation. (Note that we can use that equation because one of the time measures *is* a proper time. However, we get the same relation if we use a Lorentz transformation.)

Calculation: Solving Eq. 36-9 for Δt_0 and substituting γ from (a) and Δt from (b), we find

$$\Delta t_0 = \frac{\Delta t}{\gamma} = \frac{9.8\times10^4\text{ y}}{3.198\times10^{11}}$$
$$= 3.06\times10^{-7}\text{ y} = 9.7\text{ s}. \qquad \text{(Answer)}$$

In our frame, the trip takes 98 000 y. In the proton's frame, it takes 9.7 s! As promised at the start of this chapter, relative motion can alter the rate at which time passes, and we have here an extreme example.

REVIEW AND SUMMARY

The Postulates Einstein's **special theory of relativity** is based on two postulates:

1. The laws of physics are the same for observers in all inertial reference frames. No one frame is preferred over any other.
2. The speed of light in vacuum has the same value c in all directions and in all inertial reference frames.

The speed of light c in vacuum is an ultimate speed that cannot be exceeded by any entity carrying energy or information.

Coordinates of an Event Three space coordinates and one time coordinate specify an **event**. One task of special relativity is to relate these coordinates as assigned by two observers who are in uniform motion with respect to each other.

Simultaneous Events If two observers are in relative motion, they will not, in general, agree as to whether two events are simultaneous.

Time Dilation If two successive events occur at the same place in an inertial reference frame, the time interval Δt_0 between them, measured on a single clock where they occur, is the **proper time** between the events. *Observers in frames moving relative to that frame will measure a larger value for this interval.* For an observer moving with relative speed v, the measured time interval is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \frac{\Delta t_0}{\sqrt{1-\beta^2}} = \gamma\,\Delta t_0 \quad \text{(time dilation).} \qquad \text{(36-7 to 36-9)}$$

Here $\beta = v/c$ is the **speed parameter** and $\gamma = 1/\sqrt{1-\beta^2}$ is the **Lorentz factor**. An important result of time dilation is that moving clocks run slow as measured by an observer at rest.

Length Contraction The length L_0 of an object measured by an observer in an inertial reference frame in which the object is at rest is called its **proper length**. *Observers in frames moving relative to that frame and parallel to that length will measure a shorter length.* For an observer moving with relative speed v, the measured length is

$$L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma} \quad \text{(length contraction).} \qquad \text{(36-13)}$$

The Lorentz Transformation The *Lorentz transformation* equations relate the spacetime coordinates of a single event as seen by observers in two inertial frames, S and S', where S' is moving relative to S with velocity v in the positive x and x' direction. The four coordinates are related by

$$\begin{aligned} x' &= \gamma(x - vt),\\ y' &= y,\\ z' &= z,\\ t' &= \gamma(t - vx/c^2). \end{aligned} \qquad \text{(36-21)}$$

Relativity of Velocities When a particle is moving with speed u' in the positive x' direction in an inertial reference frame S' that itself is moving with speed v parallel to the x direction of a second inertial frame S, the speed u of the particle as measured in S is

$$u = \frac{u' + v}{1 + u'v/c^2} \quad \text{(relativistic velocity).} \qquad \text{(36-29)}$$

Relativistic Doppler Effect When a light source and a light detector move directly relative to each other, the wavelength of the light as measured in the rest frame of the source is the *proper wavelength* λ_0. The detected wavelength λ is either longer (a *red shift*) or shorter (a *blue shift*) depending on whether the source–detector separation is increasing or decreasing. When the separation is increasing, the wavelengths are related by

$$\lambda = \lambda_0\sqrt{\frac{1+\beta}{1-\beta}} \quad \text{(source and detector separating),} \qquad \text{(36-32)}$$

where $\beta = v/c$ and v is the relative radial speed (along a line connecting the source and detector). If the separation is decreasing, the signs in front of the β symbols are reversed. For speeds much less than c, the magnitude of the Doppler wavelength shift ($\Delta\lambda = \lambda - \lambda_0$) is approximately related to v by

$$v = \frac{|\Delta\lambda|}{\lambda_0}c \quad (v \ll c). \qquad \text{(36-36)}$$

Transverse Doppler Effect If the relative motion of the light source is perpendicular to a line joining the source and detector, the detected frequency f is related to the proper frequency f_0 by

$$f = f_0\sqrt{1-\beta^2}. \qquad \text{(36-37)}$$

Momentum and Energy The following definitions of linear momentum $\vec{p}$, kinetic energy K, and total energy E for a particle of mass m are valid at any physically possible speed:

$$\vec{p} = \gamma m\vec{v} \quad \text{(momentum),} \qquad \text{(36-42)}$$

$$E = mc^2 + K = \gamma mc^2 \quad \text{(total energy),} \qquad \text{(36-47, 36-48)}$$

$$K = mc^2(\gamma - 1) \quad \text{(kinetic energy).} \qquad \text{(36-52)}$$

Here γ is the Lorentz factor for the particle's motion, and mc^2 is the *mass energy*, or *rest energy*, associated with the mass of the particle. These equations lead to the relationships

$$(pc)^2 = K^2 + 2Kmc^2 \qquad \text{(36-54)}$$

and

$$E^2 = (pc)^2 + (mc^2)^2. \qquad \text{(36-55)}$$

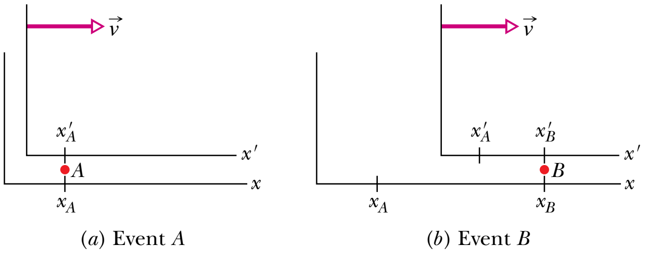
When a system of particles undergoes a chemical or nuclear reaction, the Q of the reaction is the negative of the change in the system's total mass energy:

$$Q = M_ic^2 - M_fc^2 = -\Delta M\,c^2, \qquad \text{(36-50)}$$

where M_i is the system's total mass before the reaction and M_f is its total mass after the reaction.

PROBLEMS

1. For the passing reference frames in Fig. 36-18, events A and B occur at the following spacetime coordinates: according to the unprimed frame, (x_A, t_A) and (x_B, t_B); according to the primed frame, (x'_A, t'_A) and (x'_B, t'_B). In the unprimed frame, $\Delta t = t_B - t_A = 1.00\ \mu\text{s}$ and $\Delta x = x_B - x_A = 400$ m. (a) Find an expression for $\Delta x'$ in terms of the speed parameter β and the given data. Graph $\Delta x'$ versus β for two ranges of β: (b) 0 to 0.01 and (c) 0.1 to 1. (d) At what value of β is $\Delta x'$ minimum, and (e) what is that minimum?



(*a*) Event *A* (*b*) Event *B*

Figure 36-18 Problem 1.

2. In Fig. 36-11, observer S detects two flashes of light. A big flash occurs at $x_1 = 1200$ m and, 5.00 μs later, a small flash occurs at $x_2 = 700$ m. As detected by observer S', the two flashes occur at a single coordinate x'. (a) What is the speed parameter of S', and (b) is S' moving in the positive or negative direction of the x axis? To S', (c) which flash occurs first and (d) what is the time interval between the flashes?

3. Apply the binomial theorem (Appendix E) to the last part of Eq. 36-52 for the kinetic energy of a particle. (a) Retain the first two terms of the expansion to show the kinetic energy in the form

$$K = (\text{first term}) + (\text{second term}).$$

The first term is the classical expression for kinetic energy. The second term is the first-order correction to the classical expression. Assume the particle is an electron. If its speed v is $c/20$, what is the value of (b) the classical expression and (c) the first-order correction? If the electron's speed is $0.85c$, what is the value of (d) the classical expression and (e) the first-order correction? (f) At what speed parameter β does the first-order correction become 10% or greater of the classical expression?

4. How much work must be done to increase the speed of an electron (a) from $0.18c$ to $0.20c$ and (b) from $0.97c$ to $0.99c$? Note that the speed increase is $0.01c$ in both cases.

5. (a) If m is a particle's mass, p is its momentum magnitude, and K is its kinetic energy, show that

$$m = \frac{(pc)^2 - K^2}{2Kc^2}.$$

(b) For low particle speeds, show that the right side of the equation reduces to m. (c) If a particle has $K = 55.0$ MeV when $p = 121$ MeV/c, what is the ratio m/m_e of its mass to the electron mass?

6. (a) Assuming that Eq. 36-36 holds, find how fast you would have to go through a red light to have it appear green. Take 620 nm as the wavelength of red light and 540 nm as the wavelength of green light. (b) What would the emission wavelength be if you went twice that speed?

7. A sodium light source moves in a horizontal circle at a constant speed of $0.100c$ while emitting light at the proper wavelength of $\lambda_0 = 589.00$ nm. Wavelength λ is measured for that light by a detector fixed at the center of the circle. What is the wavelength shift $\lambda - \lambda_0$?

8. The mass of an electron is $9.109\,381\,88 \times 10^{-31}$ kg. To six significant figures, find (a) γ and (b) β for an electron with kinetic energy $K = 300.000$ MeV.

9. A spaceship whose rest length is 280 m has a speed of $0.94c$ with respect to a certain reference frame. A micrometeorite, also with a speed of $0.94c$ in this frame, passes the spaceship on an antiparallel track. How long does it take this object to pass the ship as measured on the ship?

10. What is the minimum energy that is required to break a nucleus of ^{12}C (of mass 11.996 71 u) into three nuclei of ^{4}He (of mass 4.001 51 u each)?

11. An armada of spaceships that is 1.00 ly long (as measured in its rest frame) moves with speed $0.850c$ relative to a ground station in frame S. A messenger travels from the rear of the armada to the front with a speed of $0.950c$ relative to S. How long does the trip take as measured (a) in the rest frame of the messenger, (b) in the rest frame of the armada, and (c) by an observer in the ground frame S?

12. The mass of an electron is $9.109\,381\,88 \times 10^{-31}$ kg. To eight significant figures, find the following for the given electron kinetic energy: (a) γ and (b) β for $K = 1.000\,000\,0$ keV, (c) γ and (d) β for $K = 1.000\,000\,0$ MeV, and then (e) γ and (f) β for $K = 1.000\,000\,0$ GeV.

13. A spaceship is moving away from Earth at speed $0.333c$. A source on the rear of the ship emits light at wavelength 450 nm according to someone on the ship. What (a) wavelength and (b) color (blue, green, yellow, or red) are detected by someone on Earth watching the ship?

14. Stellar system Q_1 moves away from us at a speed of $0.800c$. Stellar system Q_2, which lies in the same direction in space but is closer to us, moves away from us at speed $0.400c$. What multiple of c gives the speed of Q_2 as measured by an observer in the reference frame of Q_1?

15. As you read this page (on paper or monitor screen), a cosmic ray proton passes along the left–right width of the page with relative speed v and a total energy of 23.16 nJ. According to your measurements, that left–right width is 21.0 cm. (a) What is the width according to the proton's reference frame? How much time did the passage take according to (b) your frame and (c) the proton's frame?

16. The mass of a muon is 207 times the electron mass; the average lifetime of muons at rest is 2.20 μs. In a certain experiment, muons moving through a laboratory are measured to have an average lifetime of 6.90 μs. For the moving muons, what are (a) β, (b) K, and (c) p (in MeV/c)?

17. Galaxy A is reported to be receding from us with a speed of $0.45c$. Galaxy B, located in precisely the opposite direction, is also found to be receding from us at this same speed. What multiple of c gives the recessional speed an observer on Galaxy A would find for (a) our galaxy and (b) Galaxy B?

18. How much work must be done to increase the speed of an electron from rest to (a) $0.500c$, (b) $0.990c$, and (c) $0.999\,90c$?

19. In Chapter 28, we showed that a particle of charge q and mass m will move in a circle of radius $r = mv/|q|B$ when its velocity $\vec{v}$ is perpendicular to a uniform magnetic field $\vec{B}$. We also found that the period T of the motion is independent of speed v. These two results are approximately correct if $v \ll c$. For relativistic speeds, we must use the correct equation for the radius:

$$r = \frac{p}{|q|B} = \frac{\gamma m v}{|q|B}.$$

(a) Using this equation and the definition of period ($T = 2\pi r/v$), find the correct expression for the period. (b) Is T independent of v? If a 20.0 MeV electron moves in a circular path in a uniform magnetic field of magnitude 2.20 T, what are (c) the radius according to Chapter 28, (d) the correct radius, (e) the period according to Chapter 28, and (f) the correct period?

20. Observer S reports that an event occurred on the x axis of his reference frame at $x = 3.00 \times 10^8$ m at time $t = 1.50$ s. Observer S' and her frame are moving in the positive direction of the x axis at a speed of $0.400c$. Further, $x = x' = 0$ at $t = t' = 0$. What are the (a) spatial and (b) temporal coordinate of the event according to S'? If S' were, instead, moving in the *negative* direction of the x axis, what would be the (c) spatial and (d) temporal coordinate of the event according to S'?

21. An experimenter arranges to trigger two flashbulbs simultaneously, producing a big flash located at the origin of his reference frame and a small flash at $x = 30.0$ km. An observer moving at a speed of $0.450c$ in the positive direction of x also views the flashes. (a) What is the time interval between them according to her? (b) Which flash does she say occurs first?

22. *(Come) back to the future*. Suppose that a father is 25.0 y older than his daughter. He wants to travel outward from Earth for 2.000 y and then back for another 2.000 y (both intervals as he measures them) such that he is then 25.0 y *younger* than his daughter. What constant speed parameter β (relative to Earth) is required?

23. A clock moves along an x axis at a speed of $0.700c$ and reads zero as it passes the origin of the axis. (a) Calculate the clock's Lorentz factor. (b) What time does the clock read as it passes $x = 180$ m?

24. To four significant figures, find the following when the kinetic energy is 12.00 MeV: (a) γ and (b) β for an electron ($E_0 = 0.510\ 998$ MeV), (c) γ and (d) β for a proton ($E_0 = 938.272$ MeV), and (e) γ and (f) β for an α particle ($E_0 = 3727.40$ MeV).

25. The premise of the *Planet of the Apes* movies and book is that hibernating astronauts travel far into Earth's future, to a time when human civilization has been replaced by an ape civilization. Considering only special relativity, determine how far into Earth's future the astronauts would travel if they slept for 120.0 y while traveling relative to Earth with a speed of $0.999\ 90c$, first outward from Earth and then back again.

26. In the reaction $\mathrm{p} + {}^{19}\mathrm{F} \rightarrow \alpha + {}^{16}\mathrm{O}$, the masses are

$$m(\mathrm{p}) = 1.007\ 825\ \mathrm{u}, \quad m(\alpha) = 4.002\ 603\ \mathrm{u},$$
$$m(\mathrm{F}) = 18.998\ 405\ \mathrm{u}, \quad m(\mathrm{O}) = 15.994\ 915\ \mathrm{u}.$$

Calculate the Q of the reaction from these data.

27. A rod lies parallel to the x axis of reference frame S, moving along this axis at a speed of $0.892c$. Its rest length is 1.70 m. What will be its measured length in frame S?

28. In Fig. 36-14, frame S' moves relative to frame S with velocity $0.620c\hat{\mathrm{i}}$ while a particle moves parallel to the common x and x' axes. An observer attached to frame S' measures the particle's velocity to be $0.47c\hat{\mathrm{i}}$. In terms of c, what is the particle's velocity as measured by an observer attached to frame S according to the (a) relativistic and (b) classical velocity transformation? Suppose, instead, that the S' measure of the particle's velocity is $-0.47c\hat{\mathrm{i}}$. What velocity does the observer in S now measure according to the (c) relativistic and (d) classical velocity transformation?

29. The center of our Milky Way galaxy is about 23 000 ly away. (a) To eight significant figures, at what constant speed parameter would you need to travel exactly 23 000 ly (measured in the Galaxy frame) in exactly 40 y (measured in your frame)? (b) Measured in your frame and in light-years, what length of the Galaxy would pass by you during the trip?

30. (a) The energy released in the explosion of 1.00 mol of TNT is 3.40 MJ. The molar mass of TNT is 0.227 kg/mol. What weight of TNT is needed for an explosive release of 1.80×10^{14} J? (b) Can you carry that weight in a backpack, or is a truck or train required? (c) Suppose that in an explosion of a fission bomb, 0.080% of the fissionable mass is converted to released energy. What weight of fissionable material is needed for an explosive release of 1.80×10^{14} J? (d) Can you carry that weight in a backpack, or is a truck or train required?

31. A 7.00-grain aspirin tablet has a mass of 448 mg. For how many kilometers would the energy equivalent of this mass power an automobile? Assume 12.75 km/L and a heat of combustion of 3.65×10^7 J/L for the gasoline used in the automobile.

32. What is β for a particle with (a) $K = 3.00E_0$ and (b) $E = 3.00E_0$?

33. Quasars are thought to be the nuclei of active galaxies in the early stages of their formation. If a quasar radiates energy at the rate of 2.00×10^{41} W, at what rate is the mass of this quasar being reduced to supply this energy? Express your answer in solar mass units per year, where one solar mass unit (1 smu $= 2.0 \times 10^{30}$ kg) is the mass of our Sun.

34. A rod is to move at constant speed v along the x axis of reference frame S, with the rod's length parallel to that axis. An observer in frame S is to measure the length L of the rod. Figure 36-19 gives length L versus speed parameter β for a range of values for β. The vertical axis scale is set by $L_a = 1.00$ m. What is L if $v = 0.90c$?

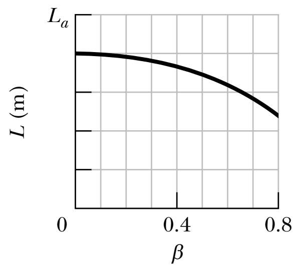


Figure 36-19 Problem 34.

35. A spaceship of rest length 270 m races past a timing station at a speed of $0.600c$. (a) What is the length of the spaceship as measured by the timing station? (b) What time interval will the station clock record between the passage of the front and back ends of the ship?

36. Certain wavelengths in the light from a galaxy in the constellation Virgo are observed to be 0.4% longer than the corresponding light from Earth sources. (a) What is the radial speed of this galaxy with respect to Earth? (b) Is the galaxy approaching or receding from Earth?

37. What must be the momentum of a particle with mass m so that the total energy of the particle is 4.00 times its rest energy?

38. A space traveler takes off from Earth and moves at speed $0.9950c$ toward the star Vega, which is 26.00 ly distant. How much time will have elapsed by Earth clocks (a) when the traveller reaches Vega and (b) when Earth observers receive word from the traveler that she has arrived? (c) How much older will Earth observers calculate the traveler to be (measured from her frame) when she reaches Vega than she was when she started the trip?

39. Bullwinkle in reference frame S' passes you in reference frame S along the common direction of the x' and x axes, as in Fig. 36-9. He carries three meter sticks: meter stick 1 is parallel to the x' axis, meter stick 2 is parallel to the y' axis, and meter stick 3 is parallel to the z' axis. On his wristwatch he counts off 10.0 s, which takes 30.0 s according to you. Two events occur during his passage. According to you, event 1 occurs at $x_1 = 33.0$ m and $t_1 = 22.0$ ns, and event 2 occurs at $x_2 = 53.0$ m and $t_2 = 62.0$ ns. According to your measurements, what is the length of (a) meter stick 1, (b) meter stick 2, and (c) meter stick 3? According to Bullwinkle, what are (d) the spatial separation and (e) the temporal separation between events 1 and 2, and (f) which event occurs first?

40. An alpha particle with kinetic energy 7.70 MeV collides with an ^{14}N nucleus at rest, and the two transform into an ^{17}O nucleus and a proton. The proton is emitted at 90° to the direction of the incident alpha particle and has a kinetic energy of 4.44 MeV. The masses of the various particles are alpha particle, 4.002 60 u; ^{14}N, 14.003 07 u; proton, 1.007 825 u; and ^{17}O, 16.999 14 u. In MeV, what are (a) the kinetic energy of the oxygen nucleus and (b) the Q of the reaction? (*Hint:* The speeds of the particles are much less than c.)

41. Reference frame S' is to pass reference frame S at speed v along the common direction of the x' and x axes, as in Fig. 36-11. An observer who rides along with frame S' is to count off a certain time interval on his wristwatch. The corresponding time interval Δt is to be measured by an observer in frame S. Figure 36-20 gives Δt versus speed parameter β for a range of values for β. The vertical axis scale is set by $\Delta t_0 = 14.0$ s. What is interval Δt if $v = 0.96c$?

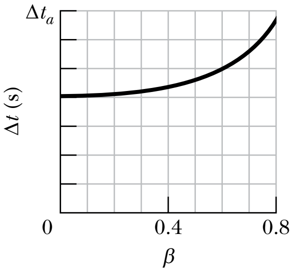


Figure 36-20 Problem 41.

42. In Fig. 36-11, the origins of the two frames coincide at $t = t' = 0$ and the relative speed is $0.980c$. Two micrometeorites collide at coordinates $x = 100$ km and $t = 200$ μs according to an observer in frame S. What are the (a) spatial and (b) temporal coordinate of the collision according to an observer in frame S'?

43. In Fig. 36-11, observer S detects two flashes of light. A big flash occurs at $x_1 = 1000$ m and, slightly later, a small flash occurs at $x_2 = 480$ m. The time interval between the flashes is $\Delta t = t_2 - t_1$. What is the smallest value of Δt for which observer S' will determine that the two flashes occur at the same x' coordinate?

44. A certain particle of mass m has momentum of magnitude $2.00mc$. What are (a) β, (b) γ, and (c) the ratio K/E_0?

45. As in Fig. 36-11, reference frame S' passes reference frame S with a certain velocity. Events 1 and 2 are to have a certain temporal separation $\Delta t'$ according to the S' observer. However, their spatial separation $\Delta x'$ according to that observer has not been set yet. Figure 36-21 gives their temporal separation Δt according to the S observer as a function of $\Delta x'$ for a range of $\Delta x'$ values. The vertical axis scale is set by $\Delta t_a = 6.00$ μs. What is $\Delta t'$?

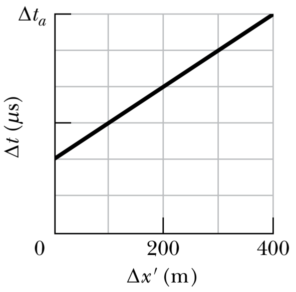


Figure 36-21 Problem 45.

46. The mean lifetime of stationary muons is measured to be 2.2000 μs. The mean lifetime of high-speed muons in a burst of cosmic rays observed from Earth is measured to be 20.000 μs. To five significant figures, what is the speed parameter β of these cosmic-ray muons relative to Earth?

47. The length of a spaceship is measured to be 25% of its rest length. (a) To three significant figures, what is the speed parameter β of the spaceship relative to the observer's frame? (b) By what factor do the spaceship's clocks run slow relative to clocks in the observer's frame?

48. In a high-energy collision between a cosmic-ray particle and a particle near the top of Earth's atmosphere, 120 km above sea level, a pion is created. The pion has a total energy E of 4.00×10^5 MeV and is traveling vertically downward. In the pion's rest frame, the pion decays 35.0 ns after its creation. At what altitude above sea level, as measured from Earth's reference frame, does the decay occur? The rest energy of a pion is 139.6 MeV.

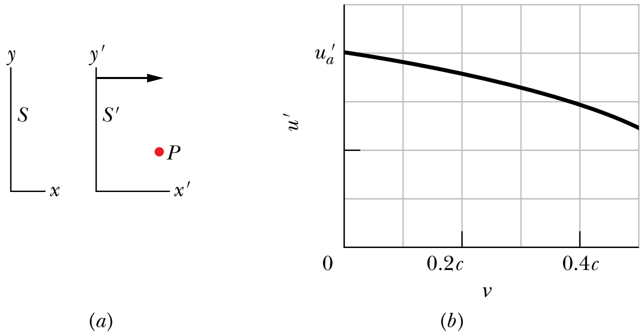
49. A meter stick in frame S' makes an angle of 30° with the x' axis. If that frame moves parallel to the x axis of frame S with speed $0.95c$ relative to frame S, what is the length of the stick as measured from S?

50. A spaceship, moving away from Earth at a speed of $0.850c$, reports back by transmitting at a frequency (measured in the spaceship frame) of 100 MHz. To what frequency must Earth receivers be tuned to receive the report?

51. Inertial frame S' moves at a speed of $0.65c$ with respect to frame S (Fig. 36-11). Further, $x = x' = 0$ at $t = t' = 0$. Two events are recorded. In frame S, event 1 occurs at the origin at $t = 0$ and event 2 occurs on the x axis at $x = 3.0$ km at $t = 4.0$ μs. According to observer S', what is the time of (a) event 1 and (b) event 2? (c) Do the two observers see the same sequence or the reverse?

52. A particle moves along the x' axis of frame S' with velocity $0.50c$. Frame S' moves with velocity $0.80c$ with respect to frame S. What is the velocity of the particle with respect to frame S?

53. In Fig. 36-22a, particle P is to move parallel to the x and x' axes of reference frames S and S', at a certain velocity relative to frame S. Frame S' is to move parallel to the x axis of frame S at velocity v. Figure 36-22b gives the velocity u' of the particle relative to frame S' for a range of values for v. The vertical axis scale is set by $u'_a = 0.800c$. What value will u' have if (a) $v = 0.95c$ and (b) $v \to c$?



(a) (b)

Figure 36-22 Problem 53.

54. An unstable high-energy particle enters a detector and leaves a track of length 0.856 mm before it decays. Its speed relative to the detector was $0.992c$. What is its proper lifetime? That is, how long would the particle have lasted before decay had it been at rest with respect to the detector?

PRACTICE QUESTIONS

Single Correct Choice Type

1. At time $t = 2.3$ s, a 4 kg block that initially moves with a constant speed of 6 m/s undergoes an inelastic collision with another block. Any two inertial observers must agree that

(a) The event took place at $t = 2.3$ s

(b) The initial speed of the block is 6 m/s

(c) The initial momentum of the block has magnitude 24 kg m/s

(d) The momentum of the two block system is conserved during the collision

2. Which one of the following is a consequence of the postulates of special relativity?

(a) There is no such thing as an inertial reference frame.

(b) Newton's laws of motion apply in every reference frame.

(c) Coulomb's law of electrostatics applies in any reference frame.

(d) The question of whether an object is at rest in the universe is meaningless.

3. Which one of the following statements is a consequence of Special Relativity?

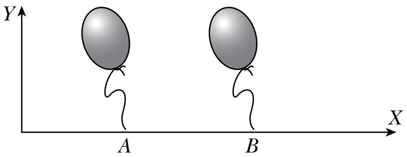
(a) Clocks that are moving run slower than when they are at rest.

(b) The length of a moving object is larger than it was at rest.

(c) Events occur at the same coordinates for observers in all inertial reference frames.

(d) Events occur at the same time for observers in all inertial reference frames.

4. Two helium-filled balloons are released simultaneously at points A and B on the X axis in an earth-based reference frame. Which one of the following statements is true for an observer moving in the $+x$ direction?



(a) The observer always sees the balloons released simultaneously.

(b) The observer could see either balloon released first depending on her speed and the distance between A and B.

(c) The observer sees balloon A released before balloon B.

(d) The observer sees balloon B released before balloon A.

5. In the year 2100, an astronaut wears an antique, but accurate, quartz wristwatch on a journey at a speed of 2.0×10^8 m/s. According to mission control in Houston, the trip lasts 12 hours. How long was the trip as measured on the watch?

(a) 6.7 hr **(b)** 12.0 hr

(c) 8.9 hr **(d)** 16.1 hr

6. In a science fiction novel, a starship takes three days to travel between two distant space stations according to its own clocks. Instruments on one of the space stations indicate that the trip took four days. How fast did the starship travel, relative to the space station?

(a) 1.98×10^8 m/s
(b) 2.51×10^8 m/s
(c) 2.24×10^8 m/s
(d) 2.83×10^8 m/s

7. The proper mean lifetime of a muon is 2.2×10^{-6} s. A beam of muons is moving with speed $0.6c$ relative to an inertial observer. How far will a muon in the beam travel, on average, before it decays?

(a) 288 m
(b) 500 m
(c) 360 m
(d) 600 m

8. Which one of the following statements concerning the proper length of a meter stick is true?

(a) The proper length is always one meter.
(b) The proper length depends upon the speed of the observer.
(c) The proper length depends upon the acceleration of the observer.
(d) The proper length depends upon the reference frame in which it is measured.

9. A meter stick is observed to be only 0.900 meters long to an inertial observer. At what speed, relative to the observer, must the meter stick be moving?

(a) 0.44×10^8 m/s
(b) 0.95×10^8 m/s
(c) 0.57×10^8 m/s
(d) 1.31×10^8 m/s

10. A UFO flies directly over a football stadium at a speed of $0.50c$. If the proper length of the field is 100 yards, what field length is measured by the crew of the UFO?

(a) 59 yards
(b) 87 yards
(c) 75 yards
(d) 113 yards

11. A proton has a mass of 1.673×10^{-27} kg. If the proton is accelerated to a speed of $0.93c$, what is the magnitude of the relativistic momentum of the proton?

(a) 6.2×10^{-17} kg·m/s
(b) 4.7×10^{-19} kg·m/s
(c) 1.3×10^{-18} kg·m/s
(d) 5.9×10^{-24} kg·m/s

12. An electron gun inside a computer monitor sends an electron toward the screen at a speed of 1.20×10^8 m/s. If the mass of the electron is 9.109×10^{-31} kg, what is the magnitude of its relativistic momentum?

(a) 9.88×10^{-23} kg·m/s
(b) 1.19×10^{-22} kg·m/s
(c) 1.09×10^{-22} kg·m/s
(d) 1.41×10^{-22} kg·m/s

13. In the distant future, a 5.40×10^5-kg intergalactic ship leaves Earth orbit and accelerates to a constant speed of $0.92c$. Determine the difference, $p - p_0$, between the relativistic and classical momenta of the ship.

(a) 3.9×10^{14} kg·m/s
(b) 8.0×10^{13} kg·m/s
(c) 2.3×10^{14} kg·m/s
(d) 5.8×10^{13} kg·m/s

14. Determine the speed at which the kinetic energy of an electron is equal to twice its rest energy.

(a) $0.45c$
(b) $0.87c$
(c) $0.63c$
(d) $0.94c$

15. A muon has rest energy 105 MeV. What is its kinetic energy when its speed is $0.95c$?

(a) 37 MeV
(b) 231 MeV
(c) 47 MeV
(d) 441 MeV

16. A space ship at rest on a launching pad has a mass of 1.00×10^5 kg. How much will its energy have increased when the ship is moving at $0.600c$?

(a) 1.12×10^{21} J
(b) 2.25×10^{21} J
(c) 1.62×10^{21} J
(d) 6.00×10^{21} J

17. Two spaceships are observed from Earth to be approaching each other along a straight line. Ship *A* moves at $0.40c$ relative to the Earth observer, while ship *B* moves at $0.50c$ relative to the same observer. What speed does the captain of ship *A* report for the speed of ship *B*?

(a) $0.10c$
(b) $0.85c$
(c) $0.75c$
(d) $0.95c$

18. Spaceship *A* travels at $0.400c$ relative to an Earth observer. According to the same observer, spaceship *A* overtakes a slower moving spaceship *B* that moves in the same direction. The captain of *B* sees *A* pass her ship at $0.114c$. Determine the speed of *B* relative to the Earth observer.

(a) $0.100c$
(b) $0.300c$
(c) $0.214c$
(d) $0.625c$

19. A passenger train travels east at high speed. One passenger is located at the east side of one car, another is located in the west side of that car. In the train's frame, these two passengers glance up at the same time. In the earth's frame,

(a) They glance up simultaneously
(b) The passenger at the east side glances up first
(c) The passenger at the west side glances up first
(d) The passengers glance sideways

20. Two events occur simultaneously on the *x* axis of reference frame *S*, one at $x = 0$ and the other at $x = +1$. According to an observer moving in the positive *x* direction

(a) The event at $x = +1$ occurs first
(b) The event at $x = 0$ occurs first
(c) Either event might occur first, depending on the value of (a) and the observer's speed
(d) The events are simultaneous

21. Two events occur simultaneously at separated points on the *y* axis of reference frame *S*. According to an observer moving in the positive *x* direction

(a) The event with the greater *y* coordinate occurs first
(b) The event with the greater *y* coordinate occurs last
(c) Either event might occur first, depending on the observer's speed
(d) The events are simultaneous

22. Two events occur on the *x* axis separated in time by Δt and in space by Δx. A reference frame, traveling at less than the speed of light, in which the two events occur at the same time

(a) Exists no matter what the values of Δx and Δt
(b) Exists only if $\Delta x/\Delta t < c$
(c) Exists only if $\Delta x/\Delta t > c$
(d) Does not exist under any condition

23. An astronaut aged 3 years when traveling at 99% of the speed of light to the Andromeda galaxy and back. The space officials who greeted her on her return have aged less than

(a) 3 years
(b) More than 2.5 years and less than 3 years
(c) More than 3 years
(d) 2.5 years

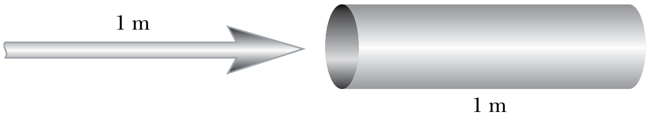
24. The spaceship Enterprise, traveling through the galaxy, sends out a smaller explorer craft that travels to a nearby planet and signals its findings back. The proper time for the trip to the planet is measured by clocks

(a) On board the Enterprise
(b) On board the explorer craft
(c) On Earth
(d) At the center of the galaxy

25. A meson when at rest decays 2 μs after it is created. If moving in the laboratory at 0.99c, its lifetime according to laboratory clocks would be

(a) the same **(b)** 0.28 s
(c) 14 μs **(d)** 4.6 s

26. A 1-meter long spear is thrown at a relativistic speed through a pipe that is 1 meter long (as shown in the following figure). Both these dimensions are measured when each is at rest. When the spear passes through the pipe, which of these statements best describes what is observed?



(a) The spear shrinks so that the pipe completely covers it at some point.
(b) The pipe shrinks so that the spear extends from both ends at some point.
(c) Both shrink equally so that the pipe completely covers it at some point.
(d) Any of the above, depending on the motion of the observer.

27. A certain automobile is 6 m long if at rest. If it is measured to be 4/5 as long, its speed is

(a) 0.1c **(b)** 0.3c
(c) 0.6c **(d)** 0.8c

28. If you were in a spaceship traveling at a speed close to the speed of light (with respect to Earth) you would notice that

(a) Your pulse rate is different than normal
(b) Some of your physical dimensions were smaller than normal
(c) Your mass is different than normal
(d) None of these effects occur

29. Frame S_1 moves in the positive x direction at 0.6c with respect to frame S_2. A particle moves in the positive x direction at 0.4c as measured by an observer in S_1. The speed of the particle as measured by an observer in S_2 is

(a) $c/5$ **(b)** $5c/19$
(c) $8c/25$ **(d)** $25c/31$

30. A spectral line of a certain star is observed to be "red shifted" from a wavelength of 500 nm to a wavelength of 1500 nm. Interpreting this as a Doppler effect, the speed of recession of this star is

(a) 0.33c **(b)** 0.50c
(c) 0.71c **(d)** 0.8c

31. The special theory of relativity predicts that there is an upper limit to the speed of a particle. It thus follows that there is also an upper limit on the following property of a particle

(a) The linear momentum
(b) The total energy
(c) The kinetic energy
(d) None of these

32. If the kinetic energy of a free particle is much greater than its rest energy then its kinetic energy is proportional to

(a) The magnitude of its momentum
(b) The square of the magnitude of its momentum
(c) The square root of the magnitude of its momentum
(d) The reciprocal of the magnitude of its momentum

33. If the kinetic energy of a free particle is much less than its rest energy then its kinetic energy is proportional to

(a) The magnitude of its momentum
(b) The square of the magnitude of its momentum
(c) The square root of the magnitude of its momentum
(d) The reciprocal of the magnitude of its momentum

Linked Comprehension

Paragraph for Questions 34–38: A subatomic particle X spontaneously decays into two particles, A and B, each of rest energy 1.40×10^2 MeV. The particles fly off in opposite directions, each with speed 0.827c relative to an inertial reference frame S.

34. Determine the total energy of particle A.

(a) 109 MeV **(b)** 200 MeV
(c) 140 MeV **(d)** 249 MeV

35. Determine the kinetic energy of particle B (relative to frame S).

(a) 109 MeV **(b)** 206 MeV
(c) 140 MeV **(d)** 249 MeV

36. Which one of the following statements concerning particle X is true?

(a) Momentum conservation requires that it was moving in frame S.
(b) Energy conservation requires that it must have been at rest in frame S.
(c) Momentum conservation requires that it must have been at rest in frame S.
(d) Energy conservation requires that its total energy was 280 MeV in frame S.

37. Which expression gives the momentum of particle A (relative to frame S)?

(a) 109 MeV/c
(b) 206 MeV/c
(c) 140 MeV/c
(d) 249 MeV/c

38. Use energy conservation to determine the rest energy of particle X.

(a) 206 MeV **(b)** 498 MeV
(c) 249 MeV **(d)** 392 MeV

Paragraph for Questions 39 and 40: The rest energy of a block is E_0. Relative to inertial observer O, the block is moving with speed v so that $\sqrt{1 - v^2/c^2} = 1/4$.

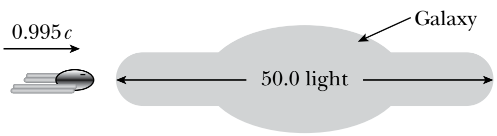
39. Which one of the following table entries is correct?

	Kinetic Energy of Block	Total Energy of Block
(a)	$4E_0$	$5E_0$
(b)	$0.75E_0$	$0.25E_0$
(c)	$4E_0$	$4E_0$
(d)	$3E_0$	$4E_0$

40. Observer O finds that the block takes 12 s to go from A to B. How long would this time interval appear to be to an observer riding on the block?

(a) 3 s **(b)** 12 s
(c) 6 s **(d)** 24 s

Paragraph for Questions 41–43: The figure shows a side view of a galaxy that is 50.0 light years in diameter. This value for the diameter is its proper length. A spaceship enters the galactic plane with speed 0.995c relative to the galaxy. Assume that the galaxy can be treated as an inertial reference frame.
Note: A light year is the distance that light travels through vacuum in one year; that is, 1 light year = $c \times$ (1 year).



41. How long does it take the spaceship to cross the galaxy according to an observer at rest in the galaxy?

(a) 5.00 years **(b)** 49.8 years
(c) 12.5 years **(d)** 50.3 years

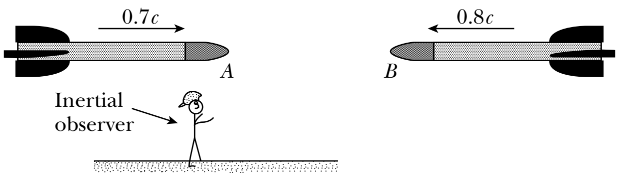
42. How long does it take the spaceship to cross the galaxy according to a clock on board the spaceship?

(a) 5.02 years **(b)** 49.8 years
(c) 12.5 years **(d)** 50.3 years

43. Determine the diameter of the galaxy as perceived by a person in the spaceship.

(a) 2.49 light years **(b)** 4.99 light years
(c) 3.63 light years **(d)** 12.4 light years

Paragraph for Questions 44–46: Two rockets, A and B, approach each other as shown in the figure. A travels to the right at 0.7c while B travels to the left at 0.8c. Both speed measurements are made relative to an inertial observer.



44. Determine the speed of A relative to B.

(a) c **(b)** 0.92c
(c) 0.96c **(d)** 0.89c

45. Both rockets have a supply of unstable mesons with a mean proper lifetime of 2.6×10^{-8} s. Which one of the following is a correct observation for the inertial observer?

(a) The mesons in A have a mean lifetime of 4.3×10^{-8} s
(b) The mesons in B have a mean lifetime of 3.6×10^{-8} s
(c) The mesons in both rockets have a mean lifetime of 2.6×10^{-8} s
(d) On average, the mesons in A will decay before the mesons in B

46. Determine the ratio of the kinetic energy of rocket B to its rest energy.

(a) 0.80 **(b)** 0.54
(c) 0.67 **(d)** 0.33

Paragraph for Questions 47–51: A spaceship traveling east flies directly over the head of an inertial observer who is at rest on the Earth's surface. The speed of the spaceship can be found from this relationship: $\sqrt{1 - v^2/c^2} = 1/2$.

47. The observer is 5 feet tall. According to the navigator of the space ship, how tall is the observer?

(a) 2.5 ft **(b)** 5 ft
(c) 3.6 ft **(d)** 8 ft

48. The navigator of the space ship observes a neon sign on a storefront. If he measures the speed of light emitted from the sign as he approaches the sign, what value will he obtain?

(a) 1.5×10^8 m/s
(b) 3.0×10^8 m/s
(c) 1.8×10^8 m/s
(d) 2.8×10^8 m/s

49. The navigator's on-board instruments indicate that the length of the spaceship is 20 m. If the length of the ship is measured by the inertial Earth-bound observer, what value will be obtained?

(a) 5 m **(b)** 20 m
(c) 10 m **(d)** 40 m

50. The pilot fires an ion gun that propels ions from the space ship at 1.0×10^8 m/s relative to the ship. What is the speed of the ions as measured by the Earth observer?

(a) 1.0×10^8 m/s
(b) 2.4×10^8 m/s
(c) 2.0×10^8 m/s
(d) 2.8×10^8 m/s

51. An apple falls from a tree and takes 4 s to reach the ground as reported by the Earth-bound observer. According to the navigator's instruments, how long did it take the apple to fall?

(a) 1 s **(b)** 4 s
(c) 8 s **(d)** 6 s

Matrix-Match

Directions for Questions 52 and 53: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

52. Lorentz transformation equations represent the relation of coordinates of space and time of two frames. In the given table, Column I shows the different inertial frames, Columns II and III show the speed of elements in two frames or relative speed between the two frames.

Column I	Column II	Column III
(I) The rocket and the Earth are inertial frames.	**(i)** $v = -0.2c$	**(J)** $u = -0.9c$
(II) The rocket and the proton are inertial frames.	**(ii)** $v = 0.8c$ relative to S	**(K)** $u' = -0.09c$
(III) The Earth and the other galaxy are inertial reference frames	**(iii)** $v = 0.9c$	**(L)** $u' = -0.9c$
(IV) The proton and the Earth are inertial frames	**(iv)** $v = +0.2c$	**(M)** $u = +0.8c$

(1) The set of values for a bullet with the speed $0.36c$ along the x-direction are

(a) (I) (iv) (M) **(b)** (IV) (ii) (L)
(c) (II) (i) (K) **(d)** (I) (ii) (L)

(2) The set of values for an electron with the speed $0.994c$ are

(a) (I) (ii) (J) **(b)** (IV) (iii) (J)
(c) (II) (iii) (L) **(d)** (I) (i) (M)

(3) The set of values for Earth with the speed $0.71c$ are

(a) (II) (ii) (J) **(b)** (III) (iv) (M)
(c) (IV) (i) (L) **(d)** (II) (iii) (M)

53. Relativistic momentum is the momentum having speed of light as one component. In the given table, Column I shows the relative speed, Column II shows the mass of the element and Column III shows the speed of element.

Column I	Column II	Column III
(I) Speed = $0.985c$	**(i)** Mass of the satellite is 2000 kg	**(J)** $u = 4$ km/s
(II) Speed = $0.200c$	**(ii)** Mass of the electron is 1×10^9 kg	**(K)** $u = 3$ km/s
(III) Speed = $0.980c$	**(iii)** Mass of the electron is 9.11×10^{-31} kg	**(L)** $u = 30$ km/s
(IV) Speed = $0.882c$	**(iv)** Mass of the helium is 6.68×10^{-27} kg	**(M)** $u = 3 \times 10^8$ m/s

(1) Conditions for an electron with a momentum of 1.56×10^{-21} kg·m/s are

(a) (I) (iii) (L) **(b)** (III) (ii) (K)
(c) (I) (iii) (M) **(d)** (I) (iii) (J)

(2) Conditions for helium with a momentum of 4.09×10^{-19} kg·m/s are

(a) (II) (iv) (M) **(b)** (III) (ii) (J)
(c) (II) (ii) (K) **(d)** (III) (i) (M)

(3) Conditions for an electron with a momentum of $3.000000015 \times 10^{13}$ kg·m/s are?

(a) (III) (i) (L) **(b)** (IV) (i) (J)
(c) (I) (iv) (J) **(d)** (III) (iii) (M)

Integer Type

54. A bomb is designed to explode 2.00 s after it is armed. The bomb is launched from Earth and accelerated to an unknown final speed. After reaching its final speed, however, the bomb is observed by people on Earth to explode 4.25 s after it is armed. What is the final speed of the bomb just before it explodes (in terms of c)?

55. Mars rotates about its axis once every 88 642 s. A spacecraft comes into the solar system and heads directly toward Mars at a speed of $0.800c$. What is the rotational period of Mars (in × 10^3 s) according to the beings on the spaceship?

56. The Milky Way galaxy is a part of a group of galaxies called the Local Group. The proper distance from the Milky Way, on one side of the Local Group, to the M31 galaxy on the other side is approximately 2.4×10^6 light-years. How long (in × 10^5 years) would it take a spaceship traveling at $0.999c$ to travel this distance according to travelers onboard?

57. The momentum of an electron is 1.60 times larger than the value computed non-relativistically. What is the speed (in × 10^8 m/s) of the electron?

58. During each hour of flight, a large jet airplane consumes 3330 gallons of fuel via combustion. Combustion releases 1.25×10^6 joules/gallon. One gallon of fuel has a mass of 2.84 kg. Calculate the energy equivalent of 3330 gallons of fuel and determine the ratio (in × 10^{-10} percent) of this energy equivalent to the amount of energy released by combustion in one hour of flight.

ANSWER KEY

Checkpoints

1. (a) Same (speed of light postulate); (b) no (the start and end of the flight are spatially separated); (c) no (because his measurement is not a proper time)

2. (a) Eq. 36-2; (b) +0.90*c*; (c) 25 ns; (d) −7.0 m

3. (a) Right; (b) more

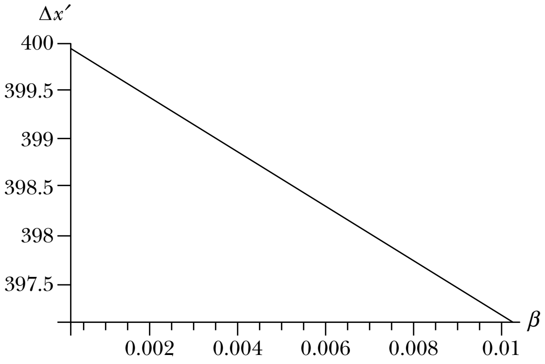
4. (a) Equal; (b) less

Problems

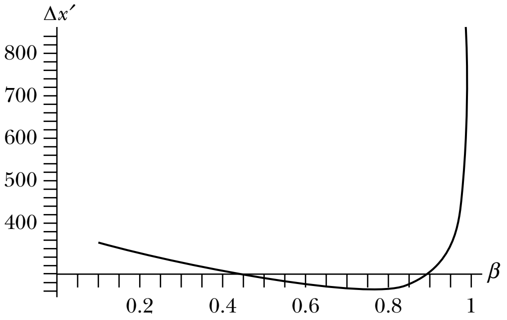
1. (a) From Table 36-2, we find

$$\Delta x' = [400\text{ m} - (299.8\text{ m})\beta] / \sqrt{1-\beta^2}$$

(b) A plot of $\Delta x'$ as a function of β with $0 < \beta < 0.01$ is shown:



(c) A plot of $\Delta x'$ as a function of β with $0.1 < \beta < 1$ is shown below:



(d) $\beta = 0.7495 \approx 0.750$; (e) $\Delta x' \approx 265$ m

2. (a) 0.334; (b) negative; (c) big flash; (d) 4.71 μs

3. (a) $1/2mv^2 + 3/8mv^4/c^2$; (b) 1.0×10^{-16} J; (c) 1.9×10^{-19} J; (d) 2.6×10^{-14} J; (e) 1.3×10^{-14} J; (f) ≈0.37

4. (a) 2.1 keV; (b) 1.5 MeV

5. (a) $m = [(pc)^2 - K^2]/2Kc^2$; (b) $m \rightarrow 1/mv = m$; (c) 105.6 MeV/c^2

6. (a) 0.137*c*; (b) 468 nm

7. +2.97 nm

8. (a) 588.084; (b) 0.999 999

9. 0.94 μs

10. −7.28 MeV

11. (a) 1.64 y; (b) 1.93 y; (c) 5.27 y

12. (a) 1.0019570; (b) 0.062469542; (c) 2.9569514; (d) 0.94107924; (e) 1957.9514; (f) 0.999 99987

13. (a) 636 nm; (b) red

14. 0.588

15. (a) 0.136 cm; (b) 700 ps; (c) 4.55 ps

16. (a) 0.948; (b) 226 MeV; (c) 314 MeV/*c*

17. (a) 0.45; (b) 0.75

18. (a) 79.1 keV; (b) 3.11 MeV; (c) 10.9 MeV

19. (a) $\gamma(2\pi m/|qB|)$; (b) no; (c) 6.86 mm; (d) 3.12 cm; (e) 16.3 ps; (f) 652 ps

20. (a) 1.3×10^8 m; (b) 1.20 s; (c) 5.24×10^8 m; (d) 2.07 s

21. (a) 50.4 μs; (b) small flash

22. 0.9973

23. (a) 1.40; (b) 6.12×10^{-7} s

24. (a) 24.48; (b) 0.9992; (c) 1.01279 ≈ 1.013; (d) 0.1597; (e) 1.003219 ≈ 1.003; (f) 0.0800

25. 8485 y

26. 8.12 MeV

27. 0.768 m

28. (a) 0.84*c*; (b) $(1.1c)\hat{\text{i}}$; (c) $(0.21c)\hat{\text{i}}$; (d) $(0.15c)\hat{\text{i}}$

29. (a) 0.999 998 49; (b) 40 ly

30. (a) 1.20×10^7 kg; (b) this is certainly more than can be carried in a backpack. Presumably, a train would be required; (c) 25 N; (d) this can be carried in a backpack

31. 1.41×10^7 km

32. (a) 0.968; (b) 0.943

33. 35 smu/y

34. 0.35 m

35. (a) 216 m; (b) 1.2 μs

36. (a) 1×10^6 m/s; (b) the galaxy is receding

37. 3.87 *mc*

38. (a) 26.13 y; (b) 52.13 y; (c) 2.610 y

39. (a) 0.333 m; (b) 1.00 m; (c) 1.00 m; (d) 26.1 m; (e) –68.7 ns; (f) event 2 occurs before event 1

40. (a) ≈2.08 MeV; (b) 2.05 MeV

41. 29 s

42. (a) 138 km; (b) –374 μs

43. 1.73×10^{-6} s

44. (a) 0.894; (b) ≈2.24; (c) 1.24

45. 6.3×10^{-7} s

46. 0.993 93

47. (a) 0.968; (b) 4.00

48. 89.9 km

49. 0.57 m

50. 28.5 MHz

51. (a) 0 s; (b) $-3.3'10^{-6}$ s; (c) reverse sequence

52. 0.93*c*

53. (a) $-0.625c \approx -0.63c$

54. 0.446 ps

Practice Questions

Single Correct Choice Type

1. (d)
2. (d)
3. (a)
4. (b)
5. (c)
6. (a)
7. (b)
8. (a)
9. (d)
10. (b)
11. (c)
12. (b)
13. (c)
14. (d)
15. (b)
16. (c)
17. (c)
18. (b)
19. (c)
20. (a)
21. (d)
22. (d)
23. (c)
24. (b)
25. (c)
26. (d)
27. (c)
28. (d)
29. (d)
30. (d)
31. (d)
32. (a)
33. (b)

Linked Comprehension

34. (d)
35. (a)
36. (c)
37. (b)
38. (b)
39. (d)
40. (a)
41. (d)
42. (a)
43. (b)
44. (c)
45. (d)
46. (c)
47. (b)
48. (b)
49. (c)
50. (d)
51. (c)

Matrix-Match

52. (1) → (d); (2) → (b); (3) → (b)

53. (1) → (c); (2) → (a); (3) → (d)

Integer Type

54. 0.882
55. 148
56. 1.1
57. 2.34
58. 4.89

CHAPTER

37

Photons and Matter Waves

37.1 | WHAT IS PHYSICS?

One primary focus of physics is Einstein's theory of relativity, which took us into a world far beyond that of ordinary experience—the world of objects moving at speeds close to the speed of light. Among other surprises, Einstein's theory predicts that the rate at which a clock runs depends on how fast the clock is moving relative to the observer: the faster the motion, the slower the clock rate. This and other predictions of the theory have passed every experimental test devised thus far, and relativity theory has led us to a deeper and more satisfying view of the nature of space and time.

Now you are about to explore a second world that is outside ordinary experience—the subatomic world. You will encounter a new set of surprises that, though they may sometimes seem bizarre, have led physicists step by step to a deeper view of reality.

Quantum physics, as our new subject is called, answers such questions as: Why do the stars shine? Why do the elements exhibit the order that is so apparent in the periodic table? How do transistors and other microelectronic devices work? Why does copper conduct electricity but glass does not? In fact, scientists and engineers have applied quantum physics in almost every aspect of everyday life, from medical instrumentation to transportation systems to entertainment industries. Indeed, because quantum physics accounts for all of chemistry, including biochemistry, we need to understand it if we are to understand life itself.

Some of the predictions of quantum physics seem strange even to the physicists and philosophers who study its foundations. Still, experiment after experiment has proved the theory correct, and many have exposed even stranger aspects of the theory. The quantum world is an amusement park full of wonderful rides that are guaranteed to shake up the commonsense world view you have developed since childhood. We begin our exploration of that quantum park with the photon.

Contents

37.1 What is Physics?
37.2 The Photon, the Quantum of Light
37.3 The Photoelectric Effect
37.4 Photons Have Momentum
37.5 Light as a Probability Wave
37.6 Electrons and Matter Waves
37.7 Schrödinger's Equation
37.8 Heisenberg's Uncertainty Principle
37.9 Reflection from a Potential Step
37.10 Tunneling Through a Potential Barrier

37.2 | THE PHOTON, THE QUANTUM OF LIGHT

Key Concepts

- An electromagnetic wave (light) is quantized (allowed only in certain quantities), and the quanta are called photons.
- For light of frequency f and wavelength λ, the photon energy is

  $$E = hf,$$

  where h is the Planck constant.

Quantum physics (which is also known as *quantum mechanics* and *quantum theory*) is largely the study of the microscopic world. In that world, many quantities are found only in certain minimum (*elementary*) amounts, or integer multiples of those elementary amounts; these quantities are then said to be *quantized*. The elementary amount that is associated with such a quantity is called the **quantum** of that quantity (*quanta* is the plural).

In a loose sense, U.S. currency is quantized because the coin of least value is the penny, or \$0.01 coin, and the values of all other coins and bills are restricted to integer multiples of that least amount. In other words, the currency quantum is \$0.01, and all greater amounts of currency are of the form n(\$0.01), where n is always a positive integer. For example, you cannot hand someone \$0.755 = 75.5(\$0.01).

In 1905, Einstein proposed that electromagnetic radiation (or simply *light*) is quantized and exists in elementary amounts (quanta) that we now call **photons**. This proposal should seem strange to you because we have just spent several chapters discussing the classical idea that light is a sinusoidal wave, with a wavelength λ, a frequency f, and a speed c such that

$$f = \frac{c}{\lambda}. \tag{37-1}$$

Furthermore, in Chapter 32 we discussed the classical light wave as being an interdependent combination of electric and magnetic fields, each oscillating at frequency f. How can this wave of oscillating fields consist of an elementary amount of something—the light quantum? What is a photon?

The concept of a light quantum, or a photon, turns out to be far more subtle and mysterious than Einstein imagined. Indeed, it is still very poorly understood. In this book, we shall discuss only some of the basic aspects of the photon concept, somewhat along the lines of Einstein's proposal. According to that proposal, the quantum of a light wave of frequency f has the energy

$$E = hf \quad \text{(photon energy).} \tag{37-2}$$

Here h is the **Planck constant**, the constant which has the value

$$h = 6.63 \times 10^{-34}\ \text{J} \cdot \text{s} = 4.14 \times 10^{-15}\ \text{eV} \cdot \text{s}. \tag{37-3}$$

The smallest amount of energy a light wave of frequency f can have is hf, the energy of a single photon. If the wave has more energy, its total energy must be an integer multiple of hf. The light cannot have an energy of, say, $0.6hf$ or $75.5hf$.

Einstein further proposed that when light is absorbed or emitted by an object (matter), the absorption or emission event occurs in the atoms of the object. When light of frequency f is absorbed by an atom, the energy hf of one photon is transferred from the light to the atom. In this *absorption event*, the photon vanishes and the atom is said to absorb it. When light of frequency f is emitted by an atom, an amount of energy hf is transferred from the atom to the light. In this *emission event*, a photon suddenly appears and the atom is said to emit it. Thus, we can have *photon absorption* and *photon emission* by atoms in an object.

For an object consisting of many atoms, there can be many photon absorptions (such as with sunglasses) or photon emissions (such as with lamps). However, each absorption or emission event still involves the transfer of an amount of energy equal to that of a single photon of the light.

When we discussed the absorption or emission of light in previous chapters, our examples involved so much light that we had no need of quantum physics, and we got by with classical physics. However, in the late 20th century, technology became advanced enough that single-photon experiments could be conducted and put to practical use. Since then quantum physics has become part of standard engineering practice, especially in optical engineering.

Before going ahead with our journey of Quantum Physics, let us consider the historical development of the photon or quantum nature of light. The physicists of late 19th century and early 20th century were quite perplexed about the radiation emitted by a black body. As you have already studied in Volume 1 of this book, at a given temperature, the spectral emissive power, dI/dl, of light emitted by a black body depends on the wavelength emitted as shown in Fig. 37-1. This could not be explained on the basis of the classical electromagnetic theory of light. Furthermore, Wein found out that if we increase the temperature of the black body, the wavelength corresponding to the most intense radiation decreases. This has been illustrated graphically in Fig. 38-1. Thus, by Wein's displacement law,

$$\lambda_m \times T = b$$

where b is Wein's constant having value of 0.288 cm·K, λ_m is the wavelength corresponding to the most intense radiation, and T is the absolute temperature of the black body.

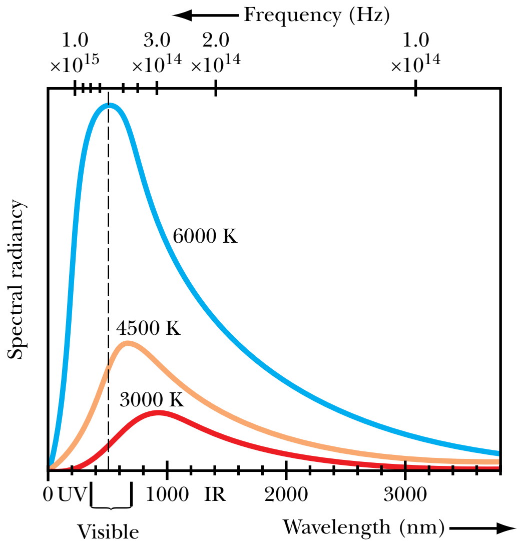


Figure 37-1 Measured spectra of wavelengths and frequencies emitted by a blackbody at three different temperatures.

Planck's Quantum Theory

In 1901, Planck showed that a successful theory of radiation was possible by making a revolutionary assumption regarding the way in which radiation is emitted or absorbed by atoms. He supposed that energy E could only be emitted or absorbed as radiation of frequency v in integral multiples of n such that

$$E = nhv.$$

It, thus, appeared that energy must also be regarded as having an atomic nature. At first, Planck and his contemporaries found this idea of a discontinuous emission of energy very hard to accept, but it soon became the only explanation of several other phenomena in physics and was recognized as the foundation of modern atomic physics.

It is worthwhile considering how this assumption accounts for the general form of the energy distribution before proceeding to the other evidence for quantum theory. In a solid we have to suppose that the atoms are fixed relative to one another and only capable of oscillation about a mean position. The thermal energy of the body will be distributed among the atoms as kinetic and potential energy, in much the same way as we supposed in our discussion of the kinetic theory of gases. Not all atoms will have the same energy, but the mean kinetic energy is dependent upon the temperature. A few atoms will have energies greatly in excess of the mean while the energies of others will be very much smaller than the mean.

The atom will emit energy in proportion to the energy it contains. If it has less energy, the quantum of energy it emits is small. The intensity I will depend on the number of quanta (or photons) emitted per unit time per unit area and the energy of one photon:

$$I = hv \times \frac{dN}{Adt}.$$

As we proceed to higher values of v the quantum hv increases until it exceeds the mean energy of the atoms in the solid. Beyond this value fewer and fewer atoms will have sufficient energy available to radiate the necessary quantum hv. Thus, the intensity radiated related to light of a particular wavelength rises to a maximum and falls away again as we proceed to higher and higher frequencies.

Moreover, as we raise the temperature of the body, the average kinetic energy of the atoms is raised, so that this energy corresponds to a larger quantum hv and a correspondingly larger frequency. At a higher temperature there will be a greater probability of an atom acquiring sufficient energy to radiate a really large quantum of energy hv.

Thus, we can see in a general way how raising the temperature of a body must lead to shift of the radiation maximum towards the region of higher frequency (i.e., shorter wavelength).

☑ CHECKPOINT 1

Rank the following radiations according to their associated photon energies, greatest first: (a) yellow light from a sodium vapor lamp, (b) a gamma ray emitted by a radioactive nucleus, (c) a radio wave emitted by the antenna of a commercial radio station, (d) a microwave beam emitted by airport traffic control radar.

SAMPLE PROBLEM 37.01

Emission and absorption of light as photons

A sodium vapor lamp is placed at the center of a large sphere that absorbs all the light reaching it. The rate at which the lamp emits energy is 100 W; assume that the emission is entirely at a wavelength of 590 nm. At what rate are photons absorbed by the sphere?

KEY IDEAS

The light is emitted and absorbed as photons. We assume that all the light emitted by the lamp reaches (and thus is absorbed by) the sphere. So, the rate R at which photons are absorbed by the sphere is equal to the rate R_{emit} at which photons are emitted by the lamp.

Calculations: That rate is

$$R_{\text{emit}} = \frac{\text{rate of energy emission}}{\text{energy per emitted photon}} = \frac{P_{\text{emit}}}{E}.$$

Next, into this we can substitute from Eq. 37-2 ($E = hf$), Einstein's proposal about the energy E of each quantum of light (which we here call a photon in modern language). We can then write the absorption rate as

$$R = R_{\text{emit}} = \frac{P_{\text{emit}}}{hf}.$$

Using Eq. 37-1 ($f = c/\lambda$) to substitute for f and then entering known data, we obtain

$$\begin{aligned} R &= \frac{P_{\text{emit}}\lambda}{hc} \\ &= \frac{(100\ \text{W})(590\times10^{-9}\ \text{m})}{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})} \\ &= 2.97\times10^{20}\ \text{photons/s}. \qquad \text{(Answer)} \end{aligned}$$

37.3 | THE PHOTOELECTRIC EFFECT

Key Concepts

- When light of high enough frequency illuminates a metal surface, electrons can gain enough energy to escape the metal by absorbing photons in the illumination, in what is called the photoelectric effect.
- The conservation of energy in such an absorption and escape is written as

$$hf = K_{\max} + \Phi,$$

where hf is the energy of the absorbed photon, $K_{\max}$ is the kinetic energy of the most energetic of the escaping electrons, and Φ (called the work function) is the least energy required by an electron to escape the electric forces holding electrons in the metal.
- If $hf = \Phi$, electrons barely escape but have no kinetic energy and the frequency is called the cutoff frequency f_0.
- If $hf < \Phi$, electrons cannot escape.

If you direct a beam of light of short enough wavelength onto a clean metal surface, the light will cause electrons to leave that surface (the light will *eject* the electrons from the surface). This **photoelectric effect** is used in many devices, including camcorders. Einstein's photon concept can explain it.

Let us analyze two basic photoelectric experiments, each using the apparatus of Fig. 37-2, in which light of frequency f is directed onto target T and ejects electrons from it. A potential difference V is maintained between target T and collector cup C to sweep up these electrons, said to be **photoelectrons**. This collection produces a **photoelectric current** i that is measured with meter A.

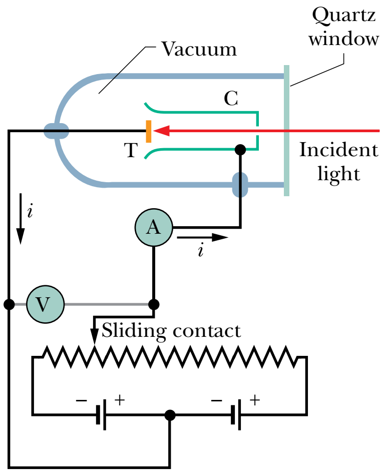


Figure 37-2 An apparatus used to study the photoelectric effect. The incident light shines on target T, ejecting electrons, which are collected by collector cup C. The electrons move in the circuit in a direction opposite the conventional current arrows. The batteries and the variable resistor are used to produce and adjust the electric potential difference between T and C.

First Photoelectric Experiment

We adjust the potential difference V by moving the sliding contact in Fig. 37-2 so that collector C is slightly negative with respect to target T. This potential difference acts to slow down the ejected electrons. We then vary V until it reaches a certain value, called the **stopping potential** V_{stop}, at which point the reading of meter A has just dropped to zero. When $V = V_{\text{stop}}$, the most energetic ejected electrons are turned back just before reaching the collector. Then $K_{\max}$, the kinetic energy of these most energetic electrons, is

$$K_{\max} = eV_{\text{stop}}, \tag{37-4}$$

where e is the elementary charge.

Measurements show that for light of a given frequency, $K_{\max}$ *does not depend on the intensity of the light source*. Whether the source is dazzling bright or so feeble that you can scarcely detect it (or has some intermediate brightness), the maximum kinetic energy of the ejected electrons always has the same value.

This experimental result is a puzzle for classical physics. Classically, the incident light is a sinusoidally oscillating electromagnetic wave. An electron in the target should oscillate sinusoidally due to the oscillating electric force on it from the wave's electric field. If the amplitude of the electron's oscillation is great enough, the electron should break free of the target's surface—that is, be ejected from the target. Thus, if we increase the amplitude of the wave and its oscillating electric field, the electron should get a more energetic "kick" as it is being ejected. *However, that is not what happens*. For a given frequency, intense light beams and feeble light beams give exactly the same maximum kick to ejected electrons.

The actual result follows naturally if we think in terms of photons. Now the energy that can be transferred from the incident light to an electron in the target is that of a single photon. Increasing the light intensity increases the *number* of photons in the light, but the photon energy, given by Eq. 37-2 ($E = hf$), is unchanged because the frequency is unchanged. Thus, the energy transferred to the kinetic energy of an electron is also unchanged.

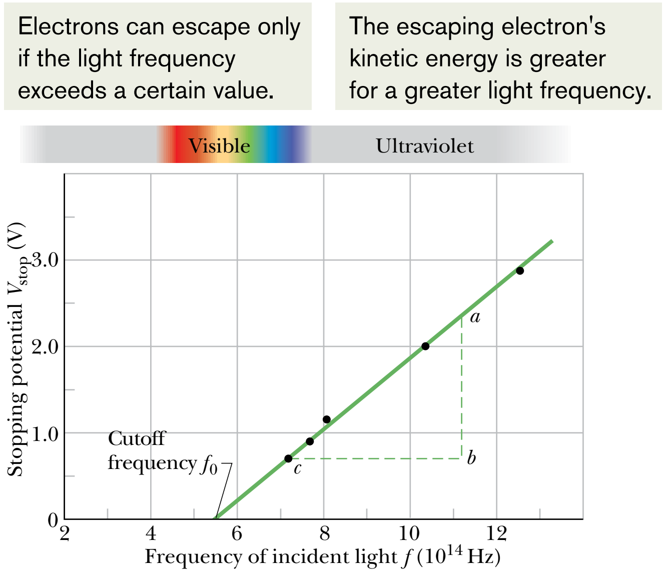


Figure 37-3 The stopping potential V_{stop} as a function of the frequency f of the incident light for a sodium target T in the apparatus of Fig. 37-2. (Data reported by R.A. Millikan in 1916.)

Second Photoelectric Experiment

Now we vary the frequency f of the incident light and measure the associated stopping potential V_{stop}. Figure 37-3 is a plot of V_{stop} versus f. Note that the photoelectric effect does not occur if the frequency is below a certain **cutoff frequency** f_0 or, equivalently, if the wavelength is greater than the corresponding **cutoff wavelength** $\lambda_0 = c/f_0$. This is so *no matter how intense the incident light is*.

This is another puzzle for classical physics. If you view light as an electromagnetic wave, you must expect that no matter how low the frequency, electrons can always be ejected by light if you supply them with enough energy—that is, if you use a light source that is bright enough. *That is not what happens*. For light below the cutoff frequency f_0, the photoelectric effect does not occur, no matter how bright the light source.

The existence of a cutoff frequency is, however, just what we should expect if the energy is transferred via photons. The electrons within the target are held there by electric forces. (If they weren't, they would drip out of the target due to the gravitational force on them.) To just escape from the target, an electron must pick up a certain minimum energy Φ, where Φ is a property of the target material called its **work function**. If the energy hf transferred to an electron by a photon exceeds the work function of the material (if $hf > \Phi$), the electron can escape the target. If the energy transferred does not exceed the work function (that is, if $hf < \Phi$), the electron cannot escape. This is what Fig. 37-3 shows.

The Photoelectric Equation

Einstein summed up the results of such photoelectric experiments in the equation

$$hf = K_{\text{max}} + \Phi \quad \text{(photoelectric equation).} \tag{37-5}$$

This is a statement of the conservation of energy for a single photon absorption by a target with work function Φ. Energy equal to the photon's energy hf is transferred to a single electron in the material of the target. If the electron is to escape from the target, it must pick up energy at least equal to Φ. Any additional energy ($hf - \Phi$) that the electron acquires from the photon appears as kinetic energy K of the electron. In the most favorable circumstance, the electron can escape through the surface without losing any of this kinetic energy in the process; it then appears outside the target with the maximum possible kinetic energy K_{max}.

Let us rewrite Eq. 37-5 by substituting for K_{max} from Eq. 37-4 ($K_{\text{max}} = eV_{\text{stop}}$). After a little rearranging, we get

$$V_{\text{stop}} = \left(\frac{h}{e}\right) f - \frac{\Phi}{e}. \tag{37-6}$$

The ratios h/e and Φ/e are constants, and so we would expect a plot of the measured stopping potential V_{stop} versus the frequency f of the light to be a straight line, as it is in Fig. 37-3. Further, the slope of that straight line should be h/e. As a check, we measure ab and bc in Fig. 37-3 and write

$$\begin{aligned} \frac{h}{e} = \frac{ab}{bc} &= \frac{2.35\ \text{V} - 0.72\ \text{V}}{(11.2 \times 10^{14} - 7.2 \times 10^{14})\ \text{Hz}} \\ &= 4.1 \times 10^{-15}\ \text{V} \cdot \text{s}. \end{aligned}$$

Multiplying this result by the elementary charge e, we find

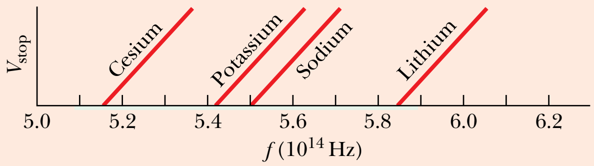
$$h = (4.1 \times 10^{-15}\ \text{V} \cdot \text{s})(1.6 \times 10^{-19}\ \text{C}) = 6.6 \times 10^{-34}\ \text{J} \cdot \text{s},$$

which agrees with values measured by many other methods.

An aside: An explanation of the photoelectric effect certainly requires quantum physics. For many years, Einstein's explanation was also a compelling argument for the existence of photons. However, in 1969 an alternative explanation for the effect was found that used quantum physics but did not need the concept of photons. As shown in countless other experiments, light *is* in fact quantized as photons, but Einstein's explanation of the photoelectric effect is not the best argument for that fact.

CHECKPOINT 2

The figure shows data like those of Fig. 37-3 for targets of cesium, potassium, sodium, and lithium. The plots are parallel. (a) Rank the targets according to their work functions, greatest first. (b) Rank the plots according to the value of h they yield, greatest first.



SAMPLE PROBLEM 37.02

Photoelectric effect and work function

Find the work function Φ of sodium from Fig. 37-3.

KEY IDEAS

We can find the work function Φ from the cutoff frequency f_0 (which we can measure on the plot). The reasoning is this: At the cutoff frequency, the kinetic energy K_{max} in Eq. 37-5 is zero. Thus, all the energy hf that is transferred from a photon to an electron goes into the electron's escape, which requires an energy of Φ.

Calculations: From that last idea, Eq. 37-5 then gives us, with $f = f_0$,

$$hf_0 = 0 + \Phi = \Phi.$$

In Fig. 37-3, the cutoff frequency f_0 is the frequency at which the plotted line intercepts the horizontal frequency axis, about 5.5×10^{14} Hz. We then have

$$\begin{aligned}\Phi = hf_0 &= (6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(5.5 \times 10^{14}\ \text{Hz})\\ &= 3.6 \times 10^{-19}\ \text{J} = 2.3\ \text{eV}. \qquad \text{(Answer)}\end{aligned}$$

SAMPLE PROBLEM 37.03

Photoelectric effect and energy absorption

A potassium foil is a distance $r = 3.5$ m from an isotropic light source that emits energy at the rate $P = 1.5$ W. The work function F of potassium is 2.2 eV. Suppose that the energy transported by the incident light were transferred to the target foil continuously and smoothly (that is, if classical physics prevailed instead of quantum physics). How long would it take for the foil to absorb enough energy to eject an electron? Assume that the foil totally absorbs all the energy reaching it and that the to-be-ejected electron collects energy from a circular patch of the foil whose radius is 5.0×10^{-11} m, about that of a typical atom.

KEY IDEAS

(1) The time interval Δt required for the patch to absorb energy ΔE depends on the rate P_{abs} at which the energy is absorbed:

$$\Delta t = \frac{\Delta E}{P_{abs}}.$$

(2) If the electron is to be ejected from the foil, the smallest amount of energy ΔE it must gain from the light is equal to the work function F of potassium. Thus,

$$\Delta t = \frac{\Phi}{P_{abs}}.$$

(3) Because the patch is totally absorbing, the rate of absorption P_{abs} is equal to the rate P_{arr} at which energy arrives at the patch; that is,

$$\Delta t = \frac{\Phi}{P_{arr}}.$$

(4) With Eq. 32-25 [I = (power/area)avg], we can relate the energy arrival rate P_{arr} to the intensity I of the light at the patch and the area A of the patch:

$$P_{arr} = IA.$$

Then, $$\Delta t = \frac{\Phi}{IA}.$$

(5) Because the light source is isotropic, the light intensity I at distance r from the source depends on the rate P_{emit} at which energy is emitted by the source, according to Eq. 32-29:

$$I = \frac{P_{emit}}{4\pi r^2}.$$

Here we are going to use the classical theory of electromagnetic waves, not the quantum physics.

Calculation: Putting these ideas together yields

$$\Delta t = \frac{4\pi r^2 \Phi}{P_{emit} A}.$$

The detection area A is $\pi(5.0 \times 10^{-11}\ \text{m})^2 = 7.85 \times 10^{-21}\ \text{m}^2$, and the work function Φ is $2.2\ \text{eV} = 3.5 \times 10^{-19}$ J. Substituting these and other data, we find that

$$\begin{aligned}\Delta t &= \frac{4\pi(3.5\ \text{m})^2(3.5 \times 10^{-19}\ \text{J})}{(1.5\ \text{W})(7.85 \times 10^{-21}\ \text{m}^2)}\\ &= 4580\ \text{s} = 1.3\ \text{h} \qquad \text{(Answer)}\end{aligned}$$

Learn: Thus, classical physics tells us that we would have to wait more than an hour after turning on the light source for a photoelectron to be ejected. The actual waiting time is less than 10^{-9} s. Apparently, then, an electron does *not* gradually absorb energy from the light arriving at the patch containing the electron. Rather, either the electron does not absorb any energy at all or it absorbs a quantum of energy instantaneously, by absorbing a photon from the light

Third Photoelectric Experiment

Let us now perform another experiment where we keep the frequency of the incident electromagnetic waves same, but we change the intensity of the incident radiations. If we increase the intensity of the incident radiations, it is seen that the maximum energy of the emitted photoelectrons does not change (So, the stopping potential remains the same.) But it seems that the number of photoelectrons emitted per unit time increase because the number of electrons reaching the collector per unit time increase. In Chapter 26, we noted that the current flowing through any cross section is given by

$$i = \frac{dq}{dt} = e\frac{dN}{dt}.$$

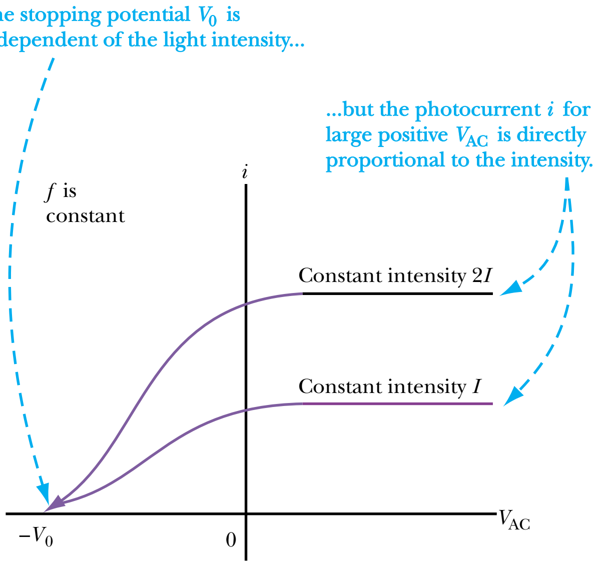


Figure 37-4 The plot of photocurrent i for a given monochromatic light. As the collector plate is made sufficiently positive, the current becomes constant, called saturation current. It doubles on doubling the intensity.

We know that the current is due to the flow of electrons only. Here e refers to the charge on the electron and N is the number of electrons emitted in a given time duration. On performing the experiment, it is seen that the current in the photoelectric tube does not follow Ohm's law. The I–V graph can be seen in Figure 37-4.

The phenomena can be easily understood in terms of the photon model. Of the photoelectrons emitted from the emitter plate, if the potential is sufficiently positive, all the electrons are pulled to the other side. So, even an electron emitted with negligible velocity is attracted to the positively charged collector plate. This maximum possible current at a given intensity and frequency is known as *saturation current.* You might be surprised at seeing that if we increase the potential of the collector plate with respect to the emitter plate, the current does not increase beyond a certain limit. However, our experience with the electrostatics revealed that when an electron is made to undergo a higher potential drop, its kinetic energy changes by a greater factor. So, the electron should arrive with a greater speed. But wait a minute! This does not necessarily mean that the number of electrons reaching the plate somehow increases. They will arrive more quickly, of course. But not in a greater number! The number of the emitted electrons will remain the same in either case, and all the emitted electrons are going to end up on the other plate. So, the current remains constant.

However, if the potential of the collector plate with respect to emitter plate is maintained at zero, some of the electrons get stuck up in between, constituting what is known as *space charge.* This building of charge decreases the probability of the electron reaching the collector. So, even at $V_{AC} = 0$, the current is not maximum.

If the electrons reach the collector plate, the circuit is complete, and a current is registered in the ammeter connected. If the potential of the collector plate is kept sufficiently negative with respect to the emitter plate (lesser than the stopping potential), electrons are emitted from the plate but fail to reach the collector. Even the most energetic photoelectron turns back before reaching the collector plate.

It is to be noted that on doubling the intensity of incident light, the number of photons incident on the plate per unit time doubles. Naturally, this would mean that the number of electrons emitted per unit time would double. Not all the photons would be successful in emitting the photons. Only a miniscule fraction of the photons is able to emit the photoelectrons. This fraction is known as *photoelectric efficiency*. It can be measured by measuring the saturation current:

$$i = \frac{dq}{dt} = e\frac{dN}{dt},$$

$$\frac{dN}{dt} = \eta \times \frac{dN_p}{dt},$$

where η is photoelectric efficiency and dN_p/dt is number of photons incident on the emitter plate per unit time.

Note that it should not be misunderstood here that the electrons are supposed to leave perpendicular to the metal plate only. They can be emitted in any direction with any velocity subject to a maximum value of the velocity.

SAMPLE PROBLEM 37.04

Calculating the number of photoelectrons liberated

A beam of light has three wavelengths 4144 Å, 4972 Å, and 6215 Å with a total intensity of 3.6×10^{-3} W·m^{-2} equally distributed amongst three wavelengths. The beam falls normally on an area 1.0 cm^2 of clean metallic surface of work function 2.3 eV. Assume that there is no loss of light by reflection and that photoelectric efficiency is 0.01% (same for all the three photons). Calculate the number of photoelectrons liberated in 2 s.

KEY IDEA

Here we can first see that this is a photoelectric effect where photons of three different frequencies are ejecting photoelectrons. Although they are incident on the same area, we can safely assume that the process of liberation of photoelectrons of them will be independent of one another.

Calculations: The three beams of photons can be independently treated in order to obtain the photoelectrons liberated. But first we will have to check out which photons are capable of liberating photoelectrons.

$$E_1 = \text{Energy of 1st photon}$$
$$= \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{(1.6 \times 10^{-19})(4144 \times 10^{-10})} = 3\ \text{eV}.$$

$$E_2 = \text{Energy of 2nd photon}$$
$$= \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{(1.6 \times 10^{-19})(4972 \times 10^{-10})} = 2.5\ \text{eV}.$$

$$E_3 = \text{Energy of 3rd photon}$$
$$= \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{(1.6 \times 10^{-19})(6216 \times 10^{-10})} = 2\ \text{eV}.$$

One convenient point to remember is that

$$h = 4.14 \times 10^{-15}\ \text{eV} \cdot \text{s}.$$

Also, only first two photons have energy more than 2.3 eV and only these photons are capable of ejecting photoelectrons. The intensity corresponding to each photon is thus

$$I = 1.2 \times 10^{-3}\ \text{W/m}^2.$$

Also,

$$\frac{dn_1}{dt} = \text{Number of photon of wavelength 4144 Å incident per unit time}$$
$$= \frac{IA}{E_1} = \left(\frac{1.2 \times 10^3}{3 \times 1.6 \times 10^{-19}}\right)(1 \times 10^{-4}) = 2.5 \times 10^{11}/\text{s}.$$

$$\frac{dn_2}{dt} = \text{Number of photons of wavelength 4144 Å incident per unit time}$$
$$= \frac{IA}{E_2} = \left(\frac{1.2 \times 10^3}{2.5 \times 1.6 \times 10^{-19}}\right)(1 \times 10^{-4}) = 3 \times 10^{11}/\text{s}.$$

Therefore, the number of photons emitted per second is

$$\eta \times \text{number of photons incident per second}$$
$$= 0.01 \times 5.5 \times 1011/100.$$

So, the number of photoelectrons emitted in 2 s = 11×10^7.

Note: Normally these liberated photoelectrons will leave the surface charged. In a photoelectric tube, as the target or the emitter plate is connected to the collector plate via a battery, therefore the charge does not build up on the emitter plate and hence we do not have to bother about the possibility of the electrons returning to the plate after the emission. But otherwise, the target plate accumulates a steady charge after some time. It becomes so very positively charge that its potential becomes equal to the stopping potential. So, the electrons liberated will land back in no time.

37.4 | PHOTONS HAVE MOMENTUM

Key Concept

- Although it is massless, a photon has momentum, which is related to its energy E, frequency f, and wavelength by

$$p = \frac{hf}{c} = \frac{h}{\lambda}.$$

In 1916, Einstein extended his concept of light quanta (photons) by proposing that a quantum of light has linear momentum. For a photon with energy hf, the magnitude of that momentum is

$$p = \frac{hf}{c} = \frac{h}{\lambda} \quad \text{(photon momentum)}, \tag{37-7}$$

where we have substituted for f from Eq. 37-1 ($f = c/\lambda$). Thus, when a photon interacts with matter, energy *and* momentum are transferred, *as if* there were a collision between the photon and matter in the classical sense.

Proof: We have just seen that the total energy of a single photon is given by $E = h\nu$. Because a photon always travels at the speed of light, it is truly a relativistic particle. Thus, we must use relativistic formulas for dealing with its mass, energy and momentum. The momentum of any particle of rest mass m is given by

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}},$$

as we had seen in Eq. 36-42. Since for a photon, the velocity is always equal to the velocity of light, the denominator is zero. To avoid having infinite momentum, we conclude that the photon's rest mass must be zero:

$$m_0 = 0.$$

A photon's kinetic energy is its total energy. The momentum of photon can be obtained from the relativistic formula:

$$E^2 = p^2c^2 + m_0^2c^4.$$

As in Eq. 36-55, putting $m_0 = 0$, we get

$$E = pc.$$

So,

$$p = \frac{E}{c}.$$

Since $E = h\nu$ for a photon, its momentum is related to its wavelength by

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}. \tag{37-8}$$

SAMPLE PROBLEM 37.05

Momentum and force of a photon

Suppose that a light bulb of power 100 W emits 10^{19} photons/s. Suppose that somehow all these photons are focused onto a piece of black paper and absorbed. (a) Calculate the momentum of one photon and (b) estimate the force all these photons could exert on the paper. Take the wavelength of the incident light as 500 nm.

KEY IDEA

The momentum of each photon is obtained from Eq. 37-7, $p = h/\lambda$. Next, each absorbed photon's momentum changes from $p = h/\lambda$ to zero. We use Newton's second law, $F = \Delta p/\Delta t$, to get the force.

Calculations: (a) Each photon has a momentum,

$$p = \frac{h}{\lambda} = \frac{6.63\times10^{-34}\,\mathrm{J\cdot s}}{500\times10^{-9}} = 1.3\times10^{-27}\,\mathrm{kg\cdot m/s}$$

Using Newton's second law for $N = 10^{19}$ photons whose momentum changes from h/λ to 0, we obtain

$$F = \frac{\Delta p}{\Delta t} = \frac{(Nh/\lambda)-0}{1\ \mathrm{s}} = N\frac{h}{\lambda}$$
$$\approx (10^{19}\,\mathrm{s}^{-1})(10^{-27}\,\mathrm{kg\cdot m/s}) \approx 10^{-8}\ \mathrm{N}.$$

Learn: This is a very small force but we can see that a very strong light source could exert measurable force. Near sun or star the force due to photons could be considerable. That's why you see a comet tail always pointing away from the Sun.

SAMPLE PROBLEM 37.06

Force on the plank due to radiation

A plank of mass m is lying on a smooth surface in situation as shown in Fig. 37-5. Assume that the plank slips over the surface and surface of plank exposed to radiation is black body. Find the force on the plank due to radiation.

KEY IDEA

First of all let us find the energy received per second by surface exposed to radiation. The area of the surface is ab. But the area of cross section of the beam incident on the surface is not ab (Fig. 37-5). From the figure, the area of cross section of the beam is $ab\cos\theta$.

Calculation: So, the power of the incident beam is

$$P = Iab\cos\theta$$

The number of photons incident per unit time can be found by

$$P = \frac{dn}{dt}\times\frac{hc}{\lambda}$$

So,

$$\frac{dn}{dt} = \frac{Iab\cos\theta\,\lambda}{hc}.$$

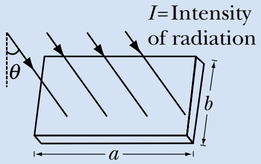


Figure 37-5 A light beam incident on a black body at an angle θ to the normal.

Each of the photon will be completely absorbed by the surface. This will produce an impulse on the slab in the horizontal direction alone. The impulse in the normal direction will be absorbed by the surface on which it is kept.

Radiation force experience by surface is

$$F = \frac{dn}{dt}\times p,$$

where p is the momentum of the photon incident. By de Broglie's hypothesis,

$$p = \frac{h}{\lambda},$$

$$F = \frac{Iab\cos\theta}{c}.$$

SAMPLE PROBLEM 37.07

Force exerted by a beam of light on a sphere

A perfectly reflecting solid sphere of radius r is kept in the path of a parallel beam of light of large aperture as in Fig. 37-6. If the beam carries intensity I, find the force exerted by the beam on the sphere.

KEY IDEA

The light beam is incident on the sphere as a parallel beam. However, since the normal at every point in not parallel to the normal at other points, the beam will not be reflected as a parallel beam.

Calculation: For the analysis, let us consider a small element of surface at angle q to horizontal direction. This element of the shape of ring subtends an angle dq at the center as shown in Fig. 37-6*a*.

The momentum of a photon is

$$p = \frac{h}{\lambda}.$$

From symmetry, we can say that the impulse on the sphere in all direction perpendicular to incident beam direction is zero. The momentum of the photon in x direction changes as can be seen in Fig. 37-6*b*. Therefore,

$$\Delta p_x = \frac{h}{\lambda} - \left(-\frac{h}{\lambda}\cos 2\theta\right)$$
$$= \frac{h}{\lambda}(1+\cos 2\theta) = \frac{h}{\lambda} \times 2\cos^2\theta.$$

The number of photons incident unit time on this element is given by

$$\frac{IdA \times \lambda}{hc}.$$

Note carefully that dA is a cross-section area of incident beam and not the element of the sphere. Area of the element is given by

$$dA_e = Rd\theta \times 2\pi R \sin\theta.$$

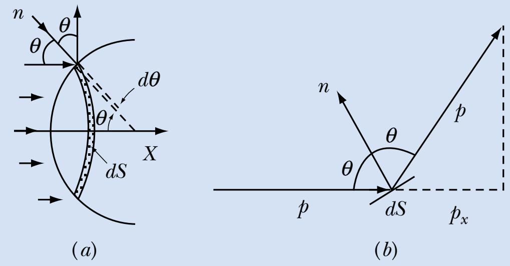


Figure 37-6 (*a*) A plane beam is incident on a perfectly reflecting sphere. (*b*) Change in momentum of a photon.

Therefore, the area of cross-section of incident beam would be lesser by a factor of cos *θ*:

$$dA = 2\pi R^2 \sin\theta d\theta \times \cos\theta.$$

Thus,

$$dF = \frac{\text{Number of photons incident}}{\text{Time}} \times \Delta p_x$$
$$= \frac{I \times 2\pi R^2 \sin\theta\cos\theta d\theta}{hc} \times \lambda \times \frac{h}{\lambda} \times 2\cos^2\theta$$
$$= \frac{4\pi R^2 I}{c}\cos^3\theta \sin\theta d\theta.$$

Also,

$$F = \int dF$$
$$= \frac{4\pi R^2 I}{c}\int_0^{\pi/2}\cos^3\theta\sin\theta d\theta$$
$$= \frac{\pi R^2 I}{c}$$

Note: If the sphere were a perfect absorber, the force exerted by light beam on the sphere would be the same.

37.5 | LIGHT AS A PROBABILITY WAVE

Key Concepts

- Photons: When light interacts with matter, the interaction is particle-like, occurring at a point and transferring energy and momentum.
- Wave: When a single photon is emitted by a source, we interpret its travel as being that of a probability wave.
- Wave: When many photons are emitted or absorbed by matter, we interpret the combined light as a classical electromagnetic wave.

A fundamental mystery in physics is how light can be a wave (which spreads out over a region) in classical physics but be emitted and absorbed as photons (which originate and vanish at points) in quantum physics. The double-slit experiment of Section 35.4 lies at the heart of this mystery. Let us discuss three versions of it.

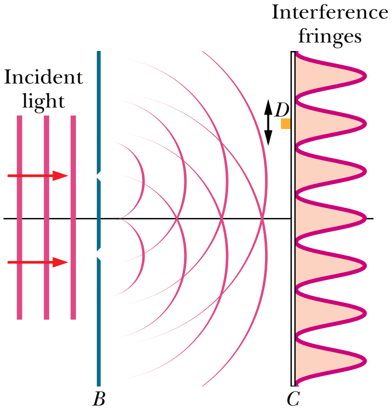


Figure 37-7 Light is directed onto screen B, which contains two parallel slits. Light emerging from these slits spreads out by diffraction. The two diffracted waves overlap at screen C and form a pattern of interference fringes. A small photon detector D in the plane of screen C generates a sharp click for each photon that it absorbs.

The Standard Version

Figure 37-7 is a sketch of the original experiment carried out by Thomas Young in 1801 (see also Fig. 35-8). Light shines on screen B, which contains two narrow parallel slits. The light waves emerging from the two slits spread out by diffraction and overlap on screen C where, by interference, they form a pattern of alternating intensity maxima and minima. In Section 35.4 we took the existence of these interference fringes as compelling evidence for the wave nature of light.

Let us place a tiny photon detector D at one point in the plane of screen C. Let the detector be a photoelectric device that clicks when it absorbs a photon. We would find that the detector produces a series of clicks, randomly spaced in time, each click signaling the transfer of energy from the light wave to the screen via a photon absorption. If we moved the detector very slowly up or down as indicated by the black arrow in Fig. 37-7, we would find that the click rate increases and decreases, passing through alternate maxima and minima that correspond exactly to the maxima and minima of the interference fringes.

The point of this thought experiment is as follows. We cannot predict when a photon will be detected at any particular point on screen C; photons are detected at individual points at random times. We can, however, predict that the relative *probability* that a single photon will be detected at a particular point in a specified time interval is proportional to the light intensity at that point.

We know from Eq. 32-28 $(I = E_{\mathrm{rms}}^2/c\mu_0)$ in Section 32.5 that the intensity I of a light wave at any point is proportional to the square of E_m, the amplitude of the oscillating electric field vector of the wave at that point. Thus,

The probability (per unit time interval) that a photon will be detected in any small volume centered on a given point in a light wave is proportional to the square of the amplitude of the wave's electric field vector at that point.

We now have a probabilistic description of a light wave, hence another way to view light. It is not only an electromagnetic wave but also a **probability wave**. That is, to every point in a light wave we can attach a numerical probability (per unit time interval) that a photon can be detected in any small volume centered on that point.

The Single-Photon Version

A single-photon version of the double-slit experiment was first carried out by G. I. Taylor in 1909 and has been repeated many times since. It differs from the standard version in that the light source in the Taylor experiment is so extremely feeble that it emits only one photon at a time, at random intervals. Astonishingly, interference fringes still build up on screen C if the experiment runs long enough (several months for Taylor's early experiment).

What explanation can we offer for the result of this single-photon double-slit experiment? Before we can even consider the result, we are compelled to ask questions like these: If the photons move through the apparatus one at a time, through which of the two slits in screen B does a given photon pass? How does a given photon even "know" that there is another slit present so that interference is a possibility? Can a single photon somehow pass through both slits and interfere with itself?

Bear in mind that the only thing we can know about photons is when light interacts with matter—we have no way of detecting them without an interaction with matter, such as with a detector or a screen. Thus, in the experiment of Fig. 37-7, all we can know is that photons originate at the light source and vanish at the screen. Between source and screen, we cannot know what the photon is or does. However, because an interference pattern eventually builds up on the screen, we can speculate that each photon travels from source to screen *as a wave* that fills up the space between source and screen and then vanishes in a photon absorption at some point on the screen, with a transfer of energy and momentum to the screen at that point.

We *cannot* predict where this transfer will occur (where a photon will be detected) for any given photon originating at the source. However, we *can* predict the probability that a transfer will occur at any given point on the screen. Transfers will tend to occur (and thus photons will tend to be absorbed) in the regions of the bright fringes in the interference pattern that builds up on the screen. Transfers will tend *not* to occur (and thus photons will tend *not* to be absorbed) in the regions of the dark fringes in the built-up pattern. Thus, we can say that the wave traveling from the source to the screen is a *probability wave*, which produces a pattern of "probability fringes" on the screen.

The Single-Photon, Wide-Angle Version

A *single* photon can take widely different paths and still interfere with itself.

In the past, physicists tried to explain the single-photon double-slit experiment in terms of small packets of classical light waves that are individually sent toward the slits. They would define these small packets as photons. However, modern experiments invalidate this explanation and definition. One of these experiments, reported in 1992 by Ming Lai and Jean-Claude Diels of the University of New Mexico, is depicted in Figure 37-8. Source S contains molecules that emit photons at well-separated times. Mirrors M_1 and M_2 are positioned to reflect light that the source emits along two distinct paths, 1 and 2, that are separated by an angle θ, which is close to 180°. This arrangement differs from the standard two-slit experiment, in which the angle between the paths of the light reaching two slits is very small.

After reflection from mirrors M_1 and M_2, the light waves traveling along paths 1 and 2 meet at beam splitter B, which transmits half the incident light and reflects the other half. On the right side of B in Fig. 37-8, the light wave traveling along path 2 and reflected by B combines with the light wave traveling along path 1 and transmitted by B. These two waves then interfere with each other at detector D (a *photomultiplier tube* that can detect individual photons).

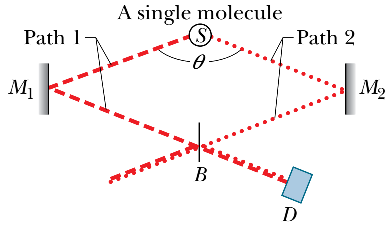


Figure 37-8 The light from a single photon emission in source S travels over two widely separated paths and interferes with itself at detector D after being recombined by beam splitter B. (Based on Ming Lai and Jean-Claude Diels, *Journal of the Optical Society of America B*, **9**, 2290–2294, December 1992.)

The output of the detector is a randomly spaced series of electronic pulses, one for each detected photon. In the experiment, the beam splitter is moved slowly in a horizontal direction (in the reported experiment, a distance of only about 50 μm maximum), and the detector output is recorded on a chart recorder. Moving the beam splitter changes the lengths of paths 1 and 2, producing a phase shift between the light waves arriving at detector D. Interference maxima and minima appear in the detector's output signal.

This experiment is difficult to understand in traditional terms. For example, when a molecule in the source emits a single photon, does that photon travel along path 1 or path 2 in Fig. 37-8 (or along any other path)? Or can it move in both directions at once? To answer, we assume that when a molecule emits a photon, a probability wave radiates in all directions from it. The experiment samples this wave in two of those directions, chosen to be nearly opposite each other.

We see that we can interpret all three versions of the double-slit experiment if we assume that (1) light is generated in the source as photons, (2) light is absorbed in the detector as photons, and (3) light travels between source and detector as a probability wave.

37.6 | ELECTRONS AND MATTER WAVES

Key Concepts

- A moving particle such as an electron can be described as a matter wave.
- The wavelength associated with the matter wave is the particle's de Broglie wavelength $\lambda = h/p$, where p is the particle's momentum.
- Particle: When an electron interacts with matter, the interaction is particle-like, occurring at a point and transferring energy and momentum.
- Wave: When an electron is in transit, we interpret it as being a probability wave.

In 1924, French physicist Louis de Broglie made the following appeal to symmetry: A beam of light is a wave, but it transfers energy and momentum to matter only at points, via photons. Why cannot a beam of particles have the same properties? That is, why cannot we think of a moving electron—or any other particle—as a **matter wave** that transfers energy and momentum to other matter at points?

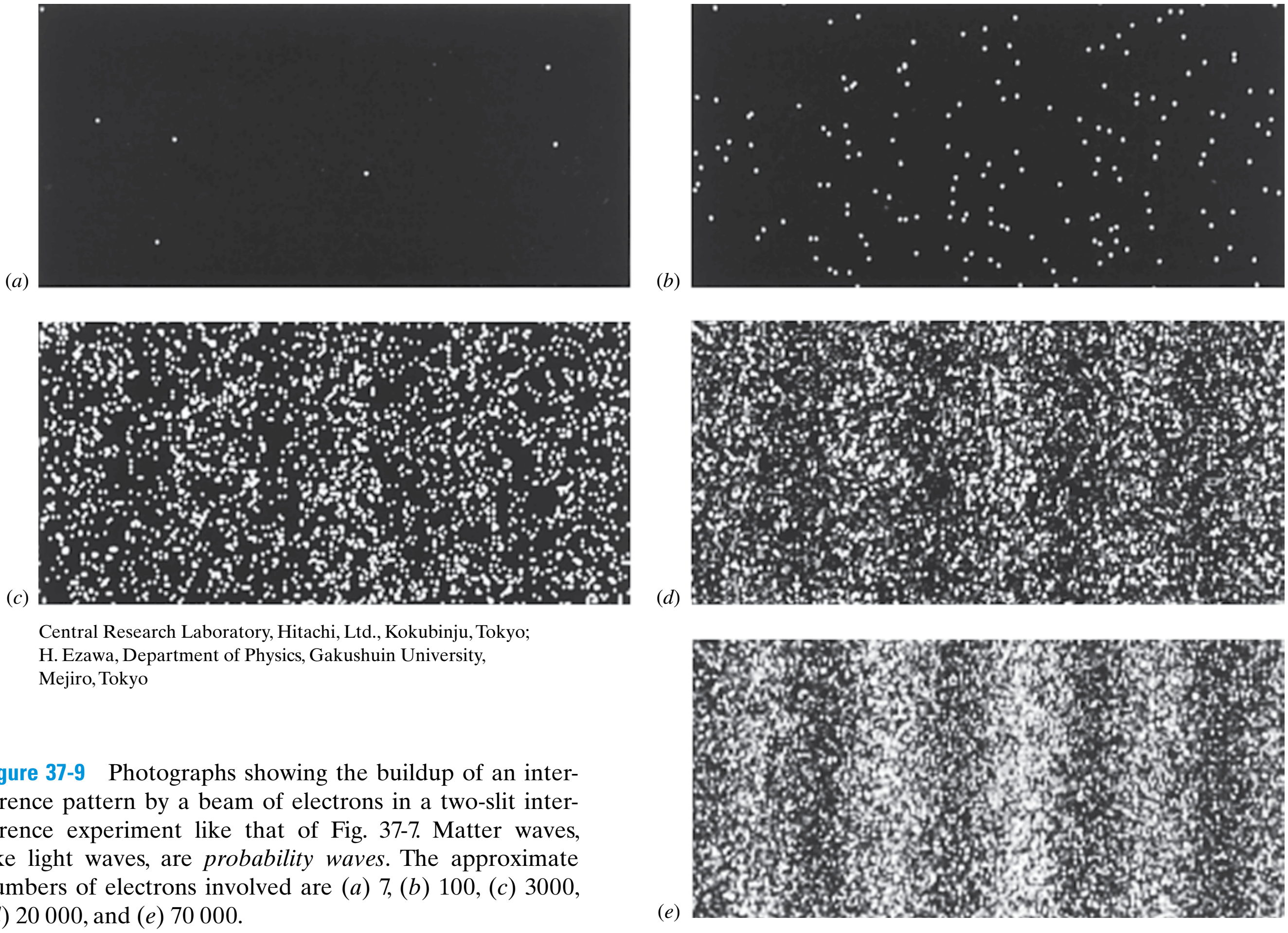
In particular, de Broglie suggested that Eq. 37-7 ($p = h/\lambda$) might apply not only to photons but also to electrons. We used that equation in Section 37.4 to assign a momentum p to a photon of light with wavelength λ. We now use it, in the form

$$\lambda = \frac{h}{p} \quad \text{(de Broglie wavelength)}, \tag{37-9}$$

to assign a wavelength λ to a particle with momentum of magnitude p. The wavelength calculated from Eq. 37-9 is called the **de Broglie wavelength** of the moving particle. De Broglie's prediction of the existence of matter waves was first verified experimentally in 1927, by C. J. Davisson and L. H. Germer of the Bell Telephone Laboratories and by George P. Thomson of the University of Aberdeen in Scotland.

Figure 37-9 shows photographic proof of matter waves in a more recent experiment. In the experiment, an interference pattern was built up when electrons were sent, *one by one*, through a double-slit apparatus. The apparatus was like the ones we have previously used to demonstrate optical interference, except that the viewing screen was similar to an old-fashioned television screen. When an electron hit the screen, it caused a flash of light whose position was recorded.

The first several electrons (top two photos) revealed nothing interesting and seemingly hit the screen at random points. However, after many thousands of electrons were sent through the apparatus, a pattern appeared on the screen, revealing fringes where many electrons had hit the screen and fringes where few had hit the screen.



Central Research Laboratory, Hitachi, Ltd., Kokubinju, Tokyo; H. Ezawa, Department of Physics, Gakushuin University, Mejiro, Tokyo

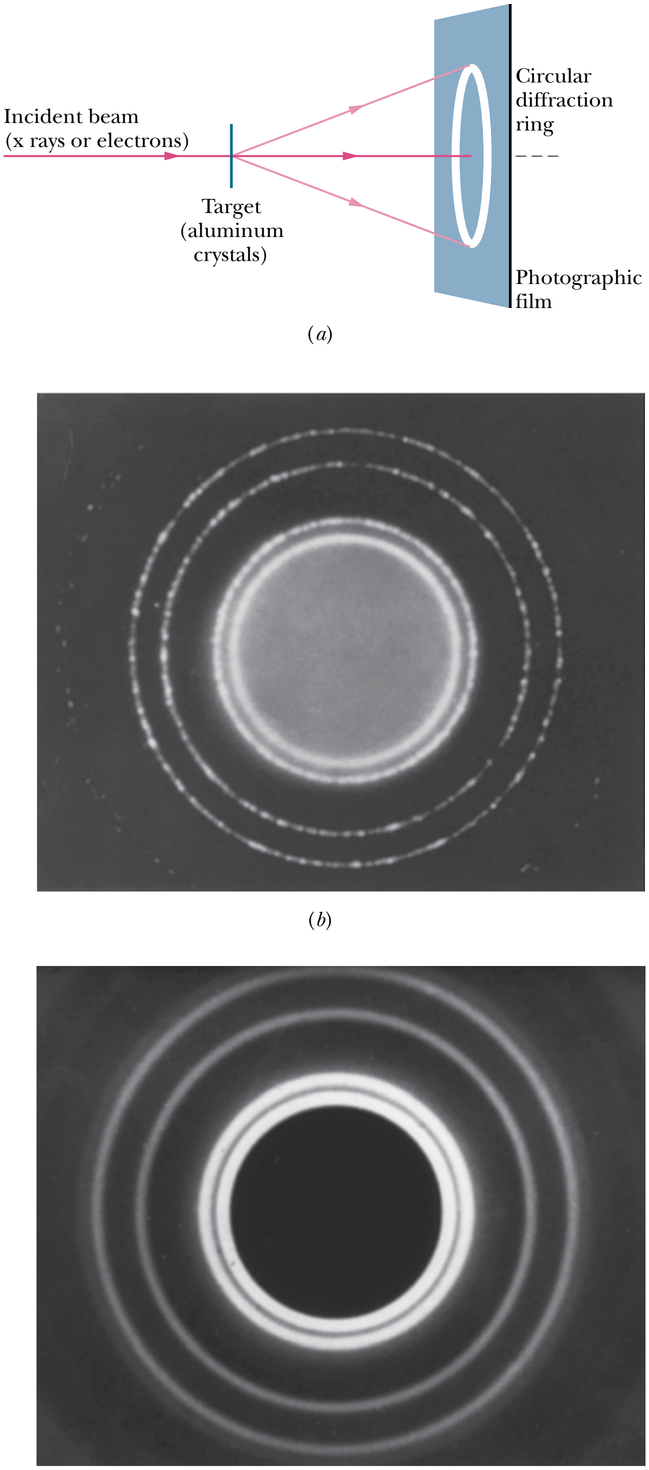
Figure 37-9 Photographs showing the buildup of an interference pattern by a beam of electrons in a two-slit interference experiment like that of Fig. 37-7. Matter waves, like light waves, are *probability waves*. The approximate numbers of electrons involved are (*a*) 7, (*b*) 100, (*c*) 3000, (*d*) 20 000, and (*e*) 70 000.

The pattern is exactly what we would expect for wave interference. Thus, *each* electron passed through the apparatus as a matter wave—the portion of the matter wave that traveled through one slit interfered with the portion that traveled through the other slit. That interference then determined the probability that the electron would materialize at a given point on the screen, hitting the screen there. Many electrons materialized in regions corresponding to bright fringes in optical interference, and few electrons materialized in regions corresponding to dark fringes.

Similar interference has been demonstrated with protons, neutrons, and various atoms. In 1994, it was demonstrated with iodine molecules I_2, which are not only 500 000 times more massive than electrons but far more complex. In 1999, it was demonstrated with the even more complex *fullerenes* (or *buckyballs*) C_{60} and C_{70}. (Fullerenes are molecules of carbon atoms that are arranged in a structure resembling a soccer ball, 60 carbon atoms in C_{60} and 70 carbon atoms in C_{70}.) Apparently, such small objects as electrons, protons, atoms, and molecules travel as matter waves. However, as we consider larger and more complex objects, there must come a point at which we are no longer justified in considering the wave nature of an object. At that point, we are back in our familiar nonquantum world, with the physics of earlier chapters of this book. In short, an electron is a matter wave and can undergo interference with itself, but a cat is not a matter wave and cannot undergo interference with itself (which must be a relief to cats).

The wave nature of particles and atoms is now taken for granted in many scientific and engineering fields. For example, electron diffraction and neutron diffraction are used to study the atomic structures of solids and liquids, and electron diffraction is used to study the atomic features of surfaces on solids.

Figure 37-10*a* shows an arrangement that can be used to demonstrate the scattering of either x rays or electrons by crystals. A beam of one or the other is directed onto a target consisting of a layer of tiny aluminum crystals. The x rays have a certain wavelength λ. The electrons are given enough energy so that their de Broglie wavelength is the same wavelength λ. The scatter of x rays or electrons by the crystals produces a circular interference pattern on a photographic film. Figure 37-10*b* shows the pattern for the scatter of x rays, and Fig. 37-10*c* shows the pattern for the scatter of electrons. The patterns are the same—both x rays and electrons are waves.



(*a*)

(*b*)

(*c*)

Parts (b) and (c) from PSSC film "Matter Waves," courtesy Education Development Center, Newton, Massachusetts

Figure 37-10 (*a*) An experimental arrangement used to demonstrate, by diffraction techniques, the wave-like character of the incident beam. Photographs of the diffraction patterns when the incident beam is (*b*) an x-ray beam (light wave) and (*c*) an electron beam (matter wave). Note that the two patterns are geometrically identical to each other.

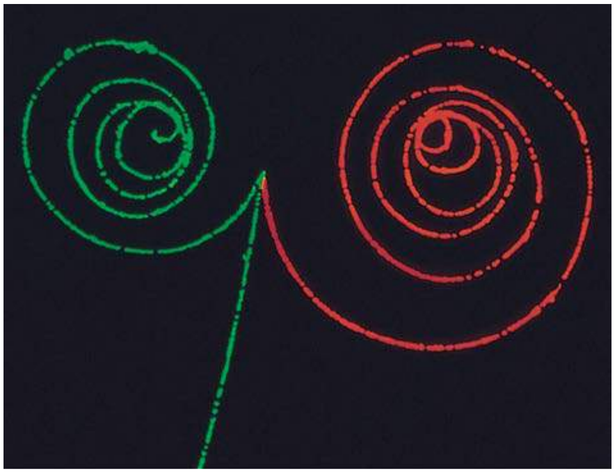
Waves and Particles

Figures 37-9 and 37-10 are convincing evidence of the *wave* nature of matter, but we have countless experiments that suggest its *particle* nature. Figure 37-11, for example, shows the tracks of particles (rather than waves) revealed in a bubble chamber. When a charged particle passes through the liquid hydrogen that fills such a chamber, the particle causes the liquid to vaporize along the particle's path. A series of bubbles thus marks the path, which is usually curved due to a magnetic field set up perpendicular to the plane of the chamber.

In Fig. 37-11, a gamma ray left no track when it entered at the top because the ray is electrically neutral and thus caused no vapor bubbles as it passed through the liquid hydrogen. However, it collided with one of the hydrogen atoms, kicking an electron out of that atom; the curved path taken by the electron to the bottom of the photograph has been color coded green. Simultaneous with the collision, the gamma ray transformed into an electron and a positron in a pair production event. Those two particles then moved in tight spirals (color coded green for the electron and red for the positron) as they gradually lost energy in repeated collisions with hydrogen atoms. Surely these tracks are evidence of the particle nature of the electron and positron, but is there any evidence of waves in Fig. 37-11?

To simplify the situation, let us turn off the magnetic field so that the strings of bubbles will be straight. We can view each bubble as a detection point for the electron. Matter waves traveling between detection points such as I and F in Fig. 37-12 will explore all possible paths, a few of which are shown.

In general, for every path connecting I and F (except the straight-line path), there will be a neighboring path such that matter waves following the two paths cancel each other by interference. For the straight-line path joining I and F, matter waves traversing all neighboring paths reinforce the wave following the direct path. You can think of the bubbles that form the track as a series of detection points at which the matter wave undergoes constructive interference.



Lawrence Berkeley Laboratory/Science Photo Library/
Photo Researchers, Inc.

Figure 37-11 A bubble-chamber image showing where two electrons (paths color coded green) and one positron (red) moved after a gamma ray entered the chamber.

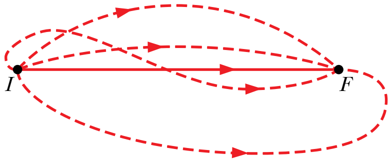


Figure 37-12 A few of the many paths that connect two particle detection points I and F. Only matter waves that follow paths close to the straight line between these points interfere constructively. For all other paths, the waves following any pair of neighboring paths interfere destructively.

CHECKPOINT 3

For an electron and a proton that have the same (a) kinetic energy, (b) momentum, or (c) speed, which particle has the shorter de Broglie wavelength?

SAMPLE PROBLEM 37.08

de Broglie wavelength of an electron

What is the de Broglie wavelength of an electron with a kinetic energy of 120 eV?

KEY IDEAS

(1) We can find the electron's de Broglie wavelength λ from Eq. 37-9 ($\lambda = h/p$) if we first find the magnitude of its momentum p. (2) We find p from the given kinetic energy K of the electron. That kinetic energy is much less than the rest energy of an electron (0.511 MeV, from Table 36-3). Thus, we can get by with the classical approximations for momentum $p(= mv)$ and kinetic energy $K(= (1/2)mv^2)$.

Calculations: We are given the value of the kinetic energy. So, in order to use the de Broglie relation, we first solve the kinetic energy equation for v and then substitute into the momentum equation, finding

$$
\begin{aligned}
p &= \sqrt{2mK} \\
&= \sqrt{(2)(9.11\times10^{-31}\,\text{kg})(120\ \text{eV})(1.60\times10^{-19}\ \text{J/eV})} \\
&= 5.91\times10^{-24}\ \text{kg}\cdot\text{m/s}.
\end{aligned}
$$

From Eq. 37-9 then

$$
\begin{aligned}
\lambda &= \frac{h}{p} \\
&= \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{5.91\times10^{-24}\ \text{kg}\cdot\text{m/s}} \\
&= 1.12\times10^{-10}\ \text{m} = 112\,\text{pm}. \qquad \text{(Answer)}
\end{aligned}
$$

This wavelength associated with the electron is about the size of a typical atom. If we increase the electron's kinetic energy, the wavelength becomes even smaller.

What we would like to emphasize is that electron and light possess both wave-like features as well as particle-like features. But essentially, both are quite different from each other. For example, the momentum of both is given by $p = h/\lambda$. But the energy of a photon is given by hc/λ whereas the energy of an electron does not have a direct relation with the wavelength. Similarly, the velocity of a photon is given by $c = f\lambda$, whereas the velocity of an electron does not have a fixed value. Electrons have charge but photons do not have any charge.

37.7 | SCHRÖDINGER'S EQUATION

Key Concepts

- A matter wave (such as for an electron) is described by a wave function $\Psi(x, y, z, t)$, which can be separated into a spacedependent part $\psi\,(x, y, z)$ and a time-dependent part $e^{-i\omega t}$, where ω is the angular frequency associated with the wave.
- For a nonrelativistic particle of mass m traveling along an x axis, with energy E and potential energy U, the space-dependent part can be found by solving Schrödinger's equation,

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0,$$

where k is the angular wave number, which is related to the de Broglie wavelength λ, the momentum p, and the kinetic energy $E - U$ by

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{2\pi\sqrt{2m(E-U)}}{h}.$$

- A particle does not have a specific location until its location is actually measured.
- The probability of detecting a particle in a small volume centered on a given point is proportional to the probability density $|\psi|^2$ of the matter wave at that point.

A simple traveling wave of any kind, be it a wave on a string, a sound wave, or a light wave, is described in terms of some quantity that varies in a wave-like fashion. For light waves, for example, this quantity is $\vec{E}\,(x, y, z, t)$, the electric field component of the wave. Its observed value at any point depends on the location of that point and on the time at which the observation is made.

What varying quantity should we use to describe a matter wave? We should expect this quantity, which we call the **wave function** $\Psi(x, y, z, t)$, to be more complicated than the corresponding quantity for a light wave because a matter wave, in addition to energy and momentum, transports mass and (often) electric charge. It turns out that Ψ, the uppercase Greek letter psi, usually represents a function that is complex in the mathematical sense; that is, we can always write its values in the form $a + ib$, in which a and b are real numbers and $i^2 = -1$.

In all the situations you will meet here, the space and time variables can be grouped separately and Ψ can be written in the form

$$\Psi(x, y, z, t) = \psi(x, y, z)\,e^{-i\omega t}, \tag{37-10}$$

where $\omega\,(=2\pi f)$ is the angular frequency of the matter wave. Note that ψ, the lowercase Greek letter psi, represents only the space-dependent part of the complete, time-dependent wave function Ψ. We shall focus on ψ. Two questions arise: What is meant by the wave function? How do we find it?

What does the wave function mean? It has to do with the fact that a matter wave, like a light wave, is a probability wave. Suppose that a matter wave reaches a particle detector that is small; then the probability that a particle will be detected in a specified time interval is proportional to $|\psi|^2$, where $|\psi|$ is the absolute value of the wave function at the location of the detector. Although ψ is usually a complex quantity, $|\psi|^2$ is always both real and positive. It is, then, $|\psi|^2$, which we call the **probability density**, and not ψ, that has *physical* meaning. Speaking loosely, the meaning is this:

The probability of detecting a particle in a small volume centered on a given point in a matter wave is proportional to the value of $|\psi|^2$ at that point.

Because ψ is usually a complex quantity, we find the square of its absolute value by multiplying ψ by ψ^*, the *complex conjugate* of ψ. (To find ψ^* we replace the imaginary number i in ψ with $-i$, wherever it occurs.)

How do we find the wave function? Sound waves and waves on strings are described by the equations of Newtonian mechanics. Light waves are described by Maxwell's equations. Matter waves for nonrelativistic particles are described by **Schrödinger's equation**, advanced in 1926 by Austrian physicist Erwin Schrödinger.

Many of the situations that we shall discuss involve a particle traveling in the x direction through a region in which forces acting on the particle cause it to have a potential energy $U(x)$. In this special case, Schrödinger's equation reduces to

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}[E - U(x)]\psi = 0 \quad \text{(Schrödinger's equation, one-dimensional motion)}, \tag{37-11}$$

in which E is the total mechanical energy of the moving particle. (We do *not* consider mass energy in this nonrelativistic equation.) We cannot derive Schrödinger's equation from more basic principles; it *is* the basic principle.

We can simplify the expression of Schrödinger's equation by rewriting the second term. First, note that $E - U(x)$ is the kinetic energy of the particle. Let's assume that the potential energy is uniform and constant (it might even be zero). Because the particle is nonrelativistic, we can write the kinetic energy classically in terms of speed v and then momentum p, and then we can introduce quantum theory by using the de Broglie wavelength:

$$E - U = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{1}{2m}\left(\frac{h}{\lambda}\right)^2. \tag{37-12}$$

By putting 2π in both the numerator and denominator of the squared term, we can rewrite the kinetic energy in terms of the angular wave number $k = 2\pi/\lambda$:

$$E - U = \frac{1}{2m}\left(\frac{kh}{2\pi}\right)^2. \tag{37-13}$$

Substituting this into Eq. 37-11 leads to

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \text{(Schrödinger's equation, uniform } U\text{)}, \tag{37-14}$$

where, from Eq. 37-13, the angular wave number is

$$k = \frac{2\pi\sqrt{2m(E-U)}}{h} \quad \text{(angular wave number)}. \tag{37-15}$$

The general solution of Eq. 37-14 is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \tag{37-16}$$

in which A and B are constants. You can show that this equation is indeed a solution of Eq. 37-14 by substituting it and its second derivative into that equation and noting that an identity results.

Equation 37-16 is the time-independent solution of Schrödinger's equation. We can assume it is the spatial part of the wave function at some initial time $t = 0$. Given values for E and U, we could determine the coefficients A and B to see how the wave function looks at $t = 0$. Then, if we wanted to see how the wave function evolves with time, we follow the guide of Eq. 37-10 and multiply Eq. 37-16 by the time dependence $e^{-i\omega t}$:

$$\begin{aligned}\Psi(x,t) &= \psi(x)e^{-i\omega t} = (Ae^{ikx} + Be^{-ikx})\,e^{-i\omega t}\\ &= Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}.\end{aligned} \tag{37-17}$$

Here, however, we will not go that far.

Finding the Probability Density $|\psi|^2$

In Section 16.3 we saw that any function F of the form $F(kx \pm \omega t)$ represents a traveling wave. In Chapter 16, the functions were sinusoidal (sines and cosines); here they are exponentials. If we wanted, we could always switch between the two forms by using the Euler formula: For a general argument θ,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta. \tag{37-18}$$

The first term on the right in Eq. 37-17 represents a wave traveling in the positive direction of x, and the second term represents a wave traveling in the negative direction of x. Let's evaluate the probability density $|\psi^2|$ for a particle with only positive motion. We eliminate the negative motion by setting B to zero, and then the solution at $t = 0$ becomes

$$\psi(x) = Ae^{ikx}. \tag{37-19}$$

To calculate the probability density, we take the square of the absolute value

$$|\psi|^2 = |Ae^{ikx}|^2 = A^2|e^{ikx}|^2.$$

Because

$$|e^{ikx}|^2 = (e^{ikx})(e^{ikx})^* = e^{ikx}e^{-ikx} = e^{ikx-ikx} = e^0 = 1,$$

we get

$$|\psi|^2 = A^2(1)^2 = A^2.$$

Now here is the point: For the condition we have set up (uniform potential energy U, including $U = 0$ for a *free particle*), the probability density is a constant (the same value A^2) for any point along the x axis, as shown in the plot of Fig. 37-13. That means that if we make a measurement to locate the particle, the location could turn out to be at any x value. Thus, we cannot say that the particle is moving along the axis in a classical way as a car moves along a street. *In fact, the particle does not have a location until we measure it.*

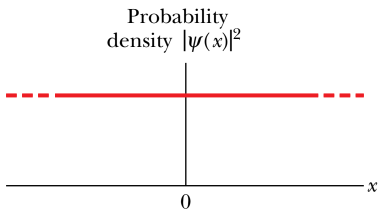


Figure 37-13 A plot of the probability density $|\psi|^2$ for a particle moving in the positive x direction with a uniform potential energy. Since $|\psi|^2$ has the same constant value for all values of x, the particle has the same probability of detection at all points along its path.

37.8 | HEISENBERG'S UNCERTAINTY PRINCIPLE

Key Concept

- The probabilistic nature of quantum physics places an important limitation on detecting a particle's position and momentum. That is, it is not possible to measure the position $\vec{r}$ and the momentum $\vec{p}$ of a particle simultaneously with unlimited precision. The uncertainties in the components of these quantities are given by

$$\Delta x \cdot \Delta p_x \geq \hbar$$
$$\Delta y \cdot \Delta p_y \geq \hbar$$
$$\Delta z \cdot \Delta p_z \geq \hbar.$$

Our inability to predict the position of a particle with a uniform electric potential energy, as indicated by Fig. 37-13, is our first example of **Heisenberg's uncertainty principle**, proposed in 1927 by German physicist Werner Heisenberg. It states that measured values cannot be assigned to the position $\vec{r}$ and the momentum $\vec{p}$ of a particle simultaneously with unlimited precision.

In terms of $\hbar = h/2\pi$ (called "h-bar"), the principle tells us

$$\begin{aligned} \Delta x \cdot \Delta p_x &\geq \hbar \\ \Delta y \cdot \Delta p_y &\geq \hbar \quad \text{(Heisenberg's uncertainty principle).} \\ \Delta z \cdot \Delta p_z &\geq \hbar \end{aligned} \tag{37-20}$$

Here Δx and Δp_x represent the intrinsic uncertainties in the measurements of the x components of $\vec{r}$ and $\vec{p}$, with parallel meanings for the y and z terms. Even with the best measuring instruments, each product of a position uncertainty and a momentum uncertainty in Eq. 37-20 will be greater than $\hbar$, *never* less.

Here we shall not derive the uncertainty relationships but only apply them. They are due to the fact that electrons and other particles are matter waves and that repeated measurements of their positions and momenta involve probabilities, not certainties. In the statistics of such measurements, we can view, say, Δx and Δp_x as the spread (actually, the standard deviations) in the measurements.

We can also justify them with a physical (though highly simplified) argument: In earlier chapters we took for granted our ability to detect and measure location and motion, such as a car moving down a street or a pool ball rolling across a table. We could locate a moving object by watching it—that is, by intercepting light scattered by the object. That scattering did not alter the object's motion. In quantum physics, however, the act of detection in itself alters the location and motion. The more precisely we wish to determine the location of, say, an electron moving along an x axis (by using light or by any other means), the more we alter the electron's momentum and thus become less certain of the momentum. That is, by decreasing Δx, we necessarily increase Δp_x. Vice versa, if we determine the momentum very precisely (less Δp_x), we become less certain of where the electron will be located (we increase Δx).

If we had an electron with a certain value of k, which, by the de Broglie relationship, means a certain momentum p_x, or $\Delta p_x = 0$, then by Eq. 37-20 it means that $\Delta x = \infty$. If we then set up an experiment to detect the electron, it could show up anywhere between $x = -\infty$ and $x = +\infty$.

You might push back on the argument: Could not we very precisely measure p_x and then next very precisely measure x wherever the electron happens to show up? Does not that mean that we have measured both p_x and x simultaneously and very precisely? No, the flaw is that although the first measurement can give us a precise value for p_x, the second measurement necessarily alters that value. Indeed, if the second measurement really does give us a precise value for x, we then have no idea what the value of p_x is.

SAMPLE PROBLEM 37.09

Uncertainty principle: position and momentum

Assume that an electron is moving along an x axis and that you measure its speed to be 2.05×10^6 m/s, which can be known with a precision of 0.50%. What is the minimum uncertainty (as allowed by the uncertainty principle in quantum theory) with which you can simultaneously measure the position of the electron along the x axis?

KEY IDEA

The minimum uncertainty allowed by quantum theory is given by Heisenberg's uncertainty principle in Eq. 37-20. We need only consider components along the x axis because we have motion only along that axis and want the uncertainty Δx in location along that axis. Since we want the minimum allowed uncertainty, we use the equality instead of the inequality in the x-axis part of Eq. 37-20, writing $\Delta x \cdot \Delta p_x = \hbar$.

Calculations: To evaluate the uncertainty Δp_x in the momentum, we must first evaluate the momentum component p_x. Because the electron's speed v_x is much less than the speed of light c, we can evaluate p_x with the classical expression for momentum instead of using a relativistic expression. We find

$$\begin{aligned} p_x = mv_x &= (9.11 \times 10^{-31}\ \text{kg})(2.05 \times 10^{6}\ \text{m/s}) \\ &= 1.87 \times 10^{-24}\ \text{kg} \cdot \text{m/s}. \end{aligned}$$

The uncertainty in the speed is given as 0.50% of the measured speed. Because p_x depends directly on speed, the uncertainty Δp_x in the momentum must be 0.50% of the momentum:

$$\Delta p_x = (0.0050)p_x$$
$$= (0.0050)(1.87 \times 10^{-24}\ \mathrm{kg \cdot m/s})$$
$$= 9.35 \times 10^{-27}\ \mathrm{kg \cdot m/s}.$$

Then the uncertainty principle gives us

$$\Delta x = \frac{\hbar}{\Delta p_x} = \frac{(6.63 \times 10^{-34}\ \mathrm{J \cdot s})/2\pi}{9.35 \times 10^{-27}\ \mathrm{kg \cdot m/s}}$$
$$= 1.13 \times 10^{-18}\ \mathrm{m} \approx 11\ \mathrm{nm}, \qquad \text{(Answer)}$$

which is about 100 atomic diameters.

37.9 | REFLECTION FROM A POTENTIAL STEP

Key Concepts

- A particle can reflect from a boundary at which its potential energy changes even when classically it would not reflect.
- The reflection coefficient R gives the probability of reflection of an individual particle at the boundary.
- For a beam of a great many particles, R gives the average fraction that will undergo reflection.
- The transmission coefficient T that gives the probability of transmission through the boundary is

$$T = 1 - R.$$

In Fig. 37-14, we send a beam of a great many nonrelativistic electrons, each of total energy E, along an x axis through a narrow tube. Initially they are in region 1 where their potential energy is $U = 0$, but at $x = 0$ they encounter a region with a negative electric potential V_b. The transition is called a *potential step* or *potential energy step*. The step is said to have a *height* U_b, which is the potential energy an electron will have once it passes through the boundary at $x = 0$, as plotted in Fig. 37-15 for potential energy as a function of position x. (Recall that $U = qV$. Here the potential V_b is negative, the electron's charge q is negative, and so the potential energy U_b is positive.)

Let's consider the situation where $E > U_b$. Classically, the electrons should all pass through the boundary—they certainly have enough energy. Indeed, we discussed such motion extensively in Chapters 22 through 24, where electrons moved into electric potentials and had changes in potential energy and kinetic energy. We simply conserved mechanical energy and noted that if the potential energy increases, the kinetic energy decreases by the same amount, and the speed thus also decreases. What we took for granted is that, because the electron energy E is greater than the potential energy U_b, all the electrons pass through the boundary. However, if we apply Schrödinger's equation, we find a big surprise—because electrons are matter waves, not tiny solid (classical) particles, some of them actually *reflect from the boundary*.

We want to find the probability that electrons reflect from the barrier. Recall that probability density is proportional to $|\psi|^2$. The probability density in the reflection is related to the probability density in the incident beam by **reflection coefficient** R.

This R gives the probability of reflection and thus is also the fraction of the incoming electrons that reflect. The **transmission coefficient** (the probability of transmission) is

$$T = 1 - R. \qquad (37\text{-}21)$$

Can the electron be reflected by the region of negative potential?

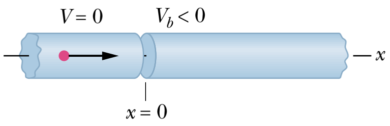


Figure 37-14 The elements of a tube in which an electron (the dot) approaches a region with a negative electric potential V_b.

Classically, the electron has too much energy to be reflected by the potential step.

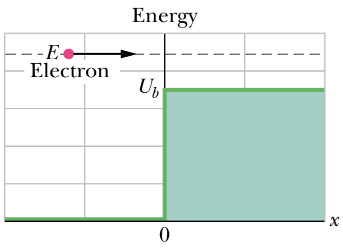


Figure 37-15 An energy diagram containing two plots for the situation of Fig. 37-14: (1) The electron's mechanical energy E is plotted. (2) The electron's electric potential energy U is plotted as a function of the electron's position x. The nonzero part of the plot (the potential step) has height U_b.

For example, suppose $R = 0.010$. Then if we send 10,000 electrons toward the barrier, we find that about 100 are reflected. However, we could never guess which 100 would be reflected. We have only the probability. The best we can say about any one electron is that it has a 1.0% chance of being reflected and a 99% chance of being transmitted. The wave nature of the electron does not allow us to be any more precise than that.

37.10 | TUNNELING THROUGH A POTENTIAL BARRIER

Key Concepts

- A potential energy barrier is a region where a traveling particle will have an increased potential energy U_b.
- The particle can pass through the barrier if its total energy $E > U_b$.
- Classically, it cannot pass through it if $E < U_b$, but in quantum physics it can, an effect called tunneling.
- For a particle with mass m and a barrier of thickness L, the transmission coefficient is

$$T \approx e^{-2bL},$$

where

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}.$$

Let's replace the potential step of Fig. 37-14 with a **potential barrier** (or **potential energy barrier**), which is a region of thickness L (the *barrier thickness* or *length*) where the electric potential is V_b (< 0) and the barrier height is U_b ($= qV$), as shown in Fig. 37-16. To the right of the barrier is region 3 with $V = 0$. As before, we'll send a beam of nonrelativistic electrons toward the barrier, each with energy E. If we again consider $E > U_b$, we have a more complicated situation than our previous potential step because now electrons can possibly reflect from two boundaries, at $x = 0$ and $x = L$.

Instead of sorting that out, let's consider the situation where $E < U_b$—that is, where the mechanical energy is less than the potential energy that would be demanded of an electron in region 2. Such a demand would require that the electron's kinetic energy ($= E - U_b$) be negative in region 2, which is, of course, simply absurd because kinetic energies must always be positive (nothing in the expression $(1/2)mv^2$ can be negative). Therefore, region 2 is *classically* forbidden to an electron with $E < U_b$.

Tunneling. However, because an electron is a matter wave, it actually has a finite probability of leaking (or, better, *tunneling*) through the barrier and materializing on the other side. Once past the barrier, it again has its full mechanical energy E as though nothing (strange or otherwise) has happened in the region $0 \le x \le L$. Figure 37-17 shows the potential barrier and an approaching electron, with an energy less than the barrier height. We are interested in the probability of the electron appearing on the other side of the barrier. Thus, we want the transmission coefficient T.

To find an expression for T we would in principle follow the procedure for finding R for a potential step. We would solve Schrödinger's equation for the general solutions in each of three regions in Fig. 37-16. We would discard the region 3 solution for a wave traveling in the $-x$ direction (there is no electron source off to the right). Then we would determine the coefficients in terms of the coefficient A of the incident electrons by applying the boundary conditions—that is, by matching the values and slopes of the wave functions at the two boundaries. Finally, we would determine the relative probability density in region 3 in terms of the incident

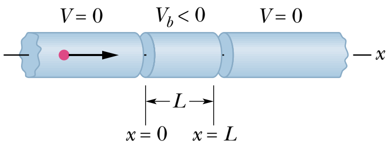

Can the electron pass through the region of negative potential?

Figure 37-16 The elements of a narrow tube in which an electron (the dot) approaches a negative electric potential V_b in the region $x = 0$ to $x = L$.

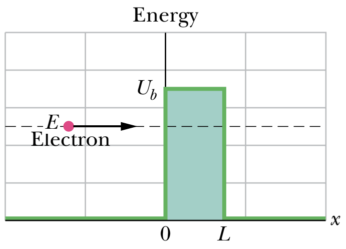

Classically, the electron lacks the energy to pass through the barrier region.

Figure 37-17 An energy diagram containing two plots for the situation of Fig. 37-16: (1) The electron's mechanical energy E is plotted when the electron is at any coordinate $x < 0$. (2) The electron's electric potential energy U is plotted as a function of the electron's position x, *assuming* that the electron can reach any value of x. The nonzero part of the plot (the potential barrier) has height U_b and thickness L.

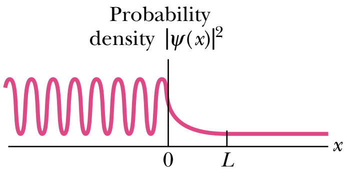


Figure 37-18 A plot of the probability density $|\psi|^2$ of the electron matter wave for the situation of Fig. 37-17. The value of $|\psi|^2$ is nonzero to the right of the potential barrier.

probability density. However, because all this requires a lot of mathematical manipulation, here we shall just examine the general results.

Figure 37-18 shows a plot of the probability densities in the three regions. The oscillating curve to the left of the barrier (for $x < 0$) is a combination of the incident matter wave and the reflected matter wave (which has a smaller amplitude than the incident wave). The oscillations occur because these two waves, traveling in opposite directions, interfere with each other, setting up a standing wave pattern.

Within the barrier (for $0 < x < L$) the probability density decreases exponentially with x. However, if L is small, the probability density is not quite zero at $x = L$.

To the right of the barrier (for $x > L$), the probability density plot describes a transmitted (through the barrier) wave with low but constant amplitude. Thus, the electron can be detected in this region but with a relatively small probability. (Compare this part of the figure with Fig. 37-13.)

As we did with a step potential, we can assign a transmission coefficient T to the incident matter wave and thebarrier. This coefficient gives the probability with which an approaching electron will be transmitted through the barrier—that is, that tunneling will occur. As an example, if $T = 0.020$, then of every 1000 electrons fired at the barrier, 20 (on average) will tunnel through it and 980 will be reflected. The transmission coefficient T is approximately

$$T \approx e^{-2bL}, \tag{37-22}$$

in which

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}, \tag{37-23}$$

and e is the exponential function. Because of the exponential form of Eq. 37-22, the value of T is very sensitive to the three variables on which it depends: particle mass m, barrier thickness L, and energy difference $U_b - E$. (Because we do not include relativistic effects here, E does not include mass energy.)

Barrier tunneling finds many applications in technology, including the tunnel diode, in which a flow of electrons produced by tunneling can be rapidly turned on or off by electronically controlling the barrier height. The 1973 Nobel Prize in physics was shared by three "tunnelers," Leo Esaki (for tunneling in semiconductors), Ivar Giaever (for tunneling in superconductors), and Brian Josephson (for the Josephson junction, a rapid quantum switching device based on tunneling). The 1986 Nobel Prize was awarded to Gerd Binnig and Heinrich Rohrer for development of the scanning tunneling microscope.

☑ **CHECKPOINT 4**

Is the wavelength of the transmitted wave in Fig. 37-18 larger than, smaller than, or the same as that of the incident wave?

SAMPLE PROBLEM 37.10

Barrier tunneling by matter wave

Suppose that the electron in Fig. 37-17, having a total energy E of 5.1 eV, approaches a barrier of height $U_b = 6.8$ eV and thickness $L = 750$ pm.

(a) What is the approximate probability that the electron will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

KEY IDEA

The probability we seek is the transmission coefficient T as given by Eq. 37-22 ($T \approx e^{-2bL}$), where

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}.$$

Calculations: The numerator of the fraction under the square-root sign is

$$(8\pi^2)(9.11 \times 10^{-31}\text{ kg})(6.8\text{ eV} - 5.1\text{ eV}) \times (1.60 \times 10^{-19}\text{ J/eV}) = 1.956 \times 10^{-47}\text{ J}\cdot\text{kg}.$$

Thus,

$$b = \sqrt{\frac{1.956 \times 10^{-47}\text{ J}\cdot\text{kg}}{(6.63 \times 10^{-34}\text{ J}\cdot\text{s})^2}} = 6.67 \times 10^9\text{ m}^{-1}.$$

The (dimensionless) quantity $2bL$ is then

$$2bL = (2)(6.67 \times 10^9\text{ m}^{-1})(750 \times 10^{-12}\text{ m}) = 10.0$$

and, from Eq. 37-22, the transmission coefficient is

$$T \approx e^{-2bL} = e^{-10.0} = 45 \times 10^{-6}. \qquad \text{(Answer)}$$

Thus, of every million electrons that strike the barrier, about 45 will tunnel through it, each appearing on the other side with its original total energy of 5.1 eV. (The transmission through the barrier does not alter an electron's energy or any other property.)

(b) What is the approximate probability that a proton with the same total energy of 5.1 eV will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

Reasoning: The transmission coefficient T (and thus the probability of transmission) depends on the mass of the particle. Indeed, because mass m is one of the factors in the exponent of e in the equation for T, the probability of transmission is very sensitive to the mass of the particle. This time, the mass is that of a proton (1.67×10^{-27} kg), which is significantly greater than that of the electron in (a). By substituting the proton's mass for the mass in (a) and then continuing as we did there, we find that $T \approx 10^{-186}$. Thus, although the probability that the proton will be transmitted is not exactly zero, it is barely more than zero. For even more massive particles with the same total energy of 5.1 eV, the probability of transmission is exponentially lower.

REVIEW AND SUMMARY

Light Quanta—Photons An electromagnetic wave (light) is quantized, and its quanta are called *photons*. For a light wave of frequency f and wavelength λ, the energy E and momentum magnitude p of a photon are

$$E = hf \quad \text{(photon energy)} \tag{37-2}$$

and

$$p = \frac{hf}{c} = \frac{h}{\lambda} \quad \text{(photon momentum).} \tag{37-7}$$

Photoelectric Effect When light of high enough frequency falls on a clean metal surface, electrons are emitted from the surface by photon–electron interactions within the metal. The governing relation is

$$hf = K_{\text{max}} + \Phi, \tag{37-5}$$

in which hf is the photon energy, K_{max} is the kinetic energy of the most energetic emitted electrons, and Φ is the **work function** of the target material—that is, the minimum energy an electron must have if it is to emerge from the surface of the target. If hf is less than Φ, electrons are not emitted.

Light Waves and Photons When light interacts with matter, energy and momentum are transferred via photons. When light is in transit, however, we interpret the light wave as a **probability wave**, in which the probability (per unit time) that a photon can be detected is proportional to E_m^2, where E_m is the amplitude of the oscillating electric field of the light wave at the detector.

Matter Waves A moving particle such as an electron or a proton can be described as a **matter wave**; its wavelength (called the **de Broglie wavelength**) is given by $\lambda = h/p$, where p is the magnitude of the particle's momentum.

The Wave Function A matter wave is described by its **wave function** $\Psi(x, y, z, t)$, which can be separated into a space-dependent part $\psi(x, y, z)$ and a time-dependent part $e^{-i\omega t}$. For a particle of mass m moving in the x direction with constant total energy E through a region in which its potential energy is $U(x)$, $\psi(x)$ can be found by solving the simplified **Schrödinger equation**:

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}[E - U(x)]\psi = 0. \tag{37-11}$$

A matter wave, like a light wave, is a probability wave in the sense that if a particle detector is inserted into the wave, the probability that the detector will register a particle during any specified time interval is proportional to $|\psi|^2$, a quantity called the **probability density**.

For a free particle—that is, a particle for which $U(x) = 0$—moving in the x direction, $|\psi|^2$ has a constant value for all positions along the x axis.

Heisenberg's Uncertainty Principle The probabilistic nature of quantum physics places an important limitation on detecting a particle's position and momentum. That is, it is not possible to measure the position $\vec{r}$ and the momentum $\vec{p}$ of a particle simultaneously with unlimited precision. The uncertainties in the components of these quantities are given by

$$\begin{aligned} \Delta x \cdot \Delta p_x &\geq \hbar \\ \Delta y \cdot \Delta p_y &\geq \hbar \\ \Delta z \cdot \Delta p_z &\geq \hbar. \end{aligned} \tag{37-20}$$

Barrier Tunneling According to classical physics, an incident particle will be reflected from a potential energy barrier whose height is greater than the particle's kinetic energy. According to quantum physics, however, the particle has a finite probability of tunneling through such a barrier, appearing on the other side unchanged. The probability that a given particle of mass m and energy E will tunnel through a barrier of height U_b and thickness L is given by the transmission coefficient T:

$$T \approx e^{-2bL}, \tag{37-22}$$

where

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}. \tag{37-23}$$

PROBLEMS

1. A 100 W sodium lamp ($\lambda = 589$ nm) radiates energy uniformly in all directions. (a) At what rate are photons emitted by the lamp? (b) At what distance from the lamp will a totally absorbing screen absorb photons at the rate of 1.00 photon/cm$^2 \cdot$s? (c) What is the photon flux (photons per unit area per unit time) on a small screen 2.00 m from the lamp?

2. In an old-fashioned television set, electrons are accelerated through a potential difference of 25.0 kV. What is the de Broglie wavelength of such electrons? (Relativity is not needed.)

3. Find the maximum kinetic energy of electrons ejected from a certain material if the material's work function is 2.3 eV and the frequency of the incident radiation is 2.5×10^{15} Hz.

4. Calculate the de Broglie wavelength of (a) a 2.00 keV electron, (b) a 2.00 keV photon, and (c) a 2.00 keV neutron.

5. Solar radiations fall on Earth's surface at a rate of 1400 W/m^2. Assuming that the radiation has an average wavelength of 550 nm, how many photons per square meter per second fall on the surface?

6. How fast must an electron move to have a kinetic energy equal to the photon energy of sodium light at wavelength 500 nm?

7. 1.5 mW of 400 nm light is directed on a photoelectric cell. If 0.10% of the incident photons produce photoelectrons, find the current in the cell.

8. The uncertainty in the position of an electron along an x axis is given as 50 pm, which is about equal to the radius of a hydrogen atom. What is the least uncertainty in any simultaneous measurement of the momentum component p_x of this electron?

9. Suppose the *fractional efficiency* of a cesium surface (with work function 1.80 eV) is 1.0×10^{-16}; that is, on average one electron is ejected for every 10^{16} photons that reach the surface. What would be the current of electrons ejected from such a surface if it were illuminated with 600 nm light from a 3.00 mW laser and all the ejected electrons took part in the charge flow?

10. An electron and a photon each have a wavelength of 0.25 nm. What is the momentum (in kg$\cdot$m/s) of the (a) electron and (b) photon? What is the energy (in eV) of the (c) electron and (d) photon?

11. The function $\psi(x)$ displayed in Eq. 37-19 can describe a free particle, for which the potential energy is $U(x) = 0$ in Schrödinger's equation (Eq. 37-11). Assume now that $U(x) = U_0$ = a constant in that equation. Show that Eq. 37-19 is a solution of Schrödinger's equation, with

$$k = \frac{2\pi}{h}\sqrt{2m(E - U_0)}$$

giving the angular wave number k of the particle.

12. Show that the angular wave number k for a nonrelativistic free particle of mass m can be written as

$$k = \frac{2\pi\sqrt{2mK}}{h},$$

in which K is the particle's kinetic energy.

13. An electron with total energy $E = 5.1$ eV approaches a barrier of height $U_b = 6.8$ eV and thickness $L = 750$ pm. What percentage change in the transmission coefficient T occurs for a 1.0% change in (a) the barrier height, (b) the barrier thickness, and (c) the kinetic energy of the incident electron?

14. An ultraviolet lamp emits light of wavelength 400 nm at the rate of 400 W. An infrared lamp emits light of wavelength 700 nm, also at the rate of 400 W. (a) Which lamp emits photons at the greater rate and (b) what is that greater rate?

15. The smallest dimension (resolving power) that can be resolved by an electron microscope is equal to the de Broglie wavelength of its electrons. What accelerating voltage would be required for the electrons to have the same resolving power as could be obtained using 120 keV gamma rays?

16. The wavelength of the yellow spectral emission line of sodium is 590 nm. At what kinetic energy would an electron have that wavelength as its de Broglie wavelength?

17. Consider a potential energy barrier like that of Fig. 37-17 but whose height U_b is 4.0 eV and whose thickness L is 0.70 nm. What is the energy of an incident electron whose transmission coefficient is 0.0010?

18. A nonrelativistic particle is moving three times as fast as an electron. The ratio of the de Broglie wavelength of the particle to that of the electron is 1.813×10^{-4}. By calculating its mass, identify the particle.

19. (a) Suppose a beam of 4.5 eV protons strikes a potential energy barrier of height 6.0 eV and thickness 0.70 nm, at a rate equivalent to a current of 1000 A. How long would you have to wait—on average—for one proton to be transmitted? (b) How long would you have to wait if the beam consisted of electrons rather than protons?

20. An isolated copper sphere of radius 5.0 cm, initially uncharged, is illuminated by ultraviolet light of wavelength 200 nm. What charge will the photoelectric effect induce on the sphere. The work function for copper is 4.70 eV.

21. An orbiting satellite can become charged by the photoelectric effect when sunlight ejects electrons from its outer surface. Satellites must be designed to minimize such charging because it can ruin the sensitive microelectronics. Suppose a satellite is coated with platinum, a metal with a very large work function ($\Phi = 5.32$ eV). Find the longest wavelength of incident sunlight that can eject an electron from the platinum.

22. The work function of tungsten is 4.50 eV. Calculate the speed of the fastest electrons ejected from a tungsten surface when light whose photon energy is 5.80 eV shines on the surface.

23. A helium–neon laser emits red light at wavelength $\lambda = 633$ nm in a beam of diameter 3.0 mm and at an energy-emission rate of 5.0 mW. A detector in the beam's path totally absorbs the beam. At what rate per unit area does the detector absorb photons?

24. A light detector (your eye) has an area of 2.00×10^{-6} m^2 and absorbs 80% of the incident light, which is at wavelength 500 nm. The detector faces an isotropic source, 3.00 m from the source. If the detector absorbs photons at the rate of exactly 4.000 s^{-1}, at what power does the emitter emit light?

25. If the de Broglie wavelength of a proton is 100 fm, (a) what is the speed of the proton and (b) through what electric potential would the proton have to be accelerated to acquire this speed?

26. What is the wavelength of (a) a photon with energy 1.00 eV, (b) an electron with energy 1.00 eV and (c) a photon of energy 1.00 GeV?

27. The wavelength associated with the cutoff frequency for silver is 325 nm. Find the maximum kinetic energy of electrons ejected from a silver surface by ultraviolet light of wavelength 275 nm.

28. A light detector has an absorbing area of 3.00×10^{-6} m^2 and absorbs 50% of the incident light, which is at wavelength 600 nm. The detector faces an isotropic source, 12.0 m from the source. The energy E emitted by the source versus time t is given in Fig. 37-19 ($E_s = 7.2$ nJ, $t_s = 2.0$ s). At what rate are photons absorbed by the detector?

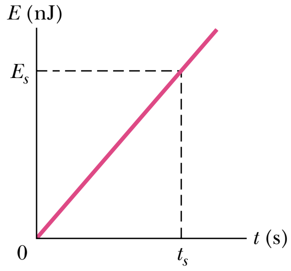


Figure 37-19 Problem 28.

29. At what rate does the Sun emit photons? For simplicity, assume that the Sun's entire emission at the rate of 3.9×10^{26} W is at the single wavelength of 550 nm.

30. What (a) frequency, (b) photon energy, and (c) photon momentum magnitude (in keV/c) are associated with x rays having wavelength 35.0 pm?

31. The beam emerging from a 1.5 W argon laser ($\lambda = 515$ nm) has a diameter d of 3.5 mm. The beam is focused by a lens system with an effective focal length f_L of 2.5 mm. The focused beam strikes a totally absorbing screen, where it forms a circular diffraction pattern whose central disk has a radius R given by $1.22 f_L \lambda / d$. It can be shown that 84% of the incident energy ends up within this central disk. At what rate are photons absorbed by the screen in the central disk of the diffraction pattern?

32. An electron moves through a region of uniform electric potential of −200 V with a (total) energy of 500 eV. What are its (a) kinetic energy (in electron-volts), (b) momentum, (c) speed, (d) de Broglie wavelength, and (e) angular wave number?

33. (a) If the work function for a certain metal is 1.8 eV, what is the stopping potential for electrons ejected from the metal when light of wavelength 400 nm shines on the metal? (b) What is the maximum speed of the ejected electrons?

34. How fast must an electron move to have a kinetic energy equal to the photon energy of sodium light at wavelength 588 nm?

35. A special kind of lightbulb emits monochromatic light of wavelength 630 nm. Electrical energy is supplied to it at the rate of 60 W, and the bulb is 93% efficient at converting that energy to light energy. How many photons are emitted by the bulb during its lifetime of 730 h?

36. The stopping potential for electrons emitted from a surface illuminated by light of wavelength 491 nm is 0.710 V. When the incident wavelength is changed to a new value, the stopping potential is 1.43 V. (a) What is this new wavelength? (b) What is the work function for the surface?

37. What is the photon energy for yellow light from a highway sodium lamp at a wavelength of 589 nm?

38. You wish to pick an element for a photocell that will operate via the photoelectric effect with visible light. Which of the following are suitable (work functions are in parentheses): tantalum (4.2 eV), tungsten (4.5 eV), aluminum (4.2 eV), barium (2.5 eV), lithium (2.3 eV)?

39. Light strikes a sodium surface, causing photoelectric emission. The stopping potential for the ejected electrons is 5.0 V, and the work function of sodium is 2.2 eV. What is the wavelength of the incident light?

40. In a photoelectric experiment using a sodium surface, you find a stopping potential of 1.85 V for a wavelength of 300 nm and a stopping potential of 0.820 V for a wavelength of 400 nm. From these data find (a) a value for the Planck constant, (b) the work function Φ for sodium, and (c) the cutoff wavelength λ_0 for sodium.

41. Singly charged sodium ions are accelerated through a potential difference of 300 V. (a) What is the momentum acquired by such an ion? (b) What is its de Broglie wavelength?

42. A satellite in Earth orbit maintains a panel of solar cells of area 2.90 m^2 perpendicular to the direction of the Sun's light rays. The intensity of the light at the panel is 1.39 kW/m^2. (a) At what rate does solar energy arrive at the panel? (b) At what rate are solar photons absorbed by the panel? Assume that the solar radiation is monochromatic, with a wavelength of 550 nm, and that all the solar radiation striking the panel is absorbed. (c) How long would it take for a "mole of photons" to be absorbed by the panel?

43. Light of wavelength 200 nm shines on an aluminum surface; 4.20 eV is required to eject an electron. What is the kinetic energy of (a) the fastest and (b) the slowest ejected electrons? (c) What is the stopping potential for this situation? (d) What is the cutoff wavelength for aluminum?

44. The meter was once defined as 1 650 763.73 wavelengths of the orange light emitted by a source containing krypton-86 atoms. What is the photon energy of that light?

45. X rays with a wavelength of 71 pm are directed onto a gold foil and eject tightly bound electrons from the gold atoms. The ejected electrons then move in circular paths of radius r in a region of uniform magnetic field $\vec{B}$. For the fastest of the ejected electrons, the product Br is equal to 1.88×10^{-4} T·m. Find (a) the maximum kinetic energy of those electrons and (b) the work done in removing them from the gold atoms.

46. What are (a) the energy of a photon corresponding to wavelength 1.00 nm, (b) the kinetic energy of an electron with de Broglie wavelength 1.00 nm, (c) the energy of a photon corresponding to wavelength 1.00 fm, and (d) the kinetic energy of an electron with de Broglie wavelength 1.00 fm?

47. Find the (a) maximum kinetic energy of photoelectrons ejected from potassium surface, for which $\phi_0 = 2.1$ eV, by photons of wavelengths 2000 Å and 5000 Å. (b) What is the threshold frequency ν_0 and (c) the corresponding wavelength?

48. Radiation from the Sun reaching Earth (just outside the atmosphere) has an intensity of 1.4 kW/m^2. (a) Assuming that Earth (and its atmosphere) behaves like a flat disk perpendicular to the Sun's rays and that all the incident energy is absorbed, calculate the force on Earth due to radiation pressure. (b) For comparison, calculate the force due to the Sun's gravitational attraction.

49. A parallel beam of monochromatic light of wavelength 500 nm is incident normally on a perfectly absorbing surface. The power through any cross section of the beam is 10 W. Find (a) the number of photons absorbed per second by the surface and (b) the force exerted by the light beam on the surface.

50. A parallel beam of monochromatic light of wavelength 663 nm is incident on a totally reflecting plane mirror. The angle of incidence is 60° and the number of photons striking the mirror per second is 1.0×10^{19}. Calculate the force exerted by the light beam on the mirror.

51. A beam of white light is incident normally on a plane surface absorbing 50% of the light and reflecting the rest. If the incident beam carries 10 W of power, find the force exerted by it on the surface.

52. A small laser emits light at power 5.00 mW and wavelength 633 nm. The laser beam is focused (narrowed) until its diameter matches the 1266 nm diameter of a sphere placed in its path. The sphere is perfectly absorbing and has density 5.00×10^3 kg/m^3. What are (a) the beam intensity at the sphere's location, (b) the radiation pressure on the sphere, (c) the magnitude of the corresponding force, and (d) the magnitude of the acceleration that force alone would give the sphere?

PRACTICE QUESTIONS

Single Correct Choice Type

1. Protons and α-particles have the same de Broglie wavelength. What is same for both of them?

(a) Energy **(b)** Time
(c) Frequency **(d)** Momentum

2. The specific charge of proton is 9.6×10^7 C/kg. What is the specific charge of α-particle?

(a) 9.6×10^7 C/kg
(b) 19.2×10^7 C/kg
(c) 4.8×10^7 C/kg
(d) 2.4×10^7 C/kg

3. An electron of mass m when accelerated through a potential difference V has de Broglie wavelength λ. The de Broglie wavelength associated with a proton of mass M accelerated through the same potential difference will be

(a) $\dfrac{\lambda m}{M}$ **(b)** $\lambda\sqrt{\dfrac{m}{M}}$
(c) $\dfrac{\lambda M}{m}$ **(d)** $\lambda\sqrt{\dfrac{M}{m}}$

4. What is the de Broglie wavelength of an electron accelerated from rest through a potential difference of V volts?

(a) $\dfrac{12.3}{\sqrt{V}}$ Å **(b)** $\dfrac{12.3}{V}$ Å
(c) $\dfrac{12.3}{V^2}$ Å **(d)** None of these

5. The frequency of incident light falling on a photosensitive metal plate is doubled, the kinetic energy of the emitted photoelectrons is

(a) Double the earlier value
(b) Unchanged
(c) More than doubled
(d) Less than doubled

6. A light of wavelength 5000 Å falls on a photosensitive plate with photoelectric work function 190 eV. Kinetic energy of the emitted photoelectrons will be ($h = 6.63 \times 10^{-34}$ J · s)
(a) 0.1 eV
(b) 0.59 eV
(c) 1.581 eV
(d) 2 eV

7. Light of two different frequencies whose photos have energies 1 and 2.5 eV, respectively, successively illuminate a metal whose work function is 0.5 eV. The ratio of the maximum speeds of the emitted electrons will be
(a) 1 : 5 **(b)** 1 : 4
(c) 1 : 2 **(d)** 1 : 1

8. Photoelectric work function of a metal is 1 eV. Light of wavelength λ = 3000 Å falls on it. The photoelectrons come out with velocity
(a) 10 m/s **(b)** 10^3 m/s
(c) 10^4 m/s **(d)** 10^6 m/s

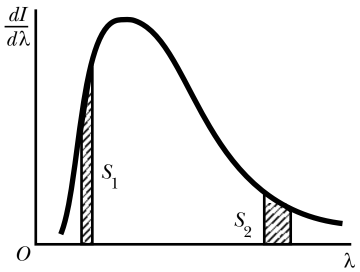
9. The de Broglie wavelength of a proton (charge = 1.6×10^{-19} C, mass = 1.6×10^{-27} kg) accelerated through a potential difference of 1 kV is
(a) 600 Å **(b)** 0.9×10^{-12} m
(c) 7 Å **(d)** 0.9 nm

10. In Davisson–Germer experiment, the correct relation between angle of diffraction ϕ and glancing angle θ is

(a) $\theta = 90^0 - \dfrac{\phi}{2}$ **(b)** $\theta = 90^0 + \dfrac{\phi}{2}$

(c) $\theta = \dfrac{\phi}{2}$ **(d)** $\theta = \phi$

11. Two separate segments of equal area are isolated in the energy distribution of black body radiation (As shown in the following figure). Let us assume that the number of photons emitted by the body per unit time in the segments is n_1 and n_2, and the energy of the photons are E_1 and E_2, respectively. Then



(a) $n_1 > n_2$ **(b)** $n_1 < n_2$
(c) $E_1 < E_2$ **(d)** $E_1 > E_2$

12. Let n_r and n_b be the number of photons emitted by a red bulb and a blue bulb, respectively, of equal power in a given time. Then
(a) $n_r = n_b$
(b) $n_r > n_b$
(c) $n_r < n_b$
(d) The information is insufficient to get a relation between n_r and n_b

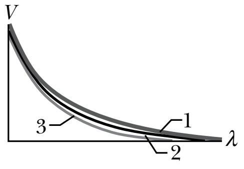
13. Consider tuning a radio to higher frequency station. The energy of radio photons detected
(a) Is zero because radio waves do not carry energy
(b) Is smaller
(c) Is greater
(d) Is the same as before, but the radio waves are of higher frequency

14. A beam of ultraviolet light is incident on the metal ball of an electroscope that is initially uncharged. Does the electroscope acquire a charge?
(a) Yes, it acquires a positive charge
(b) Yes, it acquires a negative charge
(c) No, it does not acquire a charge
(d) Its acquiring the charge depends on the work function of the metal

15. A xenon arc lamp is covered with an interference filter that only transmits light of 400 nm wavelength. When the transmitted light strikes a metal surface, a stream of electrons emerges from the metal. If the intensity of the light striking the surface is doubled, then
(a) More electrons are emitted in a given time interval
(b) The electrons that are emitted are more energetic
(c) Both of the above
(d) Neither of the above

16. A girl is performing a photoelectric experiment but no photoelectrons are being ejected. To eject electrons, she should change the light by
(a) Decreasing the frequency
(b) Increasing the frequency
(c) Increasing the intensity
(d) Increasing the wavelength

17. In the following figure gives the stopping voltage V versus the wavelength λ of light for three different materials. Rank the materials according to their work function, greatest first.

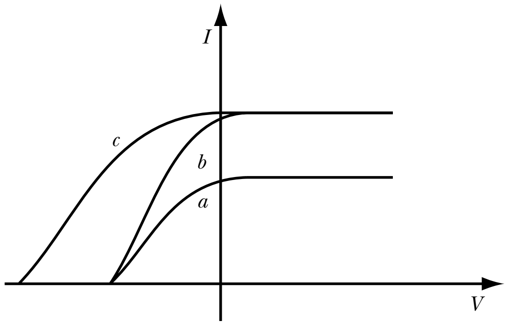


(a) $\phi_3 > \phi_2 > \phi_1$ **(b)** $\phi_1 > \phi_2 > \phi_3$
(c) $\phi_2 > \phi_3 > \phi_1$ **(d)** $\phi_3 > \phi_1 > \phi_2$

18. A non-monochromatic light is used in an experiment on photoelectric effect. The stopping potential is
(a) Related to the mean wavelength
(b) Related to the longest wavelength
(c) Related to the shortest wavelength
(d) Not related to the wavelength

19. In the following figure shows the variation of photocurrent with potential of the collector plate for a photosensitive surface for three different radiations. Let I_a, I_b, and

I_c be the intensities and f_a, f_b, and f_c be the frequencies for the curves a, b, and c, respectively.



(a) $f_a = f_b$ and $I_a = I_b$ **(b)** $f_a = f_c$ and $I_a = I_c$
(c) $f_a = f_b$ and $I_a < I_b$ **(d)** $f_a = f_c$ and $I_b = I_c$

20. In a photoelectric tube, the voltage applied is such that the photoelectrons emitted reverse in direction after traveling half the distance. If we keep the voltage same but halve the distance, then
- **(a)** The photoelectrons reverse the direction at half of the new distance
- **(b)** The photoelectrons are just able to reach the other electrode
- **(c)** The photoelectrons reverse the direction at one fourth of the new distance
- **(d)** The photoelectrons reach the other electrode with some kinetic energy

21. An image of the Sun is formed by a lens of focal length of 30 cm on the metal surface of a photoelectric cell and a photoelectric current i is produced. The lens forming the image is then replaced by another of the same diameter but of focal length of 15 cm. The photoelectric current in this case is
(a) $i/2$ **(b)** i
(c) $2i$ **(d)** $4i$

22. If a parallel beam of light having intensity I is incident normally on a perfectly reflecting surface, the force exerted on the surface, equal F. When the surface is held at an angle θ, the force is
(a) $2F \tan\theta$ **(b)** $F\cos\theta$
(c) $F\cos^2\theta$ **(d)** $2F$

23. The difference between the photons and other material particles that we know of is that
- **(a)** The photon is always chargeless whereas all the material particles have some charge
- **(b)** The rest mass of photon is zero whereas all the material particles have some mass
- **(c)** Photon cannot be accelerated by applying some force
- **(d)** Photons have a wave associated with them but material particles cannot be associated with a wave

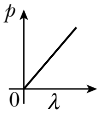
24. A beta particle, gamma ray, and alpha particle all have the same momentum. Which has the longest wavelength?
(a) Beta particle **(b)** Gamma ray
(c) Alpha particle **(d)** All the same

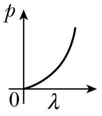
25. If a proton and an electron have the same kinetic energy, which has the longer de Broglie wavelength?
- **(a)** Electron
- **(b)** Proton
- **(c)** Both have the same wavelength
- **(d)** None of the above

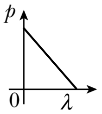
26. The equation $E = pc$ is valid
- **(a)** For an electron
- **(b)** For a photon
- **(c)** For a photon as well as for an electron
- **(d)** Neither for an electron nor for a photon

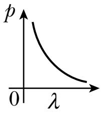
27. A particle of mass $4m$ at rest decays into two particles of masses m and $3m$ having nonzero velocities. The ratio of the de Broglie wavelengths of the particles 1 and 2 is
(a) 1/2 **(b)** 1/4
(c) 2 **(d)** 1

28. Which of the following graph represent the variation of particle momentum and associated de Broglie wavelength?

(a)


(b)


(c)


(d)


29. The frequency of matter waves is expressed as
(a) hf **(b)** E/h
(c) $1/T$ **(d)** None of these

30. $\psi(x)$ is the wave function for a particle moving along the x axis. The probability that the particle is in the interval from $x = a$ to $x = b$ is given by
(a) $\psi(b) - \psi(a)$ **(b)** $|\psi(b)|/|\psi(a)|$
(c) $|\psi(b)|^2/|\psi(a)|^2$ **(d)** $\int_a^b \psi(x)dx$

31. The significance of $|\psi|^2 x$ is
(a) Probability **(b)** Energy
(c) Probability density **(d)** Energy density

32. A free electron in motion along the x axis has a localized wave function. The uncertainty in its momentum is decreased if
- **(a)** The wave function is made more narrow
- **(b)** The wave function is made less narrow
- **(c)** The wave function remains the same but the energy of the electron is increased
- **(d)** The wave function remains the same but the energy of the electron is decreased
- **(e)** None of the above

33. The uncertainty in position of an electron in a certain state is 5×10^{-10} m. The uncertainty in its momentum might be
(a) 5.0×10^{-24} kg·m/s **(b)** 4.0×10^{-24} kg·m/s
(c) 3.0×10^{-24} kg·m/s **(d)** All of the above

34. The reflection coefficient R for a certain barrier tunneling problem is 0.80. The corresponding transmission coefficient T is

(a) 0.80 **(b)** 0.60
(c) 0.50 **(d)** 0.20

35. An electron with energy E is incident on a potential energy barrier of height E_{pot} and thickness L. The probability of tunneling increases if

(a) E decreases without any other changes
(b) E_{pot} increases without any other changes
(c) L decreases without any other changes
(d) E and E_{pot} increase by the same amount
(e) E and E_{pot} decrease by the same amount

36. An electron with energy E is incident upon a potential energy barrier of height $E_{\text{pot}} > E$ and thickness L. The transmission coefficient T

(a) Is zero
(b) Decreases exponentially with L
(c) Is proportional to $1/L$
(d) Is proportional to $1/L^2$

37. A laser emits a single, 2.0 ms pulse of light that has a frequency of 2.83×10^{11} Hz and a total power of 75 000 W. How many photons are in the pulse?

(a) 8.0×10^{23} **(b)** 2.4×10^{25}
(c) 1.6×10^{24} **(d)** 3.2×10^{25}

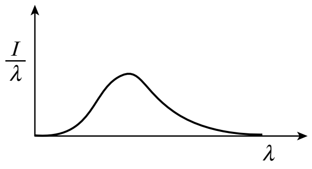
38. A laser emits a pulse of light with energy 5.0×10^3 J. Determine the number of photons in the pulse if the wavelength of light is 480 nm.

(a) 5.2×10^{16} **(b)** 1.2×10^{22}
(c) 2.5×10^{19} **(d)** 3.1×10^{22}

39. An x-ray generator produces photons with energy 49 600 eV or less. Which one of the following phrases most accurately describes the wavelength of these photons?

(a) 0.025 nm or longer **(b)** 0.75 nm or longer
(c) 0.050 nm or longer **(d)** 0.25 nm or shorter

40. The graph shows the variation in radiation intensity per unit wavelength versus wavelength for a perfect blackbody at temperature T. Complete the following statement: As the blackbody temperature is increased, the peak in intensity of this curve



(a) Will remain constant
(b) Will be shifted to longer wavelengths and its magnitude will increase
(c) Will be shifted to shorter wavelengths and its magnitude will increase
(d) Will be shifted to longer wavelengths and its magnitude will decrease

41. Which one of the following quantities is the same for all photons in vacuum?

(a) Speed **(b)** Kinetic energy
(c) Frequency **(d)** Wavelength

42. Complete the following statement: The term *photon* applies

(a) Only to x-rays
(b) To any form of particle motion
(c) Only to visible light
(d) To any form of electromagnetic radiation

43. Which one of the following statements concerning photons is false?

(a) Photons have zero mass.
(b) The rest energy of all photons is zero.
(c) Photons travel at the speed of light in a vacuum.
(d) Photons have been brought to rest by applying a strong magnetic field to them.

44. Photons of what minimum frequency are required to remove electrons from gold?

Note: The work function for gold is 4.8 eV.

(a) 7.3×10^{14} Hz **(b)** 3.8×10^{17} Hz
(c) 1.2×10^{15} Hz **(d)** 6.5×10^{15} Hz

45. White light consisting of wavelengths $380 \text{ nm} \le \lambda \le 750 \text{ nm}$ is incident on a lead surface. For which one of the following ranges of wavelengths will photoelectrons be emitted from the lead surface that has a work function $W_0 = 6.63 \times 10^{-19}$ J?

(a) $380 \text{ nm} \le \lambda \le 750 \text{ nm}$
(b) $380 \text{ nm} \le \lambda \le 410 \text{ nm}$
(c) $380 \text{ nm} \le \lambda \le 630 \text{ nm}$
(d) No photoelectrons will be emitted

46. A photon of wavelength 200 nm is scattered by an electron that is initially at rest. Which one of the following statements concerning the wavelength of the scattered photon is true?

(a) The wavelength is zero nm.
(b) The wavelength is greater than 200 nm.
(c) The wavelength is 200 nm.
(d) The wavelength is 100 nm.

47. The de Broglie wavelength of an electron ($m = 9.11 \times 10^{-31}$ kg) is 1.2×10^{-10} m. Determine the kinetic energy of the electron.

(a) 1.5×10^{-15} J **(b)** 1.7×10^{-17} J
(c) 1.6×10^{-16} J **(d)** 1.8×10^{-18} J

More than One Correct Choice Type

48. In which of the following features, the heavier of the two particles has a smaller de Broglie wavelength? The two particles

(a) Move with the same speed
(b) Move with the same linear momentum
(c) Move with the same kinetic energy
(d) Have the same change of potential energy in a conservative field

49. When photon of energy 4.25 eV strikes the surface of a metal A, the ejected photoelectrons have maximum kinetic energy T_A eV and de Broglie wavelength λ_A. The maximum kinetic energy of photoelectrons liberated from another metal B by photon of energy 4.70 eV is

$T_B = (T_A - 1.50)$ eV. If the de Broglie wavelength of these photoelectrons is $\lambda_B = 2\lambda_A$, then
(a) The work function of A is 2.25 eV
(b) The work function of A is 4.20 eV
(c) $T_A = 2.00$ eV
(d) $T_B = 2.75$ eV

50. When a monochromatic point source of light is at a distance of 0.2 m from a photoelectric cell, the cut-off voltage and the saturation current are 0.6 V and 18.0 mA, respectively. If the same source is placed 0.6 m away from the photocell, then
(a) The stopping potential will be 0.2 V
(b) The stopping potential will be 0.6 V
(c) The stopping potential will be 2.0 mA
(d) The stopping potential will be 6.0 mA

51. Photoelectric effect support quantum nature of light because
(a) There is minimum frequency of light below which no photoelectrons are emitted
(b) Electric charge of photoelectrons is quantized
(c) The maximum kinetic energy of photoelectrons depends only on the frequency of light and not on its intensity
(d) Even when the metal surface is faintly illuminated, the photoelectrons leave the surface immediately

52. At the stopping potential
(a) The electrons are not at all emitted by the plate
(b) The electrons are emitted by the plate but reverse their direction of motion just before reaching the collector plate
(c) The most energetic electrons emitted reverse their direction of motion just before reaching the other plate
(d) The collector plate is negatively charged with respect to the emitter

53. Which of the following are not dependent on the intensity of the incident radiation in a photoelectric experiment?
(a) Amount of photoelectric current
(b) Stopping potential to reduce the photoelectric current to zero
(c) Work function of the surface
(d) Maximum kinetic energy of photoelectrons

Linked Comprehension

Paragraph for Questions 54 and 55: It is desired to obtain a diffraction pattern for electrons using a diffraction grating with lines separated by 10 nm. The mass of an electron is 9.11×10^{-31} kg.

54. What is the approximate kinetic energy of electrons that would be diffracted by such a grating?
(a) 1.5×10^{-6} eV
(b) 1.5×10^{-2} eV
(c) 1.5×10^{-4} eV
(d) 1.5×10^{2} eV

55. Suppose it is desired to observe diffraction effects for a beam of electromagnetic radiation using the same grating. Roughly, what is the required energy of the individual photons in the beam?
(a) 10^{-6} eV
(b) 10^{-2} eV
(c) 10^{-4} eV
(d) 10^{2} eV

Paragraph for Questions 56 and 57: A physicist wishes to produce electrons by shining light on a metal surface. The light source emits light with a wavelength of 450 nm. The table lists the only available metals and their work functions.

Metal	**W_0 (eV)**
Barium	2.5
Lithium	2.3
Tantalum	4.2
Tungsten	4.5

56. Which metal(s) can be used to produce electrons by the photoelectric effect?
(a) Barium only
(b) Barium or lithium
(c) Tungsten only
(d) Lithium, tantalum, or tungsten

57. Which entry in the table below correctly identifies the metal that will produce the most energetic electrons and their energies?

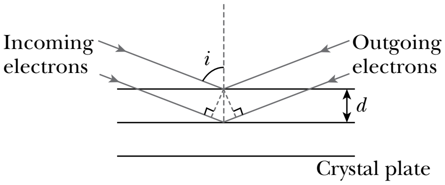
	Metal	**Maximum electron energy observed**
(a)	Lithium	2.3 eV
(b)	Lithium	0.5 eV
(c)	Tungsten	1.8 eV
(d)	Tungsten	2.8 eV

Paragraph for Questions 58 and 59: When a particle is restricted to move along x-axis between $x = 0$ and $x = a$, where a is of nanometer dimension, its energy can take only certain specific values. The allowed energies of the particle moving in such a restricted region, correspond to the formation of standing waves with nodes at its ends $x = 0$ and $x = a$. The wavelength of this standing wave is related to the linear momentum p of the particle according to the de Broglie relation. The energy of the particle of mass m is related to its linear momentum as $E = p^2/2m$. Thus, the energy of the particle can be denoted by a quantum number n taking values 1, 2, 3,… (n = 1, called the ground state) corresponding to the number of loops in the standing wave. Use the model described above to answer the following three questions for a particle moving in the line $x = 0$ to $x = a$. Take $h = 6.6 \times 10^{-34}$ Js and $e = 1.6 \times 10^{-19}$ C.

58. The allowed energy for the particle for a particular value of n is proportional to
(a) a^{-2}
(b) $a^{-3/2}$
(c) a^{-1}
(d) a^{2}

59. The speed of the particle, that can take discrete values, is proportional to
(a) $n^{-3/2}$
(b) n^{-1}
(c) $n^{1/2}$
(d) n

Paragraph for Questions 60 and 61: Wave property of electrons implies that they will show diffraction effects. Davisson and Germer demonstrated this by diffracting electrons from crystals. The law governing the diffraction from a crystal is obtained by requiring that electron waves reflected from the planes of atoms in crystal interfere constructively (Figure below).



60. Electrons accelerated by potential V are diffracted from a crystal. If $d = 1$ Å and $i = 3$ u, then V should be about ($h = 6.6 \times 10^{-19}$ C)

(a) 1000 V **(b)** 2000 V
(c) 50 V **(d)** 500 V

61. If a strong diffraction peak is observed with electrons incident at an angle l from normal to the crystal planes with distance between them, de Broglie wavelength λ_{dB} of electrons can be calculated by relationship (n is an integer)

(a) $d \cos i = n\lambda_{dB}$ **(b)** $d \sin i = n\lambda_{dB}$
(c) $2d \cos i = n\lambda_{dB}$ **(d)** $2d \sin i = n\lambda_{dB}$

Paragraph for Questions 62–64: Photoelectrons are rejected from a surface when light of wavelength $\lambda_1 = 550$ nm is incident on it. The stopping potential for such electrons is $\lambda_{s1} =$ 01.19 V. Suppose that radiation of wavelength $\lambda_2 = 190$ nm were incident on the surface. Calculate

62. The stopping potential V_{s2}

(a) 4.47 **(b)** 3.16
(c) 2.76 **(d)** 5.28

63. The work function of the surface

(a) 3.75 eV **(b)** 2.07 eV
(c) 4.20 eV **(d)** 3.60 eV

64. The threshold frequency for the surface

(a) 500 T Hz **(b)** 480 T Hz
(c) 520 T Hz **(d)** 460 T Hz

Paragraph for Questions 65–67: When a surface is irradiated with a light of wavelength 4950 Å, a photocurrent appears which vanishes if a retarding potential greater that 0.6 V is applied across the phototube. When a different source of light is used, it is found that the critical retarding potential is changed to 1.1 V.

65. Work function of the emitting surface in the first instance is

(a) 9.1 eV **(b)** 1.9 eV
(c) 0.9 eV **(d)** 0.6 eV

66. Wavelength of the second source is

(a) 4125 Å **(b)** 4950 Å
(c) 2125 Å **(d)** 5124 Å

67. Change in two stopping potentials is

(a) Not observed
(b) Significant
(c) Nominal
(d) Not possible to calculate because of insufficient data

Paragraph for Questions 68 and 69: A beam of electrons with speed v_0 passes through double slits and then is allowed to strike a fluorescent screen. An interference pattern is observed on the screen.

68. The speed of the electrons is increased to $2v_0$. What happens to the spacing of the interference fringes on the screen?

(a) The spacing increases
(b) The spacing decreases
(c) The spacing remains the same
(d) None of the above

69. The electrons are instead replaced by protons moving with speed v_0. Compared to electrons at the same speed, what happens to the spacing of the interference fringes on the screen?

(a) The spacing increases
(b) The spacing decreases
(c) The spacing remains the same
(d) None of the above

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

70.

Column I	Column II
(a) Einstein's photoelectric equation	**(p)** $\sqrt{V} = a(Z-1)$
(b) Duane–Hunt law of continuous x-rays	**(q)** $L = \frac{nh}{2\pi}$
(c) Moseley's law for characteristic x-rays	**(r)** $\lambda_{\min} = \frac{hc}{eV}$
(d) Bohr's quantum condition	**(s)** $K_{\max} = h\nu - W_0$

71.

Column I	Column II
(a) A wooden table does not emit photoelectrons when exposed to visible light	**(p)** Photocell
(b) The stopping potential is same for both the intense and dim ultraviolet light	**(q)** Establishes particle nature of light
(c) Einstein's photoelectric equation threshold	**(r)** Frequency of visible light is less than the frequency of wood
(d) Light energy is converted into electric energy	**(s)** Stopping potential depends on the frequency and not on the intensity of incident radiation

Directions for Question 72: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

72. de-Broglie equation gives the relation between particle and wave nature of the matter. In the given table, Column I shows the mass of the matter, Column II shows the speed of the matter and column III shows the kinetic energy of matter.

Column I	Column II	Column III
(I) Mass = 9.11×10^{-31} kg	**(i)** Speed = 255,000,000 m/s	**(J)** Kinetic energy = 5.84226×10^{-27} J
(II) Mass = 1.674×10^{-27} kg	**(ii)** Speed = 100.0 miles per hour	**(K)** Kinetic energy = 1.28433×10^{-17} J
(III) Mass = 5.00 ounce	**(iii)** Speed = 698 cm/s	**(L)** Kinetic energy = 5.43934×10^{-11} J
(IV) Mass = 1.673×10^{-27} kg	**(iv)** Speed = 5.31×10^{6} m/s	**(M)** Kinetic energy = 141.6146 J

(1) What are the conditions for an electron with a wavelength at 1.37 Å?

(a) (I) (iv) (M) **(b)** (I) (iv) (K)
(c) (II) (i) (J) **(d)** (II) (i) (L)

(2) What are the conditions for a proton with a wavelength at 1.55×10^{-15} m?

(a) (I) (ii) (J) **(b)** (III) (iv) (M)
(c) (IV) (i) (L) **(d)** (I) (i) (M)

(3) What are the conditions for hydrogen ion with a wavelength at 1.50×10^{-7} m?

(a) (II) (ii) (J) **(b)** (I) (iii) (J)
(c) (IV) (i) (L) **(d)** (IV) (iv) (J)

Integer Type

73. When a beam of 10.6 eV photons of intensity 2.0 W/m^2 falls on platinum surface area 1.0×10^{-4} m^2 and work function 5.6 eV, 0.53% of incident photons eject photoelectrons. Find the number of photoelectrons emitted per second and their minimum and maximum energies in eV.

74. A sample of monatomic hydrogen gas contains n atoms all in third excited state. Find the minimum value of n so that on subsequent de-excitation, all positive different energy photons are emitted.

75. A proton ($m_p = 1.673 \times 10^{-27}$ kg) and an electron ($m_e = 9.109 \times 10^{-31}$ kg) are confined such that the position x of each is known within 1.50×10^{-10} m. What is the ratio of the minimum uncertainty in the x component of the velocity of the electron to that of the proton, $\Delta v_e/\Delta v_p$?

76. The position of a hydrogen atom ($m = 1.7 \times 10^{-27}$ kg) is known within 2.0×10^{-6} m. What is the minimum uncertainty in the atom's velocity?

(a) zero m/s **(c)** 0.011 m/s
(b) 0.0085 m/s **(d)** 0.016 m/s
(e) 0.031 m/s

ANSWER KEY

Checkpoints

1. b, a, d, c
2. (a) lithium, sodium, potassium, cesium; (b) all tie
3. (a) proton; (b) same; (c) proton
4. same

Problems

1. (a) 2.96×10^{20} photon/s; (b) 4.86×10^{7} m; (c) $5.89 \times 10^{18} \dfrac{\text{photons}}{\text{m}^2 \cdot \text{s}}$
2. 7.75 pm.
3. 8.1 eV
4. (a) 0.0388 nm; (b) 1.24 nm; (c) 9.06×10^{-13} m
5. $3.87 \times 10^{21} \dfrac{\text{photons}}{\text{m}^2 \cdot \text{s}}$
6. 8.59×10^{5} m/s
7. 0.48 μA
8. 2.1×10^{-24} kg·m/s
9. 1.45×10^{-19} A
10. (a) 3.3×10^{-24} kg·m/s; (b) 3.3×10^{-24} kg·m/s; (c) 38 eV; (d) 6.2 keV
13. (a) –20%; (b) –10%; (c) +15%
14. (a) the lamp that emits light with the longer wavelength (the 700 nm infrared lamp) emits more photons per unit time; (b) 1.41×10^{21} photon/s.
15. 9.76×10^{3} V
16. 4.3×10^{-6} eV
17. 5.1 eV
18. 1.675×10^{-27} kg; this is the mass of a neutron
19. (a) 3.37×10^{111} s $\approx 10^{104}$ y; (b) 2.1×10^{-19} s
20. 8.41 pC
21. 233 nm
22. 6.76×10^{5} m/s
23. 2.3×10^{21} photons/m^2·s
24. 1.1×10^{-24} W

25. (a) 3.96×10^6 m/s; (b) 81.8 kV

26. (a) 1240 nm; (b) 1.23 nm; (c) 1.24 fm

27. 0.694eV

28. 6.0 photons/s

29. 1.0×10^{45} photons/s

30. (a) 8.57×10^{18} Hz; (b) 3.55×10^4 eV; (c) 1.89×10^{-23} kg·m/s

31. 3.3×10^{18} photons/s

32. (a) 300 eV; (b) 9.35×10^{-24} kg·m/s; (c) 1.03×10^7 m/s; (d) 7.08×10^{-11} m; (e) 8.87×10^{10} m^{-1}

33. (a) 1.3 V; (b) 6.8×10^5 m/s

34. 8.6×10^5 m/s

35. 4.7×10^{26}

36. (a) 382 nm; (b) 1.82 eV

37. 2.11 eV

38. barium and lithium

39. 170 nm

40. (a) 4.12×10^{-15} eV·s; (b) 2.27 eV; (c) 545 nm

41. (a) 1.91×10^{-21} kg·m/s; (b) 3.46×10^{-13} m

42. (a) 3.61 kW; (b) 1.00×10^{22} photons/s; (c) 60.2 s

43. (a) 2.00 eV; (b) zero kinetic energy; (c) 2.00 V; (d) 295 nm

44. 2.047 eV

45. (a) 3.1 keV; (b) 14 keV

46. (a) 1.24 keV; (b) 1.50 eV; (c) 1.24 GeV; (d) 1.24 GeV

47. (a) 4.1 eV, 0.4 eV; (b) 5.08×10^{14} Hz; (c) 5900 Å

48. (a) 6×10^8 N; (b) 3.6×10^{22}

49. (a) 2.5×10^{19}; (b) 3.33×10^{-8} N

50. 1×10^{-8} N

51. 5×10^{-8} N

52. (a) 3.97×10^9 W/m^2; (b) 13.2 Pa; (c) 1.67×10^{-11} N; (d) 3.14×10^3 m/s^2

Practice Questions

Single Correct Choice Type

1. (d)
2. (b)
3. (b)
4. (a)
5. (c)
6. (b)
7. (c)
8. (d)
9. (b)
10. (a)
11. (d)
12. (b)
13. (c)
14. (d)
15. (a)
16. (b)
17. (a)
18. (c)
19. (c)
20. (b)
21. (b)
22. (c)
23. (b)
24. (d)
25. (a)
26. (b)
27. (d)
28. (d)
29. (b)
30. (c)
31. (c)
32. (a)
33. (d)
34. (d)
35. (c)
36. (b)
37. (a)
38. (b)
39. (a)
40. (c)
41. (a)
42. (d)
43. (d)
44. (c)
45. (d)
46. (b)
47. (b)

More than One Correct Choice Type

48. (a), (c), (d)
49. (a), (b), (c)
50. (b), (c)
51. (a), (c), (d)
52. (c), (d)
53. (b), (c), (d)

Linked Comprehension

54. (b)
55. (d)
56. (b)
57. (b)
58. (a)
59. (d)
60. (c)
61. (c)
62. (a)
63. (b)
64. (a)
65. (b)
66. (a)
67. (a)
68. (b)
69. (b)

Matrix-Match

70. (a) → (s); (b) → (r); (c) → (p); (d) → (q)

71. (a) → (r); (b) → (s); (c) → (q); (d) → (p)

72. (1) → (b); (2) → (c); (3) → (d)

Integer Type

73. 5
74. 4
75. 1837
76. 0.031

CHAPTER 38

Hydrogen Atom

Contents

38.1 What is Physics?
38.2 String Waves and Matter Waves
38.3 Energies of a Trapped Electron
38.4 Wave Functions of a Trapped Electron
38.5 The Bohr Model of the Hydrogen Atom
38.6 Quantum Numbers for the Hydrogen Atom (Optional)

38.1 | WHAT IS PHYSICS?

One of the long-standing goals of physics has been to understand the nature of atoms. Early in the 20th century nobody knew how the electrons in an atom are arranged, what their motions are, how atoms emit or absorb light, or even why atoms are stable. Without this knowledge it was not possible to understand how atoms combine to form molecules or stack up to form solids. As a consequence, the foundations of chemistry—including biochemistry, which underlies the nature of life itself—were more or less a mystery.

In 1926, all these questions and many others were answered with the development of quantum physics. Its basic premise is that moving electrons, protons, and particles of any kind are best viewed as matter waves, whose motions are governed by Schrödinger's equation. Although quantum theory also applies to larger objects, such as baseballs and planets, it yields the same results as Newtonian physics, which is easier to use and more intuitive.

Before we can apply quantum physics to the problem of atomic structure, we need to develop some insights by applying quantum ideas in a few simpler situations. Some of these situations may seem simplistic and unreal, but they allow us to discuss the basic principles of the quantum physics of atoms without having to deal with the often overwhelming complexity of atoms. Besides, with advances in nanotechnology, situations that were previously found only in textbooks are now being produced in laboratories and put to use in modern electronics and materials science applications. We are on the threshold of being able to use nanometer-scale constructions called *quantum corrals* and *quantum dots* to create "designer atoms" whose properties can be manipulated in the laboratory. For both natural atoms and these artificial ones, the starting point in our discussion is the wave nature of an electron.

38.2 | STRING WAVES AND MATTER WAVES

Key Concept

- Confinement of waves (string waves, matter waves—any type of wave) leads to quantization—that is, discrete states with certain energies. States with intermediate energies are not allowed.

In Chapter 16 we saw that waves of two kinds can be set up on a stretched string. If the string is so long that we can take it to be infinitely long, we can set up a *traveling wave* of essentially any frequency. However, if the stretched string has only a finite length, perhaps because it is rigidly clamped at both ends, we can set up only *standing waves* on it; further, these standing waves can have only discrete frequencies. In other words, confining the wave to a finite region of space leads to *quantization* of the motion—to the existence of discrete *states* for the wave, each state with a sharply defined frequency.

This observation applies to waves of all kinds, including matter waves. For matter waves, however, it is more convenient to deal with the energy E of the associated particle than with the frequency f of the wave. In all that follows we shall focus on the matter wave associated with an electron, but the results apply to any confined matter wave.

Consider the matter wave associated with an electron moving in the positive x direction and subject to no net force—a so-called *free particle*. The energy of such an electron can have any reasonable value, just as a wave traveling along a stretched string of infinite length can have any reasonable frequency.

Consider next the matter wave associated with an atomic electron, perhaps the *valence* (least tightly bound) electron. The electron—held within the atom by the attractive Coulomb force between it and the positively charged nucleus—is not a free particle. It can exist only in a set of discrete states, each having a discrete energy E. This sounds much like the discrete states and quantized frequencies that are available to a stretched string of finite length. For matter waves, then, as for all other kinds of waves, we may state a **confinement principle**:

> Confinement of a wave leads to quantization—that is, to the existence of discrete states with discrete energies.

38.3 | ENERGIES OF A TRAPPED ELECTRON

Key Concepts

- Because it is a matter wave, an electron confined to an infinite potential well can exist in only certain discrete states. If the well is one-dimensional with length L, the energies associated with these quantum states are

$$E_n = \left(\frac{h^2}{8mL^2}\right)n^2, \quad \text{for } n = 1, 2, 3, \ldots,$$

where m is the electron mass and n is a quantum number.

- The lowest energy is not zero but is given by $n = 1$.
- The electron can change (jump) from one quantum state to another only if its energy change is

$$\Delta E = E_{\text{high}} - E_{\text{low}},$$

where E_{high} is the higher energy and E_{low} is the lower energy.

- If the change is done by photon absorption or emission, the energy of the photon must be equal to the change in the electron's energy:

$$hf = \frac{hc}{\lambda} = \Delta E = E_{\text{high}} - E_{\text{low}},$$

where frequency f and wavelength λ are associated with the photon.

One-Dimensional Traps

Here we examine the matter wave associated with a nonrelativistic electron confined to a limited region of space. We do so by analogy with standing waves on a string of finite length, stretched along an x axis and confined between rigid supports. Because the supports are rigid, the two ends of the string are nodes, or points at which the string is always at rest. There may be other nodes along the string, but these two must always be present, as Fig. 16-33 shows.

The states, or discrete standing wave patterns in which the string can oscillate, are those for which the length L of the string is equal to an integer number of half-wavelengths. That is, the string can occupy only states for which

$$L = \frac{n\lambda}{2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{38-1}$$

Each value of n identifies a state of the oscillating string; using the language of quantum physics, we can call the integer n a **quantum number**.

For each state of the string permitted by Eq. 38-1, the transverse displacement of the string at any position x along the string is given by

$$y_n(x) = A \sin\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots \tag{38-2}$$

in which the quantum number n identifies the oscillation pattern and A depends on the time at which you inspect the string. (Equation 38-2 is a short version of Eq. 16-66.) We see that for all values of n and for all times, there is a point of zero displacement (a node) at $x = 0$ and at $x = L$, as there must be. Figure 16-32 shows time exposures of such a stretched string for $n = 2, 3$, and 4.

Now let us turn our attention to matter waves. Our first problem is to physically confine an electron that is moving along the x axis so that it remains within a finite segment of that axis. Figure 38-1 shows a conceivable one-dimensional *electron trap.* It consists of two semi-infinitely long cylinders, each of which has an electric potential approaching $-\infty$; between them is a hollow cylinder of length L, which has an electric potential of zero. We put a single electron into this central cylinder to trap it.

The trap of Fig. 38-1 is easy to analyze but is not very practical. Single electrons *can*, however, be trapped in the laboratory with traps that are more complex in design but similar in concept. At the University of Washington, for example, a single electron has been held in a trap for months on end, permitting scientists to make extremely precise measurements of its properties.

An electron can be trapped in the $V = 0$ region.

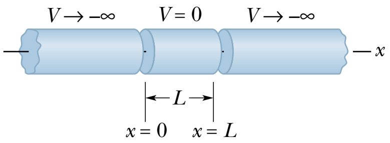


Figure 38-1 The elements of an idealized "trap" designed to confine an electron to the central cylinder. We take the semi-infinitely long end cylinders to be at an infinitely great negative potential and the central cylinder to be at zero potential.

Finding the Quantized Energies

Figure 38-2 shows the potential energy of the electron as a function of its position along the x axis of the idealized trap of Fig. 38-1. When the electron is in the central cylinder, its potential energy U $(= -eV)$ is zero because there the potential V is zero. If the electron could get outside this region, its potential energy would be positive and of infinite magnitude because there $V \to -\infty$. We call the potential energy pattern of Fig. 38-2 an **infinitely deep potential energy well** or, for short, an *infinite potential well.* It is a "well" because an electron placed in the central cylinder of Fig. 38-1 cannot escape from it. As the electron approaches either end of the cylinder, a force of essentially infinite magnitude reverses the electron's motion, thus trapping it. Because the electron can move along only a single axis, this trap can be called a *one-dimensional infinite potential well.*

Just like the standing wave in a length of stretched string, the matter wave describing the confined electron must have nodes at $x = 0$ and $x = L$. Moreover, Eq. 38-1 applies to such a matter wave if we interpret λ in that equation as the de Broglie wavelength associated with the moving electron.

An electron can be trapped in the $U = 0$ region.

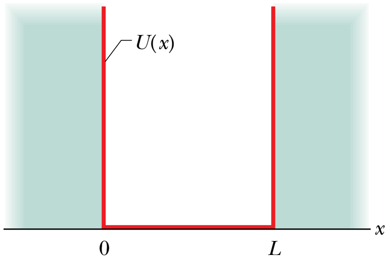


Figure 38-2 The electric potential energy $U(x)$ of an electron confined to the central cylinder of the idealized trap of Fig. 38-1. We see that $U = 0$ for $0 < x < L$, and $U \to \infty$ for $x < 0$ and $x > L$.

The de Broglie wavelength λ is defined in Eq. 37-9 as $\lambda = h/p$, where p is the magnitude of the electron's momentum. Because the electron is nonrelativistic, this momentum magnitude p is related to the kinetic energy K by $p = \sqrt{2mK}$, where m is the mass of the electron. For an electron moving within the central cylinder of Fig. 38-1, where $U = 0$, the total (mechanical) energy E is equal to the kinetic energy. Hence, we can write the de Broglie wavelength of this electron as

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}. \tag{38-3}$$

If we substitute Eq. 38-3 into Eq. 38-1 and solve for the energy E, we find that E depends on n according to

$$E_n = \left(\frac{h^2}{8mL^2}\right)n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{38-4}$$

The positive integer n here is the quantum number of the electron's quantum state in the trap.

Equation 38-4 tells us something important: Because the electron is confined to the trap, it can have only the energies given by the equation. It *cannot* have an energy that is, say, halfway between the values for $n = 1$ and $n = 2$. Why this restriction? Because an electron is a matter wave. Were it, instead, a particle as assumed in classical physics, it could have *any* value of energy while it is confined to the trap.

Figure 38-3 is a graph showing the lowest five allowed energy values for an electron in an infinite well with $L = 100$ pm (about the size of a typical atom). The values are called *energy levels,* and they are drawn in Fig. 38-3 as levels, or steps, on a ladder, in an *energy-level diagram.* Energy is plotted vertically; nothing is plotted horizontally.

The quantum state with the lowest possible energy level E_1 allowed by Eq. 38-4, with quantum number $n = 1$, is called the *ground state* of the electron. The electron tends to be in this lowest energy state. All the quantum states with greater energies (corresponding to quantum numbers $n = 2$ or greater) are called *excited states* of the electron. The state with energy level E_2, for quantum number $n = 2$, is called the *first excited state* because it is the first of the excited states as we move up the energy-level diagram. The other states have similar names.

These are the lowest five energy levels allowed the electron. (No intermediate levels are allowed.)

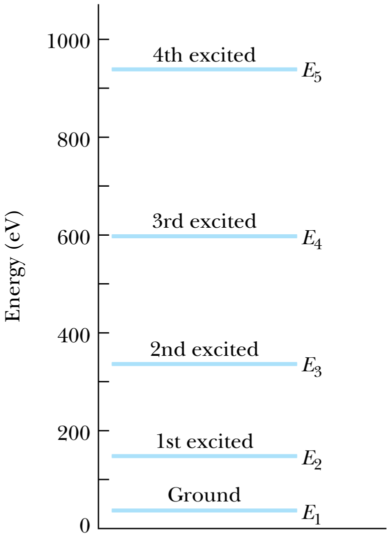


Figure 38-3 Several of the allowed energies for an electron confined to the infinite well of Fig. 38-2, with width $L = 100$ pm.

Energy Changes

A trapped electron tends to have the lowest allowed energy and thus to be in its ground state. It can be changed to an excited state (in which it has greater energy) only if an external source provides the additional energy that is required for the change. Let E_{low} be the initial energy of the electron and E_{high} be the greater energy in a state that is higher on its energy-level diagram. Then the amount of energy that is required for the electron's change of state is

$$\Delta E = E_{\text{high}} - E_{\text{low}}. \tag{38-5}$$

An electron that receives such energy is said to make a *quantum jump* (or *transition*), or to be *excited* from the lower-energy state to the higher-energy state. Figure 38-4*a* represents a quantum jump from the ground state (with energy level E_1) to the third excited state (with energy level E_4). As shown, the jump *must* be from one energy level to another, but it can bypass one or more intermediate energy levels.

Photons. One way an electron can gain energy to make a quantum jump up to a greater energy level is to absorb a photon. However, this absorption and quantum jump can occur only if the following condition is met:

> If a confined electron is to absorb a photon, the energy hf of the photon must equal the energy difference ΔE between the initial energy level of the electron and a higher level.

Thus, excitation by the absorption of light requires that

$$hf = \frac{hc}{\lambda} = \Delta E = E_{\text{high}} - E_{\text{low}}. \tag{38-6}$$

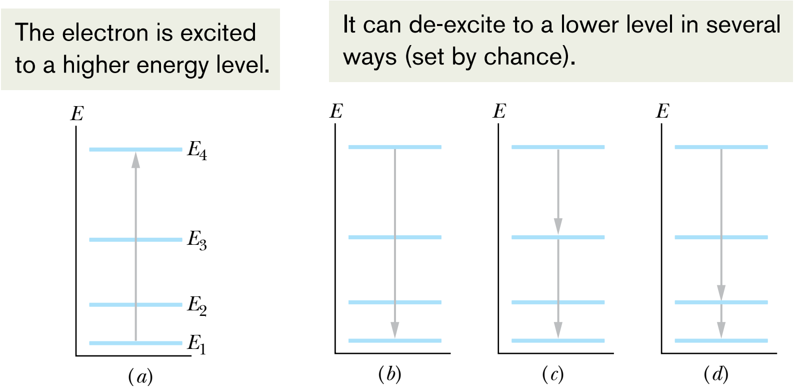

The electron is excited to a higher energy level.
It can de-excite to a lower level in several ways (set by chance).
E
E_4
E_3
E_2
E_1
(a)
(b)
(c)
(d)

Figure 38-4 (*a*) Excitation of a trapped electron from the energy level of its ground state to the level of its third excited state. (*b*)–(*d*) Three of four possible ways the electron can de-excite to return to the energy level of its ground state. (Which way is not shown?)

When an electron reaches an excited state, it does not stay there but quickly *de-excites* by decreasing its energy. Figures 38-4*b* to *d* represent some of the possible quantum jumps down from the energy level of the third excited state. The electron can reach its ground-state level either with one direct quantum jump (Fig. 38-4*b*) or with shorter jumps via intermediate levels (Figs. 38-4*c* and *d*).

An electron can decrease its energy by emitting a photon but only this way:

If a confined electron emits a photon, the energy hf of that photon must equal the energy difference ΔE between the initial energy level of the electron and a lower level.

Thus, Eq. 38-6 applies to both the absorption and the emission of light by a confined electron. That is, the absorbed or emitted light can have only certain values of hf and thus only certain values of frequency f and wavelength λ.

CHECKPOINT 1

Rank the following pairs of quantum states for an electron confined to an infinite well according to the energy differences between the states, greatest first: (a) $n = 3$ and $n = 1$, (b) $n = 5$ and $n = 4$, (c) $n = 4$ and $n = 3$.

SAMPLE PROBLEM 38.01

Energy levels in a 1D infinite potential well

An electron is confined to a one-dimensional, infinitely deep potential energy well of width $L = 100$ pm. (a) What is the smallest amount of energy the electron can have? (A trapped electron cannot have zero energy.)

KEY IDEA

Confinement of the electron (a matter wave) to the well leads to quantization of its energy. Because the well is infinitely deep, the allowed energies are given by Eq. 38-4 ($E_n = (h^2/8mL^2)n^2$), with the quantum number n a positive integer.

Lowest energy level: Here, the collection of constants in front of n^2 in Eq. 38-4 is evaluated as

$$\frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34}\,\mathrm{J\cdot s})^2}{(8)(9.11\times10^{-31}\,\mathrm{kg})(100\times10^{-12}\,\mathrm{m})^2}$$
$$= 6.031\times10^{-18}\,\mathrm{J}. \qquad (38\text{-}7)$$

The smallest amount of energy the electron can have corresponds to the lowest quantum number, which is $n = 1$ for the ground state of the electron. Thus, Eqs. 38-4 and 38-7 give us

$$E_1 = \left(\frac{h^2}{8mL^2}\right)n^2 = (6.031\times10^{-18}\text{ J})(1^2)$$
$$\approx 6.03\times10^{-18}\text{ J} = 37.7\text{ eV}. \qquad \text{(Answer)}$$

(b) How much energy must be transferred to the electron if it is to make a quantum jump from its ground state to its second excited state?

KEY IDEA

First a caution: Note that, from Fig. 38-3, the *second* excited state corresponds to the *third* energy level, with quantum number $n = 3$. Then if the electron is to jump from the $n = 1$ level to the $n = 3$ level, the required change in its energy is, from Eq. 38-5,

$$\Delta E_{31} = E_3 - E_1. \qquad (38\text{-}8)$$

Upward jump: The energies E_3 and E_1 depend on the quantum number n, according to Eq. 38-4. Therefore, substituting that equation into Eq. 38-8 for energies E_3 and E_1 and using Eq. 38-7 lead to

$$\begin{aligned}\Delta E_{31} &= \left(\frac{h^2}{8mL^2}\right)(3)^2 - \left(\frac{h^2}{8mL^2}\right)(1)^2\\ &= \frac{h^2}{8mL^2}(3^2 - 1^2)\\ &= (6.031\times10^{-18}\text{ J})(8)\\ &= 4.83\times10^{-17}\text{ J} = 301\text{ eV}. \qquad \text{(Answer)}\end{aligned}$$

(c) If the electron gains the energy for the jump from energy level E_1 to energy level E_3 by absorbing light, what light wavelength is required?

KEY IDEAS

(1) If light is to transfer energy to the electron, the transfer must be by photon absorption. (2) The photon's energy must equal the energy difference ΔE between the initial energy level of the electron and a higher level, according to Eq. 38-6 ($hf = \Delta E$). Otherwise, a photon *cannot* be absorbed.

Wavelength: Substituting c/λ for f, we can rewrite Eq. 38-6 as

$$\lambda = \frac{hc}{\Delta E}. \qquad (38\text{-}9)$$

For the energy difference ΔE_{31} we found in (b), this equation gives us

$$\begin{aligned}\lambda &= \frac{hc}{\Delta E_{31}}\\ &= \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(2.998\times10^8\text{ m/s})}{4.83\times10^{-17}\text{ J}}\\ &= 4.12\times10^{-9}\text{ m}. \qquad \text{(Answer)}\end{aligned}$$

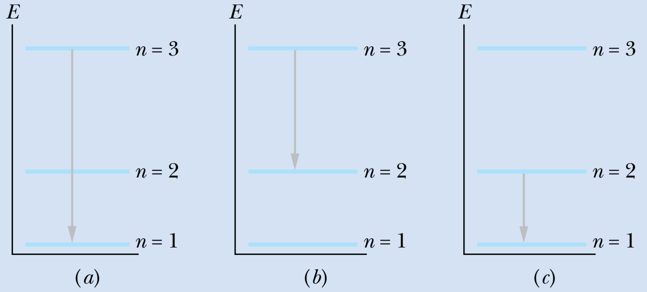
(d) Once the electron has been excited to the second excited state, what wavelengths of light can it emit by de-excitation?

KEY IDEAS

(1) The electron tends to de-excite, rather than remain in an excited state, until it reaches the ground state ($n = 1$).
(2) If the electron is to de-excite, it must lose just enough energy to jump to a lower energy level.
(3) If it is to lose energy by emitting light, then the loss of energy must be by emission of a photon.

Downward jumps: Starting in the second excited state (at the $n = 3$ level), the electron can reach the ground state ($n = 1$) by *either* making a quantum jump directly to the ground-state energy level (Fig. 38-5*a*) or by making two *separate* jumps by way of the $n = 2$ level (Figs. 38-5*b* and *c*).

The direct jump involves the same energy difference ΔE_{31} we found in (c). Then the wavelength is the same as we calculated in (c)—except now the wavelength is for light that is emitted, not absorbed. Thus, the electron



(*a*) (*b*) (*c*)

Figure 38-5 De-excitation from the second excited state to the ground state either directly (*a*) or via the first excited state (*b*, *c*).

can jump directly to the ground state by emitting light of wavelength

$$\lambda = 4.12 \times 10^{-9}\ \text{m}. \qquad \text{(Answer)}$$

Following the procedure of part (b), you can show that the energy differences for the jumps of Figs. 38-5*b* and *c* are

$$\Delta E_{32} = 3.016 \times 10^{-17}\ \text{J} \quad \text{and} \quad \Delta E_{21} = 1.809 \times 10^{-17}\ \text{J}.$$

From Eq. 38-9, we then find that the wavelength of the light emitted in the first of these jumps (from $n = 3$ to $n = 2$) is

$$\lambda = 6.60 \times 10^{-9}\ \text{m}, \qquad \text{(Answer)}$$

and the wavelength of the light emitted in the second of these jumps (from $n = 2$ to $n = 1$) is

$$\lambda = 1.10 \times 10^{-8}\ \text{m}. \qquad \text{(Answer)}$$

38.4 | WAVE FUNCTIONS OF A TRAPPED ELECTRON

Key Concepts

- The wave functions for an electron in an infinite, one-dimensional potential well with length L along an x axis are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots,$$

  where n is the quantum number.
- The product $\psi_n^2(x)\,dx$ is the probability that the electron will be detected in the interval between coordinates and $x + dx$.
- If the probability density of an electron is integrated over the entire x axis, the total probability must be 1:

$$\int_{-\infty}^{\infty} \psi_n^2(x)\,dx = 1.$$

If we solve Schrödinger's equation for an electron trapped in a one-dimensional infinite potential well of width L and impose the boundary condition that the solutions be zero at the infinite walls, we find that the wave functions for the electron are given by

$$\psi_n(x) = A \sin\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots, \qquad (38\text{-}10)$$

for $0 \le x \le L$ (the wave function is zero outside that range). We shall soon evaluate the amplitude constant A in this equation.

Note that the wave functions $\psi_n(x)$ have the same form as the displacement functions $y_n(x)$ for a standing wave on a string stretched between rigid supports (see Eq. 38-2). We can picture an electron trapped in a one-dimensional well between infinite-potential walls as being a standing matter wave.

Probability of Detection

The wave function $\psi_n(x)$ cannot be detected or directly measured in any way—we cannot simply look inside the well to see the wave the way we can see, say, a wave in a bathtub of water. All we can do is insert a probe of some kind to try to detect the electron. At the instant of detection, the electron would materialize at the point of detection, at some position along the x axis within the well.

If we repeated this detection procedure at many positions throughout the well, we would find that the probability of detecting the electron is related to the probe's position x in the well. In fact, they are related by the *probability density* $\psi_n^2(x)$. Recall from Section 37.7 that in general the probability that a particle can be detected in a specified infinitesimal volume centered on a specified point is proportional to $|\psi_n^2|$. Here, with the electron trapped in a one-dimensional well, we are concerned only with detection of the electron along the x axis. Thus, the probability density $\psi_n^2(x)$ here is a probability per unit length along the x axis. (We can omit the absolute value sign here because $\psi_n(x)$ in Eq. 38-10 is a real quantity, not a complex one.) The probability $p(x)$ that an electron can be detected at position x within the well is

$$\begin{pmatrix}\text{Probability } p(x)\\ \text{of detection in width } dx\\ \text{centered on position } x\end{pmatrix} = \begin{pmatrix}\text{Probability density } \psi_n^2(x)\\ \text{at position } x\end{pmatrix}(\text{Width } dx),$$

or

$$p(x) = \psi_n^2(x)\, dx. \tag{38-11}$$

From Eq. 38-10, we see that the probability density $\psi_n^2(x)$ is

$$\psi_n^2(x) = A^2 \sin^2\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{38-12}$$

for the range $0 \le x \le L$ (the probability density is zero outside that range). Figure 38-6 shows $\psi_n^2(x)$ for n = 1, 2, 3, and 15 for an electron in an infinite well whose width L is 100 pm.

The probability density must be zero at the infinite walls.

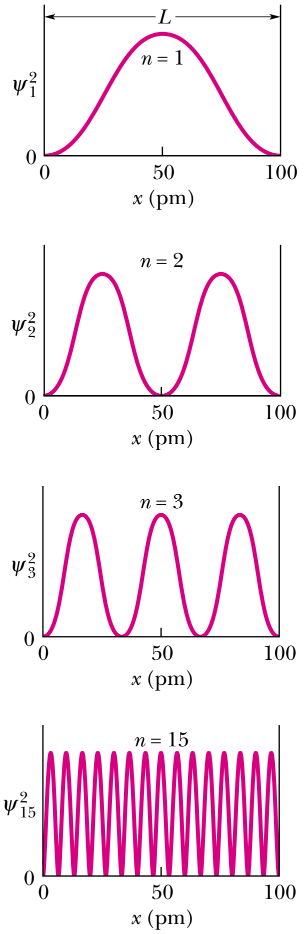


Figure 38-6 The probability density $\psi_n^2(x)$ for four states of an electron trapped in a one-dimensional infinite well; their quantum numbers are n = 1, 2, 3, and 15. The electron is most likely to be found where $\psi_n^2(x)$ is greatest and least likely to be found where $\psi_n^2(x)$ is least.

To find the probability that the electron can be detected in any finite section of the well—say, between point x_1 and point x_2—we must integrate $p(x)$ between those points. Thus, from Eqs. 38-11 and 38-12,

$$\begin{aligned}\begin{pmatrix}\text{Probability of detection}\\ \text{between } x_1 \text{ and } x_2\end{pmatrix} &= \int_{x_1}^{x_2} p(x)\\ &= \int_{x_1}^{x_2} A^2 \sin^2\left(\frac{n\pi}{L}x\right) dx.\end{aligned} \tag{38-13}$$

If the range Δx in which we search for the electron is much smaller than the well length L, then we can usually approximate the integral in Eq. 38-13 as being equal to the product $p(x)\,\Delta x$, with $p(x)$ evaluated in the center of Δx.

If classical physics prevailed, we would expect the trapped electron to be detectable with equal probabilities in all parts of the well. From Fig. 38-6 we see that it is not. For example, inspection of that figure or of Eq. 38-12 shows that for the state with n = 2, the electron is most likely to be detected near x = 25 pm and x = 75 pm. It can be detected with near-zero probability near x = 0, x = 50 pm, and x = 100 pm.

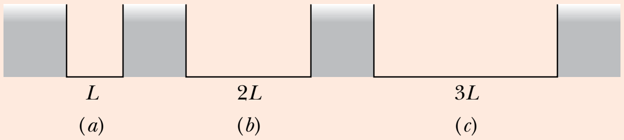
The case of n = 15 in Fig. 38-6 suggests that as n increases, the probability of detection becomes more and more uniform across the well. This result is an instance of a general principle called the **correspondence principle**:

At large enough quantum numbers, the predictions of quantum physics merge smoothly with those of classical physics.

This principle, first advanced by Danish physicist Niels Bohr, holds for all quantum predictions.

CHECKPOINT 2

The figure shows three infinite potential wells of widths L, $2L$, and $3L$; each contains an electron in the state for which $n = 10$. Rank the wells according to (a) the number of maxima for the probability density of the electron and (b) the energy of the electron, greatest first.



Normalization

The product $\psi_n^2(x)\,dx$ gives the probability that an electron in an infinite well can be detected in the interval of the x axis that lies between x and $x + dx$. We know that the electron must be *somewhere* in the infinite well; so it must be true that

$$\int_{-\infty}^{+\infty} \psi_n^2(x)dx = 1 \quad \text{(normalization equation)}, \qquad (38\text{-}14)$$

because the probability 1 corresponds to certainty. Although the integral is taken over the entire x axis, only the region from $x = 0$ to $x = L$ makes any contribution to the probability. Graphically, the integral in Eq. 38-14 represents the area under each of the plots of Fig. 38-6. If we substitute $\psi_n^2(x)$ from Eq. 38-12 into Eq. 38-14, we find that $A = \sqrt{2/L}$. This process of using Eq. 38-14 to evaluate the amplitude of a wave function is called **normalizing** the wave function. The process applies to *all* one-dimensional wave functions.

Zero-Point Energy

Substituting $n = 1$ in Eq. 38-4 defines the state of lowest energy for an electron in an infinite potential well, the ground state. That is the state the confined electron will occupy unless energy is supplied to it to raise it to an excited state.

The question arises: Why can not we include $n = 0$ among the possibilities listed for n in Eq. 38-4? Putting $n = 0$ in this equation would indeed yield a ground-state energy of zero. However, putting $n = 0$ in Eq. 38-12 would also yield $\psi_n^2(x) = 0$ for all x, which we can interpret only to mean that there is no electron in the well. We know that there is; so $n = 0$ is not a possible quantum number.

It is an important conclusion of quantum physics that confined systems cannot exist in states with zero energy. They must always have a certain minimum energy called the **zero-point energy**.

We can make the zero-point energy as small as we like by making the infinite well wider—that is, by increasing L in Eq. 38-4 for $n = 1$. In the limit as $L \to \infty$, the zero-point energy $E_1 \to 0$. However, the electron is then a free particle, no longer confined in the x direction. Also, because the energy of a free particle is not quantized, that energy can have any value, including zero. Only a confined particle must have a finite zero-point energy and can never be at rest.

CHECKPOINT 3

Each of the following particles is confined to an infinite well, and all four wells have the same width: (a) an electron, (b) a proton, (c) a deuteron, and (d) an alpha particle. Rank their zero-point energies, greatest first. The particles are listed in order of increasing mass.

SAMPLE PROBLEM 38.02

Detection probability in a 1D infinite potential well

A ground-state electron is trapped in the one-dimensional infinite potential well of Fig. 38-2, with width $L = 100$ pm.

(a) What is the probability that the electron can be detected in the left one-third of the well ($x_1 = 0$ to $x_2 = L/3$)?

KEY IDEAS

(1) If we probe the left one-third of the well, there is no guarantee that we will detect the electron. However, we can calculate the probability of detecting it with the integral of Eq. 38-13.
(2) The probability very much depends on which state the electron is in—that is, the value of quantum number n.

Calculations: Because here the electron is in the ground state, we set $n = 1$ in Eq. 38-13. We also set the limits of integration as the positions $x_1 = 0$ and $x_2 = L/3$ and set the amplitude constant A as $\sqrt{2/L}$ (so that the wave function is normalized). We then see that

$$\begin{pmatrix}\text{Probability of detection}\\ \text{in left one-third}\end{pmatrix} = \int_0^{L/3} \frac{2}{L}\sin^2\left(\frac{1\pi}{L}x\right)dx.$$

We could find this probability by substituting 100×10^{-12} m for L and then using a graphing calculator or a computer math package to evaluate the integral. Here, however, we shall evaluate the integral "by hand." First we switch to a new integration variable y:

$$y = \frac{\pi}{L}x \quad \text{and} \quad dx = \frac{L}{\pi}dy.$$

From the first of these equations, we find the new limits of integration to be $y_1 = 0$ for $x_1 = 0$ and $y_2 = \pi/3$ for $x_2 = L/3$. We then must evaluate

$$\text{Probability} = \left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_0^{\pi/3}(\sin^2 y)\,dy.$$

Using integral 11 in Appendix E, we then find

$$\text{Probability} = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)_0^{\pi/3} = 0.20.$$

Thus, we have

$$\begin{pmatrix}\text{Probability of detection}\\ \text{in left one-third}\end{pmatrix} = 0.20. \qquad \text{(Answer)}$$

That is, if we repeatedly probe the left one-third of the well, then on average we can detect the electron with 20% of the probes.

(b) What is the probability that the electron can be detected in the middle one-third of the well?

Reasoning: We now know that the probability of detection in the left one-third of the well is 0.20. By symmetry, the probability of detection in the right one-third of the well is also 0.20. Because the electron is certainly in the well, the probability of detection in the entire well is 1. Thus, the probability of detection in the middle one-third of the well is

$$\begin{aligned}\begin{pmatrix}\text{Probability of detection}\\ \text{in middle one-third}\end{pmatrix} &= 1 - 0.20 - 0.20\\ &= 0.60. \qquad \text{(Answer)}\end{aligned}$$

SAMPLE PROBLEM 38.03

Normalizing wave functions in a 1D infinite potential well

Evaluate the amplitude constant A in Eq. 38-10 for an infinite potential well extending from $x = 0$ to $x = L$.

KEY IDEA

The wave functions of Eq. 38-10 must satisfy the normalization requirement of Eq. 38-14, which states that the probability that the electron can be detected somewhere along the x axis is 1.

Calculations: Substituting Eq. 38-10 into Eq. 38-14 and taking the constant A outside the integral yield

$$A^2\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx = 1. \qquad (38\text{-}15)$$

We have changed the limits of the integral from $-\infty$ and $+\infty$ to 0 and L because the "outside" wave function is zero.

We can simplify the indicated integration by changing the variable from x to the dimensionless variable y, where

$$y = \frac{n\pi}{L}x, \tag{38-16}$$

hence

$$dx = \frac{L}{n\pi}dy.$$

When we change the variable, we must also change the integration limits (again). Equation 38-16 tells us that $y = 0$ when $x = 0$ and that $y = n\pi$ when $x = L$; thus 0 and $n\pi$ are our new limits. With all these substitutions, Eq. 38-15 becomes

$$A^2 \frac{L}{n\pi}\int_0^{n\pi} (\sin^2 y)\, dy = 1.$$

We can use integral 11 in Appendix E to evaluate the integral, obtaining the equation

$$\frac{A^2 L}{n\pi}\left[\frac{y}{2} - \frac{\sin 2y}{4}\right]_0^{n\pi} = 1.$$

Evaluating at the limits yields

$$\frac{A^2 L}{n\pi}\frac{n\pi}{2} = 1;$$

thus

$$A = \sqrt{\frac{2}{L}}. \quad \text{(Answer)} \tag{38-17}$$

This result tells us that the dimension for A^2, and thus for $\psi_n^2(x)$, is an inverse length. This is appropriate because the probability density of Eq. 38-12 is a probability *per unit length*.

38.5 | THE BOHR MODEL OF THE HYDROGEN ATOM

Key Concepts

- The Bohr model of the hydrogen atom successfully derived the energy levels for the atom, to explain the emission/absorption spectrum of the atom, but it is incorrect in almost every other aspect.
- The Bohr model is a planetary model in which the electron orbits the central proton with an angular momentum L that is limited to values given by

  $$L = n\hbar, \quad \text{for } n = 1, 2, 3, \ldots,$$

  where n is a quantum number. The value $L = 0$ is incorrectly disallowed.
- Application of the Schrödinger equation gives the correct values of L and the quantized energies:

  $$E_n = -\frac{me^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} = -\frac{13.60\text{ eV}}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots.$$
- The atom (or the electron in the atom) can change energy only by jumping between these allowed energies.
- If the jump is by photon absorption (the atom's energy increases) or photon emission (the atom's energy decreases), this restriction in energy changes leads to

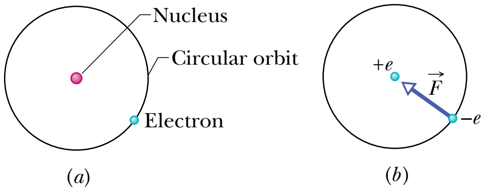
  $$\frac{1}{\lambda} = R\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right)$$

  for the wavelength of the light, where R is the Rydberg constant,

  $$R = \frac{me^4}{8\varepsilon_0^2 h^3 c} = 1.097\,373 \times 10^7\text{ m}^{-1}.$$

We have now discussed at length that confinement of an electron means that the electron's energy E is quantized and thus so is any change ΔE in its energy. In this module we want to calculate the quantized energies of the electron confined to a hydrogen atom. We shall, in principle at least, apply Schrödinger's equation to the trap, to find those energies and the associated wave functions. However, at the discretion of your instructor, let's take an historical aside to examine how the quantizing of atoms began, back when quantization was a revolutionary concept.

By the early 1900s, scientists understood that matter came in tiny pieces called atoms and that an atom of hydrogen contained positive charge $+e$ at its center and negative charge $-e$ (an electron) outside that center. However, no one understood why the electrical attraction between the electron and the positive charge did not simply cause the two to collapse together.



Bohr's model for hydrogen resembles the orbital model of a planet around a star.

Figure 38-7 (*a*) Circular orbit of an electron in the Bohr model of the hydrogen atom. (*b*) The Coulomb force $\vec{F}$ on the electron is directed radially inward toward the nucleus.

Visible Wavelengths. One clue lay in the experimental fact that a hydrogen atom can emit and absorb only four wavelengths in the visible spectrum (656 nm, 486 nm, 434 nm, and 410 nm). Why did it not emit all wavelengths as, say, a hot blackbody radiator? In 1913, Niels Bohr had a remarkable idea that simultaneously explained not only the four visible wavelengths but also why the atom did not simply collapse. Based on this, Johann Balmer devised a formula that gave these wavelengths

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \quad \text{for } n = 3, 4, 5, \text{ and } 6 \tag{38-18}$$

where R is a constant. However, he was not able to explain why the visible emission and absorption by hydrogen atom was limited to the wavelengths given by this formula.

Assumptions. To build his model, Bohr made two bold (completely unjustified) assumptions: (1) The electron in a hydrogen atom orbits the nucleus in a circle much like Earth orbits the Sun (Fig. 38-7*a*). (2) The magnitude of the angular momentum $\vec{L}$ of the electron in its orbit is restricted (quantized) to the values

$$L = n\hbar, \quad \text{for } n = 1, 2, 3, \ldots, \tag{38-19}$$

where $\hbar$ (h-bar) is $h/2\pi$ and n is a positive integer (a quantum number). We are going to follow Bohr's relatively simple arguments to get an equation for the quantized energies of the hydrogen atom, but let's be explicit here: The electron is *not* simply a particle in a planetary orbit and Eq. 38-19 does *not* correctly give the angular momentum values. (For example, $L = 0$ is missing.)

In the orbit picture of Fig. 38-7*a*, the electron is in uniform circular motion and thus experiences a centripetal force (Fig. 38-7*b*), which causes a centripetal acceleration.

Here, Bohr was just fitting in the empirical data, just like his predecessor Balmer. He did not know why the angular momentum was quantized at all. In fact, is this not what most of us do all the time? When we are unable to obtain an answer logically, we try to somehow manipulate the calculations to get the answer. In most of the cases, we are unable to justify this at all. Neither was Bohr able to!

However, in 1924, 11 years after Bohr presented his theory, de Broglie gave an interpretation of the Bohr's rule for quantization of angular momentum. If we represent the circulating electron in terms of its de Broglie wave, then the stable states are those in which the electron's de Broglie wave joins onto itself with the same phase in each revolution; otherwise, the wave would destroy itself by destructive interference. This is presented in Fig. 38-8.

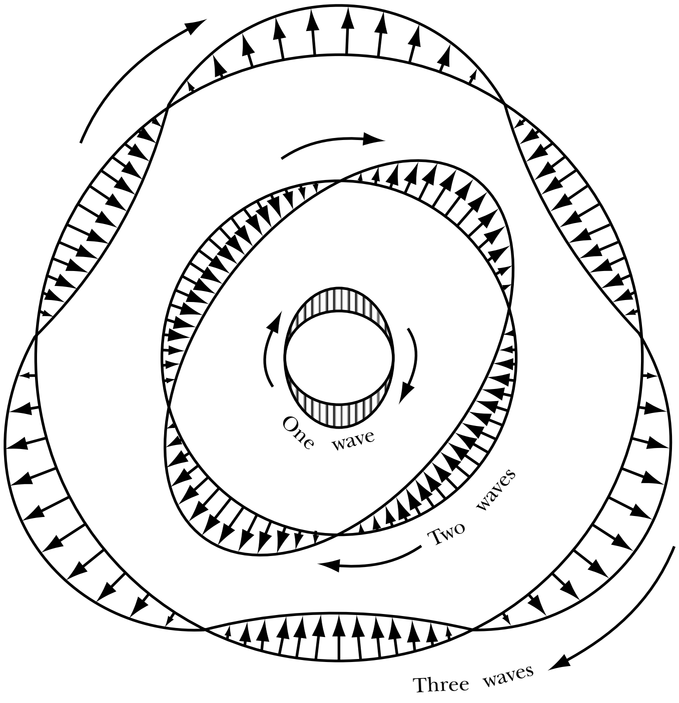


Figure 38-8 Wave pattern for the electron of the hydrogen atom in its $n = 1, 2,$ and 3 states.

In other words, the de Broglie wavelength must fit around the circumference of the orbit an integral number of times, or

$$n\lambda = 2\pi r.$$

Also,
$$\lambda = \frac{h}{mv}.$$

Therefore,
$$mv_n r_n = \frac{nh}{2\pi}.$$

The Orbital Radius Is Quantized in the Bohr Model

Let's examine the orbital motion of the electron in the Bohr model. The force holding the electron in an orbit of radius r is the Coulomb force. From Eq. 22-1, we know that the magnitude of this force is

$$F = k\frac{|q_1||q_2|}{r}$$

with $k = 1/4\pi\varepsilon_0$. Here q_1 is the charge $-e$ of the electron and q_2 is the charge Ze of the nucleus. For generality, let us assume that the nucleus has a charge Ze. This would make the model applicable to all hydrogen-like atoms, that is atoms having a single electron; for example, a helium atom stripped of an electron, a lithium atom stripped of 2 electrons and so on. The electron's acceleration is the centripetal acceleration, with a magnitude given by $a = v^2/r$, where v is the electron's speed. Both force $\vec{F}$ and acceleration $-a$ are radially inward (the negative direction on a radial axis), toward the nucleus (Fig. 38-7*b*). Thus, we can write Newton's second law ($F = ma$) for a radial axis as

$$-\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2} = m\left(\frac{v^2}{r}\right). \tag{38-20}$$

where m is the electron mass.

We next introduce quantization by using Bohr's assumption expressed in Eq. 38-19. From Eq. 10-88, the magnitude ℓ of the angular momentum of a particle of mass m and speed v moving in a circle of radius r is $\ell = rmv \sin\phi$, where ϕ (the angle between $\vec{r}$ and $\vec{v}$) is 90°. Replacing L in Eq. 38-19 with $rmv \sin 90°$ gives us

$$rmv = n\hbar,$$

or
$$v = \frac{n\hbar}{rm}. \tag{38-21}$$

Substituting this equation into Eq. 38-20, replacing $\hbar$ with $h/2\pi$, and rearranging, we find

$$r = \frac{h^2\varepsilon_0}{\pi me^2}n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{38-22}$$

We can rewrite this as

$$r = an^2, \quad \text{for } n = 1, 2, 3, \ldots, \tag{38-23}$$

where
$$a = \frac{h^2\varepsilon_0}{\pi me^2} = 5.291\,772 \times 10^{-11}\,\text{m} \approx 52.92\ \text{pm}. \tag{38-24}$$

These last three equations tell us that, in the *Bohr model of the hydrogen atom,* the electron's orbital radius r is quantized and the smallest possible orbital radius (for $n = 1$) is a, which is called the *Bohr radius.* According to the Bohr model, the electron cannot get any closer to the nucleus than orbital radius a, and that is why the attraction between electron and nucleus does not simply collapse them together.

For hydrogen like atoms

$$r = \frac{an^2}{Z}.$$

Speed of Electron in Bohr Orbit

We know that the electron speed and the radius of Bohr orbit are related by Bohr's condition of angular momentum quantization in Eq. 38-19. We have

$$v_r = \frac{nh}{2\pi mr}.$$

By substituting the value of r from Eq. 38-22, we get

$$v = \frac{Ze^2}{2nh\varepsilon_0} \tag{38-25}$$

The interesting point about the speed of the electron in the orbit is that it does not depend on the mass of the electron. So, if we substitute the electron with a charged particle with the same charge as the electron but different mass, its radius will change but not the speed.

Orbital Energy Is Quantized

Let's next find the energy of the hydrogen atom according to the Bohr model. The electron has kinetic energy $K = (1/2)mv^2$, and the electron–nucleus system has electric potential energy $U = q_1q_2/4\pi\varepsilon_0 r$ (Eq. 24-59). Again, let q_1 be the electron's charge $-e$ and q_2 be the nuclear charge $+e$. Then the mechanical energy is

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2}mv^2 + \left(-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\right). \end{aligned} \tag{38-26}$$

Solving Eq. 38-20 for mv^2 and substituting the result in Eq. 38-26 lead to

$$E = -\frac{1}{8\pi\varepsilon_0}\frac{e^2}{r}. \tag{38-27}$$

Next, replacing r with its equivalent from Eq. 38-22, we have

$$E_n = -\frac{me^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots, \tag{38-28}$$

where the subscript n on E signals that we have now quantized the energy. Evaluating the constants in Eq. 38-28 gives us

$$E_n = -\frac{2.180\times10^{-18}\,\text{J}}{n^2} = -\frac{13.61\,\text{eV}}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{38-29}$$

This equation tells us that the energy E_n of the hydrogen atom is quantized; that is, E_n is restricted by its dependence on the quantum number n. Because the nucleus is assumed to be fixed in place and only the electron has motion, we can assign the energy values of Eq. 38-29 either to the atom as a whole or to the electron alone.

Thus, the allowed energy levels of hydrogen atom are

$$E_1 = -13.6\ \text{eV}$$
$$E_2 = -3.4\ \text{eV}$$
$$E_3 = -1.51\ \text{eV}$$
$$E_4 = -0.85\ \text{eV}$$
$$E_5 = -0.54\ \text{eV}$$

Energy Changes

The energy of a hydrogen atom (or, equivalently, of its electron) changes when the atom emits or absorbs light. As we have seen several times since Eq. 38-6, emission and absorption involve a quantum of light according to

$$hf = \Delta E = E_{\text{high}} - E_{\text{low}}. \tag{38-30}$$

Let's make three changes to Eq. 38-30. On the left side, we substitute c/λ for f. On the right side, we use Eq. 38-28 twice to replace the energy terms. Then, with a simple rearrangement, we have

$$\frac{1}{\lambda} = -\frac{me^4}{8\varepsilon_0^2 h^3 c}\left(\frac{1}{n_{\text{high}}^2} - \frac{1}{n_{\text{low}}^2}\right). \tag{38-31}$$

We can rewrite this as

$$\frac{1}{\lambda} = R\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right), \tag{38-32}$$

in which R is the *Rydberg constant*:

$$R = \frac{me^4}{8\varepsilon_0^2 h^3 c} = 1.097\,373 \times 10^7\ \text{m}^{-1}. \tag{38-33}$$

Compare Eq. 38-32 from the Bohr model with Eq. 38-18 from Balmer's work. If we replace n_{low} with 2 and then restrict n_{high} to be 3, 4, 5, and 6, we have Balmer's equation generate the four visible wavelengths at which hydrogen can emit or absorb light: 656 nm, 486 nm, 434 nm, and 410 nm. This match was a triumph for Bohr and ushered in the quantum physics of atoms. However, even though the Bohr model gives the correct emission and absorption wavelengths for the hydrogen atom, the model is not correct because the electron does not orbit the nucleus like a planet orbiting the Sun. The model was even lesser successful when applied to atoms more complicated than hydrogen. The reason being that an electron trapped in any atom is a matter wave confined to a potential well, and to find the resulting quantized energy values, we must apply Schrödinger's equation to the electron. Nonetheless, Bohr's theory is a good stepping stone for entry into the world of quantum physics.

The Hydrogen Atom Spectrum

The potential well of a hydrogen atom depends on the electrical potential energy function

$$U(r) = \frac{-e^2}{4\pi\varepsilon_0 r}. \tag{38-34}$$

Because this well is three-dimensional, it is more complex than our previous one-dimensional well. Moreover, it does not have sharply defined walls. Rather, its walls vary in depth with radial distance r. Figure 38-9 is probably the best we can do in drawing the hydrogen potential well, but even that drawing takes much effort to interpret.

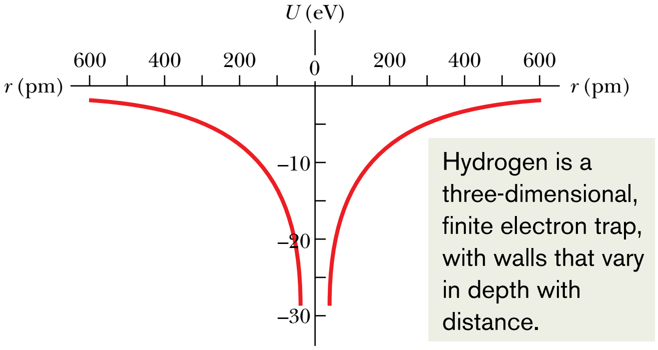


Figure 38-9 The potential energy U of a hydrogen atom as a function of the separation r between the electron and the central proton. The plot is shown twice (on the left and on the right) to suggest the three-dimensional spherically symmetric trap in which the electron is confined.

Energy Levels and Spectra of the Hydrogen Atom

Although we shall not do so here, we can apply Schrödinger's equation for an electron trapped in the potential well given by Eq. 38-34. In doing so, we would find that the energy values are quantized and that, amazingly, those values are given by Eq. 38-29 just as for the (incorrect) Bohr model. Thus, changes ΔE in energy due to emission or absorption of light are given by Eq. 38-30, and the wavelengths corresponding to ΔE are given by Eq. 38-32. Let's explore these results.

Figure 38-10*a* shows the energy levels corresponding to various values of n in Eq. 38-29. The lowest level, for $n = 1$, is the ground state of hydrogen. Higher levels correspond to excited states, just as we saw for our simpler potential traps. Note several differences, however. (1) The energy levels now have negative values rather than the positive values we previously chose in, for instance, Fig. 38-3. (2) The levels now become progressively closer as we move to higher levels. (3) The energy for the greatest value of n—namely, $n = \infty$—is now $E_\infty = 0$. For any energy greater than $E_\infty = 0$, the electron and proton are not bound together (there is no hydrogen atom), and the $E > 0$ region in Fig. 38-10*a* is like the nonquantized region.

A hydrogen atom can jump between quantized energy levels by emitting or absorbing light at the wavelengths given by Eq. 38-31. Any such wavelength is often called a *line* because of the way it is detected with a spectroscope; thus, a hydrogen atom has *absorption lines* and *emission lines.* A collection of such lines, such as in those in the visible range, is called a **spectrum** of the hydrogen atom.

Series. The lines for hydrogen are said to be grouped into *series,* according to the level at which upward jumps start and downward jumps end. For example, the emission and absorption lines for all possible jumps up from the $n = 1$ level and down to the $n = 1$ level are said to be in the *Lyman series* (Fig. 38-10*b*), named after the person who first studied those lines. Further, we can say that the Lyman series has a *home-base level* of $n = 1$. Similarly, the *Balmer series* has a home-base level of $n = 2$ (Fig. 38-10*c*), and the *Paschen series* has a home-base level of $n = 3$ (Fig. 38-10*d*).

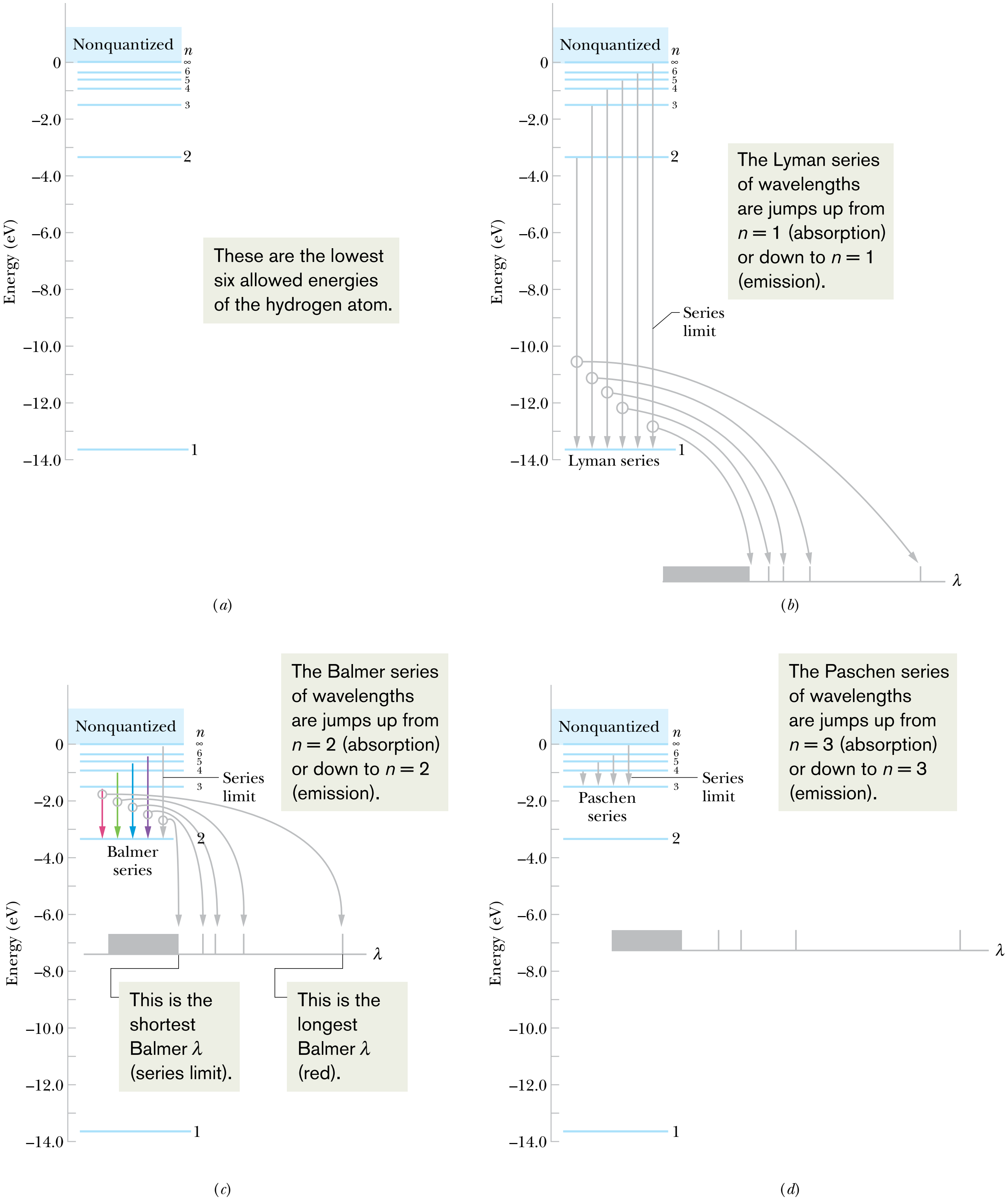
Some of the downward quantum jumps for these three series are shown in Fig. 38-10. Four lines in the Balmer series are in the visible range and are represented in Fig. 38-10*c* with arrows corresponding to their colors. The shortest of those arrows represents the shortest jump in the series, from the $n = 3$ level to the $n = 2$ level. Thus, that jump involves the smallest change in the electron's energy and the smallest amount of emitted photon energy for the series. The emitted light is red. The next jump in the series, from $n = 4$ to $n = 2$, is longer, the photon energy is greater, the wavelength of the emitted light is shorter, and the light is green. The third, fourth, and fifth arrows represent longer jumps and shorter wavelengths. For the fifth jump, the emitted light is in the ultraviolet range and thus is not visible.

The *series limit* of a series is the line produced by the jump between the home-base level and the highest energy level, which is the level with the limiting quantum number $n = \infty$. Thus, the series limit corresponds to the shortest wavelength in the series.

If a jump is upward into the nonquantized portion of Fig. 38-10, the electron's energy is no longer given by Eq. 38-29 because the electron is no longer trapped in the atom. That is, the hydrogen atom has been *ionized,* meaning that the electron has been removed to a distance so great that the Coulomb force on it from the nucleus is negligible. The atom can be ionized if it absorbs any wavelength shorter than the series limit. The free electron then has only kinetic energy K $(=(1/2)mv^2$, assuming a nonrelativistic situation).

It is now clear that when an electron jumps from a higher energy level to a lower energy level, the radiation is emitted in the form of photons. The radiation emitted in such a transition corresponds to spectral lines in atomic spectra of hydrogen atom. In general, the number of spectral lines emitted when an electron falls from state n_2 to state n_1 is

$$\frac{(n_2 - n_1)(n_2 - n_1 + 1)}{2}$$



These are the lowest six allowed energies of the hydrogen atom.

The Lyman series of wavelengths are jumps up from $n = 1$ (absorption) or down to $n = 1$ (emission).

The Balmer series of wavelengths are jumps up from $n = 2$ (absorption) or down to $n = 2$ (emission).

This is the shortest Balmer λ (series limit).

This is the longest Balmer λ (red).

The Paschen series of wavelengths are jumps up from $n = 3$ (absorption) or down to $n = 3$ (emission).

Figure 38-10 (*a*) An energy-level diagram for the hydrogen atom. Some of the transitions for (*b*) the Lyman series, (*c*) the Balmer series, and (*d*) the Paschen series. For each, the longest four wavelengths and the series-limit wavelength are plotted on a wavelength axis. Any wavelength shorter than the series-limit wavelength is allowed.

The spectral lines are defined for hydrogen atom only and not for hydrogen-like ions.

Lyman series: It lies in ultraviolet region. When an electron jumps from a higher energy state n to ground state, $n = 1$, the series of spectral lines emitted is called Lyman series. All of them lie in near ultraviolet region.

$$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n_2^2}\right). \tag{38-35}$$

Balmer series: When an electron jumps from a higher energy state n to state 2, the series of spectral lines emitted is called Balmer series.

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n_2^2}\right). \tag{38-36}$$

Paschen series: When an electron jumps from a higher energy state n to state 3, the series of spectral lines emitted is called Paschen series. All of them lie in near infrared region.

$$\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n_2^2}\right). \tag{38-37}$$

Brackett series: State n to 4 (all of them lie in far infrared region)

$$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n_2^2}\right). \tag{38-38}$$

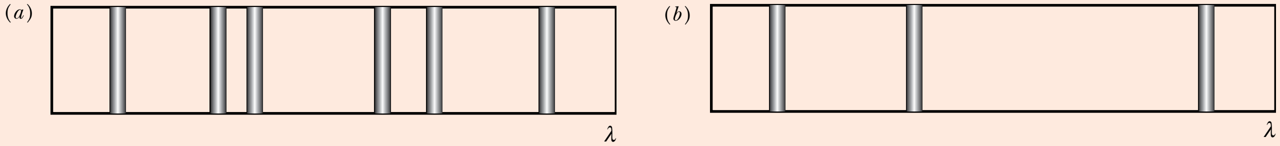
Pfund series: State n to 5 (all of them lie in far–far infrared region)

$$\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n_2^2}\right). \tag{38-39}$$

We have to understand that the absorption spectra are just a subset of the emission spectra. Most of the atoms at the room temperature are in ground state, so the atom absorbs photons corresponding to Lyman series only. But during emission, the atom can emit photons belonging to other series as well.

☑ CHECKPOINT 4

In the following figure, one of the spectra is emission spectra and another is absorption spectra of the same element. Identify which is which?



SAMPLE PROBLEM 38.04

Light emission from a hydrogen atom

(a) What is the wavelength of light for the least energetic photon emitted in the Lyman series of the hydrogen atom spectrum lines?

KEY IDEAS

(1) For any series, the transition that produces the least energetic photon is the transition between the home-base

level that defines the series and the level immediately above it. (2) For the Lyman series, the home-base level is at $n = 1$ (Fig. 38-10*b*). Thus, the transition that produces the least energetic photon is the transition from the $n = 2$ level to the $n = 1$ level.

Calculations: From Eq. 38-29 the energy difference is

$$\Delta E = E_2 - E_1 = -(13.60\text{ eV})\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.20\text{ eV}.$$

Then from Eq. 38-6 ($\Delta E = hf$), with c/λ replacing f, we have

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^{8}\text{ m/s})}{(10.20\text{ eV})(1.60\times10^{-19}\text{ J/eV})}$$
$$= 1.22\times10^{-7}\text{ m} = 122\text{ nm}. \qquad \text{(Answer)}$$

Light with this wavelength is in the ultraviolet range.

(b) What is the wavelength of the series limit for the Lyman series?

KEY IDEA

The series limit corresponds to a jump between the home-base level ($n = 1$ for the Lyman series) and the level at the limit $n = \infty$.

Calculations: Now that we have identified the values of n for the transition, we could proceed as in (a) to find the corresponding wavelength λ. Instead, let's use a more direct procedure. From Eq. 38-32, we find

$$\frac{1}{\lambda} = R\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right)$$
$$= 1.097\,373\times10^{7}\text{ m}^{-1}\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right),$$

which yields

$$\lambda = 9.11\times10^{-8}\text{ m} = 91.1\text{ nm}. \qquad \text{(Answer)}$$

Light with this wavelength is also in the ultraviolet range.

Excitation by Collision

Besides the excitation by photons, the atom can also be excited by collision. This is shown schematically in Fig. 38-11 and Fig. 38-12. For example, in a discharge tube filled with hydrogen at a low pressure, the electron is accelerated under the influence of an electric field. Near the end of the tube, the electron will have the maximum kinetic energy. It can collide with a hydrogen atom there and excite the electron in ground state to a higher excited state.

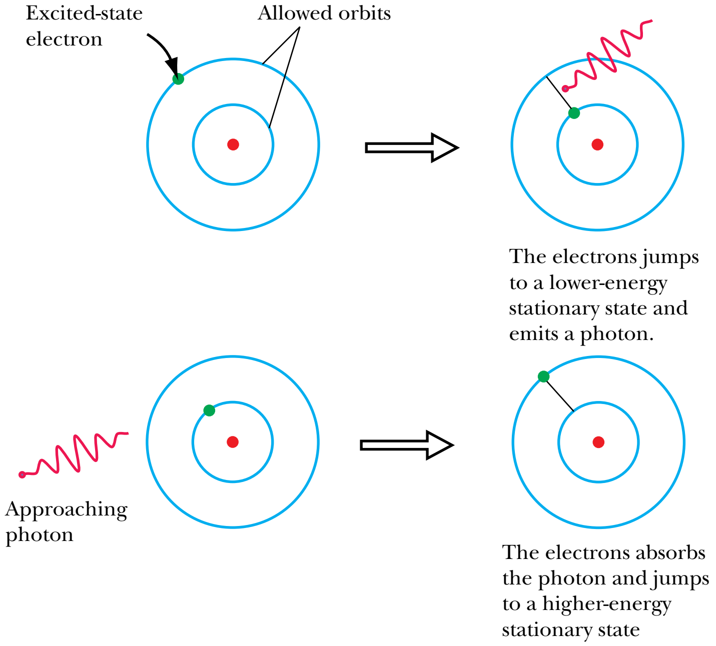

Excited-state electron
Allowed orbits
The electrons jumps to a lower-energy stationary state and emits a photon.
Approaching photon
The electrons absorbs the photon and jumps to a higher-energy stationary state

Figure 38-11 An atom can change stationary states by emitting or absorbing a photon.

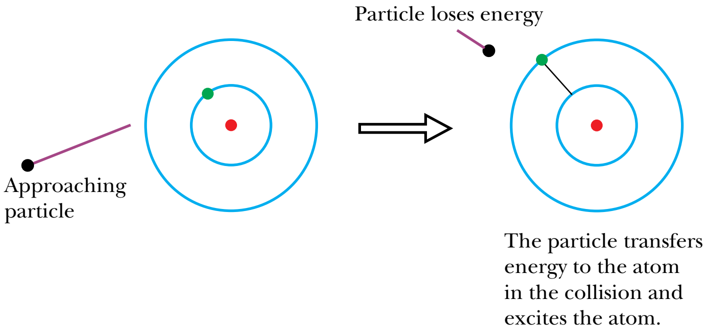

Particle loses energy
Approaching particle
The particle transfers energy to the atom in the collision and excites the atom.

Figure 38-12 An atom can change stationary states by undergoing a collision.

The state to which the electron within the atom is excited depends on the kinetic energy of the incident electron. Let us see how.

We have seen in the chapter on momentum that the kinetic energy of two particle system from laboratory frame of reference is

$$K_L = \frac{1}{2}M_{\text{tot}}V_{\text{cm}}^2 + \frac{1}{2}\mu V_{\text{rel}}^2, \tag{38-40}$$

where μ is the reduced mass of the system given by $\mu = mM/(m+M)$; M_{tot} is the total mass of the system; V_{rel} is the velocity of the one particle relative to the other; and V_{cm} is the velocity of the center of mass. We are aware that in absence of external force, V_{cm} remains constant. The only variable part in this equation for the kinetic energy of the system is $(1/2)\mu V_{\text{rel}}^2$. This is known as the kinetic energy of the system in the center of mass frame of reference.

As we have seen in momentum conservation, for an elastic collision, the magnitude of relative velocity before and after the collision remains the same. So, $(1/2)\mu V_{\text{rel}}^2$ remains the same. But for an inelastic collision, V_{rel} decreases. So, $(1/2)\mu V_{\text{rel}}^2$ decreases. For a perfectly inelastic collision, after the collision, $V_{\text{rel}} = 0$. So, the energy loss will be maximum in this case; it will be equal to $(1/2)\mu V_{\text{rel}}^2$. This energy is lost in the form of heat.

However, in the case of atomic collisions, the heat liberated has no meaning. Furthermore, the collision can be elastic and inelastic depending on $(1/2)\mu V_{\text{rel}}^2$. How? We have seen that the energy available to be lost is only $(1/2)\mu V_{\text{rel}}^2$. But an inelastic collision on atomic level means that the electron in the atom is excited to a higher state. This is the only mechanism the energy can be lost in this case. So, if

$$\frac{1}{2}\mu V_{\text{rel}}^2 \geq \text{energy required for excitation}$$

then the collision can be inelastic; otherwise the collision will be elastic.

Let us show this by an example. Consider an electron moving with energy E. We want to find the maximum value of this energy so that it does not excite any electron in a hydrogen atom it strikes:

$$\mu = \frac{m_e m_h}{m_e + m_h} \approx m_e.$$

So, the kinetic energy of the striking electron is almost entirely available for excitation. We know that to excite an electron in hydrogen atom from ground state to $n = 2$ at least requires 10.2 eV. So, if we do not want the excitation to occur, then

$$E < 10.2 \text{ eV}.$$

In other words, the potential drop in the discharge tube should not be greater than 10.2 V.

Since a discharge tube can also be used to produce the emission spectra of the gas, many times the terms related to discharge tube are also used in spectroscopy. Thus, *excitation potential* of an excited state refers to the potential difference across the discharge tube, required to excite the atom from state $n = 1$ to that state. Its numerical value is equal to the excitation energy in electron volts. *Ionization potential* of an atom refers to the potential difference across the discharge tube required to ionize the atom. Its numerical value is equal to the ionization energy in electron volts. This spectrum is characteristic of the hydrogen atom. It is like a signature which can be used to identify the atom. In the case of multielectron atom, the spectra are much more complicated, and their pattern can be predicted using only quantum theory.

SAMPLE PROBLEM 38.05

Excitation of hydrogen atom by collision with neutron

A neutron moving with a speed v strikes a hydrogen atom in the ground state moving toward it with the same speed. Find the minimum kinetic energy of the neutron for which inelastic (completely or partially) collision might take place. (Take $m_n = m_h$.)

KEY IDEA

Here the neutron has no electronic structure, but the hydrogen atom has. So, it can be excited to a higher state. As some mechanical energy will be lost from the system, we can consider the collision to be inelastic.

Calculation: We have

$$\mu = \frac{m_n m_h}{m_n + m_h} = \frac{m_h}{2}.$$

For excitation, collision should be inelastic. So,

$$\frac{1}{2}\mu V_{\text{rel}}^2 \ge 10.2\ \text{eV}$$

$$\frac{1}{2}m_n v^2 \ge 20.4\ \text{eV}.$$

Learn: The interesting part is that the striking neutron needs to have a kinetic energy greater than the energy required for excitation, in fact double that energy. Why? Consider that the collision is inelastic. In that case, by the law of conservation of momentum, the two will stick together and move with the same velocity. So, the entire energy cannot be lost in the form of a photon. Some energy will remain in form of the kinetic energy of the system.

SAMPLE PROBLEM 38.06

Frequency of radiation emitted by an excited hydrogen atom

Consider an excited hydrogen atom in state n moving with a velocity $v\,(v \ll c)$. It emits a photon in the direction of its motion and changes its state to a lower state m. Apply momentum and energy conservation principles to calculate the frequency of emitted radiation. Compare this with the frequency emitted if the atom were at rest.

KEY IDEA

Normally, we ignore the recoil of the atom that is emitting the photon. But here the atom that is emitting the photon will also change its momentum due to recoil.

Calculation: Let us assume that the atom emits a photon of frequency ν and has a velocity of v_0 after the emission.

By the law of conservation of energy, we have

$$\frac{1}{2}mv^2 + E_n = \frac{1}{2}mv_0^2 + E_m + h\nu. \tag{38-41}$$

By the law of conservation of momentum, we have

$$mv = mv_0 + h\frac{\nu}{c}. \tag{38-42}$$

In absence of recoil, the atom would have emitted a photon of frequency ν_0:

$$E_n - E_m = h\nu_0. \tag{38-43}$$

So, we have the equations as

$$\frac{1}{2}m(v^2 - v_0^2) = h(\nu_0 - \nu)$$

and

$$m(v - v_0) = h\frac{\nu}{c}.$$

By dividing the two equations, we get

$$\frac{v + v_0}{2c} = \frac{\nu_0 - \nu}{\nu}.$$

Since v and $v_0 \ll c$, we get

$$\nu \approx \nu_0\left(1+\frac{v}{c}\right).$$

Learn: If you notice, there is a striking similarity between this expression and the expression for the frequency of the wave emitted by a moving source. There we get the formula for the frequency as

$$\nu = \nu_0\left(1-\frac{v}{c}\right)^{-1}.$$

By applying binomial approximation, we get

$$\nu \approx \nu_0\left(1+\frac{v}{c}\right).$$

Effect of Motion of the Nucleus on the Spectra

It has been assumed in the above considerations that the electron moved around a fixed nucleus. This would only be strictly true if the mass of the nucleus were infinite. The fact that it is only some 1836 times as heavy as the electron implies that they each move around a common center of gravity, as shown in Fig. 38-13.

By mass moments, we know that

$$\frac{m}{M}=\frac{A}{a}.$$

Therefore,

$$a=\frac{Mr}{M+m}$$

and

$$A=\frac{mr}{M+m},$$

where A and a are the distances of the nucleus of mass M and electron of mass m, respectively, from the common center of gravity and r is the separation of the nucleus and the electron. When ω is the angular velocity of the system, the total kinetic energy is given by

$$\begin{aligned}\frac{1}{2}MV^2+\frac{1}{2}mv^2 &= \frac{1}{2}MA^2\omega^2+\frac{1}{2}ma^2\omega^2\\ &= \frac{1}{2}\frac{Mm}{M+m}r^2\omega^2\\ &= \frac{1}{2}\mu r^2\omega^2.\end{aligned}$$

It, thus, appears that to allow for the motion of the nucleus we must replace the mass m of the electron by μ, the reduced mass. The Rydberg constant then becomes

$$R=\frac{2\pi^2\mu e^4}{ch^3(4\pi\varepsilon_0)^2}=\frac{2\pi^2me^4}{ch^3(4\pi\varepsilon_0)^2}=\frac{1}{1+m/M}=\frac{R_\infty}{1+m/M}.$$

Isotopes carry the same nuclear charge, but have different nuclear masses. This gives rise to slightly different values for the Rydberg constant. So, different isotopes of a given element would absorb or emit light of slightly different wavelengths while undergoing the same transition.

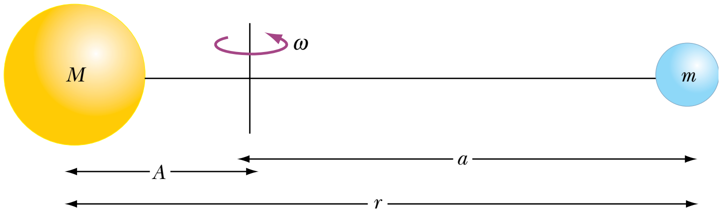


Figure 38-13 Electron and proton moving around a common center of gravity.

CHECKPOINT 5

What would be the effect of nuclear motion on the radius of the electronic orbit? Would the radius be the same as that of the predicted Bohr orbit by Eq. 38-22, more or less?

SAMPLE PROBLEM 38.07

Formation of muonic atom and its ionization energy

A muon is unstable elementary particle whose mass is $207m_e$ and whose charge is either $+e$ or $-e$. A negative muon (μ^-) can be captured by a proton to form a muonic atom. (a) Find the distance between the muon and proton for this atom. (b) Find the ionization energy of the atom.

KEY IDEAS

Here, we should replace the electronic mass by the reduced mass. Also, we can see that the mass of the nucleus and the mass of the muon are comparable. So, the motion of nucleus cannot be neglected.

Calculations:

(a) Here $m = 207\ m_e$ and $M = 1836\ m_e$, so the reduced mass is

$$\mu = \frac{mM}{m+M} = \frac{(207m_e)(1836m_e)}{207m_e + 1836m_e} = 186m_e.$$

According to Eq. 38-22, the orbit radius corresponding to $n = 1$ is

$$r_1 = \frac{h^2\varepsilon_0}{\pi m_e e^2},$$

where $r_1 = a_0 = 5.29 \times 10^{-11}$ m. Hence, the distance between the muon and the proton r that corresponds to the reduced mass μ is

$$r_1' = \left(\frac{m}{\mu}\right) r_1 = \left(\frac{m_e}{186m_e}\right) a_0 = 2.85 \times 10^{-13}\,\text{m}.$$

The muon is 186 times closer to the proton than an electron would be.

(b) From Eq. 38-28, we have, with $n = 1$ and $E_1 = -13.6$ eV:

$$E_1' = \left(\frac{\mu}{m}\right) E_1 = 186E_1 = -2.53 \times 10^3\ \text{eV} = -2.53\ \text{keV}.$$

The ionization energy is therefore 2.53 keV, 186 times that for an ordinary hydrogen atom.

Note: If we did not take the effect of reduced mass into account, the radius would have been 207 times smaller, but here we see the effect is only 186 times smaller. Similarly, we can say about the energy. But the velocity of muon in first Bohr orbit would be the same because the velocity of the electron does not depend on the electronic mass.

38.6 | QUANTUM NUMBERS FOR THE HYDROGEN ATOM (OPTIONAL)

Key Concepts

- The radial probability density $P(r)$ for a state of the hydrogen atom is defined so that $P(r)$ is the probability that the electron will be detected somewhere in the space between two spherical shells of radii r and $r + dr$ that are centered on the nucleus.
- Normalization requires that

$$\int_0^\infty P(r)\,dr = 1.$$

- The probability that the electron will be detected between any two given radii r_1 and r_2 is

$$\begin{pmatrix}\text{Probability of detection}\\ \text{between } r_1 \text{ and } r_2\end{pmatrix} = \int_{r_1}^{r_2} P(r)\,dr.$$

Table 38-1 Quantum Numbers for the Hydrogen Atom

Symbol	Name	Allowed Values
n	Principal quantum number	$1, 2, 3, \ldots$
ℓ	Orbital quantum number	$0, 1, 2, \ldots, n-1$
m_ℓ	Orbital magnetic quantum number	$-\ell, -(\ell-1), \ldots, +(\ell-1), +\ell$

Although the energies of the hydrogen atom states can be described by the single quantum number n, the wave functions describing these states require three quantum numbers, corresponding to the three dimensions in which the electron can move. The three quantum numbers, along with their names and the values that they may have, are shown in Table 38-1.

Each set of quantum numbers (n, ℓ, m_ℓ) identifies the wave function of a particular quantum state. The quantum number n, called the **principal quantum number**, appears in Eq. 38-29 for the energy of the state. The **orbital quantum number** ℓ is a measure of the magnitude of the angular momentum associated with the quantum state. The **orbital magnetic quantum number** m_ℓ is related to the orientation in space of this angular momentum vector. The restrictions on the values of the quantum numbers for the hydrogen atom, as listed in Table 38-1, are not arbitrary but come out of the solution to Schrödinger's equation. Note that for the ground state ($n = 1$), the restrictions require that $\ell = 0$ and $m_\ell = 0$. That is, the hydrogen atom in its ground state has zero angular momentum, which is not predicted by Eq. 38-19 in the Bohr model.

CHECKPOINT 6

(a) A group of quantum states of the hydrogen atom has $n = 5$. How many values of ℓ are possible for states in this group? (b) A subgroup of hydrogen atom states in the $n = 5$ group has $\ell = 3$. How many values of m_ℓ are possible for states in this subgroup?

The Wave Function of the Hydrogen Atom's Ground State

The wave function for the ground state of the hydrogen atom, as obtained by solving the three-dimensional Schrödinger equation and normalizing the result, is

$$\psi(r) = \frac{1}{\sqrt{\pi}\,a^{3/2}}\,e^{-r/a} \quad \text{(ground state)}, \tag{38-44}$$

where a ($= 5.291\,772 \times 10^{-11}$ m) is the Bohr radius. This radius is loosely taken to be the effective radius of a hydrogen atom and turns out to be a convenient unit of length for other situations involving atomic dimensions.

As with other wave functions, $\psi(r)$ in Eq. 38-44 does not have physical meaning but $\psi^2(r)$ does, being the probability density—the probability per unit volume—that the electron can be detected. Specifically, $\psi^2(r)\,dV$ is the probability that the electron can be detected in any given (infinitesimal) volume element dV located at radius r from the center of the atom:

$$\begin{pmatrix}\text{Probability of detection}\\ \text{in volume } dv\\ \text{at radius } r\end{pmatrix} = \begin{pmatrix}\text{Volume probability}\\ \text{density } \psi^2(r)\\ \text{at radius } r\end{pmatrix}(\text{Volume } dV). \tag{38-45}$$

Because $\psi^2(r)$ here depends only on r, it makes sense to choose, as a volume element dV, the volume between two concentric spherical shells whose radii are r and $r + dr$. That is, we take the volume element dV to be

$$dV = (4\pi r^2)\,dr, \tag{38-46}$$

in which $4\pi r^2$ is the surface area of the inner shell and dr is the radial distance between the two shells. Then, combining Eqs. 38-44, 38-45, and 38-46 gives us

$$\begin{pmatrix}\text{Probability of detection}\\ \text{in volume } dV\\ \text{at radius } r\end{pmatrix} = \psi^2(r)\,dV = \frac{4}{a^3}e^{-2r/a}r^2\,dr. \tag{38-47}$$

Describing the probability of detecting an electron is easier if we work with a **radial probability density** $P(r)$ instead of a volume probability density $\psi^2(r)$. This $P(r)$ is a linear probability density such that

$$\begin{pmatrix}\text{Radial probability}\\ \text{density } P(r)\\ \text{at radius } r\end{pmatrix}\begin{pmatrix}\text{Radial}\\ \text{width } dr\end{pmatrix}=\begin{pmatrix}\text{Volume probability}\\ \text{density } \psi^2(r)\\ \text{at radius } r\end{pmatrix}(\text{Volume } dV)$$

or

$$P(r)\,dr = \psi^2(r)\,dV. \tag{38-48}$$

Substituting for $\psi^2(r)\,dV$ from Eq. 38-47, we obtain

$$P(r) = \frac{4}{a^3}r^2e^{-2r/a} \quad \begin{pmatrix}\text{Radial probability density,}\\ \text{hydrogen atom ground state}\end{pmatrix}. \tag{38-49}$$

To find the probability of detecting the ground-state electron between any two radii r_1 and r_2 (that is, between a spherical shell of radius r_1 and another of radius r_2), we integrate Eq. 38-49 between those two radii:

$$\begin{pmatrix}\text{Probability of detection}\\ \text{between } r_1 \text{ and } r_2\end{pmatrix} = \int_{r_1}^{r_2} P(r)\,dr. \tag{38-50}$$

If the radial range Δr $(= r_2 - r_1)$ in which we search for the electron is small enough such that $P(r)$ does not vary by much over the range, then we can usually approximate the integral in Eq. 38-50 as being equal to the product $P(r)\,\Delta r$, with $P(r)$ evaluated in the center of Δr.

Figure 38-14 is a plot of Eq. 38-49. The area under the plot is unity; that is,

$$\int_0^\infty P(r)\,dr = 1. \tag{38-51}$$

This equation states that in a hydrogen atom, the electron must be *somewhere* in the space surrounding the nucleus.

The triangular marker on the horizontal axis of Fig. 38-14 is located one Bohr radius from the origin. The graph tells us that in the ground state of the hydrogen atom, the electron is most likely to be found at about this distance from the center of the atom.

Figure 38-14 conflicts sharply with the popular view that electrons in atoms follow well-defined orbits like planets moving around the Sun. *This popular view, however familiar, is incorrect.* Figure 38-14 shows us all that we can ever know about the location of the electron in the ground state of the hydrogen atom. The appropriate question is not "When will the electron arrive at such-and-such a point?" but "What are the odds that the electron will be detected in a small volume centered on such-and-such a point?" Figure 38-15, which we call a dot plot, suggests the probabilistic nature of the wave function: The density of dots represents the probability density of detection of the electron with the hydrogen atom in its ground state. Think of the atom in this state as a fuzzy ball with no sharply defined boundary and no hint of orbits.

It is not easy for a beginner to envision subatomic particles in this probabilistic way. The difficulty is our natural impulse to regard an electron as something like a tiny jelly bean, located at certain places at certain times and following a well-defined path. Electrons and other subatomic particles simply do not behave in this way.

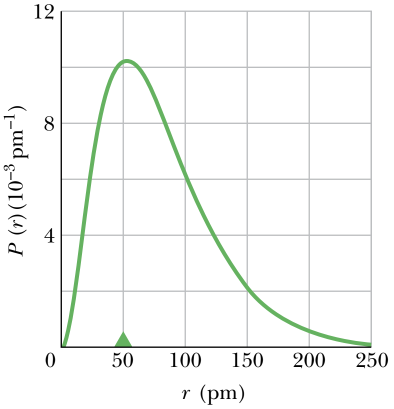


Figure 38-14 A plot of the radial probability density $P(r)$ for the ground state of the hydrogen atom. The triangular marker is located at one Bohr radius from the origin, and the origin represents the center of the atom.

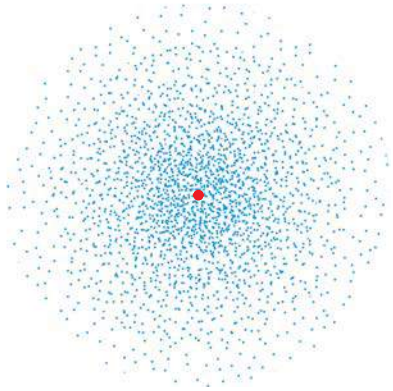


Figure 38-15 A "dot plot" showing the volume probability density $\psi^2(r)$—not the *radial* probability density $P(r)$—for the ground state of the hydrogen atom. The density of dots drops exponentially with increasing distance from the nucleus, which is represented here by a red spot.

The energy of the ground state, found by putting $n = 1$ in Eq. 38-29, is $E_1 = -13.60$ eV. The wave function of Eq. 38-44 results if you solve Schrödinger's equation with this value of the energy. Actually, you can find a solution of Schrödinger's equation for *any* value of the energy—say, $E = -11.6$ eV or -14.3 eV. This may suggest that the energies of the hydrogen atom states are not quantized—but we know that they are.

The puzzle was solved when physicists realized that such solutions of Schrödinger's equation are not physically acceptable because they yield increasingly large values as $r \to \infty$. These "wave functions" tell us that the electron is more likely to be found very far from the nucleus rather than closer to it, which makes no sense. We discard such solutions and accept only solutions that meet the boundary condition $\psi(r) \to 0$ as $r \to \infty$; that is, we agree to deal only with *confined* electrons. With this restriction, the solutions of Schrödinger's equation form a discrete set, with quantized energies given by Eq. 38-29.

Hydrogen Atom States with $n = 2$

According to the requirements of Table 38-2, there are four states of the hydrogen atom with $n = 2$; their quantum numbers are listed in Table 38-2. Consider first the state with $n = 2$ and $\ell = m_\ell = 0$; its probability density is represented by the dot plot of Fig. 38-16. Note that this plot, like the plot for the ground state shown in Fig. 38-15, is spherically symmetric. That is, in a spherical coordinate system like that defined in Fig. 38-17, the probability density is a function of the radial coordinate r only and is independent of the angular coordinates θ and ϕ.

It turns out that all quantum states with $\ell = 0$ have spherically symmetric wave functions. This is reasonable because the quantum number ℓ is a measure of the angular momentum associated with a given state. If $\ell = 0$, the angular momentum is also zero, which requires that the probability density representing the state have no preferred axis of symmetry.

Table 38-2 Quantum Numbers for Hydrogen Atom States with $n = 2$

n	ℓ	m_ℓ
2	0	0
2	1	+1
2	1	0
2	1	−1

Dot plots of ψ^2 for the three states with $n = 2$ and $\ell = 1$ are shown in Fig. 38-18. The probability densities for the states with $m_\ell = +1$ and $m_\ell = -1$ are identical. Although these plots are symmetric about the z axis, they are *not* spherically symmetric. That is, the probability densities for these three states are functions of both r and the angular coordinate θ.

Here is a puzzle: What is there about the hydrogen atom that establishes the axis of symmetry that is so obvious in Fig. 38-18? The answer: *absolutely nothing.*

The solution to this puzzle comes about when we realize that all three states shown in Fig. 38-18 have the same energy. Recall that the energy of a state, depends only on the principal quantum number n and is independent of ℓ and m_ℓ. In fact, for an *isolated* hydrogen atom there is no way to differentiate experimentally among the three states of Fig. 38-18.

If we add the volume probability densities for the three states for which $n = 2$ and $\ell = 1$, the combined probability density turns out to be spherically symmetrical, with no unique axis. One can, then, think of the electron as spending one-third of its time in each of the three states of Fig. 38-18, and one can think of the weighted sum of the three

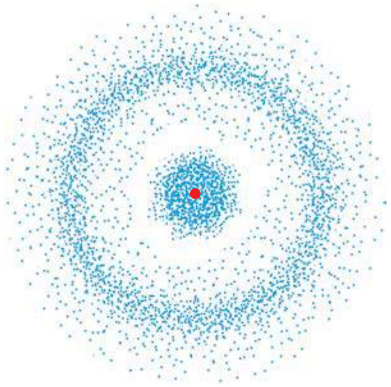


Figure 38-16 A dot plot showing the volume probability density $\psi^2(r)$ for the hydrogen atom in the quantum state with $n = 2$, $\ell = 0$ and $m_\ell = 0$. The plot has spherical symmetry about the central nucleus. The gap in the dot density pattern marks a spherical surface over which $\psi^2(r) = 0$.

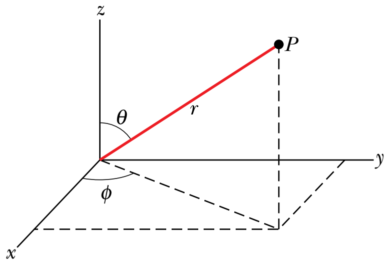


Figure 38-17 The relationship between the coordinates x, y, and z of the rectangular coordinate system and the coordinates r, θ, and ϕ of the spherical coordinate system. The latter are more appropriate for analyzing situations involving spherical symmetry, such as the hydrogen atom.

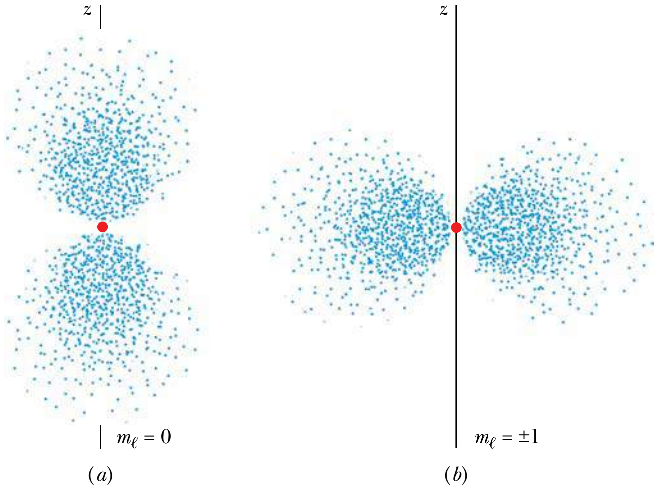


Figure 38-18 Dot plots of the volume probability density $\psi^2(r, \theta)$ for the hydrogen atom in states with $n = 2$ and $\ell = 1$. (*a*) Plot for $m_\ell = 0$. (*b*) Plot for $m_\ell = +1$ and $m_\ell = -1$. Both plots show that the probability density is symmetric about the z axis.

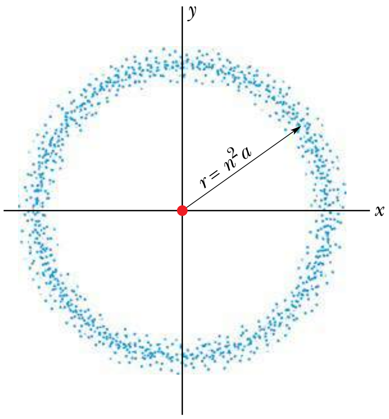


Figure 38-19 A dot plot of the radial probability density $P(r)$ for the hydrogen atom in a quantum state with a relatively large principal quantum number—namely, $n = 45$—and angular momentum quantum number $\ell = n - 1 = 44$. The dots lie close to the xy plane, the ring of dots suggesting a classical electron orbit.

independent wave functions as defining a spherically symmetric **subshell** specified by the quantum numbers $n = 2$, $\ell = 1$. The individual states will display their separate existence only if we place the hydrogen atom in an external electric or magnetic field. The three states of the $n = 2$, $\ell = 1$ subshell will then have different energies, and the field direction will establish the necessary symmetry axis.

The $n = 2$, $\ell = 0$ state, whose volume probability density is shown in Fig. 38-16, *also* has the same energy as each of the three states of Fig. 38-18. We can view all four states whose quantum numbers are listed in Table 38-2 as forming a spherically symmetric **shell** specified by the single quantum number n. The importance of shells and subshells will become evident in Chapter 39, where we discuss atoms having more than one electron.

To round out our picture of the hydrogen atom, we display in Fig. 38-19 a dot plot of the *radial* probability density for a hydrogen atom state with a relatively high quantum number ($n = 45$) and the highest orbital quantum number that the restrictions of Table 38-1 permit ($\ell = n - 1 = 44$). The probability density forms a ring that is symmetrical about the z axis and lies very close to the xy plane. The mean radius of the ring is n^2a, where a is the Bohr radius. This mean radius is more than 2000 times the effective radius of the hydrogen atom in its ground state.

Figure 38-19 suggests the electron orbit of classical physics—it resembles the circular orbit of a planet around a star. Thus, we have another illustration of Bohr's correspondence principle—namely, that at large quantum numbers the predictions of quantum mechanics merge smoothly with those of classical physics. Imagine what a dot plot like that of Figure 38-19 would look like for *really* large values of n and ℓ—say, $n = 1000$ and $\ell = 999$.

SAMPLE PROBLEM 38.08

Radial probability density for the electron in a hydrogen atom

Show that the radial probability density for the ground state of the hydrogen atom has a maximum at $r = a$.

KEY IDEAS

(1) The radial probability density for a ground-state hydrogen atom is given by Eq. 38-49,

$$P(r) = \frac{4}{a^3} r^2 e^{-2r/a}.$$

(2) To find the maximum (or minimum) of any function, we must differentiate the function and set the result equal to zero.

Calculation: If we differentiate $P(r)$ with respect to r, using derivative 7 of Appendix E and the chain rule for differentiating products, we get

$$\frac{dp}{dr} = \frac{4}{a^3} r^2 \left(\frac{-2}{a}\right) e^{-2r/a} + \frac{4}{a^3} 2re^{-2r/a}$$
$$= \frac{8r}{a^3} e^{-2r/a} - \frac{8r^2}{a^4} e^{-2r/a} = \frac{8}{a^4} r(a-r)e^{-2r/a}.$$

If we set the right side equal to zero, we obtain an equation that is true if $r = a$, so that the term $(a - r)$ in the middle of the equation is zero. In other words, dP/dr is equal to zero when $r = a$. (Note that we also have $dP/dr = 0$ at $r = 0$ and at $r = \infty$. However, these conditions correspond to a *minimum* in $P(r)$, as you can see in Fig. 38-14.)

SAMPLE PROBLEM 38.09

Probability of detection of the electron in a hydrogen atom

It can be shown that the probability $p(r)$ that the electron in the ground state of the hydrogen atom will be detected inside a sphere of radius r is given by

$$p(r) = 1 - e^{-2x}(1 + 2x + 2x^2),$$

in which x, a dimensionless quantity, is equal to r/a. Find r for $p(r) = 0.90$.

KEY IDEA

There is no guarantee of detecting the electron at any particular radial distance r from the center of the hydrogen atom. However, with the given function, we can calculate the probability that the electron will be detected *somewhere* within a sphere of radius r.

Calculation: We seek the radius of a sphere for which $p(r) = 0.90$. Substituting that value in the expression for $p(r)$, we have

$$0.90 = 1 - e^{-2x}(1 + 2x + 2x^2)$$

or

$$10e^{-2x}(1 + 2x + 2x^2) = 1.$$

We must find the value of x that satisfies this equality. It is not possible to solve explicitly for x, but an equation solver on a calculator yields $x = 2.66$. This means that the radius of a sphere within which the electron will be detected 90% of the time is $2.66a$. Mark this position on the horizontal axis of Fig. 38-14. The area under the curve from $r = 0$ to $r = 2.66a$ gives the probability of detection in that range and is 90% of the total area under the curve.

REVIEW AND SUMMARY

Confinement Confinement of waves (string waves, matter waves—any type of wave) leads to quantization—that is, discrete states with certain energies. States with intermediate energies are not allowed.

Electron in an Infinite Potential Well Because it is a matter wave, an electron confined to an infinite potential well can exist in only certain discrete states. If the well is one-dimensional with length L, the energies associated with these quantum states are

$$E_n = \left(\frac{h^2}{8mL^2}\right)n^2, \quad \text{for } n = 1, 2, 3, \ldots, \tag{38-4}$$

where m is the electron mass and n is a *quantum number*. The lowest energy, said to be the *zero-point energy*, is not zero but is given by $n = 1$. The electron can change (jump) from one state to another only if its energy change is

$$\Delta E = E_{\text{high}} - E_{\text{low}}, \tag{38-5}$$

where E_{high} is the higher energy and E_{low} is the lower energy. If the change is done by photon absorption or emission, the energy of the photon must be equal to the change in the electron's energy:

$$hf = \frac{hc}{\lambda} = \Delta E = E_{\text{high}} - E_{\text{low}}, \tag{38-6}$$

where frequency f and wavelength λ are associated with the photon.

The wave functions for an electron in an infinite, one-dimensional potential well with length L along an x axis are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{38-10}$$

where n is the quantum number and the factor $\sqrt{2/L}$ comes from normalizing the wave function. The wave function $\psi_n(x)$

does not have physical meaning, but the probability density $\psi_n^2(x)$ does have physical meaning: The product $\psi_n^2(x)\ dx$ is the probability that the electron will be detected in the interval between x and $x + dx$. If the probability density of an electron is integrated over the entire x axis, the total probability must be 1, which means that the electron will be detected somewhere along the x axis:

$$\int_{-\infty}^{\infty} \psi_n^2(x)\, dx = 1. \tag{38-14}$$

The Hydrogen Atom The Bohr model of the hydrogen atom successfully derived the energy levels for the atom, to explain the emission/absorption spectrum of the atom, but it is incorrect in almost every other aspect. It is a planetary model in which the electron orbits the central proton with an angular momentum L that is limited to values given by

$$L = n\hbar, \quad \text{for } n = 1, 2, 3, \ldots, \tag{38-19}$$

where n is a quantum number. The equation is, however, incorrect. Application of the Schrödinger equation gives the correct values of L and the quantized energies:

$$E_n = -\frac{me^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} = -\frac{13.60\text{ eV}}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{38-28, 38-29}$$

The atom (or, the electron in the atom) can change energy only by jumping between these allowed energies. If the jump is by photon absorption (the atom's energy increases) or photon emission (the atom's energy decreases), this restriction in energy changes leads to

$$\frac{1}{\lambda} = R\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right), \tag{38-32}$$

for the wavelength of the light, where R is the Rydberg constant,

$$R = \frac{me^4}{8\varepsilon_0^2 h^3 c} = 1.097\,373 \times 10^7\,\text{m}^{-1}. \tag{38-33}$$

The radial probability density $P(r)$ for a state of the hydrogen atom is defined so that $P(r)$ is the probability that the electron will be detected somewhere in the space between two spherical shells of radii r and $r + dr$ that are centered on the nucleus. The probability that the electron will be detected between any two given radii r_1 and r_2 is

$$(\text{Probability of detection}) = \int_{r_1}^{r_2} P(r)\, dr. \tag{38-50}$$

PROBLEMS

1. An atom (not a hydrogen atom) absorbs a photon whose associated wavelength is 400 nm and then immediately emits a photon whose associated wavelength is 580 nm. How much net energy is absorbed by the atom in this process?
2. Calculate the radial probability density $P(r)$ for the hydrogen atom in its ground state at (a) $r = 0$, (b) $r = a$, and (c) $r = 2a$, where a is the Bohr radius.
3. What is the ratio of the shortest wavelength of the Balmer series to the shortest wavelength of the Lyman series?
4. A neutron with a kinetic energy of 6.0 eV collides with a stationary hydrogen atom in its ground state. Explain why the collision must be elastic—that is, why kinetic energy must be conserved. (*Hint:* Show that the hydrogen atom cannot be excited as a result of the collision.)
5. An atom (not a hydrogen atom) absorbs a photon whose associated frequency is 5.6×10^{14} Hz. By what amount does the energy of the atom increase?
6. What is the probability that an electron in the ground state of the hydrogen atom will be found between two spherical shells whose radii are r and $r + \Delta r$, (a) if $r = 0.500a$ and $\Delta r = 0.010a$ and (b) if $r = 1.00a$ and $\Delta r = 0.01a$, where a is the Bohr radius? (*Hint:* Δr is small enough to permit the radial probability density to be taken to be constant between r and $r + \Delta r$.)
7. In the ground state of the hydrogen atom, the electron has a total energy of −13.6 eV. What are (a) its kinetic energy and (b) its potential energy if the electron is one Bohr radius from the central nucleus?
8. A hydrogen atom, initially at rest in the $n = 3$ quantum state, undergoes a transition to the ground state, emitting a photon in the process. What is the speed of the recoiling hydrogen atom?
9. How much work must be done to pull apart the electron and the proton that make up the hydrogen atom if the atom is initially in (a) its ground state and (b) the state with $n = 2$?
10. An electron is in a certain energy state in a one-dimensional, infinite potential well from $x = 0$ to $x = L = 180$ pm. The electron's probability density is zero at $x = 0.300L$, and $x = 0.400L$; it is not zero at intermediate values of x. The electron then jumps to the next lower energy level by emitting light. What is the change in the electron's energy?
11. The ground-state energy of an electron trapped in a one-dimensional infinite potential well is 2.6 eV. What will this quantity be if the width of the potential well is doubled?
12. Monochromatic radiation of wavelength λ is incident on a hydrogen sample in ground state. Hydrogen atom absorbs a fraction of light and subsequently emit radiation of six different wavelengths find the value of λ.
13. How much energy is required to cause an electron of hydrogen to move from $n = 1$ state to $n = 2$ state?

14. A particle is confined to the one-dimensional infinite potential well of Fig. 38-2. If the particle is in its ground state, what is its probability of detection between (a) $x = 0$ and $x = 0.25L$, (b) $x = 0.75L$ and $x = L$, and (c) $x = 0.25L$ and $x = 0.75L$?

15. For the hydrogen atom in its ground state, calculate (a) the probability density $\psi^2(r)$ and (b) the radial probability density $P(r)$ for $r = a$, where a is the Bohr radius.

16. Light of wavelength 102.6 nm is emitted by a hydrogen atom. What are the (a) higher quantum number and (b) lower quantum number of the transition producing this emission? (c) What is the name of the series that includes the transition?

17. What are the (a) energy, (b) magnitude of the momentum, and (c) wavelength of the photon emitted when a hydrogen atom undergoes a transition from a state with $n = 3$ to a state with $n = 1$?

18. What are the (a) wavelength range and (b) frequency range of the Lyman series? What are the (c) wavelength range and (d) frequency range of the Balmer series?

19. Calculate the probability that the electron in the hydrogen atom, in its ground state, will be found between spherical shells whose radii are a and $2a$, where a is the Bohr radius.

20. An electron is trapped in a one-dimensional infinite potential well. For what (a) higher quantum number and (b) lower quantum number is the corresponding energy difference equal to the energy of the $n = 5$ level? (c) Show that no pair of adjacent levels has an energy difference equal to the energy of the $n = 6$ level.

21. An electron with mass m collides inelastically with an atom of mass M initially at rest and excites it to an energy E above the ground state. Show that the initial energy of the electron must be at least $(1 + m/M)E$.

22. Suppose that a hydrogen atom in the ground state absorbs photons of wavelength 15 nm. (a) Will the atom be ionized? (b) If so, what will be the kinetic energy of the electron when it gets far away from its atom of origin?

23. What is the probability that in the ground state of the hydrogen atom, the electron will be found at a radius greater than the Bohr radius?

24. A proton is confined to a one-dimensional infinite potential well 120 pm wide. What is its ground-state energy?

25. An electron is trapped in a one-dimensional infinite potential well. For what (a) higher quantum number and (b) lower quantum number is the corresponding energy difference equal to the energy difference ΔE_{43} between the levels $n = 4$ and $n = 3$? (c) Show that no pair of adjacent levels has an energy difference equal to $2\Delta E_{43}$.

26. For what value of the principal quantum number n would the effective radius, as shown in a probability density dot plot for the hydrogen atom, be 1.0 mm? Assume that ℓ has its maximum value of $n - 1$. (*Hint:* See Fig. 38-19.)

27. A hydrogen atom is excited from its ground state to the state with $n = 4$. (a) How much energy must be absorbed by the atom? Consider the photon energies that can be emitted by the atom as it de-excites to the ground state in the several possible ways. (b) How many different energies are possible; what are the (c) highest, (d) second highest, (e) third highest, (f) lowest, (g) second lowest, and (h) third lowest energies?

28. An electron is trapped in a one-dimensional infinite potential well that is 100 pm wide; the electron is in its ground state. What is the probability that you can detect the electron in an interval of width $\Delta x = 5.0$ pm centered at $x =$ (a) 25 pm, (b) 50 pm, and (c) 90 pm? (*Hint:* The interval Δx is so narrow that you can take the probability density to be constant within it.)

29. An electron in a one-dimensional infinite potential well of length L has ground-state energy E_1. The length is changed to L' so that the new ground-state energy is $E_1' = 0.500E_1$. What is the ratio L'/L?

30. Consider an atomic nucleus to be equivalent to a one-dimensional infinite potential well with $L = 1.4 \times 10^{-14}$ m, a typical nuclear diameter. What would be the ground-state energy of an electron if it were trapped in such a potential well? (*Note:* Nuclei do not contain electrons.)

31. An electron, trapped in a one-dimensional infinite potential well 250 pm wide, is in its ground state. How much energy must it absorb if it is to jump up to the state with $n = 5$?

32. (a) What is the energy E of the hydrogen-atom electron whose probability density is represented by the dot plot of Fig. 38-16? (b) What minimum energy is needed to remove this electron from the atom?

33. What is the ground-state energy of (a) an electron and (b) a proton if each is trapped in a one-dimensional infinite potential well that is 300 pm wide?

34. What must be the width of a one-dimensional infinite potential well if an electron trapped in it in the $n = 3$ state is to have an energy of 4.7 eV?

35. An electron is trapped in a one-dimensional infinite well of width 200 pm and is in its ground state. What are the (a) longest, (b) second longest, and (c) third longest wavelengths of light that can excite the electron from the ground state via a single photon absorption?

36. Hydrogen atoms are in an excited state with $n = 5$. In terms of the Bohr's model, (a) how many spectral lines can be radiated as these atoms return to be ground state? (b) Find the energies of the line with the longest and shortest wavelengths.

37. Hydrogen atoms are initially in the $n = 4$ energy level. How many different photon energies will be emitted if the atoms go to the ground state, with all possible transition represented? Assuming that from any given excited state, all possible downward transitions are equally probable (this assumption is not true), what is the total number of photons emitted if there are 600 atoms initially in the $n = 4$ state?

38. Calculate the smallest kinetic energy which an electron may have and still excite a hydrogen atom initially at rest. What minimum kinetic energy is necessary to ionize the hydrogen atom?

39. Using the Bohr model, (a) calculate the speed of the electrons in a hydrogen atom in $n = 1, 2$, and 3 level and (b) calculate the orbital period in each of these levels.

40. Average lifetime of a hydrogen atom excited to $n = 2$ state is 10^{-8} s. Find the number of revolutions made by the electron on the average before it jumps to the ground state.

41. The energy needed to detach the electron of a hydrogen-like ion in ground state is 54.4 eV. (a) What is the wavelength of the radiation emitted when the electron jumps from the first excited state to the ground state? (b) What is the radius of the first orbit for this atom?

42. A beam of electrons bombards a sample of hydrogen. Through what potential difference must the electron have been accelerated if the first line of the Balmer series is to be emitted?

PRACTICE QUESTIONS

Single Correct Choice Type

1. If λ_1 and λ_2 are the wavelengths of the first members of Lyman and Paschen series respectively, then $\lambda_1 : \lambda_2$ is

(a) 1:3
(b) 1:30
(c) 7:50
(d) 7:108

2. A hydrogen atom of mass m emits a photon corresponding to the fourth line of Brackett series and recoils. If R is the Rydberg's constant and h is the Planck's constant, then recoiling velocity is

(a) $\dfrac{3Rh}{36m}$
(b) $\dfrac{3Rh}{64m}$
(c) $\dfrac{7Rh}{124m}$
(d) $\dfrac{7Rh}{3m}$

3. In Bohr's model of hydrogen atom, let PE represents potential energy and TE represents the total energy. In going to a higher level,

(a) PE decreases, TE increases
(b) PE increases, TE increases
(c) PE decreases, TE decreases
(d) PE increases, TE decreases

4. In hydrogen atom, which quantity is integral multiple of $h/2\pi$?

(a) Angular momentum
(b) Angular velocity
(c) Angular acceleration
(d) Momentum

5. The ratio of longest wavelength and the shortest wavelength observed in the five spectral series of emission spectrum of hydrogen is

(a) $\dfrac{4}{3}$
(b) $\dfrac{525}{376}$
(c) 25
(d) $\dfrac{900}{11}$

6. The ratio of minimum and maximum wavelength or radiation emitted by electron in ground state of Bohr's hydrogen atom is

(a) $\dfrac{3}{4}$
(b) $\dfrac{8}{9}$
(c) $\dfrac{9}{8}$
(d) $\dfrac{24}{25}$

7. The frequency f of certain line of the Lyman series of the atomic spectrum of hydrogen satisfies the following conditions: (i) it is the sum of the frequencies of another Lyman line and a Balmer line; (ii) it is the sum of the frequencies of a Lyman line, a Balmer line and a Paschen line; (iii) it is the sum of the frequencies of a Lyman and a Paschen line but no Brackett line. To what transition does f correspond?

(a) $n_2 = 3$ to $n_1 = 1$
(b) $n_2 = 3$ to $n_1 = 2$
(c) $n_2 = 2$ to $n_1 = 1$
(d) $n_2 = 4$ to $n_1 = 1$

8. The radius of the Bohr orbit in the ground state of hydrogen atom is 0.5 Å. The radius of the orbit of the electron in the third excited state of He^+ will be

(a) 8 Å
(b) 4 Å
(c) 0.5 Å
(d) 0.25 Å

9. The extreme wavelengths of Paschen series are

(a) 0.365 μm and 0.565 μm
(b) 0.818 μm and 1.89 μm
(c) 1.45 μm and 4.04 μm
(d) 2.27 μm and 7.43 μm

10. The wavelength or radiations emitted is λ_0 when an electron in hydrogen atom jumps from the third orbit to second. If in the H-atom itself, the electron jumps from fourth orbit to second orbit, the wavelength of emitted radiation will be

(a) $\dfrac{20}{27}\lambda_0$
(b) $\dfrac{16}{25}\lambda_0$
(c) $\dfrac{27}{20}\lambda_0$
(d) $\dfrac{29}{16}\lambda_0$

11. In hydrogen atom, if the difference in the energy of the electron in $n = 2$ and $n = 3$ orbits is E, the ionization energy of hydrogen atom is

(a) $13.2E$
(b) $7.2E$
(c) $5.6E$
(d) $3.2E$

12. Which of the following is true?

(a) Lyman series is a continuous spectrum
(b) Paschen series is a line spectrum in the infrared
(c) Balmer series is a line spectrum in the ultraviolet
(d) The spectral series formula can be derived from the Rutherford model of the hydrogen atom

13. According to Bohr's theory, the variation of perimeter(s) of the electronic orbit with the order of orbit (n) in a particular atom is

(a) Linear **(b)** Parabolic
(c) Exponential **(d)** Rectangular hyperbolic

14. An excited hydrogen atom emits a photon of wavelength, λ, in returning to the ground state. The quantum number n of the excited state is given by (R = Rydberg's constant)

(a) $\sqrt{\lambda R(\lambda R-1)}$ **(b)** $\sqrt{\lambda R/(\lambda R-1)}$
(c) $\sqrt{(\lambda R-1)/\lambda R}$ **(d)** $\dfrac{1}{\sqrt{\lambda R(\lambda R-1)}}$

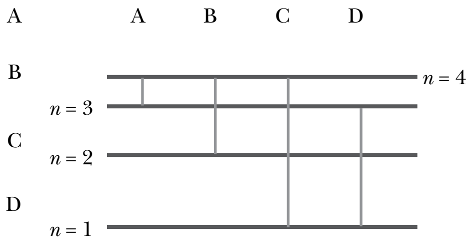
15. If the shortest wavelength of Lyman series of hydrogen atom is x, then the wavelength of first member of Balmer series of hydrogen atom will be

(a) $9x/5$ **(b)** $36x/5$
(c) $5x/9$ **(d)** $5x/36$

16. A particle in a box has quantum states with energies $E = E_0 n^2$, with $n = 1, 2, 3, 4, \ldots$ and $E_0 = 1$ eV. Which of these photons could in principle be absorbed?

(a) 1 eV **(b)** 2 eV
(c) 4 eV **(d)** 5 eV

17. Which of the following transitions produce the longest wavelength photon (as shown in the following figure)?

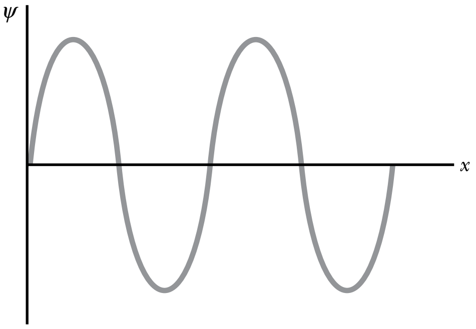


(a) $n = 4 \rightarrow 3$ **(b)** $n = 4 \rightarrow 2$
(c) $n = 3 \rightarrow 1$ **(d)** $n = 3 \rightarrow 2$

18. For a quantum particle in a box, the lowest energy quantum state has 1/2 of the particle de Broglie wave fitting in the box. The next highest state corresponds to

(a) 1/4 de Broglie wavelength
(b) 1/2 de Broglie wavelength
(c) 1 de Broglie wavelength
(d) 2 de Broglie wavelength

19. An electron is in a one-dimensional trap with zero potential energy in the interior and infinite potential energy at the walls. A graph of its wave function $y(x)$ versus x is shown in the following figure. The value of quantum number n is



(a) 0 **(b)** 2
(c) 4 **(d)** 6

20. A particle is trapped in an infinite potential energy well. It is in the state with quantum number $n = 14$. How many nodes does the probability density have (counting the nodes at the ends of the well)?

(a) None **(b)** 7
(c) 13 **(d)** 15

21. A particle is confined to a one-dimensional trap by infinite potential energy walls. Of the following states, designed by the quantum number n, for which one is the probability density greatest near the center of the well?

(a) $n = 2$ **(b)** $n = 3$
(c) $n = 4$ **(d)** $n = 5$

22. In the Bohr model of hydrogen, why is the atom 9 times larger in the $n = 3$ state than in the ground state?

(a) The electrons are moving 9 times slower
(b) The de Broglie wavelength is 9 times longer
(c) There are 9 times as many wavelengths in the orbit
(d) The de Broglie wavelength is 3 times longer, and there are 3 times as many wavelengths

23. Orbital electrons do not spiral into the nucleus because of

(a) Electromagnetic forces
(b) Angular momentum conservation
(c) The large nuclear size compared to the electron's size
(d) The wave nature of the electron

24. Bohr's model cannot explain the spectrum of neutral Lithium atoms because

(a) Lithium has neutrons in its nucleus
(b) Lithium has 3 protons in its nucleus
(c) Each lithium atom has more than one electron
(d) Neutral lithium atoms have no spectral lines at all

25. Compared to hydrogen the atom of helium has

(a) More mass and is larger in size
(b) More mass and is about the same in size
(c) More mass and is smaller in size
(d) None of the above

26. If the value of Planck's constant were to increase by a factor of two, the size of a hydrogen atom would

(a) Decrease by a factor of 4
(b) Decrease by a factor of 2
(c) Increase by a factor of 2
(d) Increase by a factor of 4

27. In the Bohr model, if an electron moves in an orbit of greater radius

(a) Its total energy increases but its kinetic energy decreases
(b) Its total energy decreases but its kinetic energy increases
(c) Its total energy as well as its kinetic energy decreases
(d) Its total energy as well as its kinetic energy increases

28. Which of the following statements is (are) wrong for hydrogen atom?

(a) The rotational frequency of an electron in an orbit of radius r proportional to the cube of the radius of the orbit

(b) The linear momentum of the revolving electron is inversely proportional to the square root of the radius of the orbit

(c) The total energy of the system is inversely proportional to the radius of the orbit

(d) The magnetic moment of the electron is proportional to square of the radius of the orbit

29. The total energy of an electron in the excited state corresponding to $n = 3$ state is E. What is its potential energy with proper sign?

(a) $-2E$ **(b)** $2E$

(c) $-E$ **(d)** E

30. Identify the highest energy photons of the following

(a) Industrial strength microwave cookers

(b) High power AM radio transmitters that cover thousands of square miles

(c) Waves from atomic electron transitions among outer orbits

(d) Waves from atomic transitions to inner orbits

31. A "toy" atom consists of three energy levels given by a ground state with energy $E_0 = 0$, the first excited state with energy $E_1 = e$ and the second excited state with energy $E_2 = 2e$ where $e > 0$. The atom is initially in the ground state. Light from a laser that emits photons with energy $1.5e$ is incident on the atom. What happens

(a) Two photons are absorbed, putting one atom in a state E_1 and one atom in a state E_2

(b) A photon will always be absorbed, but half the time the atom will go into the state will energy e and the other half into the state with energy $2e$. In this way energy will be conserved on the average

(c) The atom absorbs a photon, goes into the first excited state with energy e, and emits a photon with energy $0.5e$ to conserve energy

(d) The atom does not absorb any photons and stays in the ground state

32. When radiation with a continuous spectrum is passed through a volume of hydrogen gas whose atoms are all in the ground state, which spectral series will be present in the resulting absorption spectrum?

(a) Lyman **(b)** Balmer

(c) Paschen **(d)** Brackett

33. Which of the following statement is correct in connection with hydrogen spectrum

(a) The longest wavelength in the Balmer series is longer than the longest wavelength in Lyman series

(b) The shortest wavelength in the Balmer series is shorter than the shortest wavelength in the Lyman series

(c) The longest wavelength in both Balmer and Lyman series is equal

(d) The longest wavelength in Balmer series is shorter than the longest wavelength in the Lyman series

34. A hydrogen atom is excited up to 9th level. The total number of possible spectral lines emitted by the hydrogen atom is

(a) 36 **(b)** 35

(c) 37 **(d)** 38

35. A hydrogen atom makes a transition from $n = 2$ to $n = 1$ and emits a photon. This photon strikes a doubly ionized lithium atom in the excited state and completely removes the orbiting electron. The least quantum number for the excited state of the ion required for the process is

(a) 2 **(b)** 5

(c) 3 **(d)** 4

36. In a sample of hydrogen-like atoms all of which are in ground state, a photon beam containing photons of various energies is passed. In absorption spectrum, five dark lines are observed. The number of bright lines in the emission spectrum will be (assume that all transitions take place)

(a) 5 **(b)** 10

(c) 15 **(d)** None of these

37. At some place in universe an atom consists of a positron revolving round an antiproton. The ratio of the wavelength of the corresponding spectral lines from this atom and ordinary hydrogen atoms is

(a) 1 **(b)** $\gg 1$

(c) $\ll 1$ **(d)** Infinite

38. A hydrogen atom in its ground state absorbs a photon of energy 10.2 eV and gets excited. The change in angular momentum of the electrons in this process is (in kg/m^2/s) nearly

(a) 1×10^{-34}

(b) 2×10^{-34}

(c) 0.5×10^{-34}

(d) Cannot be determined

39. An electron with kinetic energy E eV collides with a hydrogen atom in the ground state. The collision will be elastic

(a) For all values of E

(b) For $E < 10.2$ eV

(c) For $E < 13.6$ eV

(d) Only for $E < 3.4$ eV

40. Consider the following:

(i) The probability density for an $l = 0$ state

(ii) The probability density for a state with $l \neq 0$

(iii) The average of the probability densities for all states in an $l \neq 0$ subshell

Of these, which are spherically symmetric?

(a) Only (i)

(b) Only (ii)

(c) Only (i) and (ii)

(d) Only (i) and (iii)

41. If $P(r)$ is the radial probability density for a hydrogen atom then the probability that the separation of the electron and proton is between r and $r + dr$ is

(a) $P\,dr$ **(b)** $|P|^2\,dr$

(c) $4\pi r^2 P\,dr$ **(d)** $4\pi r^2 |P|\,dr$

42. Which of the following sets of quantum numbers is possible for an electron in a hydrogen atom?

(a) $n = 4, l = 3, m_l = -3$

(b) $n = 4, l = 4, m_l = -2$

(c) $n = 5, l = -1, m_l = 2$

(d) $n = 3, l = 1, m_l = -2$

43. Which one of the following pairs of characteristics of light is best explained by assuming that light can be described in terms of photons?
(a) Photoelectric effect and the effect observed in Young's experiment
(b) Diffraction and the formation of atomic spectra
(c) Polarization and the photoelectric effect
(d) Existence of line spectra and the photoelectric effect

44. Complete the following statement: For the ground state of the hydrogen atom, the Bohr model correctly predicts
(a) Only the energy
(b) Only the angular momentum
(c) Only the angular momentum and the spin
(d) The angular momentum and the energy

45. Which one of the following will result in an electron transition from the $n = 4$ level to the $n = 7$ level in a hydrogen atom?
(a) Emission of a 0.28 eV photon
(b) Absorption of a 0.28 eV photon
(c) Emission of a 0.57 eV photon
(d) Absorption of a 0.57 eV photon

46. Determine the wavelength of incident electromagnetic radiation required to cause an electron transition from the $n = 6$ to the $n = 8$ level in a hydrogen atom.
(a) 1.2×10^3 nm
(b) 7.5×10^3 nm
(c) 2.2×10^3 nm
(d) 5.9×10^3 nm

47. The kinetic energy of the ground state electron in hydrogen is +13.6 eV. What is its potential energy?
(a) –13.6 eV
(b) –27.2 eV
(c) +27.2 eV
(d) +56.2 eV

48. Determine the kinetic energy of an electron that has a de Broglie wavelength equal to twice the diameter of the hydrogen atom. Assume that the hydrogen atom is a sphere of radius 5.3×10^{-11} m.
(a) 13.6 eV **(b)** 33.6 eV
(c) 27.2 eV **(d)** 48.9 eV

49. According to the quantum mechanical picture of the atom, which one of the following statements is true concerning the magnitude of the angular momentum L of an electron in the $n = 3$ level of the hydrogen atom?
(a) L is $0.318h$
(b) L could be $0.225h$ or $0.276h$
(c) L is $0.477h$
(d) L could be $0.225h$ or $0.390h$

50. Which quantum number applies to most of the electrons in a collection of hydrogen atoms at room temperature?
(a) $n = 1$ **(b)** $n = 3$
(c) $n = 2$ **(d)** $n = 4$

51. How many electron states (including spin states) are possible in a hydrogen atom if its energy is –3.4 eV?
(a) 2 **(b)** 6
(c) 4 **(d)** 8

More than One Correct Choice Type

52. An electron is revolving around a nucleus of hydrogen atom in the first orbit. The radius of this orbit is 0·53 Å. Choose the correct option(s)
(a) The radius of the first orbit of hydrogen-like atom He^+ is 1·06 Å
(b) The radius of the first orbit of hydrogen-like atom He^+ is 0·265 Å
(c) The energy of the electron of hydrogen-like atom He^+ in the first orbit is –13 eV
(d) The energy of the electron of hydrogen-like atom He^+ in the first orbit is –54·4 eV

53. According to Bohr's theory, hydrogen atom for an electron in the nth permissible orbit
(a) Radius of orbit $\propto n^2$
(b) Angular momentum $\propto n$
(c) Velocity $\propto n$
(d) Linear momentum $\propto (1/n)$

54. In an electron transition inside a hydrogen atom, orbital angular momentum may change by
(a) h **(b)** h/π
(c) $h/2\pi$ **(d)** $h/4\pi$

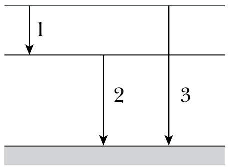
55. Whenever a hydrogen atom emits a photon in the Balmer series,
(a) It may emit another photon in Balmer series
(b) It must emit another photon in Lyman series
(c) The second photon, if emitted, will have a wavelength of about 122 nm
(d) It may emit a second photon, but the wavelength of this photon cannot be predicted

56. An electron is excited from a lower energy state to a higher energy state in a hydrogen atom. Which of the following quantity/quantities decreases/decrease in the excitation?
(a) Potential energy
(b) Angular speed
(c) Kinetic energy
(d) Angular momentum

57. An electron in hydrogen atom first jumps from second excited state to first excited state and then from first excited state to ground state. Let the ratio of wavelength, momentum, and energy of photons emitted in these two cases be a, b, and c, respectively. Then
(a) $c = 1/a$ **(b)** $a = 9/4$
(c) $b = 5/27$ **(d)** $c = 5/27$

58. A neutron collides head on with a stationary hydrogen atom in ground state. Which of the following statements are correct? (Assume that the hydrogen atom and neutron has same mass.)
(a) If kinetic energy of the neutron is less than 20.4 eV, collision must be elastic.
(b) If kinetic energy of the neutron is less than 20.4 eV, collision may be inelastic.
(c) Inelastic collision may be take place only when initial kinetic energy of neutron is greater than 20.4 eV.
(d) Perfectly inelastic collision cannot take place.

59. The wavelengths and frequencies of photons in transitions 1, 2 and 3, as shown in figure given below, for hydrogen-like atom are λ_1, λ_2, λ_3, v_1, v_2, and v_3, respectively. Then



(a) $v_3 = v_1 + v_2$ **(b)** $v_3 = \frac{v_1 v_2}{v_1 + v_2}$

(c) $\lambda_3 = \lambda_1 + \lambda_3$ **(d)** $\lambda_3 = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$

60. A particular hydrogen-like atom has its ground state binding energy 122.4 eV. It is in ground state. Then
(a) Its atomic number is 3
(b) An electron of 90 eV can excite it
(c) An electron of kinetic energy nearly 91.8 eV can be brought to almost rest by this atom
(d) An electron of kinetic energy 2.6 eV may emerge from the atom when electron of kinetic energy 125 eV collides with this atom

61. In the Bohr hydrogen atom model, R, V, and E represent the radius of the orbit, speed of the electron, and total energy of the electron, respectively. Which of the following quantities are proportional to the quantum number n?
(a) VR **(b)** RE
(c) VE^{-1} **(d)** RE^{-1}

62. When Z is doubled in an atom, which of the following statements are consistent with Bohr's theory?
(a) Energy of a state is doubled
(b) Radius of an orbit is doubled
(c) Velocity of electrons in an orbit is doubled
(d) Radius of an orbit is halved

63. If electron of the hydrogen atom is replaced by another particle of same charge but of double the mass, then
(a) Bohr radius will increase
(b) Ionization energy of the atom will be doubled
(c) Speed of the new particle in a given state will be lesser than the electron's speed in same orbit
(d) Gap between energy levels will now be doubled

64. If the potential energy is taken zero at ground state of hydrogen atom in place of infinity then
(a) Total energy of atom will get changed
(b) Binding energy of hydrogen atom in ground state will remain unchanged
(c) Kinetic energy of electron in ground state will get changed
(d) Energy of photon generated because of de-excitation of atom from first excited state will remain unchanged

65. A beam of ultraviolet light of a mixture of several wavelengths passes through hydrogen gas at room temperature, in the x direction. Assume that all the photons emitted due to electron transitions inside the gas emerge in the y direction. Let A and B denote the lights emerging from the gas in the x and y directions.
(a) Some of the incident wavelengths will be absent in A
(b) Only those wavelengths will be present in B which are absent in A
(c) B will contain some visible light
(d) B will contain some infrared light

66. A free hydrogen atom in ground state is at rest. A neutron of kinetic energy K collides with the hydrogen atom. After collision hydrogen atom emits two photons in succession one of which has energy 2.55 eV. (Assume that the hydrogen atom and neutron has same mass.)
(a) Minimum value of K is 25.5 eV
(b) Minimum value of K is 12.75 eV
(c) The other photon has energy 10.2 eV
(d) The upper energy level is of excitation energy 12.75 eV

Linked Comprehension

Paragraph for Questions 67–69: Hydrogen atom in its ground state is excited by means of a monochromatic radiation of wavelength 970·6 Å. Different wavelengths are possible in the spectrum. After absorbing the energy of radiation, hydrogen atom goes to the excited state. After 10^{-8} s, the hydrogen atom will come to the ground state by emitting the absorbed energy.

67. Energy absorbed by hydrogen atom is
(a) 1·82 eV **(b)** 2·18 eV
(c) 8·12 eV **(d)** 12·8 eV

68. The electron of excited hydrogen atom is in which of the following energy states?
(a) 2 **(b)** 3
(c) 4 **(d)** 5

69. Number of different wavelengths present in the spectrum is
(a) 4 **(b)** 6
(c) 8 **(d)** 5

Paragraph for Questions 70–72: A beam of α-particles is incident on a target of lead. A particular α-particle comes in "head-on" to a particular lead nucleus and stops 6.50×10^{-14} m away from the center of the nucleus. (This point is well outside the nucleus). Assume that the lead nucleus which has 82 protons remains at rest. The mass of α particle is 6.64×10^{-27} kg.

70. Calculate the electrostatic potential energy at the instant that the α-particle stops
(a) 36.3 MeV **(b)** 45.0 MeV
(c) 3.63 MeV **(d)** 40.0 MeV

71. What initial kinetic energy (in Joules and in MeV) did the α-particle have
(a) 36.3 **(b)** 0.36
(c) 3.63 **(d)** 2.63

72. What was the initial speed of the α-particle
(a) 132×10^2 m/s
(b) 1.32×10^7 m/s
(c) 13.2×10^2 m/s
(d) 0.13×10^7 m/s

Paragraph for Questions 73–75: Two hydrogen-like atoms A and B are of different masses and each atom contains equal number of protons and neutrons. The difference in the energies between the first Balmer lines emitted by A and B is 5.667 eV, when the atoms A and B, moving with the same velocity, strike a heavy target and rebound back with the same velocity. In the process, atom B imparts twice the momentum to the target than that A imparts.

73. The series emitted is
(a) Lyman (b) Balmer
(c) Pfund (d) Paschen

74. Atom A is
(a) $^{3}_{2}H$ (b) $^{2}_{1}H$
(c) $^{2}_{3}H$ (d) $^{1}_{2}H$

75. Atom B is
(a) $^{4}_{2}H$ (b) $^{2}_{4}H$
(c) $^{1}_{1}H$ (d) $^{3}_{2}H$

Matrix-Match

76. The ground state and first excited state energies of hydrogen atom are –13.6 eV and –3.4 eV, respectively. If potential energy in ground state is taken to be zero, then

Column I	Column II
(a) Potential energy in the first excited state would be	**(p)** 3.4 eV
(b) Total energy is the first excited state would be	**(q)** 23.8 eV
(c) Kinetic energy in the first excited state would be	**(r)** 20.4 eV
(d) Total energy in the ground state would be	**(s)** 13.6 eV

77. The question is for spectral series of hydrogen atom.

Column I	Column II
(a) Lyman series	**(p)** Infrared region
(b) Balmer series	**(q)** Ultraviolet region
(c) Paschen series	**(r)** Visible region
(d) Brackett series	**(s)** Invisible region

Directions for Questions 78 and 79: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

78. In a hydrogen atom, whenever there is transition of electron between different energy levels, energy is emitted or absorbed, and hydrogen spectrum is obtained. In the given table, Column I shows the names of the different series of hydrogen atom spectrum, Column II shows the energy levels from where electron transition takes place and Column III shows the EM wave region where these different series exists.

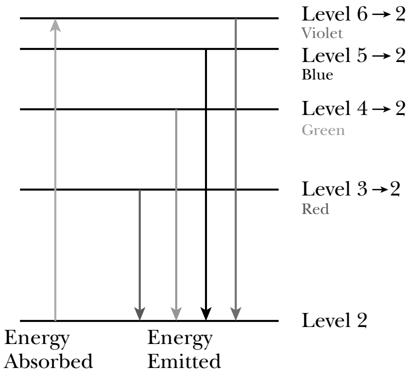
Column I	Column II	Column III
(I) Brackett series	**(i)** Electron jumps from any of the higher states to the ground state or first state	**(J)** The series of spectral lines emitted lies in far infra-red region
(II) Lyman series	**(ii)** When an electron jumps from any of the higher states to the state with $n = 3$	**(K)** The series of spectral lines emitted lies in ultra-violet region
(III) Balmer series	**(iii)** Electron jumps from any of the higher states to the state with $n = 4$	**(L)** The series of spectral lines emitted lies in near infra-red region
(IV) Paschen series	**(iv)** Electron jumps from any of the higher states to the state with $n = 2$	**(M)** The series of spectral lines emitted lies in visible region

(1) Which series has the following wavelength

$$v = RZ^2\left(\frac{1}{1^2} - \frac{1}{n_2^2}\right), n_2 = 2, 3, 4, 5, \ldots?$$

(a) (I) (iv) (M) (b) (IV) (ii) (L)
(c) (II) (i) (K) (d) (II) (i) (K)

(2) Which series does the diagram depict?



(a) (I) (ii) (J) (b) (III) (iv) (M)
(c) (II) (iii) (L) (d) (I) (i) (M)

(3) Which series has the following wavelengths

$$v = RZ^2\left(\frac{1}{4^2} - \frac{1}{n_2^2}\right)?$$

(a) (II) (ii) (J) (b) (I) (iii) (J)
(c) (IV) (i) (L) (d) (II) (iii) (M)

79. Different hydrogen spectral series have different wavelengths. In the given table, Column I shows the values of energy levels between which electronic transition takes place, Column II shows the formula of maximum wavelength of different spectral series and Column III shows the formula of minimum wavelength of different spectral series

Column I	Column II	Column III
(I) $n_2 = 2, 3, 4, \ldots, \infty$; $n_1 = 1$	**(i)** $\lambda_{\max} = \dfrac{36}{5R}$	**(J)** $\lambda_{\min} = \dfrac{25}{R}$
(II) $n_2 = 3, 4, 5, \ldots, \infty$; $n_1 = 2$	**(ii)** $\lambda_{\max} = \dfrac{4}{3R}$	**(K)** $\lambda_{\min} = \dfrac{1}{R}$
(III) $n_2 = 6, 7, 8, \ldots, \infty$; $n_1 = 5$	**(iii)** $\lambda_{\max} = \dfrac{144}{7R}$	**(L)** $\lambda_{\min} = \dfrac{4}{R}$
(IV) $n_2 = 4, 5, 6, \ldots, \infty$; $n_1 = 3$	**(iv)** $\lambda_{\max} = \dfrac{900}{11R}$	**(M)** $\lambda_{\min} = \dfrac{9}{R}$

(1) Determine the wavelengths, $\lambda_{\max}$ and $\lambda_{\min}$ for Balmer series.

- **(a)** (II) (i) (L)
- **(b)** (III) (ii) (K)
- **(c)** (I) (i) (L)
- **(d)** (I) (iii) (J)

(2) Determine the wavelengths $\lambda_{\max}$ and $\lambda_{\min}$ for Pfund series.

- **(a)** (I) (ii) (M)
- **(b)** (III) (iv) (J)
- **(c)** (II) (ii) (K)
- **(d)** (II) (iv) (M)

(3) Determine the wavelength, $\lambda_{\max}$ and $\lambda_{\min}$ for Paschen series.

- **(a)** (III) (i) (L)
- **(b)** (IV) (i) (J)
- **(c)** (III) (iv) (J)
- **(d)** (IV) (iii) (M)

Integer Type

80. The radius of hydrogen atom in its ground state is 5.3×10^{-11} m. After collision with an electron, it is found to have a radius of 21.2×10^{-11} m. What is the principal quantum number of the final state of the atom?

81. An electron is placed in an orbit about a nucleus of charge $+Ze$. It requires 47.2 eV energy to excite an electron from second Bohr orbit to third Bohr orbit. What is the value of Z?

82. A hydrogen-like atom (described by Bohr's model) is observed to emit six wavelengths originating from all possible transitions between a group of levels. These levels have energies between –0.85 eV and –0.544 eV (including both these values). Find the atomic number of the atom.

83. Determine the energy of the photon (in eV) emitted when the electron in a hydrogen atom undergoes a transition from the $n = 8$ level to the $n = 6$ level.

84. An electron is in the ground state of a hydrogen atom. A photon is absorbed by the atom and the electron is excited to the $n = 2$ state. What is the energy in eV of the photon?

85. Two possible states for the hydrogen atom are labeled A and B. The maximum magnetic quantum number for state A is +3. For state B, the maximum value is +1. What is the ratio of the magnitudes of the orbital angular momenta, L_A/L_B, of an electron in these two states?

ANSWER KEY

Checkpoints

1. (b), (a), (c)
2. (a) all tie; (b) *a*, *b*, *c*
3. (a), (b), (c), (d)
4. (a) emission spectra; (b) absorption spectra
5. The radius would be the same
6. (a) 5; (b) 7

Problems

1. 0.962 eV
2. (a) 0; (b) 10.2 nm^{-1}; (c) 5.54 nm^{-1}
3. 4.0
5. 2.3 eV
6. (a) 3.7×10^{-3}; (b) 5.4×10^{-3}
7. (a) 13.6 eV; (b) –27.2 eV
8. 3.9 m/s
9. 3.40 eV
10. 3.54×10^{-17} J
11. 0.65 eV
12. 97.5 nm
13. 10.2 eV
14. (a) 0.091; (b) 0.091; (c) 0.82
15. (a) 291 nm^{-3}; (b) 10.2 nm^{-1}
16. (a) $n = 3$; $n = 1$; (c) Layman series
17. (a) 12.1 eV; (b) 6.45×10^{-27} kg × m/s; (c) 102 nm
18. (a) 31 nm; (b) 8.2×10^{14} Hz; (c) 292 nm; (d) 3.7×10^{14} Hz
19. 0.439
20. (a) 13; (b) 12
22. Yes, 69 eV
23. 68%
24. 0.0143 eV
25. (a) 11; (b) 10
26. 4.3×10^{3}
27. (a) 12.8 eV; (b) 6; (c) 12.8 eV; (d) 12.1 eV; (e) 10.2 eV; (f) 0.661 eV; (g) 1.89 eV; (h) 2.55 eV
28. (a) 0 050; (b) 010; (c) 0 0095
29. 1.41
30. 3.07×10^{-10} J
31. 144 eV
32. (a) –3.4 eV; (b) 3.4 eV
33. (a) 1.51×10^{-18} J; (b) 8.225×10^{-22} J
34. 0.85 nm
35. (a) 4.37×10^{-8} m; (b) 1.65×10^{-8} m; (c) 8.77×10^{-9} m
36. (a) 10; (b) 4000 nm, 95 nm
37. 6, 1100
38. 10.2 eV, 13.6 eV
39. (a) 2.18×10^{6} m/s, 1.09×10^{6} m/s, 7.27×10^{6} m/s; (b) 1.53×10^{-16} s, 1.22×10^{-15} s, 4.13×10^{-15} s
40. 8.2×10^{6}
41. (a) 30.4 nm (b) 26.5 pm
42. 12.1 V

Practice Questions

Single Correct Choice Type

1. (d)
2. (b)
3. (b)
4. (a)
5. (c)
6. (a)
7. (d)
8. (b)
9. (d)
10. (a)
11. (b)
12. (b)
13. (b)
14. (b)
15. (b)
16. (d)
17. (a)
18. (c)
19. (c)
20. (d)
21. (b)
22. (d)
23. (d)
24. (c)
25. (b)
26. (d)
27. (a)
28. (a)
29. (b)
30. (d)
31. (a)
32. (a)
33. (a)
34. (a)
35. (d)
36. (c)
37. (a)
38. (a)
39. (b)
40. (d)
41. (a)
42. (a)
43. (d)
44. (a)
45. (d)
46. (b)
47. (b)
48. (b)
49. (d)
50. (a)
51. (d)

More than One Correct Choice Type

52. (b), (d)
53. (a), (b), (d)
54. (b), (c)
55. (b), (c)
56. (b), (c)
57. (a), (c), (d)
58. (a), (c)
59. (a), (d)
60. (a), (c), (d)
61. (a), (c)
62. (c), (d)
63. (b), (d)
64. (a), (b), (d)
65. (a), (c), (d)
66. (a), (c), (d)

Linked Comprehension

67. (d)
68. (d)
69. (a)
70. (c)
71. (c)
72. (b)
73. (b)
74. (b)
75. (a)

Matrix-Match

76. (a) → (r); (b) → (q); (c) → (p); (d) → (s)
77. (a) → (q); (b) → (r); (c) → (p); (d) → (p)
78. (1) → (d); (2) → (b); (3) → (b)
79. (1) → (a); (2) → (b); (3) → (d)

Integer Type

80. 2
81. 5
82. 3
83. 0.17
84. 10.2
85. 2.45

CHAPTER

39

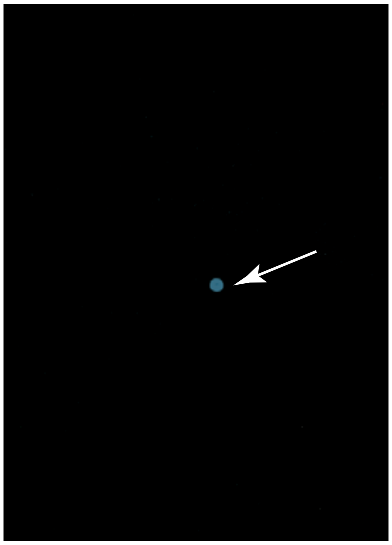
All About Atoms

Contents

39.1 What is Physics?
39.2 Some Properties of Atoms
39.3 Angular Momentum, Magnetic Dipole Moments
39.4 The Pauli Exclusion Principle
39.5 Building the Periodic Table
39.6 X Rays and the Ordering of the Elements
39.7 Lasers and Laser Light
39.8 How Lasers Work

39.1 | WHAT IS PHYSICS?

In this chapter we continue with a primary goal of physics—discovering and understanding the properties of atoms. About 100 years ago, researchers struggled to find experiments that would prove the existence of atoms. Now we take their existence for granted and even have photographs (scanning tunneling microscope images) of atoms. We can drag them around on surfaces, and even hold an individual atom indefinitely in a trap (Fig. 39-1) so as to study its properties when it is completely isolated from other atoms.



Courtesy Warren Nagourney

Figure 39-1 The blue dot is a photograph of the light emitted from a single barium ion held for a long time in a trap at the University of Washington. Special techniques caused the ion to emit light over and over again as it underwent transitions between the same pair of energy levels. The dot represents the cumulative emission of many photons.

39.2 | SOME PROPERTIES OF ATOMS

Key Concept

- Atoms have quantized energies and can make quantum jumps between them. If a jump between a higher energy and a lower energy involves the emission or absorption of a photon, the frequency associated with the light is given by

$$hf = E_{\text{high}} - E_{\text{low}}.$$

You may think the details of atomic physics are remote from your daily life. However, consider how the following properties of atoms—so basic that we rarely think about them—affect the way we live in our world.

Atoms are stable. Essentially all the atoms that form our tangible world have existed without change for billions of years. What would the world be like if atoms continually changed into other forms, perhaps every few weeks or every few years?

Atoms combine with each other. They stick together to form stable molecules and stack up to form rigid solids. An atom is mostly empty space, but you can stand on a floor—made up of atoms—without falling through it.

These basic properties of atoms can be explained by quantum physics, as can the three less apparent properties that follow.

Atoms Are Put Together Systematically

Figure 39-2 shows an example of a repetitive property of the elements as a function of their position in the periodic table (Appendix G). The figure is a plot of the **ionization energy** of the elements; the energy required to remove the most loosely bound electron from a neutral atom is plotted as a function of the position in the periodic table of the element to which the atom belongs. The remarkable similarities in the chemical and physical properties of the elements in each vertical column of the periodic table are evidence enough that the atoms are constructed according to systematic rules.

The elements are arranged in the periodic table in six complete horizontal **periods** (and a seventh incomplete period): except for the first, each period starts at the left with a highly reactive alkali metal (lithium, sodium, potassium, and so on) and ends at the right with a chemically inert noble gas (neon, argon, krypton, and so on). Quantum physics accounts for the chemical properties of these elements. The numbers of elements in the six periods are

2, 8, 8, 18, 18, and 32.

Quantum physics predicts these numbers.

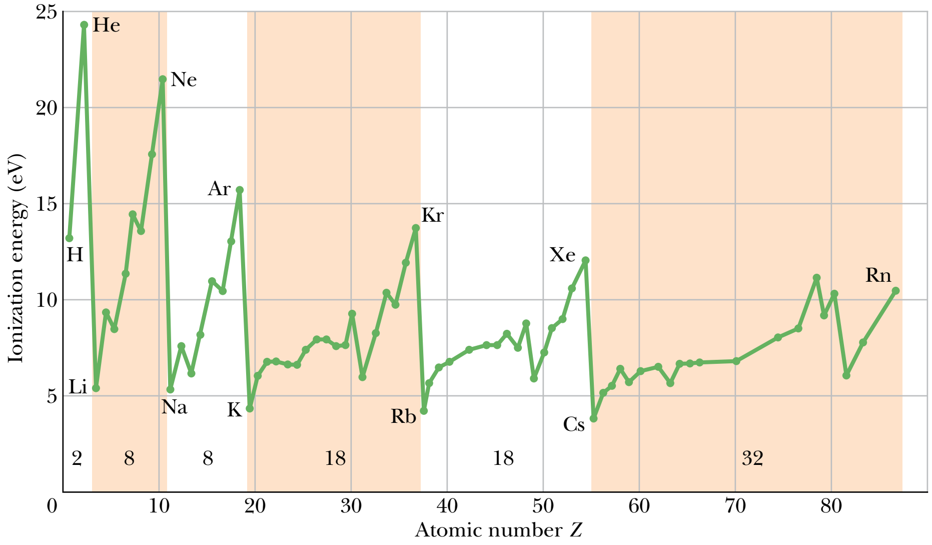


Figure 39-2 A plot of the ionization energies of the elements as a function of atomic number, showing the periodic repetition of properties through the six complete horizontal periods of the periodic table. The number of elements in each of these periods is indicated.

Atoms Emit and Absorb Light

We have already seen that atoms can exist only in discrete quantum states, each state having a certain energy. An atom can make a transition from one state to another by emitting light (to jump to a lower energy level E_{low}) or by absorbing light (to jump to a higher energy level E_{high}). As we first discussed in Section 38.3, the light is emitted or absorbed as a photon with energy

$$hf = E_{\text{high}} - E_{\text{low}}. \tag{39-1}$$

Thus, the problem of finding the frequencies of light emitted or absorbed by an atom reduces to the problem of finding the energies of the quantum states of that atom. Quantum physics allows us—in principle at least—to calculate these energies.

Atoms Have Angular Momentum and Magnetism

Figure 39-3 shows a negatively charged particle moving in a circular orbit around a fixed center. The orbiting particle has both an angular momentum $\vec{L}$ and (because its path is equivalent to a tiny current loop) a magnetic dipole moment $\vec{\mu}$. As Fig. 39-3 shows, vectors $\vec{L}$ and $\vec{\mu}$ are both perpendicular to the plane of the orbit but, because the charge is negative, they point in opposite directions.

The model of Fig. 39-3 is strictly classical and does not accurately represent an electron in an atom. In quantum physics, the rigid orbit model has been replaced by the probability density model, best visualized as a dot plot. In quantum physics, however, it is still true that in general, each quantum state of an electron in an atom involves an angular momentum $\vec{L}$ and a magnetic dipole moment $\vec{\mu}$ that have opposite directions (those vector quantities are said to be *coupled*).

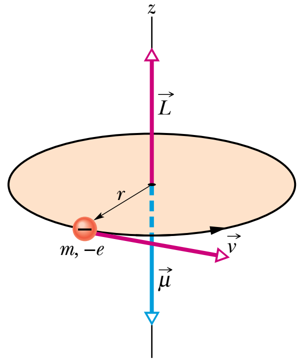
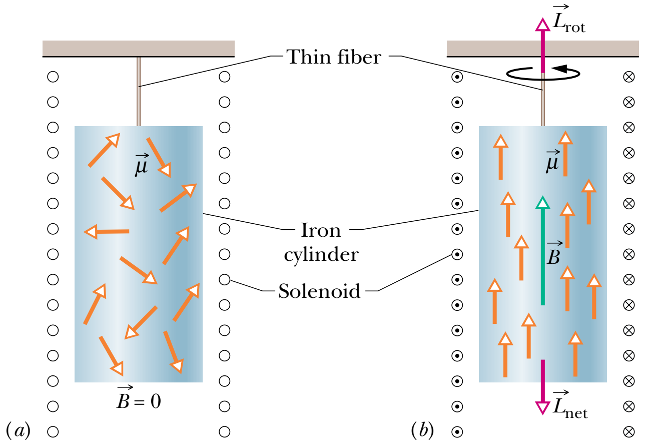


Figure 39-3 A classical model showing a particle of mass m and charge $-e$ moving with speed v in a circle of radius r. The moving particle has an angular momentum $\vec{L}$ given by $\vec{r} \times \vec{P}$, where $\vec{P}$ is its linear momentum $m\vec{v}$. The particle's motion is equivalent to a current loop that has an associated magnetic moment $\vec{\mu}$ that is directed opposite $\vec{L}$.

The Einstein–de Haas Experiment

In 1915, well before the discovery of quantum physics, Albert Einstein and Dutch physicist W. J. de Haas carried out a clever experiment designed to show that the angular momentum and magnetic moment of individual atoms are coupled.

Einstein and de Haas suspended an iron cylinder from a thin fiber, as shown in Fig. 39-4. A solenoid was placed around the cylinder but not touching it. Initially, the magnetic dipole moments $\vec{\mu}$ of the atoms of the cylinder point in random directions, and so their external magnetic effects cancel (Fig. 39-4*a*). However, when a current is switched on in the solenoid (Fig. 39-4*b*) so that a magnetic field $\vec{B}$ is set up parallel to the long axis of the cylinder, the magnetic dipole moments of the atoms of the cylinder reorient themselves, lining up with that field. If the angular momentum $\vec{L}$ of each atom is coupled to its magnetic moment $\vec{\mu}$, then this alignment of the atomic magnetic moments must cause an alignment of the atomic angular momenta opposite the magnetic field.



Aligning the magnetic moment vectors rotates the cylinder.

Figure 39-4 The Einstein–de Haas experimental setup. (*a*) Initially, the magnetic field in the iron cylinder is zero and the magnetic dipole moment vectors $\vec{\mu}$ of its atoms are randomly oriented. (*b*) When a magnetic field $\vec{B}$ is set up along the cylinder's axis, the magnetic dipole moment vectors line up parallel to $\vec{B}$ and the cylinder begins to rotate.

No external torques initially act on the cylinder; thus, its angular momentum must remain at its initial zero value. However, when $\vec{B}$ is turned on and the atomic angular momenta line up antiparallel to $\vec{B}$, they tend to give a net angular momentum $\vec{L}_{\text{net}}$ to the cylinder as a whole (directed downward in Fig. 39-4*b*). To maintain zero angular momentum, the cylinder begins to rotate around its central axis to produce an angular momentum $\vec{L}_{\text{rot}}$ in the opposite direction (upward in Fig. 39-4*b*).

The twisting of the fiber quickly produces a torque that momentarily stops the cylinder's rotation and then rotates the cylinder in the opposite direction as the twisting is undone. Thereafter, the fiber will twist and untwist as the cylinder oscillates about its initial orientation in angular simple harmonic motion.

Observation of the cylinder's rotation verified that the angular momentum and the magnetic dipole moment of an atom are coupled in opposite directions. Moreover, it dramatically demonstrated that the angular momenta associated with quantum states of atoms can result in *visible* rotation of an object of everyday size.

39.3 | ANGULAR MOMENTUM, MAGNETIC DIPOLE MOMENTS

Key Concepts

- States with the same value of quantum number n form a shell.
- States with the same values of quantum numbers n and ℓ form a subshell.
- The magnitude of the orbital angular momentum of an electron trapped in an atom has quantized values given by
$$L = \sqrt{\ell(\ell+1)}\,\hbar, \quad \text{for } \ell = 0, 1, 2, \ldots, (n-1),$$
where $\hbar$ is $h/2\pi$, ℓ is the orbital quantum number, and n is the electron's principal quantum number.
- The component L_z of the orbital angular momentum on a z axis is quantized and given by
$$L_z = m_\ell \hbar, \quad \text{for } m_\ell = 0, \pm 1, \pm 2, \ldots, \pm \ell,$$
where m_ℓ is the orbital magnetic quantum number.
- The magnitude μ_{orb} of the orbital magnetic moment of the electron is quantized with the values given by
$$\mu_{\text{orb}} = \frac{e}{2m}\sqrt{\ell(\ell+1)}\,\hbar,$$
where m is the electron mass.
- The component $\mu_{\text{orb},z}$ on a z axis is also quantized according to
$$\mu_{\text{orb},z} = -\frac{e}{2m} m_\ell \hbar = -m_\ell \mu_{\text{B}},$$
where μ_{B} is the Bohr magneton:
$$\mu_{\text{B}} = \frac{eh}{4\pi m} = \frac{e\hbar}{2m} = 9.274 \times 10^{-24}\ \text{J/T}.$$
- Every electron, whether trapped or free, has an intrinsic spin angular momentum $\vec{S}$ with a magnitude that is quantized as
$$S = \sqrt{s(s+1)}\,\hbar, \quad \text{for } s = \frac{1}{2},$$
where s is the spin quantum number. An electron is said to be a spin 1/2 particle.
- The component S_z on a z axis is also quantized according to
$$S_z = m_s \hbar, \quad \text{for } m_s = \pm s = \pm\frac{1}{2},$$
where m_s is the spin magnetic quantum number.
- Every electron, whether trapped or free, has an intrinsic spin magnetic dipole moment $\vec{\mu}_s$ with a magnitude that is quantized as
$$\mu_s = \frac{e}{m}\sqrt{s(s+1)}\hbar, \quad \text{for } s = \frac{1}{2}.$$
- The component $\mu_{s,z}$ on a z axis is also quantized according to
$$\mu_{s,z} = -2m_s \mu_{\text{B}}, \quad \text{for } m_s = \pm\frac{1}{2}.$$

Every quantum state of an electron in an atom has an associated orbital angular momentum and orbital magnetic dipole moment. Every electron, whether trapped in an atom or free, has a spin angular momentum and a spin magnetic dipole moment that are as intrinsic as its mass and charge. Let's next discuss these various quantities.

Orbital Angular Momentum

Classically, a moving particle has an angular momentum $\vec{L}$ with respect to any given reference point. In Chapter 10 we wrote this as the cross product $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is a position vector extending to the particle from the reference point and $\vec{p}$ is the particle's linear momentum $(m\vec{v})$. Although an electron in an atom is not a classical moving particle, it too has angular momentum given by $\vec{L} = \vec{r} \times \vec{p}$, with the reference point being the nucleus. However,

Table 39-1 Electron States for an Atom

Quantum Number	Symbol	Allowed Values	Related to
Principal	n	$1, 2, 3, \ldots$	Distance from the nucleus
Orbital	ℓ	$0, 1, 2, \ldots, (n-1)$	Orbital angular momentum
Orbital magnetic	m_ℓ	$0, \pm 1, \pm 2, \ldots, \pm \ell$	Orbital angular momentum (z component)
Spin	s	$\frac{1}{2}$	Spin angular momentum
Spin magnetic	m_s	$\pm\frac{1}{2}$	Spin angular momentum (z component)

unlike the classical particle, the electron's *orbital angular momentum* $\vec{L}$ is quantized. For the electron in a hydrogen atom, we can find the quantized (allowed) values by solving Schrödinger's equation. For that situation and any other, we can also find the quantized values by using the appropriate mathematics for a cross product in a quantum situation. Either way we find that the allowed magnitudes of $\vec{L}$ are given by

$$L = \sqrt{\ell(\ell+1)}\,\hbar, \quad \text{for } \ell = 0, 1, 2, \ldots, (n-1), \tag{39-2}$$

where $\hbar$ is $h/2\pi$, ℓ is the orbital quantum number (Table 39-1), and n is the electron's principal quantum number.

The electron can have a definite value of L as given by one of the allowed states in Eq. 39-2, but it cannot have a definite direction for the vector $\vec{L}$. However, we can measure (detect) definite values of a component L_z along a chosen measurement axis (usually taken to be a z axis) as given by

$$L_z = m_\ell \hbar, \quad \text{for } m_\ell = 0, \pm 1, \pm 2, \ldots, \pm \ell, \tag{39-3}$$

where m_ℓ is the orbital magnetic quantum number (Table 39-1). However, if the electron has a definite value of L_z, it does not have definite values for L_x and L_y. We cannot get around this uncertainty by, say, first measuring L_z (getting a definite value) and then measuring L_x (getting a definite value) because the second measurement can change L_z and thus we no longer have a definite value for it. Also, we can never find $\vec{L}$ aligned with an axis because then it would have a definite direction and definite components along the other axes (namely, zero components).

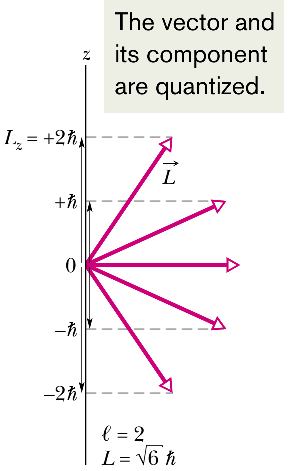

The vector and its component are quantized.

Figure 39-5 The allowed values of L_z for an electron in a quantum state with $\ell = 2$. For every orbital angular momentum vector $\vec{L}$ in the figure, there is a vector pointing in the opposite direction, representing the magnitude and direction of the orbital magnetic dipole moment $\vec{\mu}_{\text{orb}}$.

A common way to depict the allowed values for L_z is shown in Fig. 39-5 for the situation in which $\ell = 2$. However, do not take the figure literally because it implies (incorrectly) that $\vec{L}$ has the definite direction of the drawn vector. Still, it allows us to relate the five possible z components to the full vector (which has a magnitude of $\hbar\sqrt{6}$) and to define the *semi-classical angle* θ given by

$$\cos\theta = \frac{L_z}{L}. \tag{39-4}$$

Orbital Magnetic Dipole Moment

Classically, an orbiting charged particle sets up the magnetic field of a magnetic dipole. The dipole moment is related to the angular momentum of the classical particle by

$$\vec{\mu}_{\text{orb}} = -\frac{e}{2m}\vec{L}, \tag{39-5}$$

where m is the mass of the particle, here an electron. The minus sign means that the two vectors in Eq. 39-5 are in opposite directions, which is due to the fact that an electron is negatively charged.

An electron in an atom also has an orbital magnetic dipole moment given by Eq. 39-5, but $\vec{\mu}_{\text{orb}}$ is quantized. We find allowed values of the magnitude by substituting from Eq. 39-2:

$$\mu_{\text{orb}} = \frac{e}{2m}\sqrt{\ell(\ell+1)}\ \hbar. \tag{39-6}$$

As with the angular momentum, $\vec{\mu}_{\text{orb}}$ can have a definite magnitude but does not have a definite direction. The best we can do is to measure its component on a z axis, and that component can have a definite value as given by

$$\mu_{\text{orb},z} = -m_\ell \frac{e\hbar}{2m} = -m_\ell \mu_{\text{B}}, \tag{39-7}$$

where μ_{B} is the *Bohr magneton*:

$$\mu_{\text{B}} = \frac{eh}{4\pi m} = \frac{e\hbar}{2m} = 9.274 \times 10^{-24}\ \text{J/T} \quad \text{(Bohr magneton).} \tag{39-8}$$

If the electron has a definite value of $\mu_{\text{orb},z}$, it cannot have definite values of $\mu_{\text{orb},x}$ and $\mu_{\text{orb},y}$.

Spin Angular Momentum

Every electron, whether in an atom or free, has an intrinsic angular momentum that has no classical counterpart (it is *not* of the form $\vec{r} \times \vec{p}$). It is called *spin angular momentum* $\vec{S}$ (or simply *spin*), but the name is misleading because the electron is not spinning. Indeed there is nothing at all rotating in an electron, and yet the electron has angular momentum. The magnitude of $\vec{S}$ is quantized, with values restricted to

$$S = \sqrt{s(s+1)}\ \hbar, \quad \text{for } s = \frac{1}{2}, \tag{39-9}$$

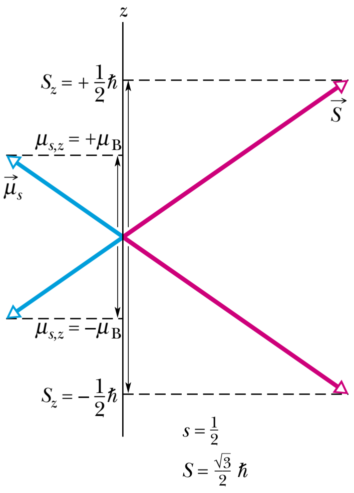
where s is the *spin quantum number*. For every electron, s = 1/2 and the electron is said to be a spin-1/2 particle. (Protons and neutrons are also spin-1/2 particles.) The language here can be confusing, because both $\vec{S}$ and s are often referred to as spin.

As with the angular momentum associated with motion, this intrinsic angular momentum can have a definite magnitude but does not have a definite direction. The best we can do is to measure its component on a z axis, and that component can have only the definite values given by

$$S_z = m_s\hbar, \quad \text{for } m_s = \pm s = \pm\frac{1}{2}. \tag{39-10}$$

Here m_s is the *spin magnetic quantum number*, which can have only two values: $m_s = +s = +1/2$ (the electron is said to be *spin up*) and $m_s = -s = -1/2$ (the electron is said to be *spin down*). Also, if S_z has a definite value, then S_x and S_y do not. Figure 39-6 is another figure that you should not take literally but it serves to show the possible values of S_z.

We have now seen the full set of quantum numbers for an electron, as listed in Table 39-1. If an electron is free, it has only its intrinsic quantum numbers s and m_s. If it is trapped in an atom, it has also has the quantum numbers n, ℓ, and m_ℓ.



z
$S_z = +\frac{1}{2}\hbar$
$\vec{S}$
$\mu_{s,z} = +\mu_{\text{B}}$
$\vec{\mu}_s$
$\mu_{s,z} = -\mu_{\text{B}}$
$S_z = -\frac{1}{2}\hbar$
$s = \frac{1}{2}$
$S = \frac{\sqrt{3}}{2}\hbar$

Figure 39-6 The allowed values of S_z and μ_z for an electron.

Spin Magnetic Dipole Moment

As with the orbital angular momentum, a magnetic dipole moment is associated with the spin angular momentum:

$$\vec{\mu}_s = -\frac{e}{m}\vec{S}, \tag{39-11}$$

where the minus sign means that the two vectors are in opposite directions, which is due to the fact that an electron is negatively charged. This $\vec{\mu}_s$ is an intrinsic property of every electron. The vector $\vec{\mu}_s$ does not have a definite direction but it can have a definite magnitude, given by

$$\mu_s = \frac{e}{m}\sqrt{s(s+1)}\,\hbar. \tag{39-12}$$

The vector can also have a definite component on a z axis, given by

$$\mu_{s,z} = -2m_s\mu_B, \tag{39-13}$$

but that means that it cannot have a definite value of $\mu_{s,x}$ or $\mu_{s,y}$. Figure 39-6 shows the possible values of $\mu_{s,z}$.

Shells and Subshells

As we discussed in Section 38.5, all states with the same n form a *shell*, and all states with the same value of n and ℓ form a *subshell*. As displayed in Table 39-1, for a given ℓ, there are $2\ell + 1$ possible values of quantum number m_ℓ and, for each m_ℓ, there are two possible values for the quantum number m_s (spin up and spin down). Thus, there are $2(2\ell + 1)$ states in a subshell. If we count all the states throughout a given shell with quantum number n, we find that the total number in the shell is $2n^2$.

Orbital and Spin Angular Momenta Combined

For an atom containing more than one electron, we define a total angular momentum $\vec{J}$, which is the vector sum of the angular momenta of the individual electrons—both their orbital and their spin angular momenta. Each element in the periodic table is defined by the number of protons in the nucleus of an atom of the element. This number of protons is defined as being the *atomic number* (or *charge number*) Z of the element. Because an electrically neutral atom contains equal numbers of protons and electrons, Z is also the number of electrons in the neutral atom, and we use this fact to indicate a $\vec{J}$ value for a neutral atom:

$$\vec{J} = (\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots + \vec{L}_Z) + (\vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \cdots + \vec{S}_Z). \tag{39-14}$$

Similarly, the total magnetic dipole moment of a multielectron atom is the vector sum of the magnetic dipole moments (both orbital and spin) of its individual electrons. However, because of the factor 2 in Eq. 39-13, the resultant magnetic dipole moment for the atom does not have the direction of vector $-\vec{J}$; instead, it makes a certain angle with that vector. The **effective magnetic dipole moment** $\vec{\mu}_{\text{eff}}$ for the atom is the component of the vector sum of the individual magnetic dipole moments in the direction of $-\vec{J}$ (Fig. 39-7). In typical atoms the orbital angular momenta and the spin angular momenta of most of the electrons sum vectorially to zero. Then $\vec{J}$ and $\vec{\mu}_{\text{eff}}$ of those atoms are due to a relatively small number of electrons, often only a single valence electron.

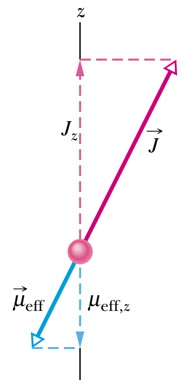


Figure 39-7 A classical model showing the total angular momentum vector $\vec{J}$ and the effective magnetic moment vector $\vec{\mu}_{\text{eff}}$.

CHECKPOINT 1

An electron is in a quantum state for which the magnitude of the electron's orbital angular momentum $\vec{L}$ is $2\sqrt{3}\hbar$. How many projections of the electron's orbital magnetic dipole moment on a z axis are allowed?

39.4 | THE PAULI EXCLUSION PRINCIPLE

Key Concept

- Electrons in atoms and other traps obey the Pauli exclusion principle, which requires that no two electrons in a trap can have the same set of quantum numbers.

In Chapter 38 we considered a variety of electron traps, from fictional one-dimensional traps to the real three-dimensional trap of a hydrogen atom. In all those examples, we trapped only one electron. However, when we discuss traps containing two or more electrons (as we shall below), we must consider a principle that governs any particle whose spin quantum number s is not zero or an integer. This principle applies not only to electrons but also to protons and neutrons, all of which have $s = 1/2$. The principle is known as the **Pauli exclusion principle** after Wolfgang Pauli, who formulated it in 1925. For electrons, it states that

> No two electrons confined to the same trap can have the same set of values for their quantum numbers.

As we shall discuss in Section 39.5, this principle means that no two electrons in an atom can have the same four values for the quantum numbers n, ℓ, m_ℓ, and m_s. All electrons have the same quantum number $s = 1/2$. Thus, any two electrons in an atom must differ in at least one of these other quantum numbers. Were this not true, atoms would collapse, and thus you and the world could not exist.

39.5 | BUILDING THE PERIODIC TABLE

Key Concepts

- In the periodic table, the elements are listed in order of increasing atomic number Z, where Z is the number of protons in the nucleus. For a neutral atom, Z is also the number of electrons.
- States with the same values of quantum number n form a shell.
- States with the same values of quantum numbers n and ℓ form a subshell.
- A closed shell and a closed subshell contain the maximum number of electrons as allowed by the Pauli exclusion principle. The net angular momentum and net magnetic moment of such closed structures are zero.

The four quantum numbers n, ℓ, m_ℓ, and m_s identify the quantum states of individual electrons in a multielectron atom. The wave functions for these states, however, are not the same as the wave functions for the corresponding states of the hydrogen atom because, in multielectron atoms, the potential energy associated with a given electron is determined not only by the charge and position of the atom's nucleus but also by the charges and positions of all the other electrons in the atom. Solutions of Schrödinger's equation for multielectron atoms can be carried out numerically—in principle at least—using a computer.

Shells and Subshells

As we discussed in Section 39.1, all states with the same n form a *shell*, and all states with the same value of n and ℓ form a *subshell*. For a given ℓ, there are $2\ell + 1$ possible values of quantum number m_ℓ and, for each m_ℓ, there are two possible values for the quantum number m_s (spin up and spin down). Thus, there are $2(2\ell + 1)$ states in a subshell. If we count all the states throughout a given shell with quantum number n, we find that the total number in the shell is $2n^2$. All states in a given subshell have about the same energy, which depends primarily on the value of n, but it also depends somewhat on the value of ℓ.

For the purpose of labeling subshells, the values of ℓ are represented by letters:

$$\begin{array}{ccccccc} \ell = 0 & 1 & 2 & 3 & 4 & 5 \ldots \\ s & p & d & f & g & h \ldots. \end{array}$$

For example, the $n = 3$, $\ell = 2$ subshell would be labeled the $3d$ subshell.

When we assign electrons to states in a multielectron atom, we must be guided by the Pauli exclusion principle of Section 39.4; that is, no two electrons in an atom can have the same set of the quantum numbers n, ℓ, m_ℓ, and m_s. If this important principle did not hold, *all* the electrons in any atom could jump to the atom's lowest energy level, which would eliminate the chemistry of atoms and molecules, and thus also eliminate biochemistry and us. Let us examine the atoms of a few elements to see how the Pauli exclusion principle operates in the building up of the periodic table.

Neon

The neon atom has 10 electrons. Only two of them fit into the lowest-energy subshell, the 1s subshell. These two electrons both have $n = 1$, $\ell = 0$, and $m_\ell = 0$, but one has $m_s = +1/2$ and the other has $m_s = -1/2$. The 1s subshell contains $2[2(0) + 1] = 2$ states. Because this subshell then contains all the electrons permitted by the Pauli principle, it is said to be **closed**.

Two of the remaining eight electrons fill the next lowest energy subshell, the 2s subshell. The last six electrons just fill the 2p subshell, which, with $\ell = 1$, holds $2[2(1) + 1] = 6$ states.

In a closed subshell, all allowed z projections of the orbital angular momentum vector $\vec{L}$ are present and, as you can verify from Fig. 39-5, these projections cancel for the subshell as a whole; for every positive projection there is a corresponding negative projection of the same magnitude. Similarly, the z projections of the spin angular momenta also cancel. Thus, a closed subshell has no angular momentum and no magnetic moment of any kind. Furthermore, its probability density is spherically symmetric. Then neon with its three closed subshells (1s, 2s, and 2p) has no "loosely dangling electrons" to encourage chemical interaction with other atoms. Neon, like the other **noble gases** that form the right-hand column of the periodic table, is almost chemically inert.

Sodium

Next after neon in the periodic table comes sodium, with 11 electrons. Ten of them form a closed neon-like core, which, as we have seen, has zero angular momentum. The remaining electron is largely outside this inert core, in the 3s subshell—the next lowest energy subshell. Because this **valence electron** of sodium is in a state with $\ell = 0$ (that is, an s state using the lettering system above), the sodium atom's angular momentum and magnetic dipole moment must be due entirely to the spin of this single electron.

Sodium readily combines with other atoms that have a "vacancy" into which sodium's loosely bound valence electron can fit. Sodium, like the other **alkali metals** that form the left-hand column of the periodic table, is chemically active.

Chlorine

The chlorine atom, which has 17 electrons, has a closed 10-electron, neon-like core, with 7 electrons left over. Two of them fill the 3s subshell, leaving five to be assigned to the 3p subshell, which is the subshell next lowest in energy. This subshell, which has $\ell = 1$, can hold $2[2(1) + 1] = 6$ electrons, and so there is a vacancy, or a "hole," in this subshell.

Chlorine is receptive to interacting with other atoms that have a valence electron that might fill this hole. Sodium chloride (NaCl), for example, is a very stable compound. Chlorine, like the other **halogens** that form column VIIA of the periodic table, is chemically active.

Iron

The arrangement of the 26 electrons of the iron atom can be represented as follows:

$$\underbrace{1s^2 \quad 2s^2\ 2p^6 \quad 3s^2\ 3p^6}\ 3d^6 \quad 4s^2.$$

The subshells are listed in numerical order and, following convention, a superscript gives the number of electrons in each subshell. From Table 39-1 we can see that an s subshell ($\ell = 0$) can hold 2 electrons, a p subshell ($\ell = 1$) can hold 6, and a d subshell ($\ell = 2$) can hold 10. Thus, iron's first 18 electrons form the five filled subshells that are marked off by the bracket, leaving 8 electrons to be accounted for. Six of the eight go into the 3d subshell, and the remaining two go into the 4s subshell.

The reason the last two electrons do not also go into the 3d subshell (which can hold 10 electrons) is that the $3d^6\ 4s^2$ configuration results in a lower-energy state for the atom as a whole than would the $3d^8$ configuration. An iron atom with 8 electrons (rather than 6) in the 3d subshell would quickly make a transition to the $3d^6\ 4s^2$ configuration, emitting electromagnetic radiation in the process. The lesson here is that except for the simplest elements, the states may not be filled in what we might think of as their "logical" sequence.

39.6 | X RAYS AND THE ORDERING OF THE ELEMENTS

Key Concepts

- When a beam of high-energy electrons impact a target, the electrons can lose their energy by scattering from atoms and emitting a continuous spectrum of x rays.
- The shortest wavelength in the spectrum is the cutoff wavelength λ_{min}, which is emitted when an incident electron loses its full kinetic energy K_0 in a single collision:

$$\lambda_{\text{min}} = \frac{hc}{K_0}.$$

- The characteristic x-ray spectrum is produced when incident electrons eject low-lying electrons in the target atoms and electrons from upper levels jump down to the resulting holes, emitting light.
- A Moseley plot is a graph of the square root of the characteristic-emission frequencies $\sqrt{f}$ versus atomic number Z of the target atoms. The straight-line plot reveals that the position of an element in the periodic table is set by Z and not the atomic weight.

When a solid target, such as solid copper or tungsten, is bombarded with electrons whose kinetic energies are in the kiloelectron-volt range, electromagnetic radiation called **x rays** is emitted. Our concern here is what these rays can teach us about the atoms that absorb or emit them.

A device used to produce x rays is generally called an x-ray tube. Figure 39-8 shows a schematic diagram of such a device. This was originally made by Coolidge and is known as Coolidge tube.

A filament F and a metallic target are fixed in an evacuated chamber C. The filament is heated electrically and thus emits electron by thermionic emission. A constant potential difference is maintained between the filament and the target using a dc power supply so that the target is at a higher potential than the filament. The electrons emitted by the filament are, therefore, accelerated by the electric field setup between the filament and the target, and in the process x rays are emitted. These x rays are brought out of the tube through a window made up of thin mica or Mylar or some such material that does not absorb x ray appreciably. In the process, a large amount of heat is developed, and thus an arrangement is provided to cool down the tube by running water. The x rays are electromagnetic waves having wavelength of the order of 0.1 nm.

Our concern here is what these rays—whose medical, dental, and industrial usefulness is so well known and widespread—can teach us about the atoms that absorb or emit them.

Figure 39-9 shows the wavelength spectrum of the x rays produced when a beam of 35 keV electrons falls on a molybdenum target. We see a broad, continuous spectrum of radiation on which are superimposed two peaks of sharply defined wavelengths. The continuous spectrum and the peaks arise in different ways, which we next discuss separately.

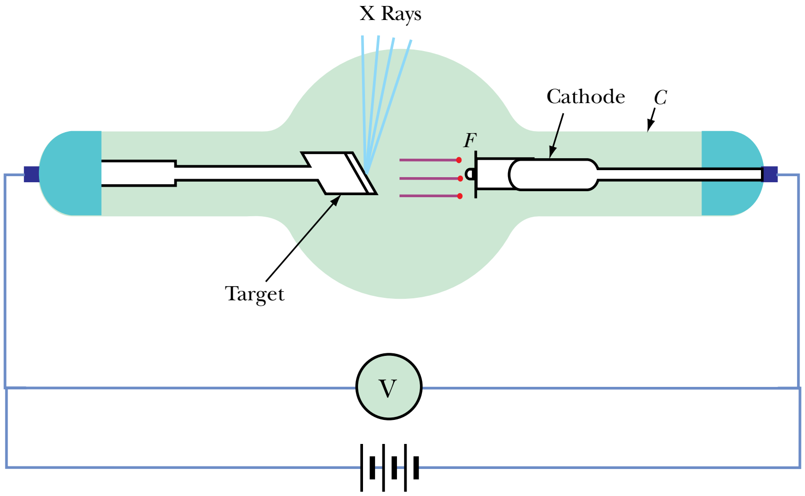


Figure 39-8 Production of x rays by stopping fast electrons in dense materials.

The Continuous X-Ray Spectrum

Here we examine the continuous x-ray spectrum of Fig. 39-9, ignoring for the time being the two prominent peaks that rise from it. Consider an electron of initial kinetic energy K_0 that collides (interacts) with one of the target atoms, as in Fig. 39-10. The electron may lose an amount of energy ΔK, which will appear as the energy of an x-ray photon that is radiated away from the site of the collision. (Very little energy is transferred to the recoiling atom because of the relatively large mass of the atom; here we neglect that transfer.)

The scattered electron in Fig. 39-10, whose energy is now less than K_0, may have a second collision with a target atom, generating a second photon, with a different photon energy. This electron-scattering process can continue until the electron is approximately stationary. All the photons generated by these collisions form part of the continuous x-ray spectrum.

A prominent feature of that spectrum in Fig. 39-9 is the sharply defined **cutoff wavelength** λ_{min}, below which the continuous spectrum does not exist. This minimum wavelength corresponds to a collision in which an incident electron loses *all* its initial kinetic energy K_0 in a single head-on collision with a target atom. Essentially all this energy appears as the energy of a single photon, whose associated wavelength—the minimum possible x-ray wavelength—is found from

$$K_0 = hf = \frac{hc}{\lambda_{\text{min}}},$$

or

$$\lambda_{\text{min}} = \frac{hc}{K_0} \quad \text{(cutoff wavelength).} \qquad (39\text{-}15)$$

The cutoff wavelength is totally independent of the target material. If we were to switch from a molybdenum target to a copper target, for example, all features of the x-ray spectrum of Fig. 39-9 would change *except* the cutoff wavelength.

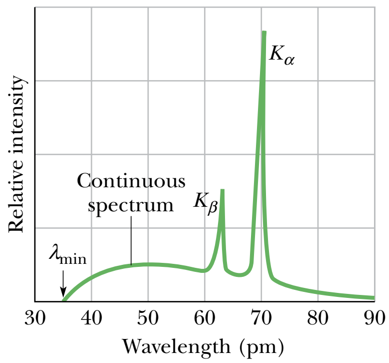


Figure 39-9 The distribution by wavelength of the x rays produced when 35 keV electrons strike a molybdenum target. The sharp peaks and the continuous spectrum from which they rise are produced by different mechanisms.

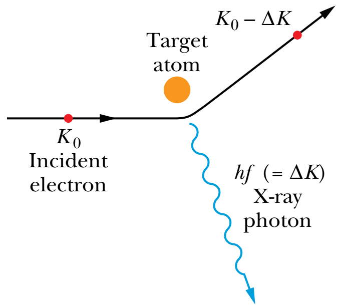


Figure 39-10 An electron of kinetic energy K_0 passing near an atom in the target may generate an x-ray photon, the electron losing part of its energy in the process. The continuous x-ray spectrum arises in this way.

CHECKPOINT 2

Does the cutoff wavelength λ_{min} of the continuous x-ray spectrum increase, decrease, or remain the same if you (a) increase the kinetic energy of the electrons that strike the x-ray target, (b) allow the electrons to strike a thin foil rather than a thick block of the target material, (c) change the target to an element of higher atomic number?

SAMPLE PROBLEM 39.01

Cut-off frequency of x-rays

A beam of 35.0 keV electrons strikes a molybdenum target, generating the x rays whose spectrum is shown in Fig. 39-9. What is the cutoff wavelength?

KEY IDEA

The cutoff wavelength λ_{min} corresponds to an electron transferring (approximately) all of its energy to an x-ray photon, thus producing a photon with the greatest possible frequency and least possible wavelength.

Calculation: From Eq. 39-15, we have

$$\lambda_{\text{min}} = \frac{hc}{K_0} = \frac{(4.14 \times 10^{-15}\,\text{eV}\cdot\text{s})(300 \times 10^{8}\,\text{m/s})}{3.50 \times 10^{3}\,\text{eV}}$$

$$= 3.35 \times 10^{-11}\,\text{m} = 35.5\,\text{pm}. \qquad \text{(Answer)}$$

The Characteristic X-Ray Spectrum

We now turn our attention to the two peaks of Fig. 39-9, labeled K_α and K_β. These (and other peaks that appear at wavelengths beyond the range displayed in Fig. 39-9) form the **characteristic x-ray spectrum** of the target material.

The peaks arise in a two-part process. (1) An energetic electron strikes an atom in the target and, while it is being scattered, the incident electron knocks out one of the atom's deep-lying (low n value) electrons. If the deep-lying electron is in the shell defined by $n = 1$ (called, for historical reasons, the K shell), there remains a vacancy, or *hole*, in this shell. (2) An electron in one of the shells with a higher energy jumps to the K shell, filling the hole in this shell. During this jump, the atom emits a characteristic x-ray photon. If the electron that fills the K-shell vacancy jumps from the shell with $n = 2$ (called the L shell), the emitted radiation is the K_α line of Fig. 39-9; if it jumps from the shell with $n = 3$ (called the M shell), it produces the K_β line, and so on. The hole left in either the L or M shell will be filled by an electron from still farther out in the atom. So you can see that a K_α line will always be accompanied by line in L-series.

In studying x rays, it is more convenient to keep track of where a hole is created deep in the atom's "electron cloud" than to record the changes in the quantum state of the electrons that jump to fill that hole. Figure 39-11 does exactly that; it is an energy-level diagram for molybdenum, the element to which Fig. 39-9 refers. The baseline ($E = 0$) represents the neutral atom in its ground state. The level marked K (at $E = 20$ keV) represents the energy of the molybdenum atom with a hole in its K shell, the level marked L (at $E = 2.7$ keV) represents the atom with a hole in its L shell, and so on.

The transitions marked K_α and K_β in Fig. 39-11 are the ones that produce the two x-ray peaks in Fig. 39-9. The K_α spectral line, for example, originates when an electron from the L shell fills a hole in the K shell. To state this transition in terms of what the arrows in Fig. 39-11 show, a hole originally in the K shell moves to the L shell.

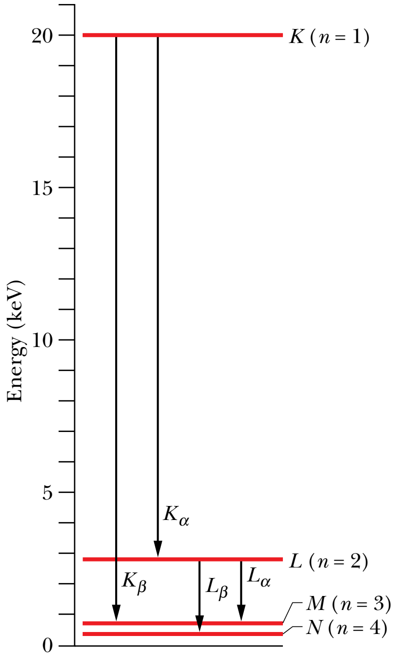


Figure 39-11 A simplified energy-level diagram for a molybdenum atom, showing the transitions (of holes rather than electrons) that give rise to some of the characteristic x rays of that element. Each horizontal line represents the energy of the atom with a hole (a missing electron) in the shell indicated.

Ordering the Elements

In 1913, British physicist H. G. J. Moseley generated characteristic x rays for as many elements as he could find—he found 38—by using them as targets for electron bombardment in an evacuated tube of his own design. By means of a trolley manipulated by strings, Moseley was able to move the individual targets into the path of an electron beam. He measured the wavelengths of the emitted x rays by the crystal diffraction method.

Moseley then sought (and found) regularities in these spectra as he moved from element to element in the periodic table. In particular, he noted that if, for a given spectral line such as K_α, he plotted for each element the square root of the frequency f against the position of the element in the periodic table, a straight line resulted. Figure 39-12 shows a portion of his extensive data. Moseley's conclusion was this:

> We have here a proof that there is in the atom a fundamental quantity, which increases by regular steps as we pass from one element to the next. This quantity can only be the charge on the central nucleus.

As a result of Moseley's work, the characteristic x-ray spectrum became the universally accepted signature of an element, permitting the solution of a number of periodic table puzzles. Prior to that time (1913), the positions of elements in the table were assigned in order of atomic *mass*, although it was necessary to invert this order for several pairs of elements because of compelling chemical evidence; Moseley showed that it is the nuclear charge (that is, atomic number Z) that is the real basis for ordering the elements.

In 1913 the periodic table had several empty squares, and a surprising number of claims for new elements had been advanced. The x-ray spectrum provided a conclusive test of such claims. The lanthanide elements, often called the rare earth elements, had been sorted out only imperfectly because their similar chemical properties made sorting difficult. Once Moseley's work was reported, these elements were properly organized.

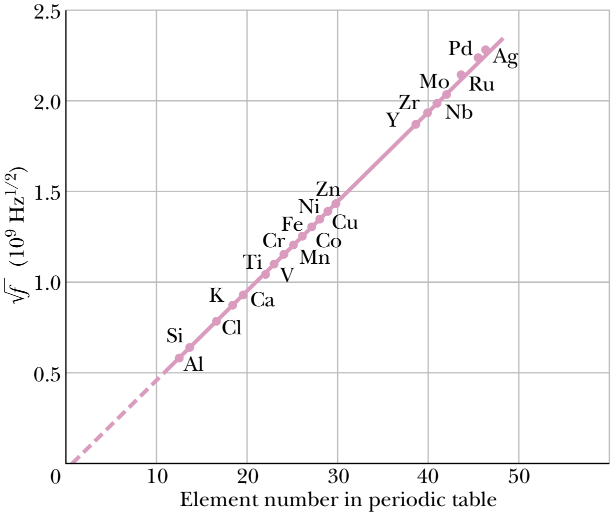


Figure 39-12 A Moseley plot of the K_α line of the characteristic x-ray spectra of 21 elements. The frequency is calculated from the measured wavelength.

It is not hard to see why the characteristic x-ray spectrum shows such impressive regularities from element to element whereas the optical spectrum in the visible and near-visible region does not: The key to the identity of an element is the charge on its nucleus. Gold, for example, is what it is because its atoms have a nuclear charge of $+79e$ (that is, $Z = 79$). An atom with one more elementary charge on its nucleus is mercury; with one fewer, it is platinum. The K electrons, which play such a large role in the production of the x-ray spectrum, lie very close to the nucleus and are thus sensitive probes of its charge. The optical spectrum, on the other hand, involves transitions of the outermost electrons, which are heavily screened from the nucleus by the remaining electrons of the atom and thus are *not* sensitive probes of nuclear charge.

Accounting for the Moseley Plot

Moseley's experimental data, of which the Moseley plot of Fig. 39-12 is but a part, can be used directly to assign the elements to their proper places in the periodic table. This can be done even if no theoretical basis for Moseley's results can be established. However, there is such a basis.

According to Eq. 38-29, the energy of the hydrogen atom is

$$E_n = -\frac{me^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} = -\frac{13.60\text{ eV}}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{39-16}$$

Consider now one of the two innermost electrons in the K shell of a multi-electron atom. Because of the presence of the other K-shell electron, our electron "sees" an effective nuclear charge of approximately $(Z - 1)e$, where e is the elementary charge and Z is the atomic number of the element. The factor e^4 in Eq. 39-16 is the product of e^2—the square of hydrogen's nuclear charge—and $(-e)^2$—the square of an electron's charge. For a multielectron atom, we can approximate the effective energy of the atom by replacing the factor e^4 in Eq. 39-16 with $(Z - 1)^2e^2 \times (-e)^2$, or $e^4(Z - 1)^2$. That gives us

$$E_n = -\frac{(13.60\text{ eV})(Z-1)^2}{n^2}. \tag{39-17}$$

We saw that the K_α x-ray photon (of energy hf) arises when an electron makes a transition from the L shell (with $n = 2$ and energy E_2) to the K shell (with $n = 1$ and energy E_1). Thus, using Eq. 39-17, we may write the energy change as

$$\begin{aligned}\Delta E &= E_2 - E_1\\ &= \frac{-(13.60\text{ eV})(Z-1)^2}{2^2} - \frac{-(13.60\text{ eV})(Z-1)^2}{1^2}\\ &= (10.2\text{ eV})(Z-1)^2.\end{aligned}$$

Then the frequency f of the K_α line is

$$f = \frac{\Delta E}{h} = \frac{(10.2\text{ eV})(Z-1)^2}{(4.14\times 10^{-15}\text{ eV}\cdot\text{s})}$$
$$= (2.46\times 10^{15}\text{ Hz})(Z-1)^2. \tag{39-18}$$

Taking the square root of both sides yields

$$\sqrt{f} = CZ - C, \tag{39-19}$$

in which C is a constant (= 4.96 × 10^7 $\text{H}^{1/2}$). Equation 30-19 is the equation of a straight line. It shows that if we plot the square root of the frequency of the K_α x-ray spectral line against the atomic number Z, we should obtain a straight line. As Fig. 39-12 shows, that is exactly what Moseley found. In general, Moseley's law is valid only for the frequency of a K_α line. If we want to find the frequency of any other line, then we can imagine that the effective nuclear charge cannot be so easily calculated because the shells other than $n = 1$ are not spherically symmetric. The general formula for any other line can be written as

$$v = \frac{\Delta E}{h} = 13.60\left(\frac{1}{1^2} - \frac{1}{n^2}\right)(Z-b)^2$$
$$\sqrt{v} = a(Z-b)$$

where a and b are constants.

SAMPLE PROBLEM 39.02

Characteristic spectrum in x-ray production

A cobalt target is bombarded with electrons, and the wavelengths of its characteristic x-ray spectrum are measured. There is also a second, fainter characteristic spectrum, which is due to an impurity in the cobalt. The wavelengths of the K_α lines are 178.9 pm (cobalt) and 143.5 pm (impurity), and the proton number for cobalt is $Z_{\text{Co}} = 27$. Determine the impurity using only these data.

KEY IDEA

The wavelengths of the K_α lines for both the cobalt (Co) and the impurity (X) fall on a K_α Moseley plot, and Eq. 39-19 is the equation for that plot.

Calculations: Substituting c/λ for f in Eq. 39-19, we obtain

$$\sqrt{\frac{c}{\lambda_{\text{Co}}}} = CZ_{\text{Co}} - C \quad \text{and} \quad \sqrt{\frac{c}{\lambda_{\text{X}}}} = CZ_{\text{X}} - C.$$

Dividing the second equation by the first neatly eliminates C, yielding

$$\sqrt{\frac{\lambda_{\text{Co}}}{\lambda_{\text{X}}}} = \frac{Z_{\text{X}} - 1}{Z_{\text{Co}} - 1}.$$

Substituting the given data yields

$$\sqrt{\frac{178.9\text{ pm}}{143.5\text{ pm}}} = \frac{Z_{\text{X}} - 1}{27 - 1}.$$

Solving for the unknown, we find that

$$Z_{\text{X}} = 30.0. \qquad \text{(Answer)}$$

Thus, the number of protons in the impurity nucleus is 30, and a glance at the periodic table identifies the impurity as zinc. Note that with a larger value of Z than cobalt, zinc has a smaller value of the K_α line. This means that the energy associated with that jump must be greater in zinc than cobalt.

Auger Effect

We have seen that when a characteristic x-ray spectrum is emitted, the atom loses an electron belonging to an inner shell. This gap in the electronic configuration is filled by an electron in an outer shell. This energy is emitted as an x-ray photon. This atom can also lose excitation energy by Auger effect. In this effect, an outer shell electron is ejected from the atom at the same time when another outer shell electron drops to the incomplete inner shell. This ejected electron carries the atom's excitation energy instead of the photon. Thus, it is like an internal photoelectric effect. Normally, both the process, x-ray emission and Auger effect, go on in metals. However, this electron does not emerge out whereas the characteristic x ray emerges out.

X-Ray Absorption Spectra

Just like the metals can radiate x rays when struck with fast moving electrons, they can also absorb the x rays. These x-ray absorption spectra enable the structures of the atomic levels to be found directly. In hydrogen, atom emission and absorption spectra are identical, except that an absorption spectrum is simpler because it is usually limited to transitions from the ground or lowest energy state. This is no longer true with x rays, which show absorption edges rather than absorption lines.

The x-ray absorption spectrum of a metal may be examined by passing a narrow beam of x rays through the metal foil. The spectrum of the incident x-rays should be continuous. Readings with and without the absorber in position at each wavelength are made and a graph of fraction of the x rays absorbed is plotted against wavelength as shown in Fig. 39-13. In general, the absorption increases with wavelength, but at certain critical wavelengths the absorption falls suddenly as shown in Fig. 39-13. This can be explained by supposing that with increasing wavelength, the energy of the x-ray photon is becoming smaller. For a wavelength lesser than that of K absorption edge, it has sufficient energy to excite an electron in the K shell. So, the probability of its absorption is high. As higher shells are already occupied, the excited electron can only be completely removed from the atom.

Close examination of the wavelength associated with the K absorption edge shows that it does not coincide with any of the characteristic lines in the K series. It does, however, coincide with the limit of the K series, which corresponds to the removal of an electron from the K shell ($n = 1$) of an atom to an infinite distance, the wavenumber of the absorbed photon being given by the formula

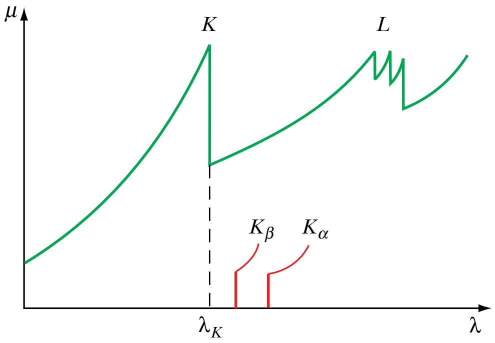


Figure 39-13 A graph showing percentage of incident x-ray photons absorbed by a thin metal foil as a function of the wavelength of the photons.

$$\bar{\nu} = R(Z-1)^2\left(\frac{1}{1^2}-\frac{1}{\infty^2}\right) = R(Z-1)^2.$$

At a wavelength higher than the K absorption edge, the energy of the photon is insufficient to remove the K-shell electron. So, there is a sudden drop in the absorption. Proceeding to greater wavelengths, the photon energy becomes still smaller until it is suddenly unable to excite an electron in the L shell. The fact that we observe three absorption edges for the L shell indicates that this level has three subshells with different energy levels. In the same way, it has been shown that M shell has five levels.

Hard and Soft X Rays

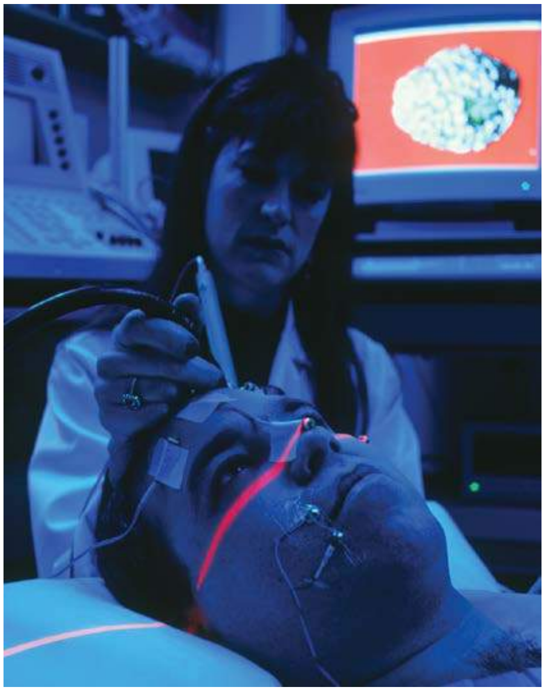
You have probably heard the terms "soft" and "hard" x-rays. This is merely a different way of indicating the wavelength. Long-wavelength x-rays are soft, that is, they have little penetrating power. These x-rays will penetrate a sheet of paper, but a piece of heavy cardboard will weaken them appreciably and a human hand will stop them. Hence, they are of no use in medicine, except when a doctor wants to irradiate the skin without affecting the tissues below. There are still softer x rays that have trouble even in penetrating ordinary air; to study these, the physicist has to pump out the air from his entire apparatus so that the x-rays can proceed unhindered. If you experiment with longer waves, you will find yourself at last dealing with rays that we earlier called ultraviolet. Although the properties of radiation change profoundly within this region, we must realize that these changes are gradual and that there is no difference in kind between x-rays and visible light.

Moreover, if you make the wavelength shorter, the x-rays get harder. Hard x-rays are useful for their power of penetrating thick steel plates and castings, making internal flaws visible; they are also used in medicine when the doctor wants to irradiate some deep-seated tumor without damaging the tissues above it. In order to produce harder x-rays, one has to increase the speed of the electrons that strikes the target. This is done by increasing the electric potential that drives the electrons toward the target. In the betatron, electrons can be brought up to the same speed as if accelerated by a potential of 20 million volts. The x-rays that this machine produces can go through several feet of solid steel.

39.7 | LASERS AND LASER LIGHT

In the early 1960s, quantum physics made one of its many contributions to technology: the **laser**. Laser light, like the light from an ordinary lightbulb, is emitted when atoms make a transition from one quantum state to a lower one. However, in a lightbulb the emissions are random, both in time and direction, and in a laser they are coordinated so that the emissions are at the same time and in the same direction. As a result, laser light has the following characteristics:

1. ***Laser light is highly monochromatic.*** Light from an ordinary incandescent lightbulb is spread over a continuous range of wavelengths and is certainly not monochromatic. The radiation from a fluorescent neon sign is monochromatic, true, to about 1 part in 10^6, but the sharpness of definition of laser light can be many times greater, as much as 1 part in 10^{15}.
2. ***Laser light is highly coherent.*** Individual long waves (*wave trains*) for laser light can be several hundred kilometers long. When two separated beams that have traveled such distances over separate paths are recombined, they "remember" their common origin and are able to form a pattern of interference fringes. The corresponding *coherence length* for wave trains emitted by a lightbulb is typically less than a meter.
3. ***Laser light is highly directional.*** A laser beam spreads very little; it departs from strict parallelism only because of diffraction at the exit aperture of the laser. For example, a laser pulse used to measure the distance to the Moon generates a spot on the Moon's surface with a diameter of only a few kilometers. Light from an ordinary bulb can be made into an approximately parallel beam by a lens, but the beam divergence is much greater than for laser light. Each point on a lightbulb's filament forms its own separate beam, and the angular divergence of the overall composite beam is set by the size of the filament.
4. ***Laser light can be sharply focused.*** If two light beams transport the same amount of energy, the beam that can be focused to the smaller spot will have the greater intensity (power per unit area) at that spot. For laser light, the focused spot can be so small that an intensity of 10^{17} W/cm^2 is readily obtained. An oxyacetylene flame, by contrast, has an intensity of only about 10^3 W/cm^2.



Sam Ogden/Photo Researchers, Inc.

Figure 39-14 A patient's head is scanned and mapped by (red) laser light in preparation for brain surgery. During the surgery, the laser-derived image of the head will be superimposed on the model of the brain shown on the monitor, to guide the surgical team into the region shown in green (lower right) on the model displayed on the screen.

Lasers Have Many Uses

The smallest lasers, used for voice and data transmission over optical fibers, have as their active medium a semiconducting crystal about the size of a pinhead. Small as they are, such lasers can generate about 200 mW of power. The largest lasers, used for nuclear fusion research and for astronomical and military applications, fill a large building. The largest such laser can generate brief pulses of laser light with a power level, during the pulse, of about 10^{14} W. This is a few hundred times greater than the total electrical power generating capacity of the United States. To avoid a brief national power blackout during a pulse, the energy required for each pulse is stored up at a steady rate during the relatively long interpulse interval.

Among the many uses of lasers are reading bar codes, manufacturing and reading compact discs and DVDs, performing surgery of many kinds (both as a surgical aid as in Fig. 39-14 and as a cutting and cauterizing tool), surveying, cutting cloth in the garment industry (several hundred layers at a time), welding auto bodies, and generating holograms.

39.8 | HOW LASERS WORK

Key Concepts

- In stimulated emission, an atom in an excited state can be induced to de-excite to a lower energy state by emitting a photon if an identical photon passes the atom.
- The light emitted in stimulated emission is in phase with and travels in the direction of the light causing the emission.
- A laser can emit light via stimulated emission provided that its atoms are in a population inversion. That is, for the pair of levels involved in the stimulated emission, more atoms must be in the upper level than the lower level so that there is more stimulated emission than just absorption.

Because the word "laser" is an acronym for "light amplification by the stimulated emission of radiation," you should not be surprised that stimulated emission is the key to laser operation. Einstein introduced this concept in 1917 in the paper where he explained the Planck formula for an ideal blackbody radiator. Although the world had to wait until 1960 to see an operating laser, the groundwork for its development was put in place decades earlier.

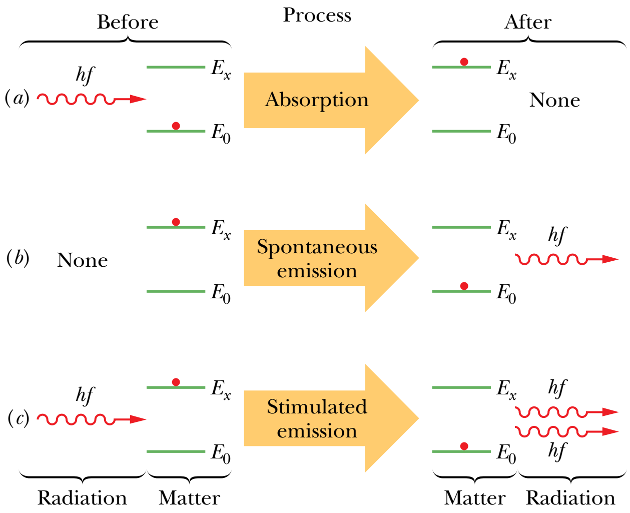
Consider an isolated atom that can exist either in its state of lowest energy (its ground state), whose energy is E_0, or in a state of higher energy (an excited state), whose energy is E_x. Here are three processes by which the atom can move from one of these states to the other:

1. ***Absorption.*** Figure 39-15*a* shows the atom initially in its ground state. If the atom is placed in an electromagnetic field that is alternating at frequency f, the atom can absorb an amount of energy hf from that field and move to the higher-energy state. From the principle of conservation of energy we have

$$hf = E_x - E_0. \tag{39-20}$$

We call this process **absorption**.

2. ***Spontaneous emission.*** In Fig. 39-15*b* the atom is in its excited state and no external radiation is present. After a time, the atom will de-excite to its ground state, emitting a photon of energy hf in the process. We call this



These are three ways that radiation (light) can interact with matter. The third way is the basis of lasing.

Figure 39-15 The interaction of radiation and matter in the processes of (*a*) absorption, (*b*) spontaneous emission, and (*c*) stimulated emission. An atom (matter) is represented by the red dot; the atom is in either a lower quantum state with energy E_0 or a higher quantum state with energy E_x. In (*a*) the atom absorbs a photon of energy hf from a passing light wave. In (*b*) it emits a light wave by emitting a photon of energy hf. In (*c*) a passing light wave with photon energy hf causes the atom to emit a photon of the same energy, increasing the energy of the light wave.

process **spontaneous emission**—*spontaneous* because the event is random and set by chance. The light from the filament of an ordinary lightbulb or any other common light source is generated in this way.

Normally, the mean life of excited atoms before spontaneous emission occurs is about 10^{-8} s. However, for some excited states, this mean life is perhaps as much as 10^5 times longer. We call such long-lived states **metastable**; they play an important role in laser operation.

3. ***Stimulated emission.*** In Fig. 39-15*c* the atom is again in its excited state, but this time radiation with a frequency given by Eq. 39-20 is present. A photon of energy hf can stimulate the atom to move to its ground state, during which process the atom emits an additional photon, whose energy is also hf. We call this process **stimulated emission**—*stimulated* because the event is triggered by the external photon. The emitted photon is in every way identical to the stimulating photon. Thus, the waves associated with the photons have the same energy, phase, polarization, and direction of travel.

Figure 39-15*c* describes stimulated emission for a single atom. Suppose now that a sample contains a large number of atoms in thermal equilibrium at temperature T. Before any radiation is directed at the sample, a number N_0 of these atoms are in their ground state with energy E_0 and a number N_x are in a state of higher energy E_x. Ludwig Boltzmann showed that N_x is given in terms of N_0 by

$$N_x = N_0 e^{-(E_x - E_0)/kT}, \tag{39-21}$$

in which k is Boltzmann's constant. This equation seems reasonable. The quantity kT is the mean kinetic energy of an atom at temperature T. The higher the temperature, the more atoms—on average—will have been "bumped up" by thermal agitation (that is, by atom–atom collisions) to the higher energy state E_x. Also, because $E_x > E_0$, Eq. 39-21 requires that $N_x < N_0$; that is, there will always be fewer atoms in the excited state than in the ground state. This is what we expect if the level populations N_0 and N_x are determined only by the action of thermal agitation. Figure 39-16*a* illustrates this situation.

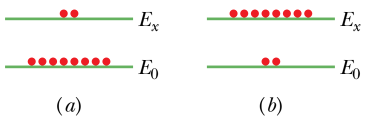


Figure 39-16 (*a*) The equilibrium distribution of atoms between the ground state E_0 and excited state E_x accounted for by thermal agitation. (*b*) An inverted population, obtained by special methods. Such a population inversion is essential for laser action.

If we now flood the atoms of Fig. 39-16*a* with photons of energy $E_x - E_0$, photons will disappear via absorption by ground-state atoms and photons will be generated largely via stimulated emission of excited-state atoms. Einstein showed that the probabilities per atom for these two processes are identical. Thus, because there are more atoms in the ground state, the *net* effect will be the absorption of photons.

To produce laser light, we must have more photons emitted than absorbed; that is, we must have a situation in which stimulated emission dominates. Thus, we need more atoms in the excited state than in the ground state, as in Fig. 39-16*b*. However, because such a **population inversion** is not consistent with thermal equilibrium, we must think up clever ways to set up and maintain one.

The Helium–Neon Gas Laser

Figure 39-17 shows a common type of laser developed in 1961 by Ali Javan and his coworkers. The glass discharge tube is filled with a 20:80 mixture of helium and neon gases, neon being the medium in which laser action occurs.

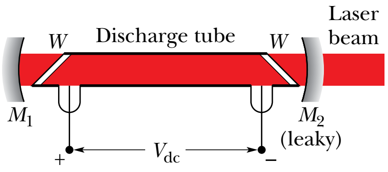
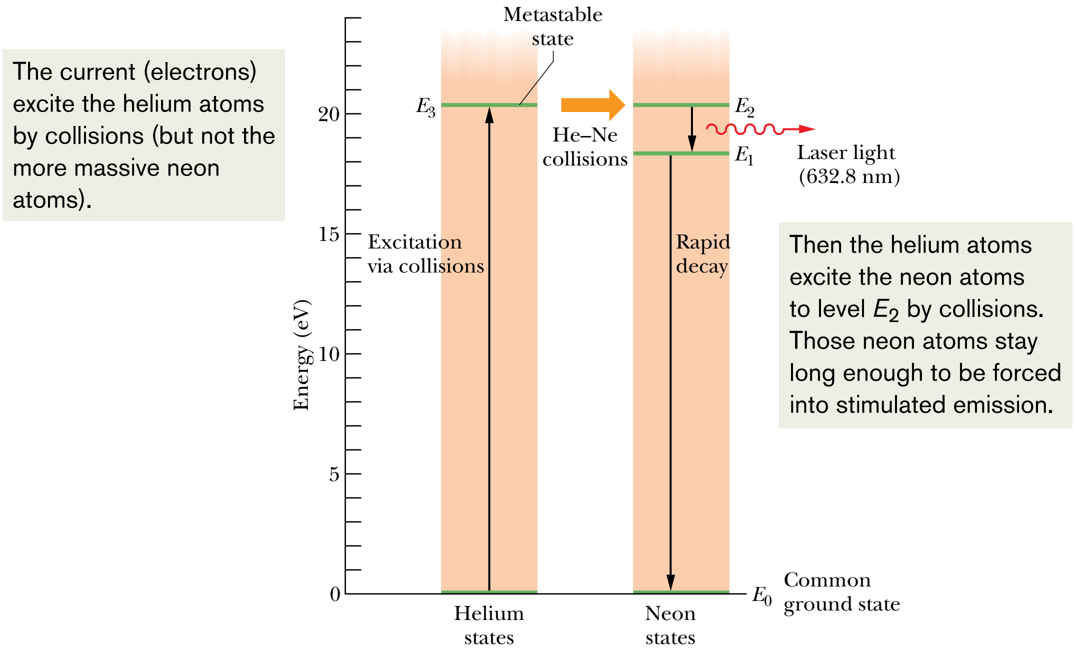


Figure 39-17 The elements of a heliumneon gas laser. An applied potential V_{DC} sends electrons through a discharge tube containing a mixture of helium gas and neon gas. Electrons collide with helium atoms, which then collide with neon atoms, which emit light along the length of the tube. The light passes through transparent windows W and reflects back and forth through the tube from mirrors M_1 and M_2 to cause more neon atom emissions. Some of the light leaks through mirror M_2 to form the laser beam.

Figure 39-18 shows simplified energy-level diagrams for the two types of atoms. An electric current passed through the helium–neon gas mixture serves—through collisions between helium atoms and electrons of the current—to raise many helium atoms to state E_3, which is metastable with a mean life of at least 1 μs. (The neon atoms are too massive to be excited by collisions with the (low-mass) electrons.)

The energy of helium state E_3 (20.61 eV) is very close to the energy of neon state E_2 (20.66 eV). Thus, when a metastable (E_3) helium atom and a ground-state (E_0) neon atom collide, the excitation energy of the helium atom is often transferred to the neon atom, which then moves to state E_2. In this manner, neon level E_2 (with a mean life of 170 ns) can become more heavily populated than neon level E_1 (which, with a mean life of only 10 ns, is almost empty).



The current (electrons) excite the helium atoms by collisions (but not the more massive neon atoms).

Then the helium atoms excite the neon atoms to level E_2 by collisions. Those neon atoms stay long enough to be forced into stimulated emission.

Figure 39-18 Five essential energy levels for helium and neon atoms in a helium–neon gas laser. Laser action occurs between levels E_2 and E_1 of neon when more atoms are at the E_2 level than at the E_1 level.

This population inversion is relatively easy to set up because (1) initially there are essentially no neon atoms in state E_1, (2) the long mean life of helium level E_3 means that there is always a good chance that collisions will excite neon atoms to their E_2 level, and (3) once those neon atoms undergo stimulated emission and fall to their E_1 level, they almost immediately fall down to their ground state (via intermediate levels not shown) and are then ready to be re-excited by collisions.

Suppose now that a single photon is spontaneously emitted as a neon atom transfers from state E_2 to state E_1. Such a photon can trigger a stimulated emission event, which, in turn, can trigger other stimulated emission events. Through such a chain reaction, a coherent beam of laser light, moving parallel to the tube axis, can build up rapidly. This light, of wavelength 632.8 nm (red), moves through the discharge tube many times by successive reflections from mirrors M_1 and M_2 shown in Fig. 39-17, accumulating additional stimulated emission photons with each passage. M_1 is totally reflecting, but M_2 is slightly "leaky" so that a small fraction of the laser light escapes to form a useful external beam.

CHECKPOINT 3

The wavelength of light from laser *A* (a helium–neon gas laser) is 632.8 nm; that from laser *B* (a carbon dioxide gas laser) is 10.6 μm; that from laser *C* (a gallium arsenide semiconductor laser) is 840 nm. Rank these lasers according to the energy interval between the two quantum states responsible for laser action, greatest first.

SAMPLE PROBLEM 39.03

Population inversion in a laser

In the helium–neon laser of Fig. 39-17, laser action occurs between two excited states of the neon atom. However, in many lasers, laser action (*lasing*) occurs between the ground state and an excited state, as suggested in Fig. 39-16*b*.

(a) Consider such a laser that emits at wavelength $\lambda = 550$ nm. If a population inversion is not generated, what is the ratio of the population of atoms in state E_x to the population in the ground state E_0, with the atoms at room temperature?

KEY IDEAS

(1) The naturally occurring population ratio N_x/N_0 of the two states is due to thermal agitation of the gas atoms (Eq. 39-21):

$$N_x/N_0 = e^{-(E_x - E_0)/kT} \tag{39-22}$$

To find N_x/N_0 with Eq. 39-22, we need to find the energy separation $E_x - E_0$ between the two states. (2) We can obtain $E_x - E_0$ from the given wavelength of 550 nm for the lasing between those two states.

Calculation: The lasing wavelength gives us

$$\begin{aligned} E_x - E_0 &= hf = \frac{hc}{\lambda} \\ &= \frac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3.00\times10^{8}\,\text{m/s})}{(550\times10^{-9}\,\text{m})(1.60\times10^{-19}\,\text{J/eV})} \\ &= 2.26\ \text{eV}. \end{aligned}$$

To solve Eq. 39-22, we also need the mean energy of thermal agitation kT for an atom at room temperature (assumed to be 300 K), which is

$$kT = (8.62\times10^{-5}\ \text{eV/K})(300\ \text{K}) = 0.0259\ \text{eV},$$

in which k is Boltzmann's constant.

Substituting the last two results into Eq. 39-22 gives us the population ratio at room temperature:

$$\begin{aligned} N_x/N_0 &= e^{-(2.26\ \text{eV})/(0.0259\ \text{eV})} \\ &\approx 1.3\times10^{-38}. \end{aligned} \qquad \text{(Answer)}$$

This is an extremely small number. It is not unreasonable, however. Atoms with a mean thermal agitation energy of only 0.0259 eV will not often impart an energy of 2.26 eV to another atom in a collision.

(b) For the conditions of (a), at what temperature would the ratio N_x/N_0 be 1/2?

Calculation: Now we want the temperature T such that thermal agitation has bumped enough neon atoms up to the higher-energy state to give $N_x/N_0 = 1/2$. Substituting that ratio into Eq. 39-22, taking the natural logarithm of both sides, and solving for T yield

$$\begin{aligned} T &= \frac{E_x - E_0}{k(\ln 2)} = \frac{2.26\ \text{eV}}{(8.62\times10^{-5}\,\text{eV/K})(\ln 2)} \\ &= 38\,000\ \text{K}. \end{aligned} \qquad \text{(Answer)}$$

This is much hotter than the surface of the Sun. Thus, it is clear that if we are to invert the populations of these two levels, some specific mechanism for bringing this about is needed—that is, we must "pump" the atoms. No temperature, however high, will naturally generate a population inversion by thermal agitation.

REVIEW AND SUMMARY

Some Properties of Atoms Atoms have quantized energies and can make quantum jumps between them. If a jump between a higher energy and a lower energy involves the emission or absorption of a photon, the frequency associated with the light is given by

$$hf = E_{\text{high}} - E_{\text{low}}. \tag{39-1}$$

States with the same value of quantum number n form a shell. States with the same values of quantum numbers n and ℓ form a subshell.

Orbital Angular Momentum and Magnetic Dipole Moments The magnitude of the orbital angular momentum of an electron trapped in an atom has quantized values given by

$$L = \sqrt{\ell(\ell+1)}\,\hbar, \quad \text{for } \ell = 0, 1, 2, \ldots, (n-1), \tag{39-2}$$

where $\hbar$ is $h/2\pi$, ℓ is the orbital magnetic quantum number, and n is the electron's principal quantum number. The component L_z of the orbital angular momentum on a z axis is quantized and given by

$$L_z = m_\ell \hbar, \quad \text{for } m_\ell = 0, \pm 1, \pm 2, \ldots, \pm\ell, \tag{39-3}$$

where m_ℓ is the orbital magnetic quantum number. The magnitude μ_{orb} of the orbital magnetic moment of the electron is quantized with the values given by

$$\mu_{\text{orb}} = \frac{e}{2m}\sqrt{\ell(\ell+1)}\,\hbar, \tag{39-6}$$

where m is the electron mass. The component $\mu_{\text{orb},z}$ on a z axis is also quantized according to

$$\mu_{\text{orb},z} = -\frac{e}{2m} m_\ell \hbar = -m_\ell \mu_B, \tag{39-7}$$

where μ_B is the Bohr magneton:

$$\mu_B = \frac{eh}{4\pi m} = \frac{e\hbar}{2m} = 9.274\times10^{-24}\ \text{J/T}. \tag{39-8}$$

Spin Angular Momentum and Magnetic Dipole Moment Every electron, whether trapped or free, has an intrinsic spin angular momentum $\vec{S}$ with a magnitude that is quantized as

$$S = \sqrt{s(s+1)}\,\hbar, \quad \text{for } s = \frac{1}{2}, \tag{39-9}$$

where s is the spin quantum number. An electron is said to be a spin-1/2 particle. The component S_z on a z axis is also quantized according to

$$S_z = m_s\hbar, \quad \text{for } m_s = \pm s = \pm\frac{1}{2}, \tag{39-10}$$

where m_s is the spin magnetic quantum number. Every electron, whether trapped or free, has an intrinsic spin magnetic dipole moment $\vec{\mu}_s$ with a magnitude that is quantized as

$$\mu_s = \frac{e}{m}\sqrt{s(s+1)}\,\hbar, \quad \text{for } s = \frac{1}{2}. \tag{39-12}$$

The component $\mu_{s,z}$ on a z axis is also quantized according to

$$\mu_{s,z} = -2m_s\mu_B, \quad \text{for } m_s = \pm\frac{1}{2}. \tag{39-13}$$

Pauli Exclusion Principle Electrons in atoms and other traps obey the Pauli exclusion principle, which requires that no two electrons in a trap can have the same set of quantum numbers.

Building the Periodic Table In the periodic table, the elements are listed in order of increasing atomic number Z, where Z is the number of protons in the nucleus. For a neutral atom, Z is also the number of electrons. States with the same value of quantum number n form a shell. States with the same values of quantum numbers n and ℓ form a subshell. A closed shell and a closed subshell contain the maximum number of electrons as allowed by the Pauli exclusion principle. The net angular momentum and net magnetic moment of such closed structures is zero.

X Rays and the Numbering of the Elements When a beam of high-energy electrons impacts a target, the electrons can lose their energy by emitting x rays when they scatter from atoms in the target. The emission is over a range of wavelengths, said to be a continuous spectrum. The shortest wavelength in the spectrum is the cutoff wavelength $\lambda_{\min}$, which is emitted when an incident electron loses its full kinetic energy K_0 in a single scattering event, with a single x-ray emission:

$$\lambda_{\min} = \frac{hc}{K_0}. \tag{39-15}$$

The characteristic x-ray spectrum is produced when incident electrons eject low-lying electrons in the target atoms and electrons from upper levels jump down to the resulting holes, emitting light. A Moseley plot is a graph of the square root of the characteristic-emission frequencies $\sqrt{f}$ versus atomic number Z of the target atoms. The straight-line plot reveals that the position of an element in the periodic table is set by Z and not by the atomic weight.

Lasers In stimulated emission, an atom in an excited state can be induced to de-excite to a lower energy state by emitting a photon if an identical photon passes the atom. The light emitted in stimulated emission is in phase with and travels in the direction of the light causing the emission.

A laser can emit light via stimulated emission provided that its atoms are in population inversion. That is, for the pair of levels involved in the stimulated emission, more atoms must be in the upper level than the lower level so that there is more stimulated emission than just absorption.

PROBLEMS

1. Show that the number of states with the same quantum number n is $2n^2$.
2. A hypothetical atom has only two atomic energy levels, separated by 3.2 eV. Suppose that at a certain altitude in the atmosphere of a star there are $6.1 \times 10^{13}/\text{cm}^3$ of these atoms in the higher-energy state and are $2.2 \times 10^{15}/\text{cm}^3$ in the lower-energy state. What is the temperature of the star's atmosphere at that altitude?
3. Suppose two electrons in an atom have quantum numbers $n = 2$ and $\ell = 1$. (a) How many states are possible for those two electrons? (Keep in mind that the electrons are indistinguishable.) (b) If the Pauli exclusion principle did not apply to the electrons, how many states would be possible?
4. An electron in a mercury atom is in the 3d subshell. Which of the following m_ℓ values are possible for it; (a) –3, (b) –1, (c) 0, (d) 1, (e) 2?
5. An atom of uranium has closed 6p and 7s subshell. Which subshell has greater number of electrons?
6. A tungsten ($Z = 74$) target is bombarded by electrons in an x-ray tube. The K, L, and M energy levels for tungsten (compare Fig. 39-11) have the energies 69.5, 11.3, and 2.30 keV, respectively. (a) What is the minimum value of the accelerating potential that will permit the production of the characteristic K_α and K_β lines of tungsten? (b) For this same accelerating potential, what is $\lambda_{\min}$? What are the (c) K_α and (d) K_β wavelengths?
7. A high-powered laser beam ($\lambda = 600$ nm) with a beam diameter of 10 cm is aimed at the Moon, 3.8×10^5 km distant. The beam spreads only because of diffraction. The angular location of the edge of the central diffraction disk (see Eq. 35-60) is given by

$$\sin\theta = \frac{1.22\lambda}{d},$$

where d is the diameter of the beam aperture. What is the diameter of the central diffraction disk on the Moon's surface?
8. What are the initial and final shells for an M_β line in an x-ray spectrum?
9. Consider the elements selenium ($Z = 34$), bromine ($Z = 35$), and krypton ($Z = 36$). In their part of the periodic table, the subshells of the electronic states are filled in the sequence

$$1s\ 2s\ 2p\ 3s\ 3p\ 3d\ 4s\ 4p\ \ldots$$

What are (a) the highest occupied subshell for selenium and (b) the number of electrons in it, (c) the highest occupied subshell for bromine and (d) the number of electrons in it, and (e) the highest occupied subshell for krypton and (f) the number of electrons in it?

10. Which element has K_α x-ray line whose wavelength is 0.180 nm?

11. An x-ray tube operates at 20 kV. A particular electron loses 5% of its kinetic energy to emit an x-ray photon at the first collision. Find the wavelength corresponding to this photon.

12. In Fig. 39-9, the x rays shown are produced when 35.0 keV electrons strike a molybdenum ($Z = 42$) target. If the accelerating potential is maintained at this value but a silver ($Z = 47$) target is used instead, what values of (a) $\lambda_{\min}$, (b) the wavelength of the K_α line, and (c) the wavelength of the K_β line result? The K, L, and M atomic x-ray levels for silver (compare Fig. 39-11) are 25.51, 3.56, and 0.53 keV.

13. (a) From Eq. 39-18, what is the ratio of the photon energies due to K_α transitions in two atoms whose atomic numbers are Z and Z'? (b) What is this ratio for uranium and aluminum? (c) For uranium and lithium?

14. Show that a moving electron cannot spontaneously change into an x-ray photon in free space. A third body (atom or nucleus) must be present. Why is it needed? (*Hint*: Examine the conservation of energy and momentum.)

15. A population inversion for two energy levels is often described by assigning a negative Kelvin temperature to the system. What negative temperature would describe a system in which the population of the upper energy level exceeds that of the lower level by 10% and the energy difference between the two levels is 2.32 eV?

16. Two of the three electrons in a lithium atom have quantum numbers (n, ℓ, m_ℓ m_s) of (1, 0, 0, +1/2) and (1, 0, 0, –1/2). What quantum numbers are possible for the third electron if the atom is (a) in the ground state and (b) in the first excited state?

17. A pulsed laser emits light at a wavelength of 694.4 nm. The pulse duration is 12 ps, and the energy per pulse is 0.150 J. (a) What is the length of the pulse? (b) How many photons are emitted in each pulse?

18. From Fig. 39-9, calculate approximately the energy difference $E_L - E_M$ for molybdenum. Compare it with the value that may be obtained from Fig. 39-11.

19. For a helium atom in its ground state, what are quantum numbers (n, ℓ, m_ℓ and m_s) for the (a) spin-up electron and (b) spin-down electron?

20. A hypothetical atom has two energy levels, with a transition wavelength between them of 580 nm. In a particular sample at 300 K, 4.0×10^{20} such atoms are in the state of lower energy. (a) How many atoms are in the upper state, assuming conditions of thermal equilibrium? (b) Suppose, instead, that 3.0×10^{20} of these atoms are "pumped" into the upper state by an external process, with 1.0×10^{20} atoms remaining in the lower state. What is the maximum energy that could be released by the atoms in a single laser pulse if each atom jumps once between those two states (either via absorption or via stimulated emission)?

21. If orbital angular momentum $\vec{L}$ is measured along, say, a z axis to obtain a value for L_z, show that

$$(L_x^2 + L_y^2)^{1/2} = [\ell(\ell+1) - m_\ell^2]^{1/2}\hbar$$

is the most that can be said about the other two components of the orbital angular momentum.

22. A laser emits at 424 nm in a single pulse that lasts 0.500 μs. The power of the pulse is 3.25 MW. If we assume that the atoms contributing to the pulse underwent stimulated emission only once during the 0.500 μs, how many atoms contributed?

23. X rays are produced in an x-ray tube by electrons accelerated through an electric potential difference of 50.0 kV. Let K_0 be the kinetic energy of an electron at the end of the acceleration. The electron collides with a target nucleus (assume the nucleus remains stationary) and then has kinetic energy $K_1 = 0.500K_0$. (a) What wavelength is associated with the photon that is emitted? The electron collides with another target nucleus (assume it, too, remains stationary) and then has kinetic energy $K_2 = 0.500K_1$. (b) What wavelength is associated with the photon that is emitted?

24. Assume that lasers are available whose wavelengths can be precisely "tuned" to anywhere in the visible range—that is, in the range 450 nm < λ < 650 nm. If every television channel occupies a bandwidth of 10 MHz, how many channels can be accommodated within this wavelength range?

25. A certain gas laser can emit light at wavelength 550 nm, which involves population inversion between ground state and an excited state. At room temperature, how many moles of neon are needed to put 12 atoms in that excited state by thermal agitation?

26. How many electron states are there in the following shells: (a) $n = 4$, (b) $n = 1$, (c) $n = 3$, (d) $n = 2$?

27. The active volume of a laser constructed of the semiconductor GaAlAs is only 200 μm^3 (smaller than a grain of sand), and yet the laser can continuously deliver 5.0 mW of power at a wavelength of 0.80 μm. At what rate does it generate photons?

28. In the subshell $\ell = 4$, (a) what is the greatest (most positive) m_ℓ value, (b) how many states are available with the greatest m_ℓ value, and (c) what is the total number of states available in the subshell?

29. A material whose K-absorption edge is 0.2 Å is irradiated by x-rays of wavelength 0.15 Å. Find the maximum energy of the photoelectrons that are emitted from the K shell.

30. The wavelength of the K_α line from iron is 193 pm. What is the energy difference between the two states of the iron atom that give rise to this transition?

31. An electron in a multielectron atom has $m_\ell = +4$. For this electron, what are (a) the value of ℓ, (b) the smallest possible value of n, and (c) the number of possible values of m_s?

32. The mirrors in the laser of Fig. 39-17, which are separated by 6.0 cm, form an optical cavity in which standing waves of laser light can be set up. Each standing wave has an integral number n of half wavelengths in the 6.0 cm length, where n is large and the waves differ slightly in wavelength. Near $\lambda = 533$ nm, how far apart in wavelength are the standing waves?

33. A hypothetical atom has energy levels uniformly separated by 1.2 eV. At a temperature of 2000 K, what is the ratio of the number of atoms in the 13th excited state to the number in the 11th excited state?

34. Ruby lases at a wavelength of 694 nm. A certain ruby crystal has 4.00×10^{19} Cr ions (which are the atoms that lase). The lasing transition is between the first excited state and the ground state, and the output is a light pulse lasting 1.50 μs. As the pulse begins, 60.0% of the Cr ions are in the first excited state and the rest are in the ground state. What is the average power emitted during the pulse?

35. Figure 39-19 shows the energy levels of two types of atoms. Atoms A are in one tube, and atoms B are in another tube. The energies (relative to a ground-state energy of zero) are indicated; the average lifetime of atoms in each level is also indicated. All the atoms are initially pumped to levels higher than the levels shown in the figure. The atoms then drop down through the levels, and many become "stuck" on certain levels, leading to population inversion and lasing. The light emitted by A illuminates B and can cause stimulated emission of B. What is the energy per photon of that stimulated emission of B?

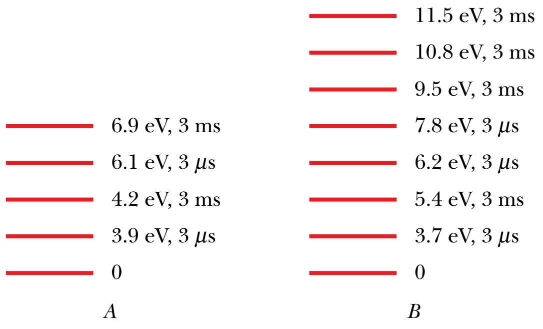


Figure 39-19 Problem 35.

36. How many electron states are there in a shell defined by the quantum number $n = 6$?

37. Through what minimum potential difference must an electron in an x-ray tube be accelerated so that it can produce x rays with a wavelength of 0.100 nm?

38. Calculate the ratio of the wavelength of the K_α line for niobium (Nb) to that for gallium (Ga). Take needed data from the periodic table of Appendix G.

39. An electron is in a state with $\ell = 3$, (a) What multiple of $\hbar$ gives the magnitude of $\vec{L}$? (b) What multiple of μ_B gives the magnitude of $\vec{\mu}$? (c) What is the largest possible value of m_ℓ, (d) what multiple of $\hbar$ gives the corresponding value of L_z, and (e) what multiple of μ_B gives the corresponding value of $\mu_{\text{orb},z}$? (f) What is the value of the semiclassical angle θ between the directions of L_z and $\vec{L}$? What is the value of angle θ for (g) the second largest possible value of m_ℓ and (h) the smallest (that is, most negative) possible value of m_ℓ?

40. When electrons bombard a molybdenum target, they produce both continuous and characteristic x rays as shown in Fig. 39-9. In that figure the kinetic energy of the incident electrons is 35.0 keV. If the accelerating potential is increased to 55.0 keV, (a) what is the value of λ_{min}, and (b) do the wavelengths of the K_α and K_β lines increase, decrease, or remain the same?

41. An electron in a hydrogen atom is in a state with $\ell = 5$. What is the minimum possible value of the semiclassical angle between $\vec{L}$ and L_z?

42. An electron is in a state with $n = 4$. What are (a) the number of possible values of ℓ, (b) the number of possible values of m_ℓ, (c) the number of possible values of m_s, (d) the number of states in the $n = 4$ shell, and (e) the number of subshells in the $n = 4$ shell?

43. (a) How many ℓ values are associated with $n = 3$? (b) How many m_ℓ, values are associated with $\ell = 1$?

44 The active medium in a particular laser that generates laser light at a wavelength of 694 nm is 6.00 cm long and 1.00 cm in diameter. (a) Treat the medium as an optical resonance cavity analogous to a closed organ pipe. How many standingwave nodes are there along the laser axis? (b) By what amount Δf would the beam frequency have to shift to increase this number by one? (c) Show that Δf is just the inverse of the travel time of laser light for one round trip back and forth along the laser axis. (d) What is the corresponding fractional frequency shift $\Delta f/f$? The appropriate index of refraction of the lasing medium (a ruby crystal) is 1.75.

45. A recently named element is darmstadtium (Ds), which has 110 electrons. Assume that you can put the 110 electrons into the atomic shells one by one and can neglect any electron–electron interaction. With the atom in ground state, what is the spectroscopic notation for the quantum number ℓ for the last electron?

46. How many electron states are in these subshells: (a) $n = 4$, $\ell = 3$; (b) $n = 3$, $\ell = 2$; (c) $n = 4$, $\ell = 1$; (d) $n = 2$, $\ell = 0$?

47. (a) What is the magnitude of the orbital angular momentum in a state with $\ell = 3$? (b) What is the magnitude of its largest projection on an imposed z axis?

48. Figure 39-20 shows the energy levels of a mercury atom. Electrons with kinetic energies of 7.0 eV are introduced into mercury vapor. (a) Convert the energy levels to the system that sets the ground state at 0 eV. (b) What is the kinetic energy of an electron after an elastic collision with a mercury atom? (c) What are the possible kinetic energies of an electron after an inelastic collision with a mercury atom? (d) What are the energies of the photons coming from the mercury vapor?

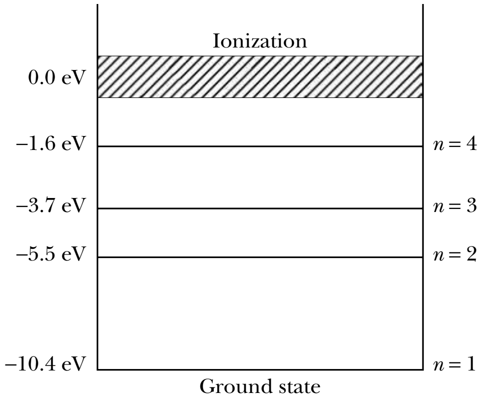


Figure 39-20 Problem 48.

49. The K_α x rays of aluminum ($Z = 13$) and zinc ($Z = 30$) have wavelengths 887 pm and 146 pm, respectively. Use Moseley's law $\sqrt{v} = a(Z - b)$ to find the wavelength of the K_α x ray of iron ($Z = 26$).

50. An imaginary atom whose energy levels are shown here undergoes an Auger transition. This implies that the x-ray photon emitted during transition from L to K state is absorbed by another electron in M shell and it comes out of the atom. Calculate the initial energy of the Auger electron as in Fig. 39-21.

51. (a) Find the energy in electron volts required to strip a calcium atom ($Z = 20$) of its last electron, assuming the other 19 have been removed. How does this compare with the energy required to excite the K_α x-ray lines of Ca, which is about 3.7 keV? (b) Why there is a difference?

52. K_α x rays of molybdenum has wavelength 71 pm. If the energy of a molybdenum atom with a K electron knocked out is 23.32 keV, what will be the energy of this atom when an L electron is knocked out?

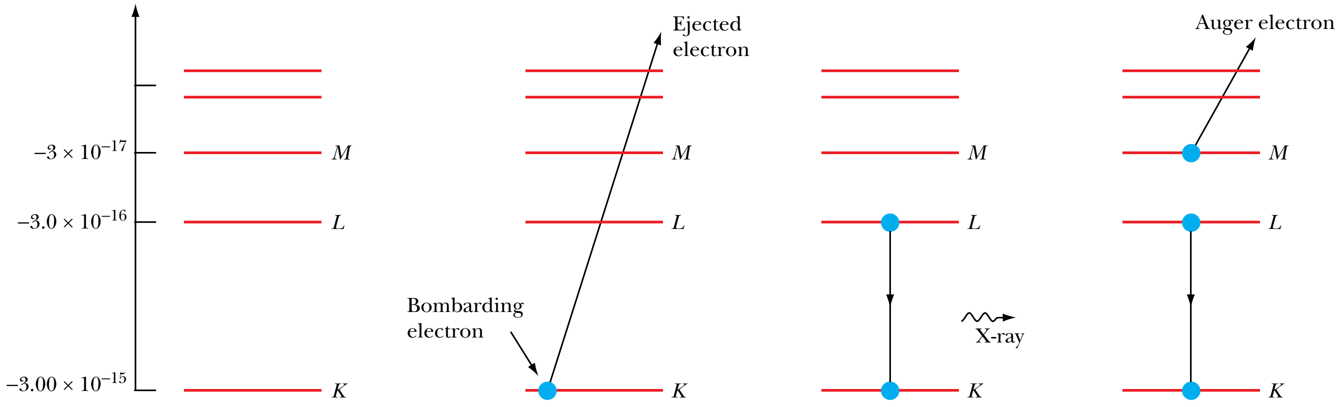


Figure 39-21 Problem 50.

PRACTICE QUESTIONS

Single Correct Choice Type

1. Electrons with energy 80 keV are incident on the tungsten target of an x-ray tube. K-shell electrons of tungsten have −72.5 keV energy. x-rays emitted by the tube contain only

(a) A continuous x-ray spectrum (Bremsstrahlung) with a minimum wavelength of ~0.155 Å

(b) A continuous x-ray spectrum (Bremsstrahlung) with all wavelengths

(c) The characteristic x-ray spectrum of tungsten

(d) A continuous x-ray spectrum (Bremsstrahlung) with a minimum wavelength of ~0.155 Å and the characteristic x-ray spectrum of tungsten

2. If the atom $^{257}_{100}\text{Fm}$ follows the Bohr model and the radius of $^{257}_{100}\text{Fm}$ is n times the Bohr radius, then the value of n is

(a) 1/4 **(b)** 4
(c) 50 **(d)** 100

3. An element with atomic number 9 emits K_α x-ray of wavelength λ. The atomic number of the element which emits K_α x-ray of wavelength 4λ is

(a) 4 **(b)** 5
(c) 6 **(d)** 7

4. Electrons with de Broglie wavelength λ fall on the target in an x-ray tube. The cut-off wavelength of the emitted x-rays is

(a) $\lambda_0 = \dfrac{2mc\lambda^2}{h}$ **(b)** $\lambda_0 = \dfrac{2h}{mc}$

(c) $\lambda_0 = \dfrac{2m^2c^2\lambda^3}{h^2}$ **(d)** $\lambda_0 = \lambda$

5. Which one of the following statements is **WRONG** in the context of x-rays generated from a x-ray tube?
(a) Wavelength of characteristic x-rays decreases when the atomic number of the target increases.
(b) Cut-off wavelength of the continuous x-rays depends on the atomic number of the target.
(c) Intensity of the characteristic x-rays depends on the electrical power given to the x-ray tube.
(d) Cut-off wavelength of the continuous x-rays depends on the energy of the electrons in the x-ray tube.

6. If λ_{Cu} is the wavelength of K_α x-ray line of copper (atomic number 29) and λ_{Mo} is the wavelength of the K_α x-ray line of molybdenum (atomic number 42), then the ratio $\lambda_{\text{Cu}}/\lambda_{\text{Mo}}$ is close to
(a) 1.99
(b) 2.14
(c) 0.50
(d) 0.48

7. A diatomic molecule is made of two masses m_1 and m_2 which are separated by a distance r. If we calculate its rotational energy by applying Bohr's rule of angular momentum quantization, its energy will be given by (n is an integer)
(a) $\dfrac{(m_1+m_2)^2n^2h^2}{2m_1^2m_2^2r^2}$
(b) $\dfrac{n^2h^2}{2(m_1+m_2)r^2}$
(c) $\dfrac{2n^2h^2}{(m_1+m_2)r^2}$
(d) $\dfrac{(m_1+m_2)n^2h^2}{2m_1m_2r^2}$

8. An atom stays in an excited state for about
(a) 10 μs
(b) 10 ms
(c) 10 ns
(d) 10 s

9. The ratio of magnetic field at the center due to the rotation of a Bohr's electron in the first state of hydrogen atom and second excited state of Li^{++} atom is
(a) 9:1
(b) 27:4
(c) 9:4
(d) 27:1

10. With increasing quantum number, the energy difference between adjacent energy levels in atoms
(a) Increases
(b) Decreases
(c) Remains constant
(d) First increases then decreases

11. If in nature, there may not be an element for which the principal quantum number $n > 4$, then the total possible number of elements will be
(a) 60
(b) 32
(c) 4
(d) 64

12. An electron has a mass of 9.1×10^{-31} kg. It revolves around the nucleus in a circular orbit of radius 0.529×10^{-10} m at a speed of 2.2×10^6 m/s. The magnitude of its linear momentum in this motion is
(a) 1.1×10^{-34} kg·m/s
(b) 2.0×10^{-24} kg·m/s
(c) 4.0×10^{-24} kg·m/s
(d) 4.0×10^{-31} kg·m/s

13. In a beryllium atom, if a_0 be the radius of the first orbit, then the radius of the second orbit will be in general
(a) na_0
(b) a_0
(c) n^2a_0
(d) $\dfrac{a_0}{n^2}$

14. The velocity of an electron in its fifth orbit, if the velocity of an electron in the second orbit of sodium atom (atomic number = 11) is v, will be
(a) v
(b) $\dfrac{22}{5}v$
(c) $\dfrac{5}{2}v$
(d) $\dfrac{2}{5}v$

15. The number of values of the orbital quantum number associated with the principal quantum number $n = 3$ is
(a) 1
(b) 2
(c) 3
(d) 4

16. The number of states in a shell with principal quantum number $n = 3$ is
(a) 3
(b) 9
(c) 15
(d) 18

17. Of the following states, $5s, 3p, 4f, 5p, 4g, 3d$, and $2p$, the one which is NOT allowed is
(a) $3p$
(b) $4f$
(c) $3d$
(d) $4g$

18. An electron is in a quantum state for which the magnitude of the orbital angular momentum is $6\sqrt{2}h/2\pi$. How many allowed values of the z component of the angular momentum are there?
(a) 4
(b) 5
(c) 7
(d) 8

19. An electron in an atom is in a state with $l = 1$. The minimum angle (in degrees) between $\vec{L}$ and the z axis is
(a) 0
(b) 18.0
(c) 45
(d) 36.7

20. The spin angular momentum of an electron is equal to
(a) $\sqrt{3}$
(b) $h\sqrt{3}/4\pi$
(c) $h/2$
(d) $\pm h/2$

21. The x-ray beam coming from an x-ray tube
(a) Is monochromatic
(b) Has all wavelengths smaller than a certain maximum wavelength
(c) Has all wavelengths greater than a certain minimum wavelength
(d) Has all wavelengths lying between a minimum and a maximum wavelength

22. The minimum wavelength of x ray that can be produced in a Coolidge tube depends on
(a) The metal used as the target
(b) The intensity of the electron beam striking the target
(c) The current flowing through the filament
(d) The potential difference between the cathode and the anode

23. If the potential difference applied to the tube is doubled and the separation between the filament and the target is also doubled, the cutoff wavelength
(a) Will remain unchanged
(b) Will be doubled
(c) Will be halved
(d) Will become four times the original

24. If the current in the circuit for heating the filament is increased, the cutoff wavelength
(a) Will increase
(b) Will decrease
(c) Will remain unchanged
(d) Will change

25. If a potential difference of 20 000 V is applied across an x-ray tube, the cutoff wavelength will be
(a) 6.21×10^{-10} m
(b) 6.21×10^{-11} m
(c) 6.21×10^{-12} m
(d) 3.1×10^{-11} m

26. The potential difference applied to an x-ray tube is 5 kV and the current through it is 3.2 mA. Then the number of electrons striking the target per second is
(a) 2×10^{16}
(b) 5×10^{6}
(c) 1×10^{17}
(d) 4×10^{15}

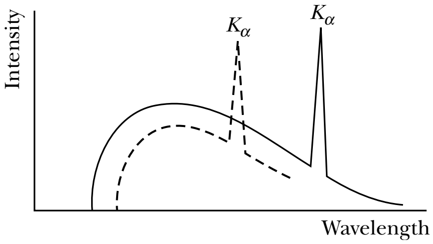
27. In an x-ray tube, the target should be made up of material having
(a) High atomic number and low thermal conductivity
(b) High atomic number and high thermal conductivity
(c) Low atomic number and low thermal conductivity
(d) Low atomic number and high thermal conductivity

28. X ray can be made harder by
(a) Increasing the potential difference between cathode and target
(b) Decreasing the potential difference between cathode and target
(c) Increasing the current through filament
(d) Decreasing the current through filament

29. Which of the following wavelength falls in an x-ray region?
(a) 10000 Å
(b) 1000 Å
(c) 1 Å
(d) 10^{-2} Å

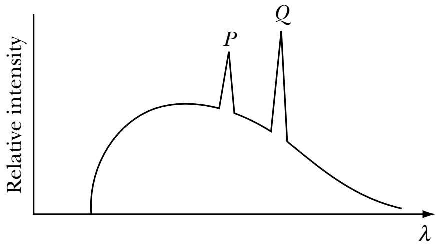
30. The energy of a photon of characteristic x ray from a Coolidge tube comes from
(a) The kinetic energy of the striking electron
(b) The kinetic energy of the free electrons of the target
(c) The kinetic energy of the ions of the target
(d) An atomic transition in the target

31. The following figure shows the intensity–wavelength relations of x-rays coming from two different Coolidge tubes. The solid curve represents the relation for tube A in which the potential difference between the target and the filament is V_A and the atomic number of the target material is Z_A. These quantities are V_B and Z_B for the other tube. Then,



(a) $V_A > V_B, Z_A > Z_B$
(b) $V_A > V_B, Z_A < Z_B$
(c) $V_A < V_B, Z_A > Z_B$
(d) $V_A < V_B, Z_A < Z_B$

32. In a characteristic x-ray spectra (as shown in the following figure), L shell of some element is superimposed on continuous x-ray spectra. Here



(a) P can represent L_α line
(b) Q represents L_α line
(c) Q and P represent L_α and L_β lines, respectively
(d) Q and P represent L_β and L_α lines, respectively

33. The K_α x-ray emission line of tungsten occurs at $\lambda = 0.021$ nm. The energy difference between K and L levels in this atom is about
(a) 0.51 MeV
(b) 1.2 MeV
(c) 59 keV
(d) 13.6 eV

34. Mosley's law for characteristic x rays is given by $\sqrt{f} = a(Z - b)$

Choose the correct statement(s).
(a) Both a and b depend on the target material
(b) Both a and b are independent of the target material
(c) Both a and b depend on the energy of the electron beam
(d) Both a and b depend on the nature of the lines (i.e., $K, L, M, \ldots$)

35. The wavelength of the K_α line for an element of atomic number 57 is α. What is the wavelength of the K_α line for the element of atomic number 29?
(a) α
(b) 2α
(c) 4α
(d) 8α

36. If the frequencies of K_α, K_β, and L_α x rays for a material $\nu_{K_\alpha}, \nu_{K_\beta}, \nu_{L_\alpha}$, respectively, then
(a) $\nu_{K_\alpha} = \nu_{K_\beta} + \nu_{L_\alpha}$
(b) $\nu_{L_\alpha} = \nu_{K_\alpha} + \nu_{K_\beta}$
(c) $\nu_{K_\beta} = \nu_{K_\alpha} + \nu_{L_\alpha}$
(d) None of these

37. The K, L, and M energy levels of platinum lie roughly at 78, 12, and 3 KeV, respectively. The ratio of wavelength of K_α line to that of K_β line in x-ray spectrum is
(a) $\frac{22}{3}$
(b) $\frac{3}{22}$
(c) $\frac{22}{25}$
(d) $\frac{25}{22}$

38. The potential difference applied to an x-ray tube is V. The ratio of de Broglie's wavelength of electron to the minimum wavelength of x ray is directly proportional to
(a) V
(b) $V^{1/2}$
(c) $1/\sqrt{V}$
(d) Independent of V

39. The energy of a tungsten atom with a vacancy in L shell is 11.3 keV. Wavelength of K_α photon for tungsten is 21.3 pm. If a potential difference of 62 kV is applied across the x-rays tube following characteristic x rays will be produced.
(a) K, L series
(b) Only K_α and L series
(c) Only L series
(d) None

40. Photons in a laser beam have the same energy, wavelength, polarization direction, and phase because
(a) Each is produced in an emission that is stimulated by another
(b) All come from the same atom
(c) The lasing material has only two quantum states
(d) All photons are alike, no matter what their source

41. Photons in a laser beam are produced by
(a) Transitions from a metastable state
(b) Transitions to a metastable state
(c) Transitions from a state that decays rapidly
(d) Pumping

42. A metastable state is important for the generation of a laser beam because it assures that
(a) Spontaneous emission does not occur more often than stimulated emission
(b) More photons are emitted than are absorbed
(c) Photons do not collide with each other
(d) Photons do not make upward transitions

43. Each atom in the periodic table has a unique set of spectral lines. Which one of the following statements is the best explanation for this observation?
(a) Each atom has a dense central nucleus.
(b) Electrons in atoms orbit the nucleus.
(c) Each atom has a unique set of energy levels that electrons can move between.
(d) Electrons in atoms are in constant motion.

44. Which one of the following pairs of characteristics of light is best explained by assuming that light can be described in terms of photons?
(a) Photoelectric effect and the effect observed in Young's experiment
(b) Diffraction and the formation of atomic spectra
(c) Polarization and the photoelectric effect
(d) Existence of line spectra and the photoelectric effect

45. Complete the following statement: An individual copper atom emits electromagnetic radiation with wavelengths that are
(a) Evenly spaced across the spectrum
(b) Unique to that particular copper atom
(c) The same as other elements in the same column of the periodic table
(d) Unique to all copper atoms

46. Determine the maximum wavelength of incident radiation that can be used to remove the remaining electron from a singly ionized helium atom He$^+$ ($Z = 2$). Assume the electron is in its ground state.
(a) 6.2 nm
(b) 22.8 nm
(c) 12.4 nm
(d) 45.6 nm

47. To which model of atomic structure does the *Pauli exclusion principle* apply?
(a) The nuclear atom
(b) The quantum mechanical atom
(c) The billiard ball atom
(d) The plum-pudding model

48. Which one of the following statements concerning the electrons specified by the notation $3p^4$ is true?
(a) The electrons are in the *M* shell.
(b) The electrons are in the $\ell = 2$ subshell.
(c) The electrons are necessarily in an excited state.
(d) They have principal quantum number 4.

49. Which one of the following subshells is not compatible with a principle quantum number of $n = 4$?
(a) *s* **(b)** *d*
(c) *g* **(d)** *p*

50. Which one of the following electronic configurations corresponds to an atomic ground state?
(a) $1s^2\,2s^1\,2p^6$
(b) $1s^1\,2s^2\,3p^1$
(c) $1s^1\,2s^1\,2p^1$
(d) $1s^2\,2s^2\,2p^1$

51. In an x-ray tube, electrons with energy 35 keV are incident on a cobalt ($Z = 27$) target. Determine the cutoff wavelength for x-ray production.
(a) 1.4×10^{-11} m **(b)** 3.6×10^{-11} m
(c) 1.8×10^{-11} m **(d)** 3.2×10^{-11} m

52. Which one of the following statements concerning the cutoff wavelength typically exhibited in x-ray spectra is true?
(a) The cutoff wavelength depends on the target material.
(b) The cutoff wavelength depends on the potential difference across the x-ray tube.
(c) The cutoff wavelength is independent of the energy of the incident electrons.
(d) The cutoff wavelength occurs because of the mutual shielding effects of *K*-shell electrons.

53. Calculate the K_α x-ray wavelength for a gold atom ($Z = 79$).
(a) 5.13×10^{-10} m
(b) 2.00×10^{-11} m
(c) 8.54×10^{-10} m
(d) 3.60×10^{-11} m

54. Electrons in an x-ray tube are accelerated through a potential difference of 40 kV. The electrons then strike a zirconium ($Z = 40$) target. Determine the cutoff frequency for x-ray production.
(a) 4.7×10^{19} Hz **(b)** 3.2×10^{18} Hz
(c) 9.7×10^{18} Hz **(d)** 6.7×10^{17} Hz

55. What is the operating voltage of a medical x-ray machine that has a cut-off wavelength of 2.20×10^{-11} m?
(a) 83 800 V **(b)** 56 500 V
(c) 10 900 V **(d)** 44 900 V

56. An argon-ion laser emits a blue-green beam of light with a wavelength of 488 nm in a vacuum. What is the difference in energy in joules between the two energy states for the atomic transition that produces this light?
(a) 4.08×10^{-19} J **(b)** 6.18×10^{-20} J
(c) 1.05×10^{-20} J **(d)** 4.76×10^{-24} J

57. A pulsed laser has an average output power of 4.0 W. Each pulse consists of light at wavelength 5.0×10^{-7} m and has a 25 ms duration. How many photons are emitted in a single pulse?
(a) 1.0×10^{17} **(b)** 3.7×10^{17}
(c) 2.5×10^{17} **(d)** 5.0×10^{17}

More than One Correct Choice Type

58. A single electron orbits a stationary nucleus of charge $+Ze$ where Z is a constant and e is the magnitude of electronic charge. It releases 47.22 eV energy if it comes from the third orbit to second orbit. (Use ionization energy of hydrogen atom = 13.6 eV)

(a) The value of Z is 5.
(b) The wavelength of electromagnetic radiation required to remove the electron from first orbit to infinity is nearly 3653 pm.
(c) The radius of the first orbit is 10.6 pm.
(d) The angular momentum of the electron in first orbit is 1.05×10^{-34} J $\cdot$ s

59. An x-ray tube operates at an accelerating potential of 20 kV. Which of the following wavelengths will be absent in the continuous spectrum of x-ray?

(a) 12 pm
(b) 45 pm
(c) 65 pm
(d) 95 pm

60. Electrons with energy 80 keV are incident on the tungsten target of an x-ray tube. K-shell electrons of tungsten have 72.5 keV energy. X-rays emitted by the tube contain

(a) A continuous x-ray spectrum (Bremsstrahlung) with a minimum wavelength of 0.155 Å
(b) A continuous x-ray spectrum (Bremsstrahlung) with all wavelengths
(c) The characteristic x-ray spectrum of tungsten
(d) A continuous x-ray spectrum (Bremsstrahlung) with a minimum wavelength of 0.155 Å and the characteristic x-ray spectrum of tungsten

61. Which (if any) of the following are essential for laser action to occur between two energy levels of an atom?

(a) There are more atoms in the upper level than in the lower
(b) The upper level is metastable
(c) The lower level is metastable
(d) The lower level is the ground state of the atom

Linked Comprehension

Paragraph for Questions 62–64: When a particle is restricted to move along x-axis between $x = 0$ and $x = a$, where a is of nanometer dimension, its energy can take only certain specific values. The allowed energies of the particle moving in such a restricted region, correspond to the formation of standing waves with nodes at its ends $x = 0$ and $x = a$. The wavelength of this standing wave is related to the linear momentum p of the particle according to the de Broglie relation. The energy of the particle of mass m is related to its linear momentum as $E = p^2/2m$. Thus, the energy of the particle can be denoted by a quantum number 'n' taking values 1, 2, 3, … (n = 1, called the ground state) corresponding to the number of loops in the standing wave.

Use the model described above to answer the following three questions for a particle moving in the line $x = 0$ to $x = a$. Take $h = 6.6 \times 10^{-34}$ J s and $e = 1.6 \times 10^{-19}$ C.

62. The allowed energy for the particle for a particular value of n is proportional to

(a) a^{-2} **(b)** $a^{-3/2}$
(c) a^{-1} **(d)** a^2

63. If the mass of the particle is $m = 1.0 \times 10^{-30}$ kg and $a = 6.6$ nm, the energy of the particle in its ground state is closest to

(a) 0.8 MeV
(b) 8 MeV
(c) 80 MeV
(d) 800 MeV

64. The speed of the particle, that can take discrete values, is proportional to

(a) $n^{-3/2}$ **(b)** n^{-1}
(c) $n^{1/2}$ **(d)** n

Paragraph for Questions 65–67: A neutral atom has the following electronic configuration: $1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^5$

65. How many electrons are in the M shell?

(a) 2 **(b)** 6
(c) 5 **(d)** 7

66. How many protons are in the atomic nucleus?

(a) 4 **(b)** 12
(c) 7 **(d)** 17

67. To which group of the periodic table does this element belong?

(a) I **(b)** VII
(c) II **(d)** VI

Paragraph for Questions 68–70: An electron in an atom has the following set of quantum numbers:

$$n = 3, \ell = 2, m_\ell = +1, m_s = +1/2.$$

68. What shell is this electron occupying?

(a) K shell **(b)** M shell
(c) L shell **(d)** N shell

69. In which subshell can the electron be found?

(a) s **(b)** d
(b) p **(d)** f

70. According to the quantum mechanical picture of the atom, which quantum number(s) *could* be different for electrons in this same atom that have exactly the same energy?

(a) n, ℓ, m_ℓ and m_S **(b)** only ℓ and m_ℓ
(c) only ℓ, m_ℓ and m_S **(d)** only m_ℓ and m_S

Paragraph for Questions 71–73: Consider the following list of electron configurations:

(1) $1s^2\, 2s^2\, 3s^2$ **(2)** $1s^2\, 2s^2\, 2p^6$
(3) $1s^2\, 2s^2\, 2p^6\, 3s^1$ **(4)** $1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^6\, 4s^2$
(5) $1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^6\, 4s^2\, 3d^6$

71. Which one of the above configurations represents a neutral atom that readily forms a singly charged positive ion?

(a) 1 **(b)** 3
(c) 2 **(d)** 4

72. Which one of the above configurations represents an excited state of a neutral atom?

(a) 1 **(b)** 3
(c) 2 **(d)** 4

73. Which one of the above configurations represents a transition element?

(a) 1	**(b)** 5
(c) 2	**(d)** 4

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

74. An electron in hydrogen atom moves from $n = 1$ to $n = 2$.

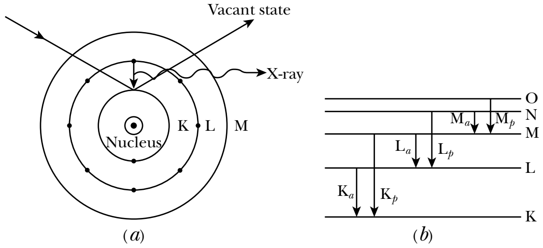
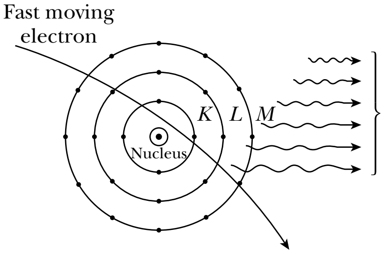
Column I	Column II
(a) Angular momentum	**(p)** One-fourth times
(b) Kinetic energy	**(q)** Two times
(c) Potential energy	**(r)** Four times
(d) Mechanical energy	**(s)** Half times

75.

Column I	Column II
(a) Lithium atom	**(p)** 54.4 eV
(b) Helium atom	**(q)** 13.6 eV
(c) Beryllium atom	**(r)** 122 eV
(d) Hydrogen atom	**(s)** 217.6 eV

Directions for Questions 76–78: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

76. The x-rays have different penetration power. In the given table, Column I shows their penetration power of x-rays, Column II shows their frequency and speed and Column III shows their wavelength range and mode of generation.

Column I	Column II	Column III
(I) More penetration power	**(i)** a few fast-moving electrons penetrate deep into the interior of the atoms of the target material	**(J)** 
(II) Consists of radiations of all possible wavelengths	**(ii)** more frequency of the order of ≈ 10^{19} Hz	**(K)** more wavelength range (4 Å to 100Å)
(III) Less penetration power	**(iii)** less frequency of the order of ≈ 10^{16} Hz	**(L)** 
(IV) Consists of definite, well defined wavelengths	**(iv)** few of the fast moving electrons having velocity of about (1/10)th of the velocity of light may penetrate the surface atoms.	**(M)** lesser wavelength range (0.1 Å to 4Å)

(1) What are the characteristics of hard x-rays?
- **(a)** (II) (iii) (M)
- **(b)** (III) (ii) (K)
- **(c)** (I) (ii) (M)
- **(d)** (I) (iii) (J)

(2) What are the characteristics of continuous x-rays?

(a) (I) (ii) (M)	**(b)** (II) (i) (L)
(c) (II) (ii) (K)	**(d)** (III) (iv) (M)

(3) What are the characteristics of soft x-rays?

(a) (III) (i) (L)	**(b)** (IV) (i) (J)
(c) (III) (iv) (J)	**(d)** (III) (iii) (K)

77. There are different types of lasers. In the given table, Column I shows the properties of different lasers or material used in lasers, Column II shows the different names of the lasers and Column III shows the uses of different type of lasers.

Column I	Column II	Column III
(I) Glass or crystalline materials are used	**(i)** Gallium Arsenide	**(J)** Increases its internal energy
(II) Similar to transistor, also known as laser diodes	**(ii)** Ruby laser	**(K)** Used in fibre optic communication
(III) Light supplies energy in the laser medium	**(iii)** Helium-neon lasers	**(L)** Used when drilling holes in metals
(IV) Laser light with very high beam quality is required.	**(iv)** Dye laser	**(M)** Used in doping

(1) What are the characteristics of solid-state laser?

(a) (III) (ii) (L) **(b)** (I) (ii) (L)
(c) (II) (i) (M) **(d)** (II) (i) (K)

(2) What are the characteristics of semiconductor laser?

(a) (I) (iii) (K) **(b)** (IV) (iii) (J)
(c) (I) (ii) (M) **(d)** (II) (i) (K)

(3) What are the characteristics of liquid state laser?

(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (III) (iv) (M) **(d)** (I) (iv) (M)

78. Four quantum numbers shows the electronic configuration of the element. In the given table, Column I shows the different values of principal quantum number, Column II shows the different values of angular momentum quantum number and Column III shows the energy levels for different orbitals.

Column I	Column II	Column III
(I) Value of $n = 2$	**(i)** Value of $l = 0$	**(J)** Lowest energy
(II) Value of $n = 1$	**(ii)** Value of $l = 1$	**(K)** $2p$ orbital have lower energy than $3s$ orbital
(III) Value of $n = 4$	**(iii)** Value of $l = 2$	**(L)** Higher energy than $1s$ orbital
(IV) Value of $n = 3$	**(iv)** Value of $l = 2$	**(M)** Higher energy than $2s$ orbital.

(1) What are the characteristics of $1s$ orbital?

(a) (I) (ii) (J) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (II) (i) (J)

(2) What are the characteristics of $2p$ orbital?

(a) (III) (ii) (J) **(b)** (I) (ii) (K)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) What are the characteristics of $2s$ orbital?

(a) (I) (i) (L) **(b)** (I) (i) (K)
(c) (III) (iii) (J) **(d)** (I) (ii) (M)

Integer Type

79. Electrons have been removed from a beryllium atom ($Z = 4$) until only one remains. Determine the energy (in eV) of the photon that can be emitted if the remaining electron is in the $n = 2$ level.

80. The second ionization energy (the energy required to remove the second outermost electron) of calcium is 11.9 eV. Determine the maximum wavelength (in nm) of incident radiation that can be used to remove the second electron from a calcium atom?

81. How many electrons could be accommodated in a g subshell?

82. The ground state electronic configuration of a neon atom is $1s^2 2s^2 2p^6$. How many of these electrons have magnetic quantum number $m_l = 0$?

83. Which electron energy (in keV) will produce the lowest cutoff wavelength for x-ray production from a nickel ($Z = 28$) surface?

84. Frequency of a photon emitted due to transition of electron of a certain element from L to K shells is found to be 4.2×10^{18} Hz. Using Moseley's law, find the atomic number of the element, given that the Rydberg's constant $R = 1.1 \times 10^7\ \text{m}^{-1}$.

ANSWER KEY

Checkpoints

1. 7 **2.** (a) decrease; (b) and (c) remains the same **3.** A, B, C

Problems

1. 72 **2.** 1.0×10^4 K **3.** (a) 15 states; (b) 21 states **4.** 2, –1, 0, and 1
5. $6p$ **6.** (a) 69.5 kV; (b) 17.8 pm; (c) 21.3 pm; (d) 18.5 pm

7. 5.6 km
8. $n = 5$ to $n = 3$
9. (a) 4p; (b) four electrons; (c) 4p; (d) five electrons;(e) 4p; (f) six electrons
10. Cobalt, atomic number 27
11. 1.24 nm
12. (a) 5.4 pm; (b) 56.5 pm; (c) 49.6 pm
13. (a) 57.5; (b) 2.07×10^3
14. to conserve momentum
15. -2.82×10^5 K
16. (a) $(n, l, m_l, m_s) = (2, 0, 0, +1/2)$ and $(2, 0, 0, -1/2)$
(b) $(n, l, m_l, m_s) = (2, 1, 1, +1/2), (2, 1, 1, -1/2), (2, 1, 0, +1/2), (2, 1, 0, -1/2),$
$(2, 1, -1, +1/2), (2, 1, -1, -1/2)$
17. (a) 3.60×10^{-3} m; (b) 5.24×10^{17} photons
18. 2.2 keV
19. (a) (1, 0, 0, +1/2); (b) (1, 0, 0, −1/2)
20. (a) $5.0 \times 10^{-16} \ll 1$, so no electron; (b) 68 J
22. 3.5×10^{18}
23. (a) 49.6 pm; (b) 99.2 pm
24. 2×10^7 channels
25. 1.5×10^{15} mol
26. (a) 32; (b) 2; (c) 18; (d) 8
27. 2.0×10^{16} s^{-1}
28. (a) 3; (b) 3; (c) 14
29. 3.31×10^{-15} J
30. 6.42 keV
31. (a) 4; (b) 5; (c) ±1/2
32. 2.4 pm
33. 9.0×10^{-7}
34. 1.5×10^6 J/s = 1.5×10^6 W
35. 3.0 eV
36. 72
37. 12.4 kV
38. 9 16 ≈ 0.563
39. (a) 3.46; (b) 3.46; (c) 3; (d) 3; (e) −3; (f) 30.0°; (g) 54.7°; (h) 150°
40. (a) 22.5 pm; (b) remain the same
41. 24.1°
42. (a) four; (b) 7; (c) 2; (d) 32; (e) four
43. (a) three (0, 1 & 2); (b) three (−1, 0 +1)
44. (a) 3.03×10^5; (b) 1.43×10^9 Hz; (d) 3.31×10^{-6}
45. g
46. (a) 14; (b) 6; (c) 6; (d) 2
47. (a) 3.65×10^{-34} J · s.; (b) 3.16×10^{-34} J · s
48. (a) 0 eV, 4.9 eV, 6.7 eV, 8.8 eV and so on; (b) 7 eV; (c) 2.12 eV, 0.3 eV; (d) 6.7 eV, 4.9 eV, 1.8 eV
49. 198 pm
50. 2.67×10^{-15} J
51. (a) 5.4 keV; (b) because other electrons change the potential
52. 5.82 keV

Practice Questions

Single Correct Choice Type

1. (c)
2. (a)
3. (b)
4. (a)
5. (b)
6. (b)
7. (d)
8. (c)
9. (a)
10. (b)
11. (a)
12. (b)
13. (c)
14. (d)
15. (c)
16. (d)
17. (d)
18. (d)
19. (a)
20. (b)
21. (c)
22. (d)
23. (c)
24. (c)
25. (b)
26. (a)
27. (b)
28. (a)
29. (c)
30. (d)
31. (b)
32. (c)
33. (c)
34. (d)
35. (c)
36. (c)
37. (d)
38. (b)
39. (b)
40. (a)
41. (a)
42. (a)
43. (c)
44. (d)
45. (d)
46. (b)
47. (b)
48. (a)
49. (c)
50. (d)
51. (b)
52. (b)
53. (b)
54. (c)
55. (b)
56. (a)
57. (c)

More than One Correct Choice Type

58. (a), (b), (c), (d)
59. (a), (b)
60. (a), (c), (d)
61. (a), (b)

Linked Comprehension

62. (a)
63. (b)
64. (d)
65. (d)
66. (d)
67. (b)
68. (b)
69. (b)
70. (c)
71. (b)
72. (a)
73. (b)

Matrix-Match

74. (a) → (q); (b) → (p); (c) → (p); (d) → (p)

75. (a) → (r); (b) → (p); (c) → (s); (d) → (q)

76. (1) → (c); (2) → (b); (3) → (d)

77. (1) → (b); (2) → (d); (3) → (c)

78. (1) → (d); (2) → (b); (3) (a)

Integer Type

79. 164

80. 104

81. 18

82. 6

83. 45

84. 42

CHAPTER

40

The Nucleus

Contents

40.1 What is Physics?
40.2 Discovering the Nucleus
40.3 Some Nuclear Properties
40.4 Radioactive Decay
40.5 Alpha Decay
40.6 Beta Decay
40.7 Gamma Decay
40.8 Radioactive Dating
40.9 Nuclear Models (Optional)
40.10 Nuclear Fission: The Basic Process
40.11 Thermonuclear Fusion: The Basic Process
40.12 Thermonuclear Fusion in the Sun and Other Stars

40.1 | WHAT IS PHYSICS?

We now turn to what lies at the center of an atom—the nucleus. For the last 90 years, a principal goal of physics has been to work out the quantum physics of nuclei, and, for almost as long, a principal goal of some types of engineering has been to apply that quantum physics with applications ranging from radiation therapy in the war on cancer to detectors of radon gas in basements.

Before we get to such applications and the quantum physics of nuclei, let's first discuss how physicists discovered that an atom has a nucleus. As obvious as that fact is today, it initially came as an incredible surprise.

40.2 | DISCOVERING THE NUCLEUS

Key Concepts

- The positive charge of an atom is concentrated in the central nucleus rather than being spread through the volume of the atom. This structure was proposed in 1910 by Ernest Rutherford of England after he conducted experiments with what we now call Rutherford scattering. Alpha particles (positively charged particles consisting of two protons and two neutrons) are directed through a thin metal foil to be scattered by the (positive) nuclei within the atoms.
- The total energy (kinetic energy plus electric potential energy) of the system of alpha particle and target nucleus is conserved as the alpha particle approaches the nucleus.

In the first years of the 20th century, not much was known about the structure of atoms beyond the fact that they contain electrons. The electron had been discovered (by J. J. Thomson) in 1897, and its mass was unknown in those early days. Thus, it was not possible even to say how many negatively charged electrons a given atom contained. Scientists reasoned that because atoms were electrically neutral, they must also contain some positive charge, but nobody knew what form this compensating positive charge took. One popular model was that the positive and negative charges were spread uniformly in a sphere.

In 1911 Ernest Rutherford proposed that the positive charge of the atom is densely concentrated at the center of the atom, forming its **nucleus**, and that, furthermore, the nucleus is responsible for most of the mass of the atom. Rutherford's proposal was no mere conjecture but was based firmly on the results of an experiment suggested by him and carried out by his collaborators, Hans Geiger (of Geiger counter fame) and Ernest Marsden, a 20-year-old student who had not yet earned his bachelor's degree.

In Rutherford's day it was known that certain elements, called **radioactive**, transform into other elements spontaneously, emitting particles in the process. One such element is radon, which emits alpha (α) particles that have an energy of about 5.5 MeV. We now know that these particles are helium nuclei.

Rutherford's idea was to direct energetic alpha particles at a thin target foil and measure the extent to which they were deflected as they passed through the foil. Alpha particles, which are about 7300 times more massive than electrons, have a charge of $+2e$.

Figure 40-1 shows the experimental arrangement of Geiger and Marsden. Their alpha source was a thin-walled glass tube of radon gas. The experiment involves counting the number of alpha particles that are deflected through various scattering angles ϕ.

Figure 40-2 shows their results. Note especially that the vertical scale is logarithmic. We see that most of the particles are scattered through rather small angles, but—and this was the big surprise—a very small fraction of them are scattered through very large angles, approaching 180°. In Rutherford's words: "It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you had fired a 15-inch shell at a piece of tissue paper and it [the shell] came back and hit you."

Why was Rutherford so surprised? At the time of these experiments, most physicists believed in the so-called plum pudding model of the atom, which had been advanced by J. J. Thomson. In this view the positive charge of the atom was thought to be spread out through the entire volume of the atom. The electrons (the "plums") were thought to vibrate about fixed points within this sphere of positive charge (the "pudding").

The maximum deflecting force that could act on an alpha particle as it passed through such a large positive sphere of charge would be far too small to deflect the alpha particle by even as much as 1°. (The expected deflection has been compared to what you would observe if you fired a bullet through a sack of snowballs.) The electrons in the atom would also have very little effect on the massive, energetic alpha particle. They would, in fact, be themselves strongly deflected, much as a swarm of gnats would be brushed aside by a stone thrown through them.

Rutherford saw that, to deflect the alpha particle backward, there must be a large force; this force could be provided if the positive charge, instead of being spread throughout the atom, were concentrated tightly at its center.

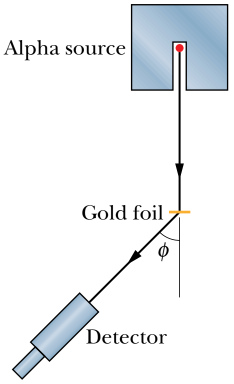


Figure 40-1 An arrangement (top view) used in Rutherford's laboratory in 1911–1913 to study the scattering of α particles by thin metal foils. The detector can be rotated to various values of the scattering angle ϕ. The alpha source was radon gas, a decay product of radium. With this simple "tabletop" apparatus, the atomic nucleus was discovered.

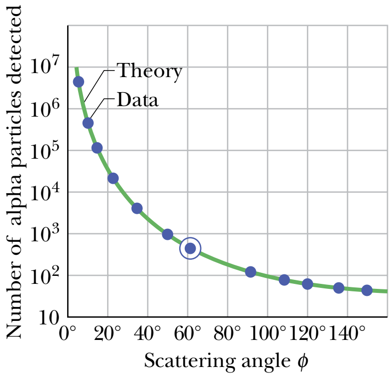


Figure 40-2 The dots are alpha-particle scattering data for a gold foil, obtained by Geiger and Marsden using the apparatus of Fig. 40-1. The solid curve is the theoretical prediction, based on the assumption that the atom has a small, massive, positively charged nucleus. The data have been adjusted to fit the theoretical curve at the experimental point that is enclosed in a circle.

Then the incoming alpha particle could get very close to the positive charge without penetrating it; such a close encounter would result in a large deflecting force.

Figure 40-3 shows possible paths taken by typical alpha particles as they pass through the atoms of the target foil. As we see, most are either undeflected or only slightly deflected, but a few (those whose incoming paths pass, by chance, very close to a nucleus) are deflected through large angles. From an analysis of the data, Rutherford concluded that the radius of the nucleus must be smaller than the radius of an atom by a factor of about 10^4. In other words, the atom is mostly empty space.

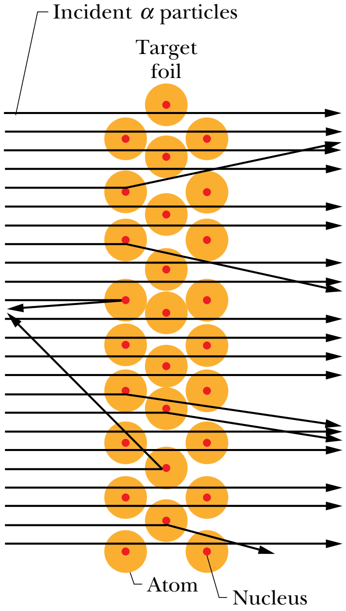


Figure 40-3 The angle through which an incident alpha particle is scattered depends on how close the particle's path lies to an atomic nucleus. Large deflections result only from very close encounters.

SAMPLE PROBLEM 40.01

Rutherford scattering of an alpha particle by a gold nucleus

An alpha particle with kinetic energy $K_i = 5.30$ MeV happens, by chance, to be headed directly toward the nucleus of a neutral gold atom (Fig. 40-4*a*). What is its *distance of closest approach d* (least center-to-center separation) to the nucleus? Assume that the atom remains stationary.

KEY IDEAS

(1) Throughout the motion, the total mechanical energy E of the particle–atom system is conserved. (2) In addition to the kinetic energy, that total energy includes electric potential energy U as given by Eq. 24-54 ($U = q_1q_2/4\pi\varepsilon_0 r$).

Calculations: The alpha particle has a charge of $+2e$ because it contains two protons. The target nucleus has a charge of $q_{\text{Au}} = +79e$ because it contains 79 protons. However, that nuclear charge is surrounded by an electron "cloud" with a charge of $q_e = -79e$, and thus the alpha particle initially "sees" a neutral atom with a net charge of $q_{\text{atom}} = 0$. The electric force on the particle is zero and the initial electric potential energy of the particle–atom system is $U_i = 0$.

Once the alpha particle enters the atom, we say that it passes through the electron cloud surrounding the nucleus. That cloud then acts as a closed conducting spherical shell and, by Gauss' law, has no effect on the (now internal) charged alpha particle. Then the alpha particle "sees" only the nuclear charge q_{Au}. Because q_α and q_{Au} are both positively charged, a repulsive electric force acts on the alpha particle, slowing it, and the particle–atom system has a potential energy

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_\alpha q_{\text{Au}}}{r}$$

that depends on the center-to-center separation r of the incoming particle and the target nucleus (Fig. 40-4*b*).

As the repulsive force slows the alpha particle, energy is transferred from kinetic energy to electric potential energy. The transfer is complete when the alpha particle momentarily stops at the distance of closest approach d to the target nucleus (Fig. 40-4*c*). Just then the kinetic energy is $K_f = 0$ and the particle–atom system has the electric potential energy

$$U_f = \frac{1}{4\pi\varepsilon_0}\frac{q_\alpha q_{\text{Au}}}{d}.$$

To find d, we conserve the total mechanical energy between the initial state i and this later state f, writing

$$K_i + U_i = K_f + U_f$$

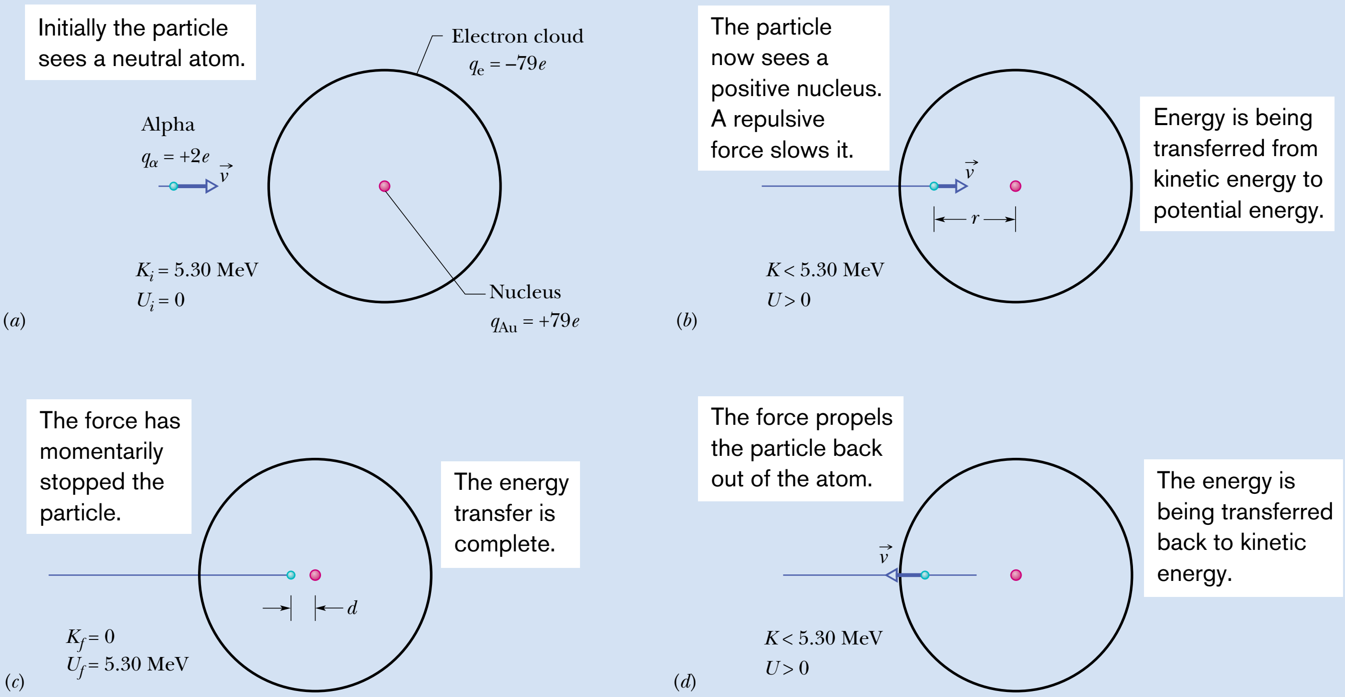
and

$$K_i + 0 = 0 + \frac{1}{4\pi\varepsilon_0}\frac{q_\alpha q_{\text{Au}}}{d}.$$

(We are assuming that the alpha particle is not affected by the force holding the nucleus together, which acts over only a short distance.) Solving for d and then substituting for the charges and initial kinetic energy lead to

$$\begin{aligned} d &= \frac{(2e)(79e)}{4\pi\varepsilon_0 K_\alpha} \\ &= \frac{(2\times 79)(1.60\times 10^{-19}\ \mathrm{C})^2}{4\pi\varepsilon_0(5.30\ \mathrm{MeV})(1.60\times 10^{-13}\ \mathrm{J/MeV})} \\ &= 4.29\times 10^{-14}\ \mathrm{m}. \qquad \text{(Answer)} \end{aligned}$$

This distance is considerably larger than the sum of the radii of the gold nucleus and the alpha particle. Thus, this alpha particle reverses its motion (Fig. 40-4*d*) without ever actually "touching" the gold nucleus.


Initially the particle sees a neutral atom.

The particle now sees a positive nucleus. A repulsive force slows it.

Energy is being transferred from kinetic energy to potential energy.

The force has momentarily stopped the particle.

The energy transfer is complete.

The force propels the particle back out of the atom.

The energy is being transferred back to kinetic energy.

Figure 40-4 An alpha particle (*a*) approaches and (*b*) then enters a gold atom, headed toward the nucleus. The alpha particle (*c*) comes to a stop at the point of closest approach and (*d*) is propelled back out of the atom.

40.3 | SOME NUCLEAR PROPERTIES

Key Concepts

- Different types of nuclei are called nuclides. Each is characterized by an atomic number Z (the number of protons), a neutron number N, and a mass number A (the total number of nucleons—protons and neutrons). Thus, $A = Z + N$. A nuclide is represented with a symbol such as $^{197}\mathrm{Au}$ or $^{197}_{79}\mathrm{Au}$, where the chemical symbol carries a superscript with the value of A and (possibly) a subscript with the value of Z.
- Nuclides with the same atomic number but different neutron numbers are isotopes of one another.
- Nuclei have a mean radius r given by

$$r = r_0 A^{1/3}$$

where $r_0 \approx 1.2$ fm.
- Atomic masses are often reported in terms of mass excess

$$\Delta = M - A,$$

where M is the actual mass of an atom in atomic mass units and A is the mass number for that atom's nucleus.

- The binding energy of a nucleus is the difference

$$\Delta E_{\text{be}} = \sum (mc^2) - Mc^2,$$

where $\Sigma(mc^2)$ is the total mass energy of the individual protons and neutrons. The binding energy of a nucleus is the amount of energy needed to break the nucleus into its constituent parts (and is *not* an energy that resides in the nucleus).

- The binding energy per nucleon is

$$\Delta E_{\text{ben}} = \frac{\Delta E_{\text{be}}}{A}.$$

- The energy equivalent of one mass unit (u) is 931.494 013 MeV.
- A plot of the binding energy per nucleon ΔE_{ben} versus mass number A shows that middle-mass nuclides are the most stable and that energy can be released both by fission of high-mass nuclei and by fusion of low-mass nuclei.

When we are interested primarily in their properties as specific nuclear species (rather than as parts of atoms), we call these particles **nuclides**.

Some Nuclear Terminology

Nuclei are made up of protons and neutrons. The number of protons in a nucleus (called the **atomic number** or **proton number** of the nucleus) is represented by the symbol Z; the number of neutrons (the **neutron number**) is represented by the symbol N. The total number of neutrons and protons in a nucleus is called its **mass number** A; thus

$$A = Z + N. \tag{40-1}$$

Neutrons and protons, when considered collectively as members of a nucleus, are called **nucleons**.

We represent nuclides with symbols such as those displayed here. Consider ^{197}Au, for example. The superscript 197 is the mass number A. The chemical symbol Au tells us that this element is gold, whose atomic number is 79. Sometimes the atomic number is explicitly shown as a subscript, as in $^{197}_{79}\text{Au}$. From Eq. 40-1, the neutron number of this nuclide is the difference between the mass number and the atomic number, namely, 197 − 79, or 118.

Nuclides with the same atomic number Z but different neutron numbers N are called **isotopes** of one another. The element gold has 36 isotopes, ranging from ^{173}Au to ^{204}Au. Only one of them (^{197}Au) is stable; the remaining 35 are radioactive. Such **radionuclides** undergo **decay** (or **disintegration**) by emitting a particle and thereby transforming to a different nuclide.

During such a nuclear reaction, the total number of nucleons before and after the reaction remains the same. Also, the total charge of the system remains conserved by the law of conservation of charge. For example, when ^{198}Au undergoes beta decay, the nuclear reaction is

$$^{198}_{79}\text{Au} \rightarrow {}^{198}_{80}\text{Hg} + {}^{0}_{-1}\text{e} + \bar{\nu}.$$

Here on left-hand side, we can see that the nucleus contains 79 protons and 119 neutrons. On decay, the number of protons increases to 80, but the number of neutrons decreases to 118. The charge is conserved because an electron is emitted during the process. The $\bar{\nu}$ (antineutrino) is an uncharged particle.

Organizing the Nuclides

The neutral atoms of all isotopes of an element (all with the same Z) have the same number of electrons and the same chemical properties, and they fit into the same box in the periodic table of the elements. The *nuclear* properties of the isotopes of a given element, however, are very different from one isotope to another. Thus, the periodic table is of limited use to the nuclear physicist, the nuclear chemist, or the nuclear engineer.

We organize the nuclides on a **nuclidic chart** like that in Fig. 40-5, in which a nuclide is represented by plotting its proton number against its neutron number. The stable nuclides in this figure are represented by the green, the radionuclides by the beige. As you can see, the radionuclides tend to lie on either side of—and at the upper end of—a well-defined band of stable nuclides. Note too that light stable nuclides tend to lie close to the line $N = Z$, which means that they have about the same numbers of neutrons and protons. Heavier nuclides, however, tend to have many more neutrons than protons. As an example, we saw that ^{197}Au has 118 neutrons and only 79 protons, a *neutron excess* of 39.

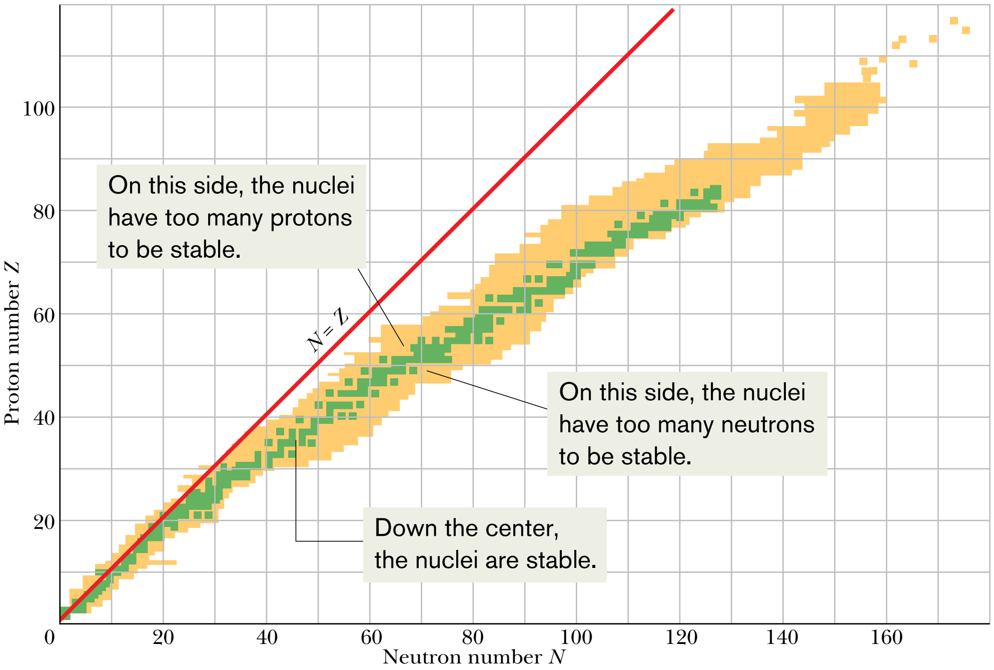
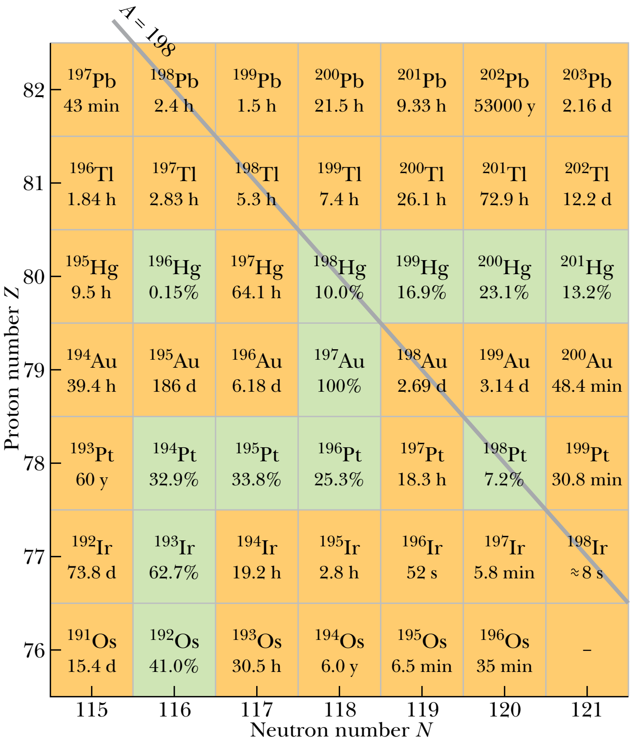


Figure 40-5 A plot of the known nuclides. The green shading identifies the band of stable nuclides, the beige shading the radionuclides. Low-mass, stable nuclides have essentially equal numbers of neutrons and protons, but more massive nuclides have an increasing excess of neutrons. The figure shows that there are no stable nuclides with $Z > 83$ (bismuth).


A = 198
Proton number Z
Neutron number N

Z \ N	115	116	117	118	119	120	121
82	^{197}Pb 43 min	^{198}Pb 2.4 h	^{199}Pb 1.5 h	^{200}Pb 21.5 h	^{201}Pb 9.33 h	^{202}Pb 53000 y	^{203}Pb 2.16 d
81	^{196}Tl 1.84 h	^{197}Tl 2.83 h	^{198}Tl 5.3 h	^{199}Tl 7.4 h	^{200}Tl 26.1 h	^{201}Tl 72.9 h	^{202}Tl 12.2 d
80	^{195}Hg 9.5 h	^{196}Hg 0.15%	^{197}Hg 64.1 h	^{198}Hg 10.0%	^{199}Hg 16.9%	^{200}Hg 23.1%	^{201}Hg 13.2%
79	^{194}Au 39.4 h	^{195}Au 186 d	^{196}Au 6.18 d	^{197}Au 100%	^{198}Au 2.69 d	^{199}Au 3.14 d	^{200}Au 48.4 min
78	^{193}Pt 60 y	^{194}Pt 32.9%	^{195}Pt 33.8%	^{196}Pt 25.3%	^{197}Pt 18.3 h	^{198}Pt 7.2%	^{199}Pt 30.8 min
77	^{192}Ir 73.8 d	^{193}Ir 62.7%	^{194}Ir 19.2 h	^{195}Ir 2.8 h	^{196}Ir 52 s	^{197}Ir 5.8 min	^{198}Ir ≈8 s
76	^{191}Os 15.4 d	^{192}Os 41.0%	^{193}Os 30.5 h	^{194}Os 6.0 y	^{195}Os 6.5 min	^{196}Os 35 min	–

Figure 40-6 An enlarged and detailed section of the nuclidic chart of Fig. 40-5, centered on ^{197}Au. Light color squares represent stable nuclides, for which relative isotopic abundances are given. Dark color squares represent radionuclides, for which halflives are given. Isobaric lines of constant mass number A slope as shown by the example line for $A = 198$.

Nuclidic charts are available as wall charts, in which each small box on the chart is filled with data about the nuclide it represents. Figure 40-6 shows a section of such a chart, centered on ^{197}Au. Relative abundances (usually, as found on Earth) are shown for stable nuclides, and half-lives (a measure of decay rate) are shown for radionuclides. The sloping line points out a line of **isobars**—nuclides of the same mass number, $A = 198$ in this case.

In recent years, nuclides with atomic numbers as high as $Z = 118$ ($A = 294$) have been found in laboratory experiments (no elements with Z greater than 92 occur naturally). Although large nuclides generally should be highly unstable and last only a very brief time, certain supermassive nuclides are relatively stable, with fairly long lifetimes. These stable supermassive nuclides and other predicted ones form an *island of stability* at high values of Z and N on a nuclidic chart like Fig. 40-5.

☑ CHECKPOINT 1

Based on Fig. 40-5, which of the following nuclides do you conclude are not likely to be detected: ^{52}Fe ($Z = 26$), ^{90}As ($Z = 33$), ^{158}Nd ($Z = 60$), ^{175}Lu ($Z = 71$), ^{208}Pb ($Z = 82$)?

Nuclear Radii

A convenient unit for measuring distances on the scale of nuclei is the *femtometer* ($= 10^{-15}$ m). This unit is often called the *fermi*; the two names share the same abbreviation. Thus,

$$1 \text{ femtometer} = 1 \text{ fermi} = 1 \text{ fm} = 10^{-15} \text{ m}. \tag{40-2}$$

We can learn about the size and structure of nuclei by bombarding them with a beam of high-energy electrons and observing how the nuclei deflect the incident electrons. The electrons must be energetic enough (at least 200 MeV) to have de Broglie wavelengths that are smaller than the nuclear structures they are to probe.

The nucleus, like the atom, is not a solid object with a well-defined surface. Furthermore, although most nuclides are spherical, some are notably ellipsoidal. Nevertheless, electron-scattering experiments (as well as experiments of other kinds) allow us to assign to each nuclide an effective radius given by

$$r = r_0 A^{1/3}, \tag{40-3}$$

in which A is the mass number and $r_0 \approx 1.2$ fm. We see that the volume of a nucleus, which is proportional to r^3, is directly proportional to the mass number A and is independent of the separate values of Z and N. That is, we can treat most nuclei as being a sphere with a volume that depends on the number of nucleons, regardless of their type.

We can picture the nucleus as made up of nucleons that are incompressible. It is like a ball of snow that children make. As more snow is added to the ball, the snow ball gets bigger, not denser. The shape of the nucleus will be the same as shown in Fig. 40-7.

In fact, we can use Eq. 40-3 to estimate the density of nuclear matter. Consider a nucleus with mass number A. Its mass within 1% is A atomic mass units. Thus,

$$\begin{aligned}\rho_{\text{nuc}} &\approx \frac{A\text{ u}}{(4/3)\pi r^3}\\ &= \frac{A\text{ u}}{(4/3)\pi r_0^3 A}\\ &= \frac{1\text{ u}}{(4/3)\pi r_0^3 A}\\ &= \frac{1.66\times10^{-27}\text{ kg}}{(4/3)\pi(1.2\times10^{-15}\text{ m})^3}\\ &= 2.3\times10^{17}\text{ kg/m}^3\end{aligned} \tag{40-4}$$

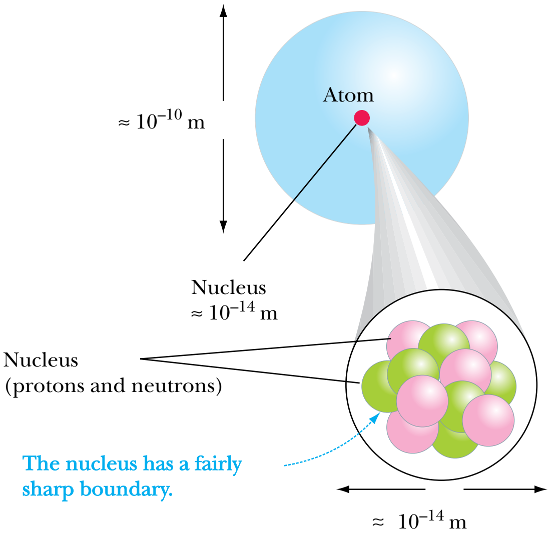


Figure 40-7 The nucleus is a tiny sphere within an atom.

Atomic Masses

Atomic masses are now measured to great precision, but usually nuclear masses are not directly measurable because stripping off all the electrons from an atom is difficult. As we briefly discussed in Chapter 36, atomic masses are often reported in *atomic mass units*, a system in which the atomic mass of neutral ^{12}C is defined to be exactly 12 u.

Precise atomic masses are available in tables on the web and in the text books. However, sometimes we need only an approximation of the mass of either a nucleus alone or a neutral atom. The mass number A of a nuclide gives such an approximate mass in atomic mass units. For example, the approximate mass of both the nucleus and the neutral atom for ^{197}Au is 197 u, which is close to the actual atomic mass of 196.966 552 u.

As we saw in Section 36.13,

$$1\ \text{u} = 1.660\ 538\ 86 \times 10^{-27}\ \text{kg}. \qquad (40\text{-}5)$$

We also saw that if the total mass of the participants in a nuclear reaction changes by an amount Δm, there is an energy release or absorption given by Eq. 36-50 ($Q = -\Delta m\ c^2$). As we shall now see, nuclear energies are often reported in multiples of 1 MeV. Thus, a convenient conversion between mass units and energy units is provided by Eq. 37-46:

$$c^2 = 931.494\ 013\ \text{MeV/u}. \qquad (40\text{-}6)$$

Scientists and engineers working with atomic masses often prefer to report the mass of an atom by means of the atom's *mass excess* Δ, defined as

$$\Delta = M - A \quad \text{(mass excess)}, \qquad (40\text{-}7)$$

where M is the actual mass of the atom in atomic mass units and A is the mass number for that atom's nucleus.

Nuclear Binding Energies

The mass M of a nucleus is *less* than the total mass Σm of its individual protons and neutrons. That means that the mass energy Mc^2 of a nucleus is *less* than the total mass energy $\Sigma(mc^2)$ of its individual protons and neutrons. The difference between these two energies is called the **binding energy** of the nucleus:

$$\sum E_{\text{be}} = \sum (mc^2) - Mc^2 \quad \text{(binding energy)}. \qquad (40\text{-}8)$$

Caution: Binding energy is not an energy that resides in the nucleus. Rather, it is a *difference* in mass energy between a nucleus and its individual nucleons: If we were able to separate a nucleus into its nucleons, we would have to transfer a total energy equal to ΔE_{be} to those particles during the separating process. Although we cannot actually tear apart a nucleus in this way, the nuclear binding energy is still a convenient measure of how well a nucleus is held together, in the sense that it measures how difficult the nucleus would be to take apart.

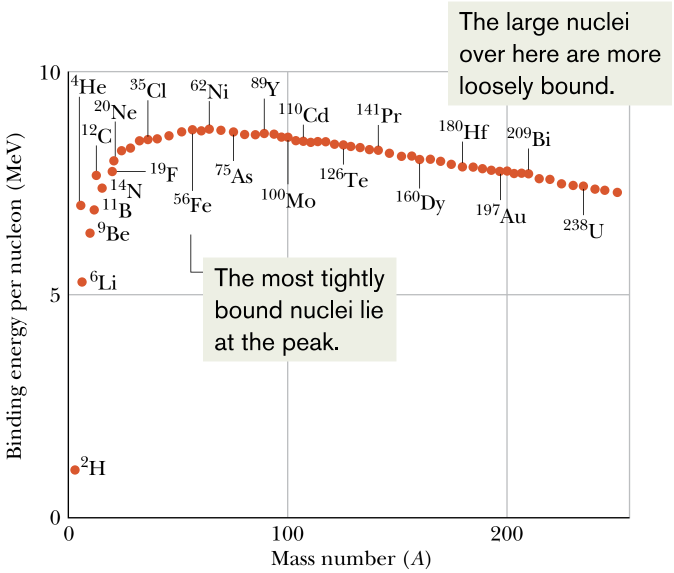
A better measure is the **binding energy per nucleon** ΔE_{ben}, which is the ratio of the binding energy ΔE_{be} of a nucleus to the number A of nucleons in that nucleus:

$$\Delta E_{\text{ben}} = \frac{\Delta E_{\text{be}}}{A} \quad \text{(binding energy per nucleon)}. \qquad (40\text{-}9)$$

We can think of the binding energy per nucleon as the average energy needed to separate a nucleus into its individual nucleons. *A greater binding energy per nucleon means a more tightly bound nucleus.*

Figure 40-8 is a plot of the binding energy per nucleon ΔE_{ben} versus mass number A for a large number of nuclei. Those high on the plot are very tightly bound; that is, we would have to supply a great amount of energy per nucleon to break apart one of those nuclei. The nuclei that are lower on the plot, at the left and right sides, are less tightly bound, and less energy per nucleon would be required to break them apart.

These simple statements about Fig. 40-8 have profound consequences. The nucleons in a nucleus on the right side of the plot would be more tightly bound if that nucleus were to split into two nuclei that lie near the top of the plot.



The large nuclei over here are more loosely bound.

The most tightly bound nuclei lie at the peak.

Figure 40-8 The binding energy per nucleon for some representative nuclides. The nickel nuclide ^{62}Ni has the highest binding energy per nucleon (about 8.794 60 MeV/nucleon) of any known stable nuclide. Note that the alpha particle (^{4}He) has a higher binding energy per nucleon than its neighbors in the periodic table and thus is also particularly stable.

Such a process, called **fission**, occurs naturally with large (high mass number A) nuclei such as uranium, which can undergo fission spontaneously (that is, without an external cause or source of energy). The process can also occur in nuclear weapons, in which many uranium or plutonium nuclei are made to fission all at once, to create an explosion.

The nucleons in any pair of nuclei on the left side of the plot would be more tightly bound if the pair were to combine to form a single nucleus that lies near the top of the plot. Such a process, called **fusion**, occurs naturally in stars. Were this not true, the Sun would not shine and thus life could not exist on Earth.

Nuclear Energy Levels

The energy of nuclei, like that of atoms, is quantized. That is, nuclei can exist only in discrete quantum states, each with a well-defined energy. Figure 40-9 shows some of these energy levels for ^{28}Al, a typical low-mass nuclide. Note that the energy scale is in millions of electron-volts, rather than the electron-volts used for atoms. When a nucleus makes a transition from one level to a level of lower energy, the emitted photon is typically in the gamma-ray region of the electromagnetic spectrum.

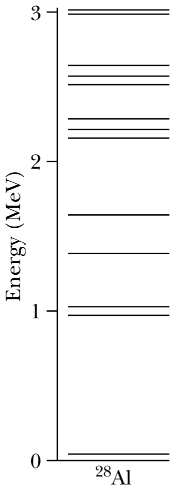


Figure 40-9 Energy levels for the nuclide ^{28}Al, deduced from nuclear reaction experiments.

Nuclear Spin and Magnetism

Many nuclides have an intrinsic *nuclear angular momentum*, or spin, and an associated intrinsic *nuclear magnetic moment*. Although nuclear angular momenta are roughly of the same magnitude as the angular momenta of atomic electrons, nuclear magnetic moments are much smaller than typical atomic magnetic moments.

The Nuclear Force

The force that controls the motions of atomic electrons is the familiar electromagnetic force. To bind the nucleus together, however, there must be a strong attractive nuclear force of a totally different kind, strong enough to overcome the repulsive force between the (positively charged) nuclear protons and to bind both protons and neutrons into the tiny nuclear volume. The nuclear force must also be of short range because its influence does not extend very far beyond the nuclear "surface." Its range is of the order of 2 fm.

The present view is that the nuclear force that binds neutrons and protons in the nucleus is not a fundamental force of nature but is a secondary, or "spillover," effect of the **strong force** that binds quarks together to form neutrons and protons. In much the same way, the attractive force between certain neutral molecules is a spillover effect of the Coulomb electric force that acts within each molecule to bind it together.

This strong force is independent of the charge. This means that the strong force of proton–proton, neutron–neutron, proton–neutron interactions is the same, apart from the additional repulsive Coulomb force for the proton–proton interaction. It is customary to talk of the potential energy when we talk of nuclear forces. Here, the potential energy of interaction of a proton and a neutron is shown in Fig. 40-10*a*.

But when we see the potential energy of interaction between a proton and another proton (Fig. 40-10*b*), the graph looks a little different. This includes the effect of Coulomb repulsion as well. We can also say from the graph that the strong nuclear force is independent of the charge as the potential well has the same shape for both the figures. For a very short distance, less than 0.4 fm, the nuclear force becomes repulsive. If it were not so, then all the nuclei would collapse to zero size.

This strong nuclear force does not affect the electrons. So, the electrons can be used as a projectile to probe the inner details of the nucleus. Based on these experiments, we have been able to conclude that its magnitude depends on the relative spin orientation of the nucleons. This will help us to understand Fig. 40-5. The black points refer to stable nuclides (about 270). In heavy stable nuclei, $N > Z$. As the number of protons increases, the strength of the Coulomb force increases which tends to break apart the nucleus. As a result, Mother Nature adds more neutrons to keep the nucleus stable because the neutrons experience only attractive force. At $Z = 83$ and beyond, the addition of more neutrons cannot compensate the repulsive force between the protons. So, none of the nuclides with $Z > 83$ are stable.

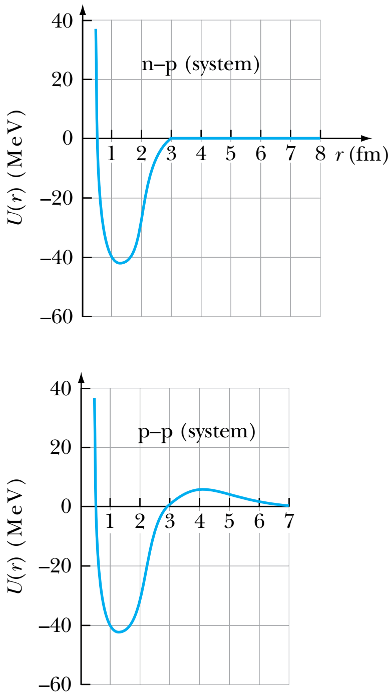


Figure 40-10 (*a*) Potential energy versus separation for neutron–proton system. (*b*) Potential energy versus separation for proton–proton system. The difference is two curves is due to the Coulomb repulsion between protons.

SAMPLE PROBLEM 40.02

Binding energy per nucleon

What is the binding energy per nucleon for ^{120}Sn?

KEY IDEAS

1. We can find the binding energy per nucleon ΔE_{ben} if we first find the binding energy ΔE_{be} and then divide by the number of nucleons A in the nucleus, according to Eq. 40-9 ($\Delta E_{ben} = \Delta E_{be}/A$).
2. We can find ΔE_{be} by finding the difference between the mass energy Mc^2 of the nucleus and the total mass energy $\Sigma(mc^2)$ of the individual nucleons that make up the nucleus, according to Eq. 40-8 ($\Delta E_{be} = \Sigma(mc^2) - Mc^2$).

Calculations: From Appendix F, we see that a ^{120}Sn nucleus consists of 50 protons ($Z = 50$) and 70 neutrons ($N = A - Z = 120 - 50 = 70$). Thus, we need to imagine a ^{120}Sn nucleus being separated into its 50 protons and 70 neutrons,

$$(^{120}\text{Sn nucleus}) \rightarrow 50\begin{pmatrix}\text{separate}\\ \text{protons}\end{pmatrix} + 70\begin{pmatrix}\text{separate}\\ \text{neutrons}\end{pmatrix}, \quad (40\text{-}10)$$

and then compute the resulting change in mass energy.

For that computation, we need the masses of a ^{120}Sn nucleus, a proton, and a neutron. However, because the mass of a neutral atom (nucleus *plus* electrons) is much easier to measure than the mass of a bare nucleus, calculations of binding energies are traditionally done with atomic masses. Thus, let's modify Eq. 40-10 so that it has a neutral ^{120}Sn atom on the left side. To do that, we include 50 electrons on the left side (to match the 50 protons in the ^{120}Sn nucleus). We must also add 50 electrons on the right side to balance Eq. 40-10. Those 50 electrons can be combined with the 50 protons, to form 50 neutral hydrogen atoms. We then have

$$(^{120}\text{Sn atom}) \rightarrow 50\begin{pmatrix}\text{separate}\\ \text{H atoms}\end{pmatrix} + 70\begin{pmatrix}\text{separate}\\ \text{neutrons}\end{pmatrix}. \qquad (40\text{-}11)$$

In Appendix F, the mass M_{Sn} of a ^{120}Sn atom is 119.902 197 u and the mass m_{H} of a hydrogen atom is 1.007 825 u; the mass m_{n} of a neutron is 1.008 665 u. Thus, Eq. 40-8 yields

$$\begin{aligned}\Delta E_{\text{be}} &= \Sigma(mc^2) - Mc^2\\ &= 50(m_{\text{H}}c^2) + 70(m_{\text{n}}c^2) - M_{\text{Sn}}c^2\\ &= 50(1.007\,825\text{ u})c^2 + 70(1.008\,665\text{ u})c^2\\ &\quad - (119.902\,197\text{ u})c^2\\ &= (1.095\,603\text{ u})c^2\\ &= (1.095\,603\text{ u})(931.494\,013\text{ MeV/u})\\ &= 1020.5\text{ MeV},\end{aligned}$$

where Eq. 40-6 (c^2 = 931.494 013 MeV/u) provides an easy unit conversion. Note that using atomic masses instead of nuclear masses does not affect the result because the mass of the 50 electrons in the ^{120}Sn atom subtracts out from the mass of the electrons in the 50 hydrogen atoms.

Now Eq. 40-9 gives us the binding energy per nucleon as

$$\begin{aligned}\Delta E_{\text{ben}} &= \frac{\Delta E_{\text{be}}}{A} = \frac{1020.5\text{ MeV}}{120}\\ &= 8.50\text{ MeV/nucleon.}\end{aligned} \qquad \text{(Answer)}$$

40.4 | RADIOACTIVE DECAY

Key Concepts

- Most nuclides spontaneously decay at a rate $R = dN/dt$ that is proportional to the number N of radioactive atoms present. The proportionality constant is the disintegration constant λ.
- The number of radioactive nuclei is given as a function of time by

$$N = N_0 e^{-\lambda t},$$

where N_0 is the number at time $t = 0$.

- The rate at which the nuclei decay is given as a function of time by

$$R = R_0 e^{-\lambda t},$$

where R_0 is the rate at time $t = 0$.

- The half-life $T_{1/2}$ and the mean life τ are measures of how quickly radioactive nuclei decay and are related by

$$T_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2.$$

As Fig. 40-5 shows, most nuclides are radioactive. A radioactive nuclide spontaneously emits a particle, transforming itself in the process into a different nuclide, occupying a different square on the nuclide chart. Radioactive decay provided the first evidence that the laws for subatomic processes are statistical. For example, in a 1 mg sample of uranium metal, with 2.5×10^{18} atoms of the very long-lived radionuclide ^{238}U, only about 12 of the nuclei will decay in a given second by emitting an alpha particle and transforming into a nucleus of ^{234}Th. However,

> There is absolutely no way to predict whether any given nucleus in a radioactive sample will be among the small number of nuclei that decay during any given second. All have the same chance.

Although we cannot predict which nuclei in a sample will decay, we can say that if a sample contains N radioactive nuclei, then the rate ($= -dN/dt$) at which nuclei will decay is proportional to N:

$$-\frac{dN}{dt} = \lambda N, \qquad (40\text{-}12)$$

in which λ, the **disintegration constant** (or **decay constant**) has a characteristic value for every radionuclide. Its SI unit is the inverse second (s^{-1}).

To find N as a function of time t, we first rearrange Eq. 40-12 as

$$\frac{dN}{N} = -\lambda\, dt, \tag{40-13}$$

and then integrate both sides, obtaining

$$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_{t_0}^{t} dt,$$

or

$$\ln N - \ln N_0 = -\lambda(t - t_0). \tag{40-14}$$

Here N_0 is the number of radioactive nuclei in the sample at some arbitrary initial time t_0. Setting $t_0 = 0$ and rearranging Eq. 40-14 give us

$$\ln \frac{N}{N_0} = -\lambda t. \tag{40-15}$$

Taking the exponential of both sides (the exponential function is the antifunction of the natural logarithm) leads to

$$\frac{N}{N_0} = e^{-\lambda t}$$

or

$$N = N_0 e^{-\lambda t} \quad \text{(radioactive decay),} \tag{40-16}$$

in which N_0 is the number of radioactive nuclei in the sample at $t = 0$ and N is the number remaining at any subsequent time t. Note that lightbulbs (for one example) follow no such exponential decay law. If we life-test 1000 bulbs, we expect that they will all "decay" (that is, burn out) at more or less the same time. The decay of radionuclides follows quite a different law.

We are often more interested in the decay rate R ($= -dN/dt$) than in N itself. Differentiating Eq. 40-16, we find

$$R = -\frac{dN}{dt} = \lambda N_0 e^{-\lambda t}$$

or

$$R = R_0 e^{-\lambda t} \quad \text{(radioactive decay),} \tag{40-17}$$

an alternative form of the law of radioactive decay (Eq. 40-16). Here R_0 is the decay rate at time $t = 0$ and R is the rate at any subsequent time t. We can now rewrite Eq. 40-12 in terms of the decay rate R of the sample as

$$R = \lambda N, \tag{40-18}$$

where R and the number of radioactive nuclei N that have not yet undergone decay must be evaluated at the same instant.

The total decay rate R of a sample of one or more radionuclides is called the **activity** of that sample. The SI unit for activity is the **becquerel**, named for Henri Becquerel, the discoverer of radioactivity:

1 becquerel = 1 Bq = 1 decay per second.

An older unit, the **curie**, is still in common use:

$$1 \text{ curie} = 1 \text{ Ci} = 3.7 \times 10^{10} \text{ Bq}.$$

Often a radioactive sample will be placed near a detector that does not record all the disintegrations that occur in the sample. The reading of the detector under these circumstances is proportional to (and smaller than) the true activity of the sample. Such proportional activity measurements are reported not in becquerel units but simply in counts per unit time.

Lifetimes. There are two common time measures of how long any given type of radionuclides lasts. One measure is the **half-life** $T_{1/2}$ of a radionuclide, which is the time at which both N and R have been reduced to one-half their initial values. The other measure is the **mean** (or **average**) **life** τ, which is the time at which both N and R have been reduced to e^{-1} of their initial values.

To relate $T_{1/2}$ to the disintegration constant λ, we put $R = 1/2R_0$ in Eq. 40-17 and substitute $T_{1/2}$ for t. We obtain

$$\frac{1}{2}R_0 = R_0 e^{-\lambda T_{1/2}}.$$

Taking the natural logarithm of both sides and solving for $T_{1/2}$, we find

$$T_{1/2} = \frac{\ln 2}{\lambda}.$$

Similarly, to relate τ to λ, we put $R = e^{-1}R_0$ in Eq. 40-17, substitute τ for t, and solve for τ, finding

$$\tau = \frac{1}{\lambda}.$$

We summarize these results with the following:

$$T_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2. \tag{40-19}$$

Probability of Decay

As the decay process is governed by statistics, it is appropriate to talk of probability of a decay or survival of a nucleus over here. Thus, if we take a sample of a radioactive element, we can say that its probability of decay in time dt is

$$\frac{dN}{N} = -\lambda dt.$$

Over a large interval of time say t, the probability of survival will be

$$\frac{N}{N_0} = e^{-\lambda t}.$$

Thus, the probability of the decay of a given nuclei will be

$$\frac{N_0 - N}{N_0} = 1 - e^{-\lambda t}.$$

CHECKPOINT 2

The nuclide ^{131}I is radioactive, with a half-life of 8.04 days. At noon on January 1, the activity of a certain sample is 600 Bq. Using the concept of half-life, without written calculation, determine whether the activity at noon on January 24 will be a little less than 200 Bq, a little more than 200 Bq, a little less than 75 Bq, or a little more than 75 Bq.

SAMPLE PROBLEM 40.03

Finding the disintegration constant and half-life from a graph

The table that follows shows some measurements of the decay rate of a sample of ^{128}I, a radionuclide often used medically as a tracer to measure the rate at which iodine is absorbed by the thyroid gland.

Time (min)	R (counts/s)	Time (min)	R (counts/s)
4	392.2	132	10.9
36	161.4	164	4.56
68	65.5	196	1.86
100	26.8	218	1.00

Find the disintegration constant λ and the half-life $T_{1/2}$ for this radionuclide.

KEY IDEAS

The disintegration constant λ determines the exponential rate at which the decay rate R decreases with time t (as indicated by Eq. 40-17, $R = R_0 e^{-\lambda t}$). Therefore, we should be able to determine λ by plotting the measurements of R against the measurement times t. However, obtaining λ from a plot of R versus t is difficult because R decreases exponentially with t, according to Eq. 40-17. A neat solution is to transform Eq. 40-17 into a linear function of t, so that we can easily find λ. To do so, we take the natural logarithms of both sides of Eq. 40-17.

Calculations: We obtain

$$\begin{aligned}\ln R &= \ln(R_0 e^{-\lambda t}) = \ln R_0 + \ln(e^{-\lambda t})\\ &= \ln R_0 - \lambda t. \qquad (40\text{-}20)\end{aligned}$$

Because Eq. 40-20 is of the form $y = b + mx$, with b and m constants, it is a linear equation giving the quantity ln R as a function of t. Thus, if we plot ln R (instead of R) versus t, we should get a straight line. Further, the slope of the line should be equal to $-\lambda$.

Figure 40-11 shows a plot of ln R versus time t for the given measurements. The slope of the straight line that fits through the plotted points is

$$\text{slope} = \frac{0 - 6.2}{225\text{ min} - 0} = -0.0276\text{ min}^{-1}.$$

Thus,

$$-\lambda = -0.0276\text{ min}^{-1}$$

or

$$\lambda = 0.0276\text{ min}^{-1} \approx 1.7\text{ h}^{-1}. \qquad \text{(Answer)}$$

The time for the decay rate R to decrease by 1/2 is related to the disintegration constant λ via Eq. 40-19 ($T_{1/2} = (\ln 2)/\lambda$). From that equation, we find

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{\ln 2}{0.0276\text{ min}^{-1}} \approx 25\text{ min}. \qquad \text{(Answer)}$$

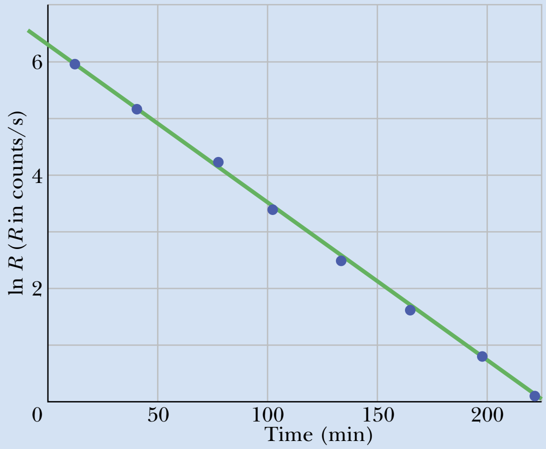


Figure 40-11 A semilogarithmic plot of the decay of a sample of ^{128}I, based on the data in the table.

SAMPLE PROBLEM 40.04

Radioactivity of the potassium in a banana

Of the 600 mg of potassium in a large banana, 0.0117% is radioactive ^{40}K, which has a half-life $T_{1/2}$ of 1.25×10^9 y. What is the activity of the banana?

KEY IDEAS

(1) We can relate the activity R to the disintegration constant λ with Eq. 40-18, but let's write it as $R = \lambda N_{40}$, where N_{40} is the number of ^{40}K nuclei (and thus atoms) in the banana. (2) We can relate the disintegration constant to the known half-life $T_{1/2}$ with Eq. 40-19 ($T_{1/2} = (\ln 2)/\lambda$).

Calculations: Combining Eqs. 40-19 and 40-18 yields

$$R = \frac{N_{40} \ln 2}{T_{1/2}}. \qquad (40\text{-}21)$$

We know that N_{40} is 0.0117% of the total number N of potassium atoms in the banana. We can find an expression for N by combining two equations that give the number of moles n of potassium in the banana. From Eq. 20-2, $n = N/N_A$, where N_A is Avogadro's number (6.02×10^{23} mol^{-1}). From Eq. 20-3, $n = M_{\text{sam}}/M$, where M_{sam} is the sample mass (here the given 600 mg of potassium) and M is the molar mass of potassium. Combining those two equations to eliminate n, we can write

$$N_{40} = (1.17 \times 10^{-4})\frac{M_{\text{sam}} N_A}{M}. \qquad (40\text{-}22)$$

From Appendix F, we see that the molar mass of potassium is 39.102 g/mol. Equation 40-22 then yields

$$N_{40} = (1.17 \times 10^{-4})\frac{(600 \times 10^{-3}\ \text{g})(6.02 \times 10^{23}\ \text{mol}^{-1})}{39.102\ \text{g/mol}}$$
$$= 1.081 \times 10^{18}.$$

Substituting this value for N_{40} and the given half-life of 1.25×10^9 y for $T_{1/2}$ into Eq. 40-21 leads to

$$R = \frac{(1.081 \times 10^{18})(\ln 2)}{(1.25 \times 10^9\ \text{y})(3.16 \times 10^7\ \text{s/y})}$$
$$= 18.96\ \text{Bq} \approx 19.0\ \text{Bq}. \qquad \text{(Answer)}$$

This is about 0.51 nCi. Your body always has about 160 g of potassium. If you repeat our calculation here, you will find that the ^{40}K component of that everyday amount has an activity of 5.06×10^3 Bq (or 0.14 μCi). So, eating a banana adds less than 1% to the radiation your body receives daily from radioactive potassium.

SAMPLE PROBLEM 40.05

Probability of decay

Suppose that we have a large sample of radioactive nuclei. Out of that sample, we pick up four nuclei. What is the probability that exactly two nuclei would have decayed after one half-life?

KEY IDEA

Here, we should remember the statement that after one half-life, half of the nuclei decay is just an estimate. In reality, the decay of each nuclei is an independent process and we have to consider the probability that half of the nuclei decay during that time period and half of the nuclei survive. *Note the word exactly*.

Calculation: Out of the four nuclei available, we can choose any two nuclei that are to decay in

$$^4_2\text{C} = \frac{4!}{2! \times 2!} = 6\ \text{ways}$$

The probability of survival of each nuclei in one half-life is *Ra*

Similarly, the probability of decay of each nuclei in one half-life is also

$$^4_2\text{C}(e^{-\lambda t})2 \times (1 - e^{-\lambda t})^2 = \tfrac{3}{8}$$

To examine it more deeply, we can say that in any time interval t, there can be two outcomes. Either the nuclei can decay or it can survive. So, the sum of the probability of survival and the probability of decay should be 1. In general, if n nuclei are being considered,

$$(P_s + P_d)^n = 1,$$

where P_s represents the probability of survival and P_d represents the probability of decay.
The different terms in this binomial expression represent the probability of different outcomes. Thus, the probability that all the nuclei will decay and none would survive will be

$$^n_0\text{C}(P_s)^0(P_d)^n$$

and so on.

40.5 | ALPHA DECAY

Key Concept

- Some nuclides decay by emitting an alpha particle (a helium nucleus, ^{4}He). Such decay is inhibited by a potential energy barrier that must be penetrated by tunneling.

When a nucleus undergoes **alpha decay**, it transforms to a different nuclide by emitting an alpha particle (a helium nucleus, ^{4}He). For example, when uranium ^{238}U undergoes alpha decay, it transforms to thorium ^{234}Th:

$$^{238}\text{U} \rightarrow {}^{234}\text{Th} + {}^{4}\text{He}. \qquad (40\text{-}23)$$

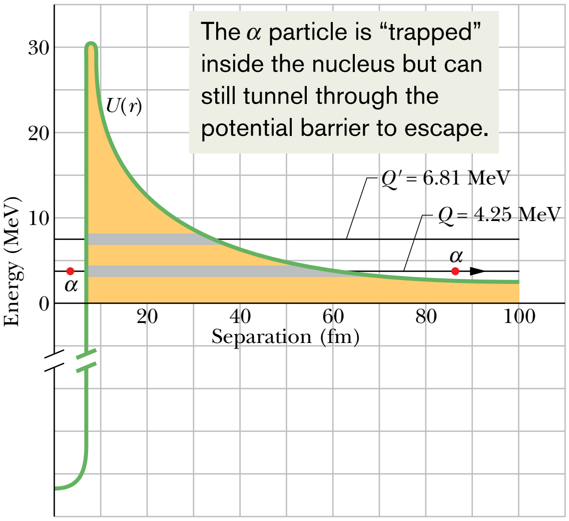


Figure 40-12 A potential energy function for the emission of an alpha particle by ^{238}U. The horizontal black line marked $Q = 4.25$ MeV shows the disintegration energy for the process. The thick gray portion of this line represents separations r that are classically forbidden to the alpha particle. The alpha particle is represented by a dot, both inside this potential energy barrier (at the left) and outside it (at the right), after the particle has tunneled through. The horizontal black line marked $Q' = 6.81$ MeV shows the disintegration energy for the alpha decay of ^{228}U. (Both isotopes have the same potential energy function because they have the same nuclear charge.)

This alpha decay of ^{238}U can occur spontaneously (without an external source of energy) because the total mass of the decay products ^{234}Th and ^{4}He is less than the mass of the original ^{238}U. Thus, the total mass energy of the decay products is less than the mass energy of the original nuclide. As defined by Eq. 36-50 ($Q = -\Delta M\ c^2$), in such a process the difference between the initial mass energy and the total final mass energy is called the Q of the process.

For a nuclear decay, we say that the difference in mass energy is the decay's *disintegration energy* Q. The Q for the decay in Eq. 40-23 is 4.25 MeV—that amount of energy is said to be released by the alpha decay of ^{238}U, with the energy transferred from mass energy to the kinetic energy of the two products.

The half-life of ^{238}U for this decay process is 4.5×10^9 y. Why so long? If ^{238}U can decay in this way, why does not every ^{238}U nuclide in a sample of ^{238}U atoms simply decay at once? To answer the questions, we must examine the process of alpha decay.

We choose a model in which the alpha particle is imagined to exist (already formed) inside the nucleus before it escapes from the nucleus. Figure 40-12 shows the approximate potential energy $U(r)$ of the system consisting of the alpha particle and the residual ^{234}Th nucleus, as a function of their separation r. This energy is a combination of (1) the potential energy associated with the (attractive) strong nuclear force that acts in the nuclear interior and (2) a Coulomb potential associated with the (repulsive) electric force that acts between the two particles before and after the decay has occurred.

The horizontal black line marked $Q = 4.25$ MeV shows the disintegration energy for the process. If we assume that this represents the total energy of the alpha particle during the decay process, then the part of the $U(r)$ curve above this line constitutes a potential energy barrier like that in Fig. 37-16. This barrier cannot be surmounted. If the alpha particle were able to be at some separation r within the barrier, its potential energy U would exceed its total energy E. This would mean, classically, that its kinetic energy K (which equals $E - U$) would be negative, an impossible situation.

Tunneling. We can see now why the alpha particle is not immediately emitted from the ^{238}U nucleus. That nucleus is surrounded by an impressive potential barrier, occupying—if you think of it in three dimensions—the volume lying between two spherical shells (of radii about 8 and 60 fm). This argument is so convincing that we now change our last question and ask: Since the particle seems permanently trapped inside the nucleus by the barrier, how can the ^{238}U nucleus *ever* emit an alpha particle? The answer is that, as you learned in Section 37.10, there is a finite probability that a particle can tunnel through an energy barrier that is classically insurmountable. In fact, alpha decay occurs as a result of barrier tunneling.

The very long half-life of ^{238}U tells us that the barrier is apparently not very "leaky." If we imagine that an already-formed alpha particle is rattling back and forth inside the nucleus, it would arrive at the inner surface of the barrier about 10^{38} times before it would succeed in tunneling through the barrier. This is about 10^{21} times per second for about 4×10^9 years (the age of Earth)! We, of course, are waiting on the outside, able to count only the alpha particles that *do* manage to escape without being able to tell what's going on inside the nucleus.

We can test this explanation of alpha decay by examining other alpha emitters. For an extreme contrast, consider the alpha decay of another uranium isotope, ^{228}U, which has a disintegration energy Q' of 6.81 MeV, about 60% higher than that of ^{238}U. (The value of Q' is also shown as a horizontal black line in Fig. 40-12.) Recall from Section 37.10 that the transmission coefficient of a barrier is very sensitive to small changes in the total energy of the particle seeking to penetrate it. Thus, we expect alpha decay to occur more readily for this nuclide than for ^{238}U. Indeed it does. As Table 40-1 shows, its half-life is only 9.1 min! An increase in Q by a factor of only 1.6 produces a decrease in half-life (that is, in the effectiveness of the barrier) by a factor of 3×10^{14}. This is sensitivity indeed.

Table 40-1 Two Alpha Emitters Compared

Radionuclide	Q	Half-Life
^{238}U	4.25 MeV	4.5×10^9 y
^{228}U	6.81 MeV	9.1 min

SAMPLE PROBLEM 40.06

Q value in an alpha decay, using masses

We are given the following atomic masses:

^{238}U	238.050 79 u	^{4}He	4.002 60 u
^{234}Th	234.043 63 u	^{1}H	1.007 83 u
^{237}Pa	237.051 21 u		

Here Pa is the symbol for the element protactinium ($Z = 91$).

(a) Calculate the energy released during the alpha decay of ^{238}U. The decay process is

$$^{238}\text{U} \rightarrow {}^{234}\text{Th} + {}^{4}\text{He}.$$

Note, incidentally, how nuclear charge is conserved in this equation: The atomic numbers of thorium (90) and helium (2) add up to the atomic number of uranium (92). The number of nucleons is also conserved: 238 = 234 + 4.

KEY IDEA

The energy released in the decay is the disintegration energy Q, which we can calculate from the change in mass ΔM due to the ^{238}U decay.

Calculation: To do this, we use Eq. 36-50,

$$Q = M_i c^2 - M_f c^2, \qquad (40\text{-}24)$$

where the initial mass M_i is that of ^{238}U and the final mass M_f is the sum of the ^{234}Th and ^{4}He masses. Using the atomic masses given in the problem statement, Eq. 40-24 becomes

$$\begin{aligned} Q &= (238.050\,79\text{ u})c^2 - (234.043\,63\text{ u} + 4.002\,60\text{ u})c^2 \\ &= (0.004\,56\text{ u})c^2 = (0.004\,56\text{ u})(931.494\,013\text{ MeV/u}) \\ &= 4.25\text{ MeV}. \qquad \text{(Answer)} \end{aligned}$$

Note that using atomic masses instead of nuclear masses does not affect the result because the total mass of the electrons in the products subtracts out from the mass of the nucleons + electrons in the original ^{238}U.

(b) Show that ^{238}U cannot spontaneously emit a proton; that is, protons do not leak out of the nucleus in spite of the proton–proton repulsion within the nucleus.

Calculation: If this happened, the decay process would be

$$^{238}\text{U} \rightarrow {}^{237}\text{Pa} + {}^{1}\text{H}.$$

(You should verify that both nuclear charge and the number of nucleons are conserved in this process.) Using the same Key Idea as in part (a) and proceeding as we did there, we would find that the mass of the two decay products

$$237.051\,21\text{ u} + 1.007\,83\text{ u}$$

would *exceed* the mass of ^{238}U by $\Delta m = 0.008\,25$ u, with disintegration energy

$$Q = -7.68\text{ MeV}.$$

The minus sign indicates that we must *add* 7.68 MeV to a ^{238}U nucleus before it will emit a proton; it will certainly not do so spontaneously.

40.6 | BETA DECAY

Key Concepts

- In beta decay, either an electron or a positron is emitted by a nucleus, along with a neutrino.
- The emitted particles share the available disintegration energy. Sometimes the neutrino gets most of the energy and sometimes the electron or positron gets most of it.

A nucleus that decays spontaneously by emitting an electron or a positron (a positively charged particle with the mass of an electron) is said to undergo **beta decay**. Like alpha decay, this is a spontaneous process, with a definite disintegration energy and half-life. Again like alpha decay, beta decay is a statistical process, governed by Eqs. 40-16 and 40-17. In *beta-minus* (β^-) decay, an electron is emitted by a nucleus, as in the decay

$$^{32}\text{P} \rightarrow {}^{32}\text{S} + \text{e}^- + \nu \quad (T_{1/2} = 14.3 \text{ d}). \tag{40-25}$$

In *beta-plus* (β^+) decay, a positron is emitted by a nucleus, as in the decay

$$^{64}\text{Cu} \rightarrow {}^{64}\text{Ni} + \text{e}^+ + \nu \quad (T_{1/2} = 12.7 \text{ h}). \tag{40-26}$$

The symbol ν represents a **neutrino**, a neutral particle which has a very small mass, that is emitted from the nucleus along with the electron or positron during the decay process. Neutrinos interact only very weakly with matter and—for that reason—are so extremely difficult to detect that their presence long went unnoticed.*

Both charge and nucleon number are conserved in the above two processes. In the decay of Eq. 40-25, for example, we can write for charge conservation

$$(+15e) = (+16e) + (-e) + (0),$$

because ^{32}P has 15 protons, ^{32}S has 16 protons, and the neutrino ν has zero charge. Similarly, for nucleon conservation, we can write

$$(32) = (32) + (0) + (0),$$

because ^{32}P and ^{32}S each have 32 nucleons and neither the electron nor the neutrino is a nucleon.

It may seem surprising that nuclei can emit electrons, positrons, and neutrinos, since we have said that nuclei are made up of neutrons and protons only. However, we saw earlier that atoms emit photons, and we certainly do not say that atoms "contain" photons. We say that the photons are created during the emission process.

It is the same with the electrons, positrons, and neutrinos emitted from nuclei during beta decay. They are created during the emission process. For beta-minus decay, a neutron transforms into a proton within the nucleus according to

$$\text{n} \rightarrow \text{p} + \text{e}^- + \nu. \tag{40-27}$$

For beta-plus decay, a proton transforms into a neutron via

$$\text{p} \rightarrow \text{n} + \text{e}^+ + \nu. \tag{40-28}$$

These processes show why the mass number A of a nuclide undergoing beta decay does not change; one of its constituent nucleons simply changes its character according to Eq. 40-27 or 40-28.

In both alpha decay and beta decay, the same amount of energy is released in every individual decay of a particular radionuclide. In the alpha decay of a particular radionuclide, every emitted alpha particle has the same sharply defined kinetic energy. However, in the beta-minus decay of Eq. 40-27 with electron emission, the disintegration energy Q is shared—in varying proportions—between the emitted electron and neutrino. Sometimes the electron gets nearly all the energy, sometimes the neutrino does. In every case, however, the sum of the electron's energy and the neutrino's energy gives the same value Q. A similar sharing of energy, with a sum equal to Q, occurs in beta-plus decay (Eq. 40-28).

Thus, in beta decay the energy of the emitted electrons or positrons may range from near zero up to a certain maximum K_{max}. Figure 40-13 shows the distribution of positron energies for the beta decay of ^{64}Cu (see Eq. 40-26). The maximum positron energy K_{max} must equal the disintegration energy Q because the neutrino has approximately zero energy when the positron has K_{max}:

$$Q = K_{\text{max}}. \tag{40-29}$$

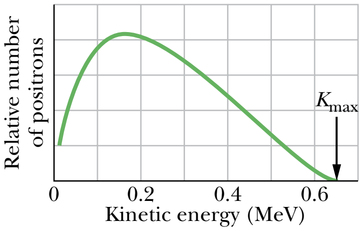


Figure 40-13 The distribution of the kinetic energies of positrons emitted in the beta decay of ^{64}Cu. The maximum kinetic energy of the distribution (K_{max}) is 0.653 MeV. In all ^{64}Cu decay events, this energy is shared between the positron and the neutrino, in varying proportions. The *most probable* energy for an emitted positron is about 0.15 MeV.

---

*Beta decay also includes *electron capture*, in which a nucleus decays by absorbing one of its atomic electrons, emitting a neutrino in the process. We do not consider that process here. Also, the neutral particle emitted in the decay process of Eq. 40-25 is actually an *antineutrino*, a distinction we shall not make in this introductory treatment.

The Neutrino

Wolfgang Pauli first suggested the existence of neutrinos in 1930. His neutrino hypothesis not only permitted an understanding of the energy distribution of electrons or positrons in beta decay but also solved another early beta-decay puzzle involving "missing" angular momentum.

The neutrino is a truly elusive particle; the mean free path of an energetic neutrino in water has been calculated as no less than several thousand light-years. At the same time, neutrinos left over from the big bang that presumably marked the creation of the universe are the most abundant particles of physics. Billions of them pass through our bodies every second, leaving no trace.

In spite of their elusive character, neutrinos have been detected in the laboratory. This was first done in 1953 by F. Reines and C. L. Cowan, using neutrinos generated in a high-power nuclear reactor. (In 1995, Reines received a Nobel Prize for this work.) In spite of the difficulties of detection, experimental neutrino physics is now a well-developed branch of experimental physics, with avid practitioners at laboratories throughout the world.

The Sun emits neutrinos copiously from the nuclear furnace at its core, and at night these messengers from the center of the Sun come up at us from below, Earth being almost totally transparent to them. In February 1987, light from an exploding star in the Large Magellanic Cloud (a nearby galaxy) reached Earth after having traveled for 170 000 years. Enormous numbers of neutrinos were generated in this explosion, and about 10 of them were picked up by a sensitive neutrino detector in Japan; Fig. 40-14 shows a record of their passage.

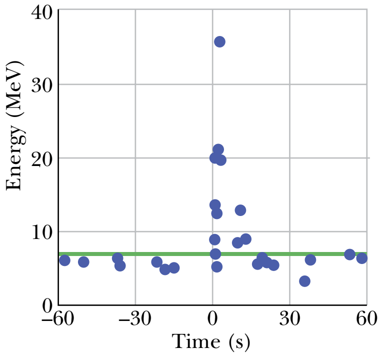


Figure 40-14 A burst of neutrinos from the supernova SN 1987A, which occurred at (relative) time 0, stands out from the usual *background* of neutrinos. (For neutrinos, 10 is a "burst.") The particles were detected by an elaborate detector housed deep in a mine in Japan. The supernova was visible only in the Southern Hemisphere; so the neutrinos had to penetrate Earth (a trifling barrier for them) to reach the detector.

Radioactivity and the Nuclidic Chart

We can increase the amount of information obtainable from the nuclidic chart of Fig. 40-5 by including a third axis showing the mass excess Δ expressed in the unit MeV/c^2. The inclusion of such an axis gives Fig. 40-15, which reveals

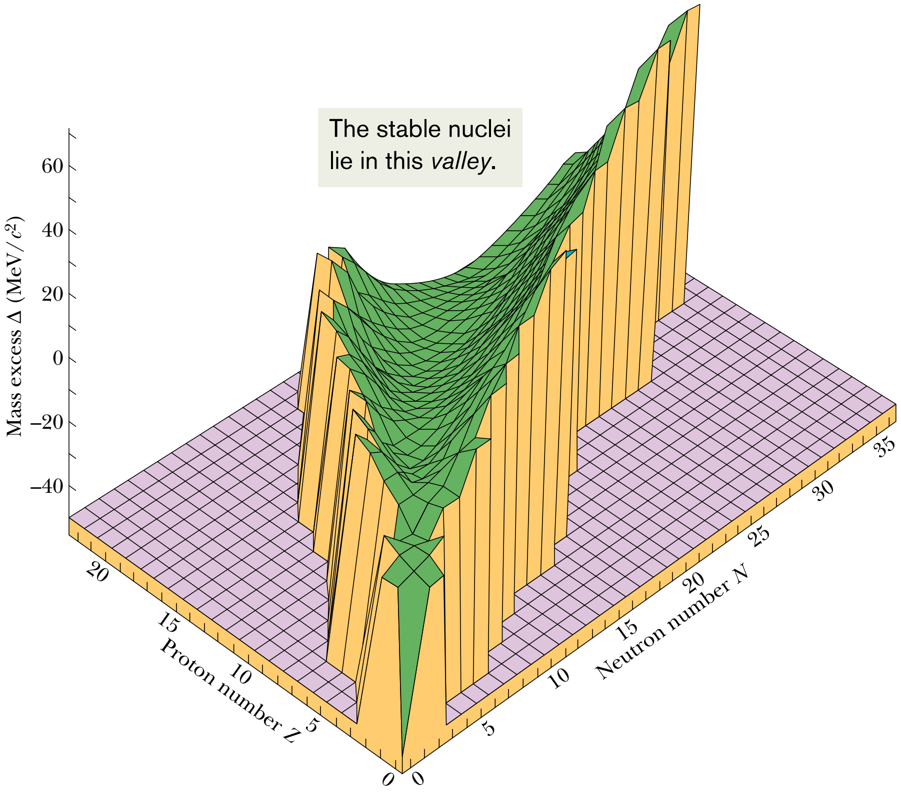


Figure 40-15 A portion of the valley of the nuclides, showing only the nuclides of low mass. Deuterium, tritium, and helium lie at the near end of the plot, with helium at the high point. The valley stretches away from us, with the plot stopping at about $Z = 22$ and $N = 35$. Nuclides with large values of A, which would be plotted much beyond the valley, can decay into the valley by repeated alpha emissions and by fission (splitting of a nuclide).

the degree of nuclear stability of the nuclides. For the low-mass nuclides, we find a "valley of the nuclides," with the stability band of Fig. 40-5 running along its bottom. Nuclides on the proton-rich side of the valley decay into it by emitting positrons, and those on the neutron-rich side do so by emitting electrons.

CHECKPOINT 3

^{238}U decays to ^{234}Th by the emission of an alpha particle. There follows a chain of further radioactive decays, either by alpha decay or by beta decay. Eventually a stable nuclide is reached and, after that, no further radioactive decay is possible. Which of the following stable nuclides is the end product of the ^{238}U radioactive decay chain: ^{206}Pb, ^{207}Pb, ^{208}Pb, or ^{209}Pb? (*Hint:* You can decide by considering the changes in mass number A for the two types of decay.)

SAMPLE PROBLEM 40.07

Q value in a beta decay, using masses

Calculate the disintegration energy Q for the beta decay of ^{32}P, as described by Eq. 40-25. The needed atomic masses are 31.973 91 u for ^{32}P and 31.972 07 u for ^{32}S.

KEY IDEA

The disintegration energy Q for the beta decay is the amount by which the mass energy is changed by the decay.

Calculations: Q is given by Eq. 36-50 ($Q = -\Delta M\ c^2$). However, we must be careful to distinguish between nuclear masses (which we do not know) and atomic masses (which we do know). Let the boldface symbols $\mathbf{m}_P$ and $\mathbf{m}_S$ represent the nuclear masses of ^{32}P and ^{32}S, and let the italic symbols m_P and m_S represent their atomic masses. Then we can write the change in mass for the decay of Eq. 40-25 as

$$\Delta m = (\mathbf{m}_S + m_e) - \mathbf{m}_P,$$

in which m_e is the mass of the electron. If we add and subtract $15m_e$ on the right side of this equation, we obtain

$$\Delta m = (\mathbf{m}_S + 16m_e) - (\mathbf{m}_P + 15m_e).$$

The quantities in parentheses are the atomic masses of ^{32}S and ^{32}P; so

$$\Delta m = m_S - m_P.$$

We thus see that if we subtract only the atomic masses, the mass of the emitted electron is automatically taken into account. (This procedure will not work for positron emission.)

The disintegration energy for the ^{32}P decay is then

$$\begin{aligned} Q &= -\Delta m\ c^2 \\ &= -(31.972\ 07\ \text{u} - 31.973\ 91\ \text{u})(931.494\ 013\ \text{MeV/u}) \\ &= 1.71\ \text{MeV}. \qquad \text{(Answer)} \end{aligned}$$

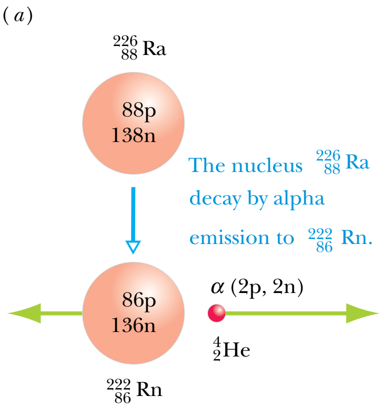
Experimentally, this calculated quantity proves to be equal to $K_{\max}$, the maximum energy the emitted electrons can have. Although 1.71 MeV is released every time a ^{32}P nucleus decays, in essentially every case the electron carries away less energy than this. The neutrino gets all the rest, carrying it stealthily out of the laboratory.

40.7 | GAMMA DECAY

Key Concept

- In gamma decay, one or more photons are emitted by a nucleus placed in an excited state. These photons are called gamma rays with typical energies of 10 keV–5 MeV.

As we have seen, the energy of internal motion of a nucleus is quantized. A typical nucleus has a set of allowed energy levels, including a ground state (state of lowest energy) and several excited states. Because of the great strength of nuclear interactions, excitation energies of nuclei are typically of the order of 1 MeV. In ordinary physical and chemical transformations, the nucleus always remains in its ground state. When a nucleus is placed in


(a)
$^{226}_{88}$Ra
88p
138n
The nucleus $^{226}_{88}$Ra decay by alpha emission to $^{222}_{86}$Rn.
86p
136n
α (2p, 2n)
$^{4}_{2}$He
$^{222}_{86}$Rn

(b) Potential-energy curve for an α particle and $^{222}_{86}$Rn nucleus

(c) Energy-level diagram for the system

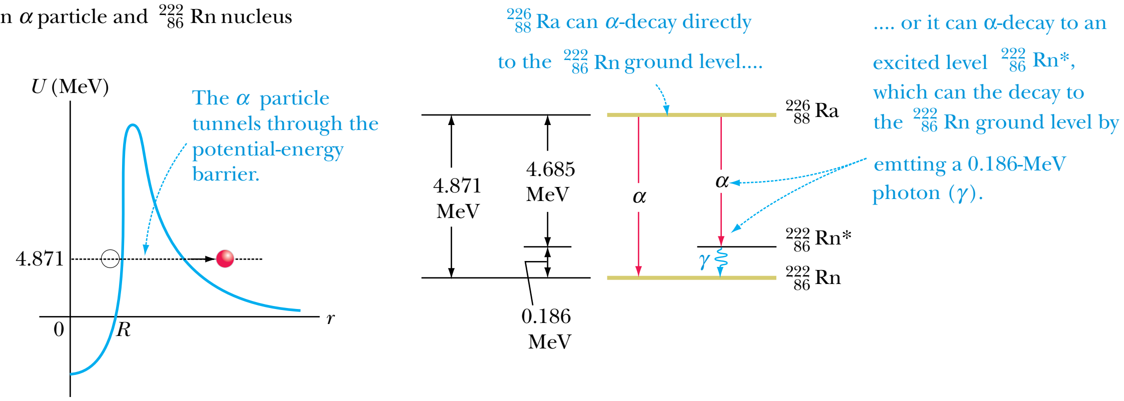

U (MeV)
The α particle tunnels through the potential-energy barrier.
4.871
0
R
r
$^{226}_{88}$Ra can α-decay directly to the $^{222}_{86}$Rn ground level....
.... or it can α-decay to an excited level $^{222}_{86}$Rn*, which can the decay to the $^{222}_{86}$Rn ground level by emtting a 0.186-MeV photon (γ).
4.871 MeV
4.685 MeV
0.186 MeV
α
α
γ
$^{226}_{88}$Ra
$^{222}_{86}$Rn*
$^{222}_{86}$Rn

Figure 40-16 Alpha decay of the unstable radium nuclide $^{226}_{88}$Ra.

an excited state, either by bombardment with high-energy particles or by a radioactive transformation, it can decay to the ground state by emission of one or more photons called gamma rays, with typical energies of 10 keV–5 MeV. This process is called gamma (g) decay. For example, alpha particles emitted from ^{226}Ra have two possible kinetic energies, either 4.784 or 4.602 MeV. Including the recoil energy of the resulting ^{222}Rn nucleus, these correspond to a total released energy of 4.871 or 4.685 MeV, respectively. When an alpha particle with the smaller energy is emitted, the ^{222}Rn nucleus is left in an excited state. It then decays to its ground state by emitting a gamma-ray photon with energy, 4.871 – 4.685 MeV = 0.186 MeV. A photon with this energy is observed during this decay (Fig. 40-16).

40.8 | RADIOACTIVE DATING

Key Concept

- Naturally occurring radioactive nuclides provide a means for estimating the dates of historic and prehistoric events. For example, the ages of organic materials can often be found by measuring their ^{14}C content, and rock samples can be dated using the radioactive ^{40}K.

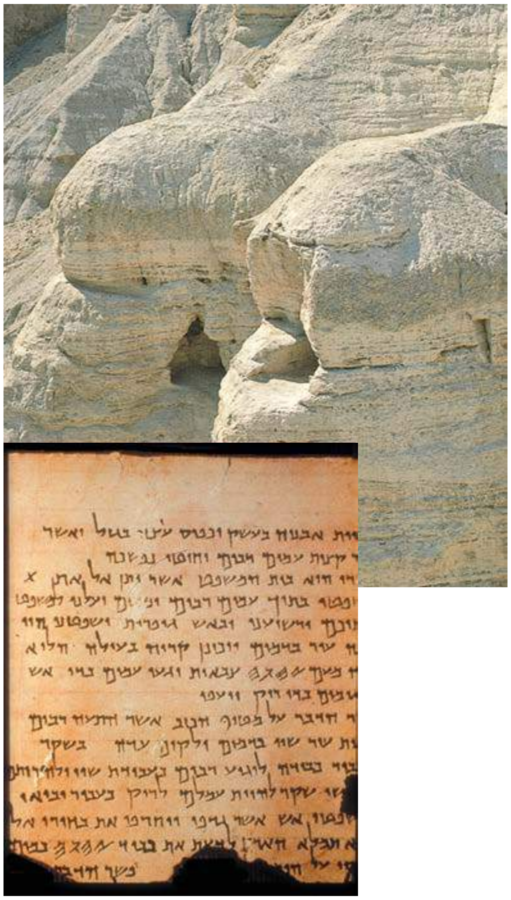
If you know the half-life of a given radionuclide, you can in principle use the decay of that radionuclide as a clock to measure time intervals. The decay of very long-lived nuclides, for example, can be used to measure the age of rocks—that is, the time that has elapsed since they were formed. Such measurements for rocks from Earth and the

Moon, and for meteorites, yield a consistent maximum age of about 4.5×10^9 y for these bodies.

The radionuclide ^{40}K, for example, decays to ^{40}Ar, a stable isotope of the noble gas argon. The half-life for this decay is 1.25×10^9 y. A measurement of the ratio of ^{40}K to ^{40}Ar, as found in the rock in question, can be used to calculate the age of that rock. Other long-lived decays, such as that of ^{235}U to ^{207}Pb (involving a number of intermediate stages of unstable nuclei), can be used to verify this calculation.

For measuring shorter time intervals, in the range of historical interest, radiocarbon dating has proved invaluable. The radionuclide ^{14}C (with $T_{1/2} = 5730$ y) is produced at a constant rate in the upper atmosphere as atmospheric nitrogen is bombarded by cosmic rays. This radiocarbon mixes with the carbon that is normally present in the atmosphere (as CO_2) so that there is about one atom of ^{14}C for every 10^{13} atoms of ordinary stable ^{12}C. Through biological activity such as photosynthesis and breathing, the atoms of atmospheric carbon trade places randomly, one atom at a time, with the atoms of carbon in every living thing, including broccoli, mushrooms, penguins, and humans. Eventually an exchange equilibrium is reached at which the carbon atoms of every living thing contain a fixed small fraction of the radioactive nuclide ^{14}C.

This equilibrium persists as long as the organism is alive. When the organism dies, the exchange with the atmosphere stops and the amount of radiocarbon trapped in the organism, since it is no longer being replenished, dwindles away with a half-life of 5730 y. By measuring the amount of radiocarbon per gram of organic matter, it is possible to measure the time that has elapsed since the organism died. Charcoal from ancient campfires, the Dead Sea scrolls (actually, the cloth used to plug the jars holding the scrolls), and many prehistoric artifacts have been dated in this way.



Top photo: George Rockwin/Bruce Coleman, Inc./Photoshot Holdings Ltd. Inset photo: www.BibleLandPictures.com/Alamy

A fragment of the Dead Sea scrolls and the caves from which the scrolls were recovered.

SAMPLE PROBLEM 40.08

Radioactive dating of a moon rock

In a Moon rock sample, the ratio of the number of (stable) ^{40}Ar atoms present to the number of (radioactive) ^{40}K atoms is 10.3. Assume that all the argon atoms were produced by the decay of potassium atoms, with a half-life of 1.25×10^9 y. How old is the rock?

KEY IDEAS

(1) If N_0 potassium atoms were present at the time the rock was formed by solidification from a molten form, the number of potassium atoms now remaining at the time of analysis is

$$N_K = N_0 e^{-\lambda t}, \qquad (40\text{-}30)$$

in which t is the age of the rock. (2) For every potassium atom that decays, an argon atom is produced. Thus, the number of argon atoms present at the time of the analysis is

$$N_{Ar} = N_0 - N_K. \qquad (40\text{-}31)$$

Calculations: We cannot measure N_0; so let's eliminate it from Eqs. 40-30 and 40-31. We find, after some algebra, that

$$\lambda t = \ln\left(1 + \frac{N_{Ar}}{N_K}\right), \qquad (40\text{-}32)$$

in which $N_{\mathrm{Ar}}/N_{\mathrm{K}}$ *can* be measured. Solving for t and using Eq. 42-18 to replace λ with $(\ln 2)/T_{1/2}$ yield

$$\begin{aligned} t &= \frac{T_{1/2}\ln(1+N_{\mathrm{Ar}}/N_{\mathrm{K}})}{\ln 2} \\ &= \frac{(1.25\times10^{9}\ \mathrm{y})[\ln(1+10.3)]}{\ln 2} \\ &= 4.37\times10^{9}\ \mathrm{y}. \qquad \text{(Answer)} \end{aligned}$$

Lesser ages may be found for other lunar or terrestrial rock samples, but no substantially greater ones. Thus, the oldest rocks were formed soon after the solar system formed, and the solar system must be about 4 billion years old.

40.9 | NUCLEAR MODELS (OPTIONAL)

Key Concepts

- The collective model of nuclear structure assumes that nucleons collide constantly with one another and that relatively long-lived compound nuclei are formed when a projectile is captured. The formation and eventual decay of a compound nucleus are totally independent events.
- The independent particle model of nuclear structure assumes that each nucleon moves, essentially without collision, in a quantized state within the nucleus. The model predicts nucleon levels and magic nucleon numbers associated with closed shells of nucleons.
- The combined model assumes that extra nucleons occupy quantized states outside a central core of closed shells.

Nuclei are more complicated than atoms. For atoms, the basic force law (Coulomb's law) is simple in form and there is a natural force center, the nucleus. For nuclei, the force law is complicated and cannot, in fact, be written down explicitly in full detail. Furthermore, the nucleus—a jumble of protons and neutrons—has no natural force center to simplify the calculations.

In the absence of a comprehensive nuclear *theory*, we turn to the construction of nuclear *models*. A nuclear model is simply a way of looking at the nucleus that gives a physical insight into as wide a range of its properties as possible. The usefulness of a model is tested by its ability to provide predictions that can be verified experimentally in the laboratory.

Two models of the nucleus have proved useful. Although based on assumptions that seem flatly to exclude each other, each accounts very well for a selected group of nuclear properties. After describing them separately, we shall see how these two models may be combined to form a single coherent picture of the atomic nucleus.

The Collective Model

In the *collective model*, formulated by Niels Bohr, the nucleons, moving around within the nucleus at random, are imagined to interact strongly with each other, like the molecules in a drop of liquid. A given nucleon collides frequently with other nucleons in the nuclear interior, its mean free path as it moves about being substantially less than the nuclear radius.

The collective model permits us to correlate many facts about nuclear masses and binding energies; it is useful (as you will see later) in explaining nuclear fission. It is also useful for understanding a large class of nuclear reactions.

Consider, for example, a generalized nuclear reaction of the form

$$X + a \rightarrow C \rightarrow Y + b. \tag{40-33}$$

We imagine that projectile a enters target nucleus X, forming a **compound nucleus** C and conveying to it a certain amount of excitation energy. The projectile, perhaps a neutron, is at once caught up by the random motions that characterize the nuclear interior. It quickly loses its identity—so to speak—and the excitation energy it carried into the nucleus is quickly shared with all the other nucleons in C.

The quasi-stable state represented by C in Eq. 40-35 may have a mean life of 10^{-16} s before it decays to Y and b. By nuclear standards, this is a very long time, being about one million times longer than the time required for a nucleon with a few million electron-volts of energy to travel across a nucleus.

The central feature of this compound-nucleus concept is that the formation of the compound nucleus and its eventual decay are totally independent events. At the time of its decay, the compound nucleus has "forgotten" how it was formed. Hence, its mode of decay is not influenced by its mode of formation. As an example, Fig. 40-17 shows three possible ways in which the compound nucleus ^{20}Ne might be formed and three in which it might decay. Any of the three formation modes can lead to any of the three decay modes.

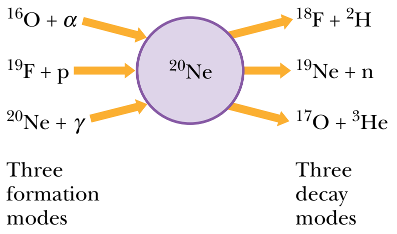


Figure 40-17 The formation modes and the decay modes of the compound nucleus ^{20}Ne.

The Independent Particle Model

In the collective model, we assume that the nucleons move around at random and bump into one another frequently. The *independent particle model*, however, is based on just the opposite assumption—namely, that each nucleon remains in a well-defined quantum state within the nucleus and makes hardly any collisions at all! The nucleus, unlike the atom, has no fixed center of charge; we assume in this model that each nucleon moves in a potential well that is determined by the smeared-out (time-averaged) motions of all the other nucleons.

A nucleon in a nucleus, like an electron in an atom, has a set of quantum numbers that defines its state of motion. Also, nucleons obey the Pauli exclusion principle, just as electrons do; that is, no two nucleons in a nucleus may occupy the same quantum state at the same time. In this regard, the neutrons and the protons are treated separately, each particle type with its own set of quantum states.

The fact that nucleons obey the Pauli exclusion principle helps us to understand the relative stability of nucleon states. If two nucleons within the nucleus are to collide, the energy of each of them after the collision must correspond to the energy of an *unoccupied* state. If no such state is available, the collision simply cannot occur. Thus, any given nucleon experiencing repeated "frustrated collision opportunities" will maintain its state of motion long enough to give meaning to the statement that it exists in a quantum state with a well-defined energy.

In the atomic realm, the repetitions of physical and chemical properties that we find in the periodic table are associated with a property of atomic electrons—namely, they arrange themselves in shells that have a special stability when fully occupied. We can take the atomic numbers of the noble gases,

$$2, 10, 18, 36, 54, 86, \ldots,$$

as *magic electron numbers* that mark the completion (or closure) of such shells.

Nuclei also show such closed-shell effects, associated with certain **magic nucleon numbers:**

$$2, 8, 20, 28, 50, 82, 126, \ldots$$

Any nuclide whose proton number Z or neutron number N has one of these values turns out to have a special stability that may be made apparent in a variety of ways.

Examples of "magic" nuclides are ^{18}O ($Z = 8$), ^{40}Ca ($Z = 20$, $N = 20$), ^{92}Mo ($N = 50$), and ^{208}Pb ($Z = 82$, $N = 126$). Both ^{40}Ca and ^{208}Pb are said to be "doubly magic" because they contain both filled shells of protons *and* filled shells of neutrons.

The magic number 2 shows up in the exceptional stability of the alpha particle (^{4}He), which, with $Z = N = 2$, is doubly magic. For example, on the binding energy curve of Fig. 40-8, the binding energy per nucleon for this nuclide stands well above those of its periodic-table neighbors hydrogen, lithium, and beryllium. The neutrons and protons making up the alpha particle are so tightly bound to one another, in fact, that it is impossible to add another proton or neutron to it; there is no stable nuclide with $A = 5$.

The central idea of a closed shell is that a single particle outside a closed shell can be relatively easily removed, but considerably more energy must be expended to remove a particle from the shell itself. The sodium atom, for example, has one (valence) electron outside a closed electron shell. Only about 5 eV is required to strip the valence electron away from a sodium atom; however, to remove a *second* electron (which must be plucked out of a closed shell) requires 22 eV. As a nuclear case, consider ^{121}Sb ($Z = 51$), which contains a single proton outside a closed shell of 50 protons. To remove this lone proton requires 5.8 MeV; to remove a *second* proton, however, requires an energy of 11 MeV. There is much additional experimental evidence that the nucleons in a nucleus form closed shells and that these shells exhibit stable properties.

We have seen that quantum theory can account beautifully for the magic electron numbers—that is, for the populations of the subshells into which atomic electrons are grouped. It turns out that, under certain assumptions, quantum theory can account equally well for the magic nucleon numbers! The 1963 Nobel Prize in physics was, in fact, awarded to Maria Mayer and Hans Jensen "for their discoveries concerning nuclear shell structure."

A Combined Model

Consider a nucleus in which a small number of neutrons (or protons) exist outside a core of closed shells that contains magic numbers of neutrons or protons. The outside nucleons occupy quantized states in a potential well established by the central core, thus preserving the central feature of the independent-particle model. These outside nucleons also interact with the core, deforming it and setting up "tidal wave" motions of rotation or vibration within it. These collective motions of the core preserve the central feature of the collective model. Such a model of nuclear structure thus succeeds in combining the seemingly irreconcilable points of view of the collective and independent-particle models. It has been remarkably successful in explaining observed nuclear properties.

SAMPLE PROBLEM 40.09

Lifetime of a compound nucleus made by neutron capture

Consider the neutron capture reaction

$$^{109}\text{Ag} + \text{n} \rightarrow {}^{110}\text{Ag} \rightarrow {}^{110}\text{Ag} + \gamma, \qquad (40\text{-}34)$$

in which a compound nucleus (^{110}Ag) is formed. Figure 40-18 shows the relative rate at which such events take place, plotted against the energy of the incoming neutron. Find the mean lifetime of this compound nucleus by using the uncertainty principle in the form

$$\Delta E \cdot \Delta t \approx \hbar. \qquad (40\text{-}35)$$

Here ΔE is a measure of the uncertainty with which the energy of a state can be defined. The quantity Δt is a measure of the time available to measure this energy. In fact, here Δt is just t_{avg}, the average life of the compound nucleus before it decays to its ground state.

Reasoning: We see that the relative reaction rate peaks sharply at a neutron energy of about 5.2 eV. This suggests that we are dealing with a single excited energy level of the compound nucleus ^{110}Ag. When the available energy (of the incoming neutron) just matches the energy of this level above the ^{110}Ag ground state, we have "resonance" and the reaction of Eq. 40-34 really "goes."

However, the resonance peak is not infinitely sharp but has an approximate half-width (ΔE in the figure) of about 0.20 eV. We can account for this resonance-peak width by saying that the excited level is not sharply defined in energy but has an energy uncertainty ΔE of about 0.20 eV.

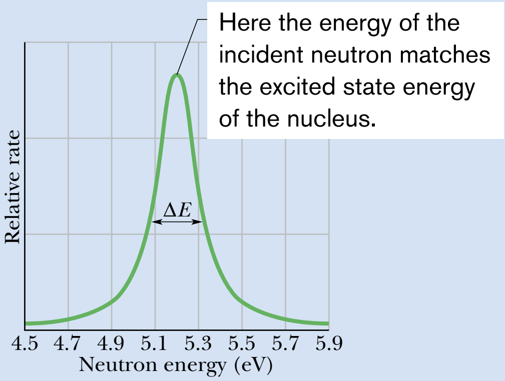


Figure 40-18 A plot of the relative number of reaction events of the type described by Eq. 40-34 as a function of the energy of the incident neutron. The half-width ΔE of the resonance peak is about 0.20 eV.

Calculation: Substituting that uncertainty of 0.20 eV into Eq. 40-35 gives us

$$\Delta t = t_{\text{avg}} \approx \frac{\hbar}{\Delta E} \approx \frac{(4.14\times10^{-15}\,\text{eV}\cdot\text{s})/2\pi}{0.20\ \text{eV}}$$

$$\approx 3\times10^{-15}\,\text{s}. \qquad \text{(Answer)}$$

This is several hundred times greater than the time a 5.2 eV neutron takes to cross the diameter of a ^{109}Ag nucleus. Therefore, the neutron is spending this time of 3×10^{-15} s *as part of* the nucleus.

40.10 | NUCLEAR FISSION: THE BASIC PROCESS

Key Concepts

- Nuclear processes are about a million times more effective, per unit mass, than chemical processes in transforming mass into other forms of energy.
- If a thermal neutron is captured by a ^{235}U nucleus, the resulting ^{236}U can undergo fission, producing two intermediatemass nuclei and one or more neutrons.
- The energy released in such a fission event is $Q \approx 200$ MeV.

In 1932 English physicist James Chadwick discovered the neutron. A few years later Enrico Fermi in Rome found that when various elements are bombarded by neutrons, new radioactive elements are produced. Fermi had predicted that the neutron, being uncharged, would be a useful nuclear projectile; unlike the proton or the alpha particle, it experiences no repulsive Coulomb force when it nears a nuclear surface. Even *thermal neutrons*, which are slowly moving neutrons in thermal equilibrium with the surrounding matter at room temperature, with a kinetic energy of only about 0.04 eV, are useful projectiles in nuclear studies.

In the late 1930s physicist Lise Meitner and chemists Otto Hahn and Fritz Strassmann, working in Berlin and following up on the work of Fermi and his co-workers, bombarded solutions of uranium salts with such thermal neutrons. They found that after the bombardment a number of new radionuclides were present. In 1939 one of the radionuclides produced in this way was positively identified, by repeated tests, as barium. But how, Hahn and Strassmann wondered, could this middle-mass element ($Z = 56$) be produced by bombarding uranium ($Z = 92$) with neutrons?

The puzzle was solved within a few weeks by Meitner and her nephew Otto Frisch. They suggested the mechanism by which a uranium nucleus, having absorbed a thermal neutron, could split, with the release of energy, into two roughly equal parts, one of which might well be barium. Frisch named the process **fission**.

Meitner's central role in the discovery of fission was not fully recognized until recent historical research brought it to light. She did not share in the Nobel Prize in chemistry that was awarded to Otto Hahn in 1944. However, in 1997 Meitner was (finally) honored by having an element named after her: meitnerium (symbol Mt, $Z = 109$).

A Closer Look at Fission

Figure 40-19 shows the distribution by mass number of the fragments produced when ^{235}U is bombarded with thermal neutrons. The most probable mass numbers, occurring in about 7% of the events, are centered around $A \approx 95$ and $A \approx 140$. Curiously, the "double-peaked" character of Fig. 40-19 is still not understood.

In a typical ^{235}U fission event, a ^{235}U nucleus absorbs a thermal neutron, producing a compound nucleus ^{236}U in a highly excited state. It is *this* nucleus that actually undergoes fission, splitting into two fragments. These fragments—between them—rapidly emit two neutrons, leaving (in a typical case) ^{140}Xe ($Z = 54$) and ^{94}Sr ($Z = 38$) as fission fragments. Thus, the stepwise fission equation for this event is

$$^{235}\text{U} + \text{n} \rightarrow {}^{236}\text{U} \rightarrow {}^{140}\text{Xe} + {}^{94}\text{Sr} + 2\text{n}. \qquad (40\text{-}36)$$

Note that during the formation and fission of the compound nucleus, there is conservation of the number of protons and of the number of neutrons involved in the process (and thus conservation of their total number and the net charge).

In Eq. 40-34, the fragments ^{140}Xe and ^{94}Sr are both highly unstable, undergoing beta decay (with the conversion of a

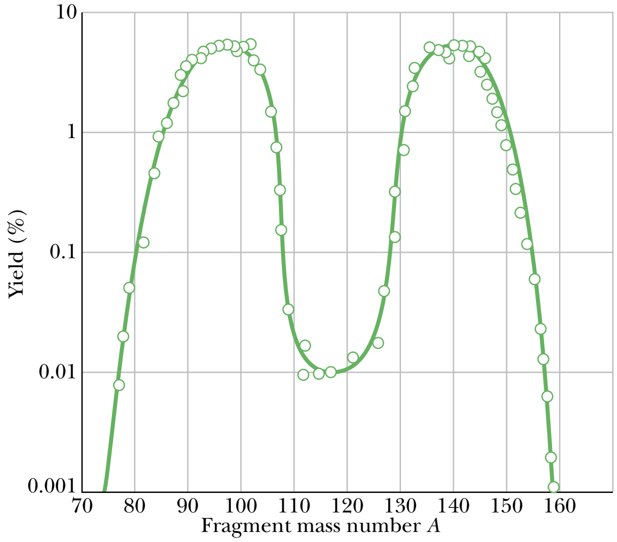


Figure 40-19 The distribution by mass number of the fragments that are found when many fission events of ^{235}U are examined. Note that the vertical scale is logarithmic.

neutron to a proton and the emission of an electron and a neutrino) until each reaches a stable end product. For xenon, the decay chain is

$$\begin{array}{c|c|c|c|c|c} & {}^{140}\mathrm{Xe} \rightarrow & {}^{140}\mathrm{Cs} \rightarrow & {}^{140}\mathrm{Ba} \rightarrow & {}^{140}\mathrm{La} \rightarrow & {}^{140}\mathrm{Ce} \\ \mathrm{T}_{1/2} & 14\text{ s} & 64\text{ s} & 13\text{ d} & 40\text{ h} & \text{Stable} \\ \hline Z & 54 & 55 & 56 & 57 & 58 \end{array} \tag{40-37}$$

For strontium, it is

$$\begin{array}{c|c|c|c} & {}^{94}\mathrm{Sr} \rightarrow & {}^{94}\mathrm{Y} \rightarrow & {}^{94}\mathrm{Zr} \\ \mathrm{T}_{1/2} & 75\text{ s} & 19\text{ min} & \text{Stable} \\ \hline Z & 38 & 39 & 40 \end{array} \tag{40-38}$$

As we should expect from Section 40.6, the mass numbers (140 and 94) of the fragments remain unchanged during these beta-decay processes and the atomic numbers (initially 54 and 38) increase by unity at each step.

Inspection of the stability band on the nuclidic chart of Fig. 40-5 shows why the fission fragments are unstable. The nuclide ^{236}U, which is the fissioning nucleus in the reaction of Eq. 40-36, has 92 protons and 236 – 92, or 144, neutrons, for a neutron/proton ratio of about 1.6. The primary fragments formed immediately after the fission reaction have about this same neutron/proton ratio. However, stable nuclides in the middle-mass region have smaller neutron/proton ratios, in the range of 1.3 to 1.4. The primary fragments are thus *neutron rich* (they have too many neutrons) and will eject a few neutrons, two in the case of the reaction of Eq. 40-36. The fragments that remain are still too neutron rich to be stable. Beta decay offers a mechanism for getting rid of the excess neutrons—namely, by changing them into protons within the nucleus.

We can estimate the energy released by the fission of a high-mass nuclide by examining the total binding energy per nucleon ΔE_{ben} before and after the fission. The idea is that fission can occur because the total mass energy will decrease; that is, ΔE_{ben} will *increase* so that the products of the fission are *more* tightly bound. Thus, the energy Q released by the fission is

$$Q = \begin{pmatrix} \text{total final} \\ \text{binding energy} \end{pmatrix} - \begin{pmatrix} \text{initial} \\ \text{binding energy} \end{pmatrix}. \tag{40-39}$$

For our estimate, let us assume that fission transforms an initial high-mass nucleus to two middle-mass nuclei with the same number of nucleons. Then we have

$$Q = \begin{pmatrix} \text{final} \\ \Delta E_{\text{ben}} \end{pmatrix} \begin{pmatrix} \text{final number} \\ \text{of nucleons} \end{pmatrix} - \begin{pmatrix} \text{initial} \\ \Delta E_{\text{ben}} \end{pmatrix} \begin{pmatrix} \text{initial number} \\ \text{of nucleons} \end{pmatrix}. \tag{40-40}$$

From Fig. 40-8, we see that for a high-mass nuclide ($A \approx 240$), the binding energy per nucleon is about 7.6 MeV/nucleon. For middle-mass nuclides ($A \approx 120$), it is about 8.5 MeV/nucleon. Thus, the energy released by fission of a high-mass nuclide to two middle-mass nuclides is

$$Q = \left(8.5 \frac{\text{MeV}}{\text{nucleon}}\right)(2\text{ nuclei})\left(120 \frac{\text{nucleons}}{\text{nucleus}}\right) - \left(7.6 \frac{\text{MeV}}{\text{nucleon}}\right)(240\text{ nucleons}) \approx 200\text{ MeV}. \tag{40-41}$$

CHECKPOINT 4

A generic fission event is

$${}^{235}\mathrm{U} + \mathrm{n} \rightarrow X + Y + 2\mathrm{n}.$$

Which of the following pairs *cannot* represent X and Y: (a) ^{141}Xe and ^{93}Sr; (b) ^{139}Cs and ^{95}Rb; (c) ^{156}Nd and ^{79}Ge; (d) ^{121}In and ^{113}Ru?

SAMPLE PROBLEM 40.10

Q value in a fission of uranium-235

Find the disintegration energy Q for the fission event of Eq. 40-36, taking into account the decay of the fission fragments as displayed in Eqs. 40-37 and 40-38. Some needed atomic and particle masses are

^{235}U	235.0439 u	^{140}Ce	139.9054 u
n	1.008 66 u	^{94}Zr	93.9063 u

KEY IDEAS

(1) The disintegration energy Q is the energy transferred from mass energy to kinetic energy of the decay products. (2) $Q = -\Delta m\, c^2$, where Δm is the change in mass.

Calculations: Because we are to include the decay of the fission fragments, we combine Eqs. 40-36, 40-37, and 40-38 to write the overall transformation as

$$^{235}\text{U} \rightarrow {}^{140}\text{Ce} + {}^{94}\text{Zr} + \text{n}. \qquad (40\text{-}42)$$

Only the single neutron appears here because the initiating neutron on the left side of Eq. 40-36 cancels one of the two neutrons on the right of that equation. The mass difference for the reaction of Eq. 40-42 is

$$\begin{aligned}\Delta m &= (139.9054\text{ u} + 93.9063\text{ u} + 1.008\,66\text{ u}) \\ &\quad - (235.0439\text{ u}) \\ &= -0.223\,54\text{ u},\end{aligned}$$

and the corresponding disintegration energy is

$$\begin{aligned}Q &= -\Delta m\, c^2 = -(-0.223\,54\text{ u})(931.494\,013\text{ MeV/u}) \\ &= 208\text{ MeV}, \qquad \text{(Answer)}\end{aligned}$$

which is in good agreement with our estimate of Eq. 40-41.

If the fission event takes place in a bulk solid, most of this disintegration energy, which first goes into kinetic energy of the decay products, appears eventually as an increase in the internal energy of that body, revealing itself as a rise in temperature. Five or six percent or so of the disintegration energy, however, is associated with neutrinos that are emitted during the beta decay of the primary fission fragments. This energy is carried out of the system and is lost.

40.11 | THERMONUCLEAR FUSION: THE BASIC PROCESS

Key Concepts

- The release of energy by fusion of two light nuclei is inhibited by their mutual Coulomb barrier (due to the electric repulsion between the two collections of protons).
- Fusion can occur in bulk matter only if the temperature is high enough (that is, if the particle energy is high enough) for appreciable barrier tunneling to occur.

The binding energy curve of Fig. 40-8 shows that energy can be released if two light nuclei combine to form a single larger nucleus, a process called nuclear **fusion**. That process is hindered by the Coulomb repulsion that acts to prevent the two positively charged particles from getting close enough to be within range of their attractive nuclear forces and thus “fusing.” The range of the nuclear force is short, hardly beyond the nuclear “surface,” but the range of the repulsive Coulomb force is long and that force thus forms an energy barrier. The height of this *Coulomb barrier* depends on the charges and the radii of the two interacting nuclei. For two protons ($Z = 1$), the barrier height is 400 keV. For more highly charged particles, of course, the barrier is correspondingly higher.

To generate useful amounts of energy, nuclear fusion must occur in bulk matter. The best hope for bringing this about is to raise the temperature of the material until the particles have enough energy—due to their thermal motions alone—to penetrate the Coulomb barrier. We call this process **thermonuclear fusion**.

In thermonuclear studies, temperatures are reported in terms of the kinetic energy K of interacting particles via the relation

$$K = kT, \qquad (40\text{-}43)$$

in which K is the kinetic energy corresponding to the *most probable speed* of the interacting particles, k is the Boltzmann constant, and the temperature T is in kelvins. Thus, rather than saying, "The temperature at the center of the Sun is 1.5×10^7 K," it is more common to say, "The temperature at the center of the Sun is 1.3 keV."

Room temperature corresponds to $K \approx 0.03$ eV; a particle with only this amount of energy could not hope to overcome a barrier as high as, say, 400 keV. Even at the center of the Sun, where $kT = 1.3$ keV, the outlook for thermonuclear fusion does not seem promising at first glance. Yet we know that thermonuclear fusion not only occurs in the core of the Sun but is the dominant feature of that body and of all other stars.

The puzzle is solved when we realize two facts: (1) The energy calculated with Eq. 40-42 is that of the particles with the *most probable* speed, as defined in Section 20.6; there is a long tail of particles with much higher speeds and, correspondingly, much higher energies. (2) The barrier heights that we have calculated represent the *peaks* of the barriers. Barrier tunneling can occur at energies considerably below those peaks, as we saw with alpha decay in Section 40.5.

CHECKPOINT 5

Which of these potential fusion reactions will *not* result in the net release of energy: (a) ^{6}Li + ^{6}Li, (b) ^{4}He + ^{4}He, (c) ^{12}C + ^{12}C, (d) ^{20}Ne + ^{20}Ne, (e) ^{35}Cl + ^{35}Cl, and (f) ^{14}N + 35Cl? (*Hint:* Consult the curve of Fig. 40-8.)

SAMPLE PROBLEM 40.11

Fusion in a gas of protons, and the required temperature

Assume a proton is a sphere of radius $R \approx 1$ fm. Two protons are fired at each other with the same kinetic energy K.

(a) What must K be if the particles are brought to rest by their mutual Coulomb repulsion when they are just "touching" each other? We can take this value of K as a representative measure of the height of the Coulomb barrier.

KEY IDEAS

The mechanical energy E of the two-proton system is conserved as the protons move toward each other and momentarily stop. In particular, the initial mechanical energy E_i is equal to the mechanical energy E_f when they stop. The initial energy E_i consists only of the total kinetic energy $2K$ of the two protons. When the protons stop, energy E_f consists only of the electric potential energy U of the system, as given by Eq. 24-54 ($U = q_1q_2/4\pi\varepsilon_0 r$).

Calculations: Here the distance r between the protons when they stop is their center-to-center distance $2R$, and their charges q_1 and q_2 are both e. Then we can write the conservation of energy $E_i = E_f$ as

$$2K = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2R}.$$

This yields, with known values,

$$\begin{aligned} K &= \frac{e^2}{16\pi\varepsilon_0 R} \\ &= \frac{(1.60\times10^{-19}\,\text{C})^2}{(16\pi)(8.85\times10^{-12}\,\text{F/m})(1\times10^{-15}\,\text{m})} \\ &= 5.75\times10^{-14}\,\text{J} = 360\ \text{keV} \approx 400\ \text{keV}. \quad \text{(Answer)} \end{aligned}$$

(b) At what temperature would a proton in a gas of protons have the average kinetic energy calculated in (a) and thus have energy equal to the height of the Coulomb barrier?

KEY IDEA

If we treat the proton gas as an ideal gas, then from Eq. 20-16, the average energy of the protons is $K_{\text{avg}} = 3/2kT$, where k is the Boltzmann constant.

Calculation: Solving that equation for T and using the result of (a) yield

$$\begin{aligned} T &= \frac{2K_{\text{avg}}}{3k} = \frac{(2)(5.75\times10^{-14}\,\text{J})}{(3)(1.38\times10^{-23}\,\text{J/K})} \\ &\approx 3\times10^{9}\,\text{K}. \qquad \text{(Answer)} \end{aligned}$$

The temperature of the core of the Sun is only about 1.5×10^7 K; thus fusion in the Sun's core must involve protons whose energies are *far* above the average energy.

40.12 | THERMONUCLEAR FUSION IN THE SUN AND OTHER STARS

Key Concepts

- The Sun's energy arises mainly from the thermonuclear burning of hydrogen to form helium by the proton–proton cycle.
- Elements up to $A \approx 56$ (the peak of the binding energy curve) can be built up by other fusion processes once the hydrogen fuel supply of a star has been exhausted.

The Sun has been radiating energy at the rate of 3.9×10^{26} W for several billion years. Where does all this energy come from? It does not come from chemical burning. (Even if the Sun were made of coal and had its own oxygen, burning the coal would last only 1000 y.) It also does not come from the Sun shrinking, transferring gravitational potential energy to thermal energy. (Its lifetime would be short by a factor of at least 500.) That leaves only thermonuclear fusion. The Sun, as you will see, burns not coal but hydrogen, and in a nuclear furnace, not an atomic or chemical one.

The fusion reaction in the Sun is a multistep process in which hydrogen is burned to form helium, hydrogen being the "fuel" and helium the "ashes." Figure 40-20 shows the **proton–proton** (p-p) **cycle** by which this occurs.

The p-p cycle starts with the collision of two protons ($^1\text{H} + {}^1\text{H}$) to form a deuteron ($^2\text{H}$), with the simultaneous creation of a positron (e^+) and a neutrino (ν). The positron immediately annihilates with any nearby electron (e^-), their mass energy appearing as two gamma-ray photons (γ) as in Section 22.5.

A pair of such events is shown in the top row of Fig. 40-20. These events are actually extremely rare. In fact, only once in about 10^{26} proton–proton collisions is a deuteron formed; in the vast majority of cases, the two protons simply rebound elastically from each other. It is the slowness of this "bottleneck" process that regulates the rate of energy production and keeps the Sun from exploding. In spite of this slowness, there are so very many protons in the huge and dense volume of the Sun's core that deuterium is produced in just this way at the rate of 10^{12} kg/s.

Once a deuteron has been produced, it quickly collides with another proton and forms a ^{3}He nucleus, as the middle row of Fig. 40-20 shows. Two such ^{3}He nuclei may eventually (within 10^5 y; there is plenty of time) find each other, forming an alpha particle (^{4}He) and two protons, as the bottom row in the figure shows.

Overall, we see from Fig. 40-20 that the p-p cycle amounts to the combination of four protons and two electrons to form an alpha particle, two neutrinos, and six gamma-ray photons. That is,

$$4\,{}^1\text{H} + 2e^- \rightarrow {}^4\text{He} + 2\nu + 6\gamma. \tag{40-44}$$

Let us now add two electrons to each side of Eq. 40-44, obtaining

$$(4\,{}^1\text{H} + 4e^-) \rightarrow ({}^4\text{He} + 2e^-) + 2\nu + 6\gamma. \tag{40-45}$$

The quantities in the two sets of parentheses then represent *atoms* (not bare nuclei) of hydrogen and of helium. That allows us to compute the energy release in the overall reaction of Eq. 40-44 (and Eq. 40-45) as

$$\begin{aligned} Q &= -\Delta m\, c^2 \\ &= -[4.002\,603\text{ u} - (4)(1.007\,825\text{ u})][931.5\text{ MeV/u}] \\ &= 26.7\text{ MeV}, \end{aligned}$$

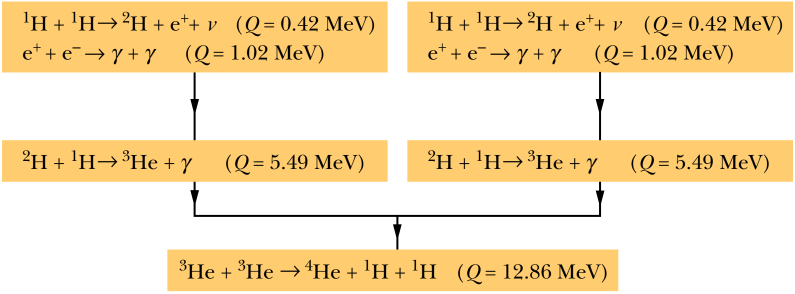

$^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + e^+ + \nu$ ($Q = 0.42$ MeV)
$e^+ + e^- \rightarrow \gamma + \gamma$ ($Q = 1.02$ MeV)
$^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + e^+ + \nu$ ($Q = 0.42$ MeV)
$e^+ + e^- \rightarrow \gamma + \gamma$ ($Q = 1.02$ MeV)
$^2\text{H} + {}^1\text{H} \rightarrow {}^3\text{He} + \gamma$ ($Q = 5.49$ MeV)
$^2\text{H} + {}^1\text{H} \rightarrow {}^3\text{He} + \gamma$ ($Q = 5.49$ MeV)
$^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^1\text{H} + {}^1\text{H}$ ($Q = 12.86$ MeV)

Figure 40-20 The proton–proton mechanism that accounts for energy production in the Sun. In this process, protons fuse to form an alpha particle (^{4}He), with a net energy release of 26.7 MeV for each event.

in which 4.002 603 u is the mass of a helium atom and 1.007 825 u is the mass of a hydrogen atom. Neutrinos have a negligibly small mass, and gamma-ray photons have no mass; thus, they do not enter into the calculation of the disintegration energy.

This same value of Q follows (as it must) from adding up the Q values for the separate steps of the proton–proton cycle in Fig. 40-20. Thus,

$$\begin{aligned} Q &= (2)(0.42\text{ MeV}) + (2)(1.02\text{ MeV}) + (2)(5.49\text{ MeV}) + 12.86\text{ MeV} \\ &= 26.7\text{ MeV}. \end{aligned}$$

About 0.5 MeV of this energy is carried out of the Sun by the two neutrinos indicated in Eqs. 40-44 and 40-45; the rest (= 26.2 MeV) is deposited in the core of the Sun as thermal energy. That thermal energy is then gradually transported to the Sun's surface, where it is radiated away from the Sun as electromagnetic waves, including visible light. Note that in fission as well as fusion, there should be at least two products, else the law of conservation of momentum as well as of energy will not hold true.

REVIEW AND SUMMARY

The Nuclides Approximately 2000 **nuclides** are known to exist. Each is characterized by an **atomic number** Z (the number of protons), a **neutron number** N, and a **mass number** A (the total number of **nucleons**—protons and neutrons). Thus, $A = Z + N$. Nuclides with the same atomic number but different neutron numbers are **isotopes** of one another. Nuclei have a mean radius r given by

$$r = r_0A^{1/3}, \qquad (40\text{-}3)$$

where $r_0 \approx 1.2$ fm.

Mass and Binding Energy Atomic masses are often reported in terms of *mass excess*

$$\Delta = M - A \quad \text{(mass excess)}, \qquad (40\text{-}7)$$

where M is the actual mass of an atom in atomic mass units and A is the mass number for that atom's nucleus. The **binding energy** of a nucleus is the difference

$$\Delta E_{\text{be}} = \Sigma(mc^2) - Mc^2 \quad \text{(binding energy)}, \qquad (40\text{-}8)$$

where $\Sigma(mc^2)$ is the total mass energy of the *individual* protons and neutrons. The **binding energy per nucleon** is

$$\Delta E_{\text{ben}} = \frac{\Delta E_{\text{be}}}{A} \quad \text{(binding energy per nucleon)}. \qquad (40\text{-}9)$$

Mass–Energy Exchanges The energy equivalent of one mass unit (u) is 931.494 013 MeV. The binding energy curve shows that middle-mass nuclides are the most stable and that energy can be released both by fission of high-mass nuclei and by fusion of low-mass nuclei.

The Nuclear Force Nuclei are held together by an attractive force acting among the nucleons, part of the **strong force** acting between the quarks that make up the nucleons.

Radioactive Decay Most known nuclides are radioactive; they spontaneously decay at a rate R $(= -dN/dt)$ that is proportional to the number N of radioactive atoms present, the proportionality constant being the **disintegration constant** λ. This leads to the law of exponential decay:

$$N = N_0e^{-\lambda t}, \quad R = \lambda N = R_0e^{-\lambda t} \quad \text{(radioactive decay)}. \qquad (40\text{-}16, 40\text{-}18, 40\text{-}17)$$

The **half-life** $T_{1/2} = (\ln 2)/\lambda$ of a radioactive nuclide is the time required for the decay rate R (or the number N) in a sample to drop to half its initial value.

Alpha Decay Some nuclides decay by emitting an alpha particle (a helium nucleus, ^{4}He). Such decay is inhibited by a potential energy barrier that cannot be penetrated according to classical physics but is subject to tunneling according to quantum physics. The barrier penetrability, and thus the half-life for alpha decay, is very sensitive to the energy of the emitted alpha particle.

Beta Decay In beta decay either an electron or a positron is emitted by a nucleus, along with a neutrino. The emitted particles share the available disintegration energy. The electrons and positrons emitted in beta decay have a continuous spectrum of energies from near zero up to a limit $K_{\max}$ $(= Q = -\Delta m\, c^2)$.

Gamma Decay When a nucleus is placed in an excited state, either by bombardment with high-energy particles or by a radioactive transfer motion, it can decay to the ground state by emission of one or more photons called gamma rays.

Radioactive Dating Naturally occurring radioactive nuclides provide a means for estimating the dates of historic and prehistoric events. For example, the ages of organic materials can often be found by measuring their ^{14}C content; rock samples can be dated using the radioactive isotope ^{40}K.

Nuclear Models The **collective** model of nuclear structure assumes that nucleons collide constantly with one another and that relatively long-lived **compound nuclei** are formed when a projectile is captured. The formation and eventual decay of a compound nucleus are totally independent events.

The **independent particle** model of nuclear structure assumes that each nucleon moves, essentially without collisions, in a quantized state within the nucleus. The model predicts nucleon levels and **magic nucleon numbers** (2, 8, 20, 28, 50, 82, and 126) associated with closed shells of nucleons; nuclides with any of these numbers of neutrons or protons are particularly stable.

The **combined** model, in which extra nucleons occupy quantized states outside a central core of closed shells, is highly successful in predicting many nuclear properties.

Energy from the Nucleus Nuclear processes are about a million times more effective, per unit mass, than chemical processes in transforming mass into other forms of energy.

Nuclear Fission Equation 40-36 shows a fission of ^{236}U induced by thermal neutrons bombarding ^{235}U. Equations 40-37 and 40-38 show the beta-decay chains of the primary fragments. The energy released in such a fission event is $Q \approx 200$ MeV.

Nuclear Fusion The release of energy by the fusion of two light nuclei is inhibited by their mutual Coulomb barrier (due to the electric repulsion between the two collections of protons). Fusion can occur in bulk matter only if the temperature is high enough (that is, if the particle energy is high enough) for appreciable barrier tunneling to occur.

The Sun's energy arises mainly from the thermonuclear burning of hydrogen to form helium by the **proton–proton cycle** outlined in Fig. 40-20. Elements up to $A \approx 56$ (the peak of the binding energy curve) can be built up by other fusion processes once the hydrogen fuel supply of a star has been exhausted. Fusion of more massive elements requires an input of energy and thus cannot be the source of a star's energy output.

PROBLEMS

1. A ^{238}U nucleus emits a 4.196 MeV alpha particle. Calculate the disintegration energy Q for this process, taking the recoil energy of the residual ^{234}Th nucleus into account.

2. The half-life of a particular radioactive isotope is 6.5 h. If there are initially 65×10^{19} atoms of this isotope, how many remain at the end of 26 h?

3. Consider the three formation processes shown for the compound nucleus ^{20}Ne in Fig. 40-14. Here are some of the atomic and particle masses:

^{20}Ne	19.992 44 u	α	4.002 60 u
^{19}F	18.998 40 u	p	1.007 83 u
^{16}O	15.994 91 u		

What energy must (a) the alpha particle, (b) the proton, and (c) the γ-ray photon have to provide 25.0 MeV of excitation energy to the compound nucleus?

4. Plutonium isotope ^{239}Pu decays by alpha decay with a half-life of 24 100 y. How many milligrams of helium are produced by an initially pure 10.0 g sample of ^{239}Pu at the end of 20 000 y? (Consider only the helium produced directly by the plutonium and not by any by-products of the decay process.)

5. Calculate the mass of a sample of (initially pure) ^{40}K that has an initial decay rate of 6.20×10^5 disintegrations/s. The isotope has a half-life of 1.28×10^9 y.

6. The nuclide ^{198}Au, with a half-life of 2.70 d, is used in cancer therapy. What mass of this nuclide is required to produce an activity of 250 Ci?

7. What is the activity of a 10 ng sample of ^{92}Kr, which has a halflife of 1.84 s?

8. Generally, more massive nuclides tend to be more unstable to alpha decay. For example, the most stable isotope of uranium, ^{238}U, has an alpha decay half-life of 4.5×10^9 y. The most stable isotope of plutonium is ^{244}Pu with an 8.0×10^7 y half-life, and for curium we have ^{248}Cm and 3.4×10^5 y. When half of an original sample of ^{238}U has decayed, what fraction of the original sample of (a) plutonium and (b) curium is left?

9. How much energy is released when a ^{238}U nucleus decays by emitting (a) an alpha particle and (b) a sequence of neutron, proton, neutron, proton? (c) Convince yourself both by reasoned argument and by direct calculation that the difference between these two numbers is just the total binding energy of the alpha particle. (d) Find that binding energy. Some needed atomic and particle masses are

^{238}U	238.050 79 u	^{234}Th	234.043 63 u
^{237}U	237.048 73 u	^{4}He	4.002 60 u
^{236}Pa	236.048 91 u	^{1}H	1.007 83 u
^{235}Pa	235.045 44 u	n	1.008 66 u

10. A 1.00 g sample of samarium emits alpha particles at a rate of 120 particles/s. The responsible isotope is ^{147}Sm, whose natural abundance in bulk samarium is 15.0%. Calculate the half-life.

11. A measurement of the energy E of an intermediate nucleus must be made within the mean lifetime Δt of the nucleus and necessarily carries an uncertainty ΔE according to the uncertainty principle

$$\Delta E \cdot \Delta t = \hbar.$$

(a) What is the uncertainty ΔE in the energy for an intermediate nucleus if the nucleus has a mean lifetime of 10^{-22} s? (b) Is the nucleus a compound nucleus?

12. A free neutron decays according to Eq. 40-27. If the neutron–hydrogen atom mass difference is 840 μu, what is the maximum kinetic energy $K_{\max}$ possible for the electron produced in a neutron decay?

13. When aboveground nuclear tests were conducted, the explosions shot radioactive dust into the upper atmosphere. Global air circulations then spread the dust worldwide before it settled out on ground and water. One such test was conducted in October 1976. What fraction of the ^{90}Sr produced by that explosion still existed in October 2006? The half-life of ^{90}Sr is 29 y.

14. The radionuclide ^{11}C decays according to

$$^{11}\text{C} \rightarrow {}^{11}\text{B} + e^{+} + \nu, \quad T_{1/2} = 20.3\ \text{min}.$$

The maximum energy of the emitted positrons is 0.960 MeV. (a) Show that the disintegration energy Q for this process is given by

$$Q = (m_{\text{C}} - m_{\text{B}} - 2m_{\text{e}})c^2,$$

where m_{C} and m_{B} are the atomic masses of ^{11}C and ^{11}B, respectively, and m_{e} is the mass of a positron. (b) Given the mass values $m_{\text{C}} = 11.011\ 434$ u, $m_{\text{B}} = 11.009\ 305$ u, and $m_{\text{e}} = 0.000\ 548\ 6$ u, calculate Q and compare it with the maximum energy of the emitted positron given above. (*Hint:* Let $\mathbf{m}_{\text{C}}$ and $\mathbf{m}_{\text{B}}$ be the nuclear masses and then add in enough electrons to use the atomic masses.)

15. After long effort, in 1902 Marie and Pierre Curie succeeded in separating from uranium ore the first substantial quantity of radium, one decigram of pure RaCl_2. The radium was the radioactive isotope ^{226}Ra, which has a half-life of 1600 y. (a) How many radium nuclei had the Curies isolated? (b) What was the decay rate of their sample, in disintegrations per second?

16. A typical kinetic energy for a nucleon in a middle-mass nucleus may be taken as 4.00 MeV. To what effective nuclear temperature does this correspond, based on the assumptions of the collective model of nuclear structure?

17. Consider an initially pure 3.4 g sample of ^{67}Ga, an isotope that has a half-life of 78 h. (a) What is its initial decay rate? (b) What is its decay rate 48 h later?

18. Under certain rare circumstances, a nucleus can decay by emitting a particle more massive than an alpha particle. Consider the decays

$$^{223}\text{Ra} \rightarrow {}^{209}\text{Pb} + {}^{14}\text{C} \quad \text{and} \quad {}^{223}\text{Ra} \rightarrow {}^{219}\text{Rn} + {}^{4}\text{He}.$$

Calculate the Q value for the (a) first and (b) second decay and determine that both are energetically possible. (c) The Coulomb barrier height for alpha-particle emission is 30.0 MeV. What is the barrier height for ^{14}C emission? (Be careful about the nuclear radii.) The needed atomic masses are

^{223}Ra	223.018 50 u	^{14}C	14.003 24 u
^{209}Pb	208.981 07 u	^{4}He	4.002 60 u
^{219}Rn	219.009 48 u		

19. Some radionuclides decay by capturing one of their own atomic electrons, a K-shell electron, say. An example is

$$^{49}\text{V} + e^{-} \rightarrow {}^{49}\text{Ti} + \nu, \quad T_{1/2} = 331\ \text{d}.$$

Show that the disintegration energy Q for this process is given by

$$Q = (m_{\text{V}} - m_{\text{Ti}})c^2 - E_K,$$

where m_{V} and m_{Ti} are the atomic masses of ^{49}V and ^{49}Ti, respectively, and E_K is the binding energy of the vanadium K-shell electron. (*Hint:* Put $\mathbf{m}_{\text{V}}$ and $\mathbf{m}_{\text{Ti}}$ as the corresponding nuclear masses and then add in enough electrons to use the atomic masses.)

20. A rock recovered from far underground is found to contain 0.86 mg of ^{238}U, 0.15 mg of ^{206}Pb, and 1.6 mg of ^{40}Ar. How much ^{40}K will it likely contain? Assume that ^{40}K decays to only ^{40}Ar with a half-life of 1.25×10^9 y. Also assume that ^{238}U has a half-life of 4.47×10^9 y.

21. In the following list of nuclides, identify (a) those with filled nucleon shells, (b) those with one nucleon outside a filled shell, and (c) those with one vacancy in an otherwise filled shell: ^{13}C, ^{18}O, ^{40}K, ^{49}Ti, ^{60}Ni, ^{91}Zr, ^{92}Mo, ^{121}Sb, ^{143}Nd, ^{144}Sm, ^{205}Tl, and ^{207}Pb.

22. A radioactive sample intended for irradiation of a hospital patient is prepared at a nearby laboratory. The sample has a halflife of 83.61 h. What should its initial activity be if its activity is to be 7.4×10^8 Bq when it is used to irradiate the patient 24 h later?

23. Large radionuclides emit an alpha particle rather than other combinations of nucleons because the alpha particle has such a stable, tightly bound structure. To confirm this statement, calculate the disintegration energies for these hypothetical decay processes and discuss the meaning of your findings:

(a) $^{235}\text{U} \rightarrow {}^{232}\text{Th} + {}^{3}\text{He}$, **(b)** $^{235}\text{U} \rightarrow {}^{231}\text{Th} + {}^{4}\text{He}$,
(c) $^{235}\text{U} \rightarrow {}^{230}\text{Th} + {}^{5}\text{He}$.

The needed atomic masses are

^{232}Th	232.0381 u	^{3}He	3.0160 u
^{231}Th	231.0363 u	^{4}He	4.0026 u
^{230}Th	230.0331 u	^{5}He	5.0122 u
^{235}U	235.0429 u		

24. The strong neutron excess (defined as $N - Z$) of high-mass nuclei is illustrated by noting that most high-mass nuclides could never fission into two stable nuclei without neutrons being left over. For example, consider the spontaneous fission of a ^{235}U nucleus into two stable *daughter nuclei* with atomic numbers 39 and 53. From Appendix F, determine the name of the (a) first and (b) second daughter nucleus. From Fig. 40-6, approximately how many neutrons are in the (c) first and (d) second? (e) Approximately how many neutrons are left over?

25. The isotope ^{40}K can decay to either ^{40}Ca or ^{40}Ar; assume both decays have a half-life of 1.26×10^9 y. The ratio of the Ca produced to the Ar produced is 8.54/1 = 8.54. A sample originally had only ^{40}K. It now has equal amounts of ^{40}K and ^{40}Ar; that is, the ratio of K to Ar is 1/1 = 1. How old is the sample? (*Hint:* Work this like other radioactive-dating problems, except that this decay has two products.)

26. In 1992, Swiss police arrested two men who were attempting to smuggle osmium out of Eastern Europe for a clandestine sale. However, by error, the smugglers had picked up ^{137}Cs. Reportedly, each smuggler was carrying a 1.0 g sample of ^{137}Cs *in a pocket*! In (a) bequerels and (b) curies, what was the activity of each sample? The isotope ^{137}Cs has a half-life of 30.2 y. (The activities of radioisotopes commonly used in hospitals range up to a few millicuries.)

27. A radiation detector records 9500 counts in 1.00 min. Assuming that the detector records all decays, what is the activity of the radiation source in (a) becquerels and (b) curies?

28. The cesium isotope ^{137}Cs is present in the fallout from aboveground detonations of nuclear bombs. Because it decays with a slow (30.2 y) half-life into ^{137}Ba, releasing considerable energy in the process, it is of environmental concern. The atomic masses of the Cs and Ba are 136.9071 and 136.9058 u, respectively; calculate the total energy released in such a decay.

29. What is the binding energy per nucleon of the europium isotope $^{152}_{63}$Eu? Here are some atomic masses and the neutron mass.

$^{152}_{63}$Eu	151.921 742 u	^{1}H	1.007 825 u
n	1.008 665 u		

30. What is the mass excess Δ_1 of ^{1}H (actual mass is 1.007 825 u) in (a) atomic mass units and (b) MeV/c^2? What is the mass excess Δ_n of a neutron (actual mass is 1.008 665 u) in (c) atomic mass units and (d) MeV/c^2? What is the mass excess Δ_{120} of ^{120}Sn (actual mass is 119.902 197 u) in (e) atomic mass units and (f) MeV/c^2?

31. A ^{7}Li nucleus with a kinetic energy of 3.00 MeV is sent toward a ^{232}Th nucleus. What is the least center-to-center separation between the two nuclei, assuming that the (more massive) ^{232}Th nucleus does not move?

32. A penny has a mass of 2.8 g. Calculate the energy that would be required to separate all the neutrons and protons in this coin from one another. For simplicity, assume that the penny is made entirely of ^{63}Cu atoms (of mass 62.929 60 u). The masses of the proton-plus-electron and the neutron are 1.007 83 u and 1.008 66 u, respectively.

33. The radionuclide ^{56}Mn has a half-life of 2.58 h and is produced in a cyclotron by bombarding a manganese target with deuterons. The target contains only the stable manganese isotope ^{55}Mn, and the manganese – deuteron reaction that produces ^{56}Mn is

$$^{55}\text{Mn} + \text{d} \rightarrow {}^{56}\text{Mn} + \text{p}.$$

If the bombardment lasts much longer than the half-life of ^{56}Mn, the activity of the ^{56}Mn produced in the target reaches a final value of 8.88×10^{10} Bq. (a) At what rate is ^{56}Mn being produced? (b) How many ^{56}Mn nuclei are then in the target? (c) What is their total mass?

34. A radioactive isotope of mercury, ^{197}Hg, decays to gold, ^{197}Au, with a disintegration constant of 0.0108 h^{-1}. (a) Calculate the half-life of the ^{197}Hg. What fraction of a sample will remain at the end of (b) three half-lives and (c) 10.0 days?

35. (a) Show that the total binding energy E_{be} of a given nuclide is

$$E_{\text{be}} = Z\Delta_{\text{H}} + N\Delta_{\text{n}} - \Delta,$$

where Δ_{H} is the mass excess of ^{1}H, Δ_{n} is the mass excess of a neutron, and Δ is the mass excess of the given nuclide. (b) Using this method, calculate the binding energy per nucleon for ^{197}Au. The needed mass excesses, rounded to three significant figures, are $\Delta_{\text{H}} = +7.29$ MeV, $\Delta_{\text{n}} = +8.07$ MeV, and $\Delta_{197} = -31.2$ MeV. Note the economy of calculation that results when mass excesses are used in place of the actual masses.

36. The radionuclide ^{64}Cu has a half-life of 12.7 h. If a sample contains 5.50 g of initially pure ^{64}Cu at $t = 0$, how much of it will decay between $t = 14.0$ h and $t = 16.0$ h?

37. A certain radionuclide is being manufactured in a cyclotron at a constant rate R. It is also decaying with disintegration constant λ. Assume that the production process has been going on for a time that is much longer than the half-life of the radionuclide. (a) Show that the number of radioactive nuclei present after such time remains constant and is given by $N = R/\lambda$. (b) Now show that this result holds no matter how many radioactive nuclei were present initially. The nuclide is said to be in *secular equilibrium* with its source; in this state its decay rate is just equal to its production rate.

38. Calculate the distance of closest approach for a head-on collision between a 4.60 MeV alpha particle and a copper nucleus.

39. Cancer cells are more vulnerable to x and gamma radiation than are healthy cells. In the past, the standard source for radiation therapy was radioactive ^{60}Co, which decays, with a half-life of 5.27 y, into an excited nuclear state of ^{60}Ni. That nickel isotope then immediately emits two gamma-ray photons, each with an approximate energy of 1.2 MeV. How many radioactive ^{60}Co nuclei are present in a 6000 Ci source of the type used in hospitals? (Energetic particles from linear accelerators are now used in radiation therapy.)

40. A source contains two phosphorus radionuclides, ^{32}P ($T_{1/2} = 14.3$ d) and ^{33}P ($T_{1/2} = 25.3$ d). Initially, 10.0% of the decays come from ^{33}P. How long must one wait until 95.0% do so?

41. The plutonium isotope ^{239}Pu is produced as a by-product in nuclear reactors and hence is accumulating in our environment. It is radioactive, decaying with a half-life of 2.41×10^4 y. (a) How many nuclei of Pu constitute a chemically lethal dose of 2.50 mg? (b) What is the decay rate of this amount?

42. When an alpha particle collides elastically with a nucleus, the nucleus recoils. Suppose a 5.00 MeV alpha particle has a head-on elastic collision with a gold nucleus that is initially at rest. What is the kinetic energy of (a) the recoiling nucleus and (b) the rebounding alpha particle?

43. An α particle (^{4}He nucleus) is to be taken apart in the following steps. Give the energy (work) required for each step: (a) remove a proton, (b) remove a neutron, and (c) separate the remaining proton and neutron. For an α particle, what are (d) the total binding energy and (e) the binding energy per nucleon? (f) Does either match an answer to (a), (b), or (c)? Here are some atomic masses and the neutron mass.

^{4}He	4.002 60 u	^{2}H	2.014 10 u
^{3}H	3.016 05 u	^{1}H	1.007 83 u
n	1.008 67 u		

44. In a Rutherford scattering experiment, assume that an incident alpha particle (radius 1.80 fm) is headed directly toward a target gold nucleus (radius 6.23 fm). What energy must the alpha particle have to just barely "touch" the gold nucleus?

45. Because the neutron has no charge, its mass must be found in some way other than by using a mass spectrometer. When a neutron and a proton meet (assume both to be almost stationary), they combine and form a deuteron, emitting a gamma ray whose energy is 2.2233 MeV. The masses of the proton and the deuteron are 1.007 276 467 u and 2.013 553 212 u, respectively. Find the mass of the neutron from these data.

46. What is the binding energy per nucleon of ^{262}Bh? The mass of the atom is 262.1231 u.

47. The nuclide ^{14}C contains (a) how many protons and (b) how many neutrons?

48. For overcoming the Coulomb barrier for fusion, methods other than heating the fusible material have been suggested. For example, if you were to use two particle accelerators to accelerate two beams of deuterons directly toward each other so as to collide head-on, (a) what voltage would each accelerator require in order for the colliding deuterons to overcome the Coulomb barrier?

49. Consider the fission of ^{238}U by fast neutrons. In one fission event, no neutrons are emitted and the final stable end products, after the beta decay of the primary fission fragments, are ^{140}Ce and ^{99}Ru. (a) What is the total of the beta-decay events in the two beta-decay chains? (b) Calculate Q for this fission process. The relevant atomic and particle masses are

^{238}U	238.050 79 u	^{140}Ce	139.905 43 u
n	1.008 66 u	^{99}Ru	98.905 94 u

50. Assume that the protons in a hot ball of protons each have a kinetic energy equal to kT, where k is the Boltzmann constant and T is the absolute temperature. If $T = 1 \times 10^7$ K, what (approximately) is the least separation any two protons can have?

51. The uranium ore mined today contains only 0.72% of fissionable ^{235}U, too little to make reactor fuel for thermal-neutron fission. For this reason, the mined ore must be enriched with ^{235}U. Both ^{235}U ($T_{1/2} = 7.0 \times 10^8$ y) and ^{238}U ($T_{1/2} = 4.5 \times 10^9$ y) are radioactive. How far back in time would natural uranium ore have been a practical reactor fuel, with a ^{235}U/^{238}U ratio of 3.0%?

52. What is the Q of the following fusion process?

$$^{2}_{1}\text{H} + ^{1}_{1}\text{H} \rightarrow ^{3}_{2}\text{He} + \text{photon}$$

Here are some atomic masses.

$^{2}_{1}$H	2.014 102 u	$^{1}_{1}$H	1.007 825 u
$^{3}_{2}$He	3.016 029 u		

53. Calculate the Coulomb barrier height for two ^{7}Li nuclei that are fired at each other with the same initial kinetic energy K. (*Hint:* Use Eq. 40-3 to calculate the radii of the nuclei.)

54. Calculate and compare the energy released by (a) the fusion of 1.0 kg of hydrogen deep within the Sun and (b) the fission of 1.0 kg of ^{235}U in a fission reactor.

55. Calculate the height of the Coulomb barrier for the head-on collision of two deuterons, with effective radius 2.1 fm.

56. Roughly 0.0150% of the mass of ordinary water is due to "heavy water," in which one of the two hydrogens in an H_2O molecule is replaced with deuterium, ^{2}H. How much average fusion power could be obtained if we "burned" all the ^{2}H in 1.00 liter of water in 1.00 day by somehow causing the deuterium to fuse via the reaction $^{2}\text{H} + ^{2}\text{H} \rightarrow ^{3}\text{He} + \text{n}$?

57. Show that the energy released when three alpha particles fuse to form ^{12}C is 7.27 MeV. The atomic mass of ^{4}He is 4.0026 u, and that of ^{12}C is 12.0000 u.

58. The fission properties of the plutonium isotope ^{239}Pu are very similar to those of ^{235}U. The average energy released per fission is 180 MeV. How much energy, in MeV, is released if all the atoms in 1.00 kg of pure ^{239}Pu undergo fission?

59. The Sun has mass 2.0×10^{30} kg and radiates energy at the rate 3.9×10^{26} W. (a) At what rate is its mass changing? (b) What fraction of its original mass has it lost in this way since it began to burn hydrogen, about 4.5×10^9 y ago?

60. In certain stars the carbon cycle is more effective than the proton–proton cycle in generating energy. This carbon cycle is

$$\begin{aligned}
^{12}\text{C} + ^{1}\text{H} &\rightarrow ^{13}\text{N} + \gamma, & Q_1 &= 1.95\ \text{MeV}\\
^{13}\text{N} &\rightarrow ^{13}\text{C} + e^{+} + \nu, & Q_2 &= 1.19,\\
^{13}\text{C} + ^{1}\text{H} &\rightarrow ^{14}\text{N} + \gamma, & Q_3 &= 7.55,\\
^{14}\text{N} + ^{1}\text{H} &\rightarrow ^{15}\text{O} + \gamma, & Q_4 &= 7.30,\\
^{15}\text{O} &\rightarrow ^{15}\text{N} + e^{+}\ \nu\ , & Q_5 &= 1.73,\\
^{15}\text{N} + ^{1}\text{H} &\rightarrow ^{12}\text{C} + ^{4}\text{He}, & Q_6 &= 4.97.
\end{aligned}$$

(a) Show that this cycle is exactly equivalent in its overall effects to the proton–proton cycle of Fig. 40-20. (b) Verify that the two cycles, as expected, have the same Q value.

61. Verify the three Q values reported for the reactions given in Fig. 40-20. The needed atomic and particle masses are

^{1}H	1.007 825 u	^{4}He	4.002 603 u
^{2}H	2.014 102 u	$e^{\pm}$	0.000 548 6 u
^{3}He	3.016 029 u		

(*Hint:* Distinguish carefully between atomic and nuclear masses, and take the positrons properly into account.)

62. A thermal neutron (with approximately zero kinetic energy) is absorbed by a ^{238}U nucleus. How much energy is transferred from mass energy to the resulting oscillation of the nucleus? Here are some atomic masses and the neutron mass.

^{237}U	237.048 723 u	^{238}U	238.050 782 u
^{239}U	239.054 287 u	^{240}U	240.056 585 u
n	1.008 664 u		

63. Calculate the disintegration energy Q for the fission of ^{52}Cr into two equal fragments. The masses you will need are

^{52}Cr 51.940 51 u ^{26}Mg 25.982 59 u.

64. Calculate the energy released in the fission reaction

$$^{235}\text{U} + \text{n} \rightarrow {}^{141}\text{Cs} + {}^{93}\text{Rb} + 2\text{n}.$$

Here are some atomic and particle masses.

^{235}U	235.043 92 u	^{93}Rb	92.921 57 u
^{141}Cs	140.919 63 u	n	1.008 66 u

65. (a)–(d) Complete the following table, which refers to the generalized fission reaction $^{235}\text{U} + \text{n} \rightarrow \text{X} + \text{Y} + b\text{n}$.

X	Y	*b*
^{140}Xe	(a)	1
^{139}I	(b)	2
(c)	^{100}Zr	2
^{141}Cs	^{92}Rb	(d)

66. The isotope ^{235}U decays by alpha emission with a half-life of 7.0×10^8 y. It also decays (rarely) by spontaneous fission, and if the alpha decay did not occur, its half-life due to spontaneous fission alone would be 3.0×10^{17} y. (a) At what rate do spontaneous fission decays occur in 1.0 g of ^{235}U? (b) How many ^{235}U alpha-decay events are there for every spontaneous fission event?

67. The binding energy of the ^{14}N nucleus is 16.76 pJ and ^{14}C nucleus is 16.86 pJ. Which nucleus is the decay product of the other?

68. For a radioactive sample, determine the ratio of the number of atoms decays of during the first half of its half-life to the number of atoms decays of during the second half of its half-cycle.

69. The half-life of a substance A is 1 h. The half-life of another substance B is 10 years. A mixture of the two substance has an activity of 1000 Bq. After 1 h, its activity drops to 900 Bq. What will be its activity after 20 years?

70. Calculate the maximum possible energy of the γ rays emitted by an excited ^{11}B nuclues formed by the capture of a slow neutron by a ^{10}B nucleus. Take mass of $^{10}\text{B} = 10.012938$ u and $^{11}\text{B} = 11.009305$ u.

PRACTICE QUESTIONS

Single Correct Choice Type

1. After how many days will 1/20th of the radioactive element remain, if the half-life of the element is 6.931 days?

(a) 23.03 days **(b)** 25.12 days
(c) 28.32 days **(d)** 29.96 days

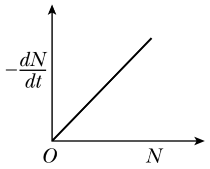
2. n number of α- particles are being emitted by N atoms of a radioactive element. The half-life of element will be

(a) $\left(\dfrac{N}{n}\right)$s **(b)** $\left(\dfrac{n}{N}\right)$s

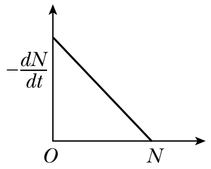
(c) $\dfrac{0.693N}{n}$s **(d)** $\dfrac{0.693n}{N}$s

3. Which of the graphs shown in the figure below best represents the relationship between the radioactive decay rate $[-(dN/dt)]$ and population N?

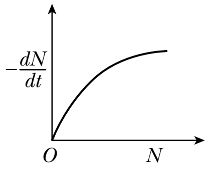
(a)



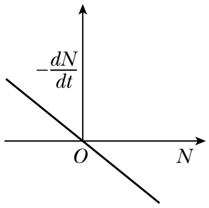
(b)



(c)



(d)



4. A fossil bone has $^{14}\text{C}:{}^{12}\text{C}$ ratio, which is 1/16 of that in a living animal's bone. If the half-life of ^{14}C is 5730 years, then the age of the fossil bone is

(a) 11460 years **(b)** 17190 years
(c) 22920 years **(d)** 4584 years

5. In a radioactive material the activity at time, t_1 is R_1 and later at time t_2, it is R_2. If the decay constant of the material is λ, then

(a) $R_1 = R_2(t_2/t_1)$ **(b)** $R_1 = R_2$
(c) $R_1 = R_2 e^{-(t_1-t_2)}$ **(d)** $R_1 = R_2 e^{(t_1-t_2)}$

6. A nuclear reaction that uses one nucleus of $^{236}_{92}$U generates 170 MeV. How much energy is released when 5.0 kg of this isotope is used?

(a) 1.4×10^{10} J **(b)** 3.5×10^{14} J
(c) 6.9×10^{13} J **(d)** 8.9×10^{14} J

7. Most suitable element for nuclear fission is the element with atomic mass near

(a) 11 **(b)** 21
(c) 52 **(d)** 92

8. A certain radioactive substance has a half-life of 5 years. Thus, for a nucleus in a sample of the element, the probability of decay in 10 years is

(a) 50% **(b)** 75%
(c) 100% **(d)** 60%

9. A sample of uranium is mixture of three isotopes $^{234}_{92}$U, $^{235}_{92}$U and $^{238}_{92}$U present in the ratio of 0.006%, 0·71% and 99.284% respectively. The half-lies of the three isotopes are $2{\cdot}5 \times 10^5$ years, $7{\cdot}1 \times 10^8$ year and $4{\cdot}5 \times 10^9$ years, respectively. The ratio of the contribution of activity of ^{238}U to ^{235}U isotopes is

(a) 10.4 **(b)** 20.4

(c) 21.4 **(d)** 22.4

10. At time $t = 0$, number of nuclei of a radioactive substance are 100. At $t = 1$ s these numbers become 90. The number of nuclei at $t = 2$ s is

(a) 50

(b) 80

(c) 70

(d) nuclei will remain undecayed

11. The decay constant of a radioactive sample is 1. The half-life and mean life of the sample, respectively, are given by

(a) $\frac{1}{\lambda}$ and $\frac{\log_e 2}{\lambda}$

(b) $\frac{\log_e 2}{\lambda}$ and $\frac{1}{\lambda}$

(c) $\lambda \log_e\ 2$ and $\frac{1}{\lambda}$

(d) $\frac{\lambda}{\log_e 2}$ and $\frac{1}{\lambda}$

12. A nucleus with $Z = 92$ emits the following in a sequence: $\alpha, \alpha, \beta^-, \beta^-, \alpha, \alpha, \alpha, \alpha, \beta^-, \beta^-, \alpha, \beta^+, \beta^+, \alpha$. The Z of the resulting nucleus is

(a) 76 **(b)** 78

(c) 82 **(d)** 74

13. The atomic mass of B^{10} is 10.811 amu. The binding energy of B^{10} nucleus is (given that the mass of electron is 0.0005498 amu, the mass of proton is $m_p = 1.007276$ amu, and the mass of neutron is $m_n = 1.008665$ amu)

(a) −678.272 MeV

(b) 678.932 MeV

(c) 378.932 MeV

(d) None of these

14. Four vessels A, B, C and D, respectively, contain 20 g atom ($T_{1/2} = 5$ h), 2 g atom ($T_{1/2} = 1$ h), 5 g atom ($T_{1/2} = 2$ h), and 10 g atom ($T_{1/2} = 3$ h) of different radio nuclides in the beginning. The maximum activity would be exhibited by the vessel is

(a) A **(b)** B

(c) C **(d)** D

15. In Rutherford's famous gold foil scattering experiment, he found that most alpha particles would pass through the foil undeflected. Which of the following nuclear properties can be inferred from this observation?

(a) The nucleus must have a positive charge

(b) Most of the mass of an atom is in the nucleus

(c) The nucleus contains both protons and neutrons

(d) The diameter of the nucleus is small compared to the diameter of the atom

16. The electron, neutron, and proton, listed in descending order of mass are

(a) Electron, neutron, proton

(b) Electron, proton, neutron

(c) Proton, neutron, electron

(d) Neutron, proton, electron

17. Which of the following is a wrong description of binding energy of a nucleus?

(a) It is the energy required to break a nucleus into its constituent nucleons

(b) It is the energy made available when free nucleons combine to form a nucleus

(c) It is proportional to the sum of the rest mass energies of its nucleons minus the rest mass energy of the nucleus

(d) It is the sum of the kinetic energy of all the nucleons in the nucleus

18. The binding energies of the atom of elements A and B are E_a and E_b, respectively. Three atoms of the element B fuse to give one atom of element A. This fusion process is accompanied by release of energy e. Then E_a and E_b are related to each other as

(a) $E_a + e = 3E_b$

(b) $E_a = 3E_b$

(c) $E_a - e = 3E_b$

(d) $E_a + 3E_b + e = 0$

19. How does the rest energy of a nucleus compare to the sum of the rest energies of the individual nucleons when they are apart from the nucleus?

(a) It is greater than the sum

(b) It is less than the sum

(c) It is equal to the sum

(d) It is in the ratio of 1:1.25

20. The binding energies of the nuclei of $^4_2$He, $^7_3$Li, $^{12}_6$C and $^{14}_7$N are 28, 52, 90, and 98 MeV, respectively. Which of these are most stable?

(a) $^4_2$He **(b)** $^7_3$Li

(c) $^{12}_6$C **(d)** $^{14}_7$N

21. In the nuclear reaction described by the equation below, an alpha particle reacts with a nucleus to form another nucleus and an additional particle X. $^4_2\text{He} + {}^{12}_6\text{C} \rightarrow {}^{13}_7\text{N} + X$. This additional particle, X, is a

(a) Positron **(b)** Neutron

(c) Proton **(d)** Tritium nucleus

22. Nucleus A is converted into C through the following reactions

$$A \rightarrow B + \alpha \text{ and } B \rightarrow C + 2\text{e}-$$

(a) A and B are isotopes

(b) A and C are isobars

(c) A and B are isobars

(d) A and C are isotopes

23. In one average life (assuming the decay to proceed according to radioactive law)

(a) Half the active nuclei decay

(b) Less than half the active nuclei decay

(c) More than half the active nuclei decay

(d) All the nuclei decay

24. In total, 90% of the active nuclei present in a radioactive sample are found to remain undecayed after 1 day. From the start, the percentage of undecayed nuclei left after 2 days will be

(a) 85% **(b)** 81%

(c) 80% **(d)** 79%

25. Let T be the mean life of a radioactive sample. 75% of the active nuclei present in the sample initially will decay in time

(a) $2T$ **(b)** $0.5T(\ln 2)$
(c) $4T$ **(d)** $2(\ln 2)T$

26. A freshly prepared radioactive source of half-life 2 h emits radiation of intensity which is 64 times the permissible safe level. The minimum time after which it would be possible to work safely with this source is

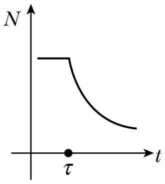
(a) 6 h **(b)** 12 h
(c) 24 h **(d)** 128 h

27. The half-life of ^{131}I is 8 days. Given a sample of ^{131}I at time $t = 0$, we can assert that

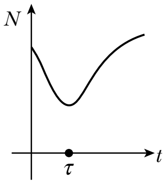
(a) No nucleus will decay before $t = 4$ days
(b) No nucleus will decay $t = 8$ days
(c) All nuclei will decay before $t = 16$ days
(d) A given nucleus may decay at any time after $t = 0$

28. A radioactive sample consists of two distinct species having equal number of atoms initially. The mean life of one species is t and that of the other is 5τ. The decay products in both cases are stable. A plot is made of the total number of radioactive nuclei as a function of time. Which of the following figure best represent the form of this plot

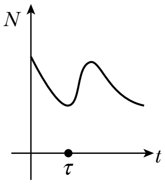
(a)



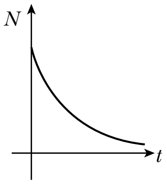
(b)



(c)



(d)



29. Half-lives of two radioactive substances A and B are 20 min and 40 min, respectively. Initially, the samples had equal number of nuclei. After 80 min, the ratio of decayed numbers of A and B nuclei is

(a) 1:16 **(b)** 4:1
(c) 1:4 **(d)** 5:4

30. The ratio of molecular mass of two radioactive substances is 3/2 and the ratio of their decay constant is 4/3. Then the ratio of their initial activity per mole will be

(a) 2 **(b)** $\frac{8}{9}$
(c) $\frac{4}{3}$ **(d)** $\frac{9}{8}$

31. An unstable nucleus decays in three different modes, each mode having a different half-life. T_1, T_2, T_3 ($T_1 \gg T_2 \gg T_3$). The overall half-life of a sample will be given by

(a) $T \approx T_1$
(b) $T \approx T_3$
(c) $T \approx (T_1 + T_2 + T_3)/3$
(d) $T \approx T_2$

32. The decay constant of a radioactive material represents the fraction of nuclei of a particular isotope that

(a) Will eventually decay if you wait long enough
(b) Will decay per unit time
(c) Are present in an element containing a mixture of isotopes
(d) Are present in a chemical compound containing elements of different atomic numbers

33. Samples of two different isotopes, X and Y. both contain the same number of radioactive atoms. Sample X has a half-life twice than that of Y. How do their activities be compared?

(a) X has a smaller activity than Y
(b) X has a greater activity than Y
(c) Activities of X and Y are equal
(d) Need to know number of neutrons in nucleus

34. A particular isotope has a half-life of 10 min. What does that statement imply about a sample of this type of nucleus?

(a) All of the nuclei will decay in 10 min
(b) All of the nuclei will decay in 20 min
(c) Every time you start with 10 nuclei at the beginning of a 10-min interval, 5 nuclei will be left at the end of the interval.
(d) The expected number of nuclei to decay in 10 min will be one-half the number at the beginning of the 10 min.

35. A radioactive substance is dissolved in a liquid and the solution is heated. The activity of the solution

(a) Is smaller than that of element
(b) Is greater than that of element
(c) Is equal to that of element
(d) Will be smaller or greater depending upon whether the solution is weak or concentrated.

36. Nuclear reactions can occur spontaneously when

(a) Q is negative
(b) Q is positive
(c) Q is negative, positive, or imaginary.
(d) Q is either positive or negative, but not when Q is imaginary

37. A nucleus with mass number 220 initially at rest emits an α particle. If the Q value of the reaction is 5.5 MeV, identify the kinetic energy of the α particle

(a) 4.4 MeV
(b) 5.4 MeV
(c) 5.6 MeV
(d) 6.5 MeV

38. During a negative beta decay,

(a) An atomic electron is ejected
(b) An electron which is already present within the nucleus is ejected
(c) A neutron in the nucleus decays emitting an electron
(d) A part of the excitation energy of the nucleus is converted into an electron

39. If ${}^{238}_{92}$U changes to ${}^{210}_{85}$Al by a series of α and β decays, the number of α and β decays undergone is

(a) 7 and 5
(b) 7 and 7
(c) 5 and 7
(d) 7 and 9

40. Ten grams of ^{57}Co is kept in an open container beta decays with a half-life of 270 days. The weight of the material inside the container after 540 days will be very nearly

(a) 10 g **(b)** 5 g

(c) 2.5 g **(d)** 1.25 g

41. Which one of the following combinations of radioactive decay results in the formation of an isotope of the original nucleus?

(a) One alpha and four beta

(b) One alpha and two beta

(c) One alpha and one beta

(d) Two alpha and one beta

42. If a beam consisting of α, β, and γ radiations is passed through an electric field perpendicular to the beam, the deflections suffered by the components, in decreasing order, are

(a) α, β, γ **(b)** α, γ, β

(c) β, α, γ **(d)** β, γ, α

43. A γ-ray photon is emitted

(a) After ionization of an atom

(b) Due to conversion of a neutron into a proton in the nucleus

(c) After de-excitation of a nucleus

(d) Due to conversion of a proton into a neutron in the nucleus

44. An archaeologist extracts a sample of carbon from an ancient ax handle and finds that it emits an average of 10 beta emissions per minute. She finds that the same mass of carbon from a living tree emits 40 beta per minute. Knowing that the half-life of carbon-14 is 5730 years, she concludes that the age of the ax handle is about

(a) 2865 years **(b)** 5730 years

(c) 11460 years **(d)** 17190 years

45. When a U-235 nucleus absorbs a neutron and undergoes nuclear fission, about 200 MeV of energy is released. But in what form? Interestingly, most of this energy initially appears in the form of

(a) γ rays

(b) Kinetic energy of emitted neutrons

(c) Kinetic energy of the fission fragments

(d) Heat

46. A star initially has 1040 deuterons. It produces energy via the processes

(i) ${}^{2}_{1}\mathrm{H} + {}^{2}_{1}\mathrm{H} \rightarrow {}^{3}_{1}\mathrm{H} + \mathrm{p}$ and (ii) ${}^{2}_{1}\mathrm{H} + {}^{3}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + \mathrm{n}$

If the average power radiated by the star is 1016 W, the deuteron supply of the star is exhausted in a time of the order of

(a) 106 s **(b)** 108 s

(c) 1012 s **(d)** 1016 s

47. Which of the following is a correct statement?

(a) In fission process, two fragments are necessary but in fusion two products are not necessary.

(b) In fission process, two fragments are not necessary but in fusion two products are necessary.

(c) In fission process as well as fusion, two products are necessary.

(d) In fission process or fusion, two products are not necessary.

48. Which of the following statements are correct?

(a) Nuclei with small mass number and atomic number are more likely to undergo fusion than fission.

(b) Nuclei having a small binding energy per nucleon are more likely to undergo fusion than fission.

(c) Nuclei having large binding energy per nucleon are more likely to undergo fusion than fission.

(d) Nuclei with large atomic number are more likely to undergo fusion than fission.

49. Osmium has atomic number 76. A particular isotope of osmium has an atomic mass of 186.956 u. Which symbol correctly represents this isotope?

(a) ${}^{187}_{111}\mathrm{Os}$ **(b)** ${}^{76}_{187}\mathrm{Os}$

(c) ${}^{111}_{76}\mathrm{Os}$ **(d)** ${}^{187}_{76}\mathrm{Os}$

50. Which of the following is *not* an assumption involved in the expression: $r = (1.2 \times 10^{-15}\ \mathrm{m})\ A^{1/3}$?

(a) Nuclei are incompressible.

(b) The nucleus is spherical in shape.

(c) All nucleons have roughly the same mass.

(d) Nuclear densities are proportional to the mass numbers.

51. What is the approximate radius of a carbon nucleus that has six protons and six neutrons?

(a) 1.2×10^{-15} m **(b)** 2.7×10^{-15} m

(c) 2.2×10^{-15} m **(d)** 2.9×10^{-15} m

52. Assuming the radius of a hydrogen atom is given by the Bohr radius, $r_{\mathrm{Bohr}} = 5.29 \times 10^{-1}$ m, what is the ratio of the nuclear density of a hydrogen atom to its atomic density?

Note: Assume for this calculation that the mass of the atom is equal to the mass of the proton.

(a) 1.2×10^{-14} **(b)** 8.6×10^{13}

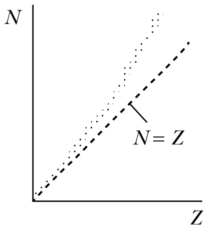
(c) 4.4×10^{4} **(d)** 3.9×10^{17}

53. ${}^{207}_{82}\mathrm{Pb}$ has a mass of 3.4368×10^{-25} kg. What is the approximate density of this lead nucleus?

(a) 2.3×10^{17} kg/m^3 **(b)** 4.8×10^{19} kg/m^3

(c) 3.5×10^{18} kg/m^3 **(d)** 5.2×10^{20} kg/m^3

54. Which one of the following concepts explains why heavy nuclei do not follow the $N = Z$ line (or trend) shown in the figure?



(a) Transmutation

(b) Coulomb repulsion

(c) Particle-wave duality

(d) Pauli exclusion principle

55. The binding energy of an isotope of chlorine is 298 MeV. What is the mass defect of this chlorine nucleus in atomic mass units?

(a) 3.13 u **(b)** 0.882 u

(c) 2.30 u **(d)** 0.320 u

56. What is the mass defect of $^{120}_{50}\text{Sn}$ (atomic mass = 119.902 200 u)? The hydrogen atom has a mass of 1.007 83 u; and the neutron has a mass of 1.008 67 u.

(a) 6.9175×10^{-28} kg
(b) 1.8202×10^{-27} kg
(c) 8.0024×10^{-28} kg
(d) 1.0687×10^{-27} kg

57. How much energy is required to remove a neutron (m_n = 1.008 665 u) from $^{15}_{7}\text{N}$ that has an atomic mass of 15.000 108 u to make $^{14}_{7}\text{N}$ that has an atomic mass of 14.003 074 u?

Note: The energy equivalent of the atomic mass unit is 931.5 MeV.

(a) 1.163 MeV **(b)** 939.6 MeV
(c) 6.423 MeV **(d)** 928.7 MeV

58. Which one of the following thicknesses of lead would be least effective in stopping β rays?

(a) 0.04 mm **(b)** 0.30 mm
(c) 0.25 mm **(d)** 0.40 mm

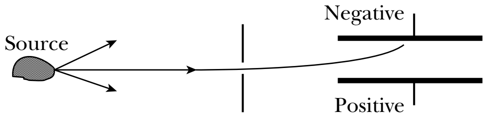
59. Consider the nuclear decay process: $^{90}_{39}\text{Y} \rightarrow {}^{90}_{38}\text{Sr} + \underline{?}$ What is(are) the missing product(s)?

(a) A photon
(b) A positron and a neutrino
(c) A proton
(d) An electron and an alpha particle

60. Which one of the following quantities is not a conserved quantity according to the laws of physics?

(a) Electric charge **(b)** Angular momentum
(c) Kinetic energy **(d)** Nucleon number

61. Radiation or(and) particles emerge(s) from a radioactive sample. These products from the sample are allowed to pass through a narrow slit and may be considered a beam. The beam is passed between two plates that carry opposite electrical charge. The experimental region contains no magnetic fields. It is observed that the beam is deflected toward the negatively charged plate.



Which one of the following statements is the best conclusion for this situation?

(a) The beam is only α rays.
(b) The beam could be either α rays or β^+ rays.
(c) The beam is only β^- rays.
(d) The beam could be α rays, β^+ rays, or γ rays.

62. The half-life of a particular isotope of iodine is 8.0 days. How much of a 10.0 g sample of this isotope will remain after 30 days?

(a) 0.37 g **(b)** 0.60 g
(c) 0.45 g **(d)** 0.74 g

63. The half-life a particular isotope of barium is 12 s. What is the activity of a 1.0×10^{-6} kg sample of this isotope?

(a) 1.2×10^{15} Bq **(b)** 2.5×10^{17} Bq
(c) 1.8×10^{16} Bq **(d)** 3.6×10^{18} Bq

64. The activity of carbon-14 in a sample of charcoal from an archaeological site is 0.04 Bq. Determine the age of the sample. The half-life of carbon-14 is 5730 years.

(a) 10 500 y **(b)** 14 500 y
(c) 12 500 y **(d)** 16 500 y

65. Which one of the following statements is true concerning the radioisotope carbon-14 that is used in carbon dating?

(a) Carbon-14 is produced by living cells.
(b) Carbon-14 is produced during β^- decay.
(c) Carbon-14 is produced by the decay of carbon-12.
(d) Carbon-14 is produced by cosmic rays striking the atmosphere.

66. The ratio of the abundance of carbon-14 to carbon-12 in a sample of dead wood is one quarter the ratio for living wood. If the half-life of carbon-14 is 5730 years, which one of the following expressions determines how many years ago the wood died?

(a) 2×5730
(b) 0.75×5730
(c) 4×5730
(d) 0.50×5730

67. Consider the following reaction: $^{14}_{7}\text{N}(\text{d},\alpha)\,^{12}_{6}\text{C}$ where $^{14}_{7}\text{N}$ has a mass of 14.003 074 u; $^{12}_{6}\text{C}$ has a mass of 12.000 000 u; d has a mass of 2.014 102 u; and α has a mass of 4.002 603 u. How much energy is released in this reaction (Note: 1 u = 931.5 MeV)?

(a) 1.2 MeV
(b) 13.6 MeV
(c) 8.3 MeV
(d) 27.4 MeV

68. Consider the reaction: $^{1}_{1}\text{H} + {}^{150}_{62}\text{Sm} \rightarrow {}^{147}_{61}\text{Pm} + {}^{4}_{2}\text{He}$ where the masses are $^{1}_{1}\text{H}$ = 1.007 825 u; $^{4}_{2}\text{He}$ = 4.002 603 u; $^{150}_{62}\text{Sm}$ = 149.917 276 u; $^{147}_{61}\text{Pm}$ = 146.915 108 u. How much energy is released in the reaction?

(a) 3.14 MeV
(b) 12.6 MeV
(c) 6.88 MeV
(d) 19.8 MeV

69. How many kilowatt·hours of energy are released from 25 g of deuterium $^{2}_{1}\text{H}$ fuel in the fusion reaction: $^{2}_{1}\text{H} + {}^{2}_{1}\text{H} \rightarrow {}^{4}_{2}\text{He} + \gamma$ where the masses are $^{2}_{1}\text{H}$ = 2.014 102 u and $^{4}_{2}\text{He}$ = 4.002 603 u.

Notes: Ignore the energy carried off by the gamma ray. Conversion factors: 1 kWh = 3.600×10^{6} J; 1 eV = 1.602×10^{-19} J.

(a) 1×10^{6} kWh
(b) 3×10^{6} kWh
(c) 2×10^{6} kWh
(d) 4×10^{6} kWh

More than One Correct Choice Type

70. For a stable nuclei, the

(a) Binding energy is large
(b) Binding energy per nucleons is large
(c) Packing fraction is very large
(d) Packing fraction is very small

71. The mass number of a nucleus is
- **(a)** Always less than its atomic number
- **(b)** Always more than its atomic number
- **(c)** Sometimes equal to its atomic number
- **(d)** Sometimes more than and sometimes equal to its atomic number

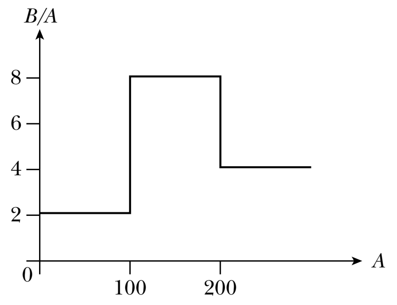
72. If A, Z, and N denote the mass number, the atomic number and the neutron number for a given nucleus, we may say that
- **(a)** $N = Z + A$
- **(b)** Isobars have the same A but different Z and N
- **(c)** Isotopes have the same Z but different N and A
- **(d)** Isotones have the same N but different A and Z

73. The decay constant of a radioactive substance is 0.173 per year. Therefore,
- **(a)** Nearly 63% of the radioactive substance will decay in 1/0.173 year
- **(b)** Half-life of the radioactive substance is 1/0.173 year
- **(c)** One-fourth of the radioactive substance will be left after nearly 8 years
- **(d)** All the above statements are true

74. Which of the following statements are correct?
- **(a)** Nuclei having an odd number of protons and an odd number of neutrons are generally unstable.
- **(b)** The mass number of a nucleus is equal to its atomic number only for the nucleus ${}^1_1\text{H}$.
- **(c)** A radioactive element of half-life 1.5 years completely disintegrates in 4.5 years.
- **(d)** The mass per nucleon in an oxygen atom is slightly less that in a hydrogen atom.

75. A radioactive sample has initial concentration N_0 of nuclei, then
- **(a)** The number of undecayed nuclei present in the sample decays exponentially with time
- **(b)** The activity (R) of the sample at any instant is directly proportional to the number of undecayed nuclei present in the sample at that time
- **(c)** The number of decayed nuclei grows linearly with time
- **(d)** The number of decayed nuclei grows exponentially with time

76. Assume that the nuclear biding energy per nucleon *(B/A)* versus mass number (A) is as shown in figure below. Use this plot shown in figure to choose the correct statements from the following.



- **(a)** Fusion of two nuclei with mass numbers lying in the range of $1 < A < 50$ will release energy
- **(b)** Fusion of two nuclei with mass numbers lying in the range of $51 < A < 100$ will release energy
- **(c)** Fission of a nucleus lying in the mass range of $100 < A < 200$ will release energy when broken into two equal fragments
- **(d)** Fission of a nucleus lying in the mass range of $200 < A < 260$ will release energy when broken into two equal fragments.

77. From the following equations, choose the possible nuclear reactions.
- **(a)** ${}^{12}_{6}\text{C} + {}^{1}_{1}\text{H} \to {}^{13}_{7}\text{N} + 2\,\text{MeV}$
- **(b)** ${}^{13}_{6}\text{C} + {}^{1}_{1}\text{H} \to {}^{14}_{6}\text{N} + 4.3\,\text{MeV}$
- **(c)** ${}^{14}_{7}\text{N} + {}^{1}_{1}\text{H} \to {}^{15}_{8}\text{O} + 7.3\text{ MeV}$
- **(d)** ${}^{235}_{92}\text{U} + {}^{1}_{0}\text{n} \to {}^{140}_{54}\text{Xe} + {}^{94}_{38}\text{Sr} + {}^{1}_{0}\text{n} + \gamma + 200\text{ MeV}$

78. In the options given below, let E denote the mass energy of a nucleus and $E(n)$ that of a neutron. The correct option is
- **(a)** $E\left({}^{236}_{92}\text{U}\right) > E\left({}^{137}_{53}\text{I}\right) + E\left({}^{97}_{39}\text{Y}\right) + 2E(n)$
- **(b)** $E\left({}^{236}_{92}\text{U}\right) < E\left({}^{137}_{53}\text{I}\right) + E\left({}^{97}_{39}\text{Y}\right) + 2E(n)$
- **(c)** $E\left({}^{236}_{92}\text{U}\right) < E\left({}^{140}_{56}\text{Ba}\right) + E\left({}^{94}_{36}\text{Kr}\right) + 2E(n)$
- **(d)** $E\left({}^{236}_{92}\text{U}\right) > E\left({}^{140}_{56}\text{Ba}\right) + E\left({}^{94}_{36}\text{Kr}\right) + 2E(n)$

79. Let m_p be the mass of proton, m_n the mass of a neutron, M_1 the mass of a ${}^{10}_{20}\text{Ne}$ nucleus and M_2 the mass of a ${}^{20}_{40}\text{Ca}$ nucleus. Then
- **(a)** $M_2 = 2M_1$
- **(b)** $M_2 > 2M_1$
- **(c)** $M_2 < 2M_1$
- **(d)** $M_1 < 10(M_\text{n} + m_\text{p})$

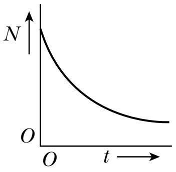
80. Which of the following are evidences for the existence of neutrons in an atom?
- **(a)** The ratio of the charge to the mass of atom's nucleus is smaller than that of a hydrogen nucleus
- **(b)** An atom is always electrically neutral
- **(c)** Isotopes are present
- **(d)** Some atoms are radioactive in nature

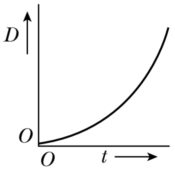
81. The instability of the nucleus can be due to various causes. An unstable nucleus emits radiations if possible to transform into less unstable state. Then the cause and the result can be
- **(a)** A nucleus of excess nucleons is α active
- **(b)** An excited nucleus of excess protons is β^- active
- **(c)** An excited nucleus of excess protons is β^+ active
- **(d)** A nucleus of excess neutrons is β^- active

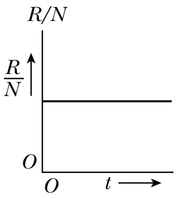
82. A fraction f_1 of a radioactive sample decays in one mean life, and a fraction f_2 decays in one half-life. Then
- **(a)** $f_1 > f_2$
- **(b)** $f_1 < f_2$
- **(c)** $f_1 = f_2$
- **(d)** $f_1 \neq f_2$

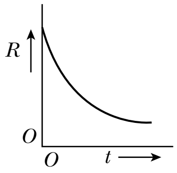
83. A large population of radioactive nucleus starts disintegrating at $t = 0$. At time t, if N = number of parent nuclei present, D = the number of daughter nuclei present, and

R = rate at which the daughter nuclei are produced, then the correct representation will be

(a)


(b)


(c)


(d)


84. In β decay, the Q-value of the process is E. Then,
- **(a)** Kinetic energy of a β-particle cannot exceed E
- **(b)** Kinetic energy of anti-neutrino emitted lies between zero and E
- **(c)** N/Z ratio of the nucleus is altered
- **(d)** Mass number (A) of the nucleus is altered

85. Consider the following nuclear reactions and select the correct statements from the option given below.

Reaction I: $n \rightarrow p + e^- + v$ Reaction II: $p \rightarrow n + e^+ + v$

- **(a)** Free neutron is unstable, therefore reaction I is possible
- **(b)** Free proton is stable, therefore reaction II is not possible
- **(c)** Inside a nucleus, both decays (reactions I and II) are possible
- **(d)** Inside a nucleus, reaction I is not possible but reaction II is possible

86. What is the essential distinction between x rays and γ rays
- **(a)** γ rays have shorter wavelength than x rays
- **(b)** γ rays are extraterrestrial, x rays are man-made
- **(c)** γ rays have less penetrating power than x rays
- **(d)** γ rays have greater ionizing powers than x rays

87. When the nucleus of an electrically neutral atom undergoes a radioactive decay process, it will remain neutral after the decay if the process is
- **(a)** α decay
- **(b)** β decay
- **(c)** γ decay
- **(d)** K-capture

88. Which of the following is an assumption in Carbon dating process?
- **(a)** The ratio of ^{14}C to ^{12}C is constant in living organisms
- **(b)** The half-life of a nuclide remains constant over the years
- **(c)** ^{14}C stops being absorbed when an organism dies
- **(d)** Fluctuation of ^{14}C and ^{12}C ratios occurs in well-known periodic intervals

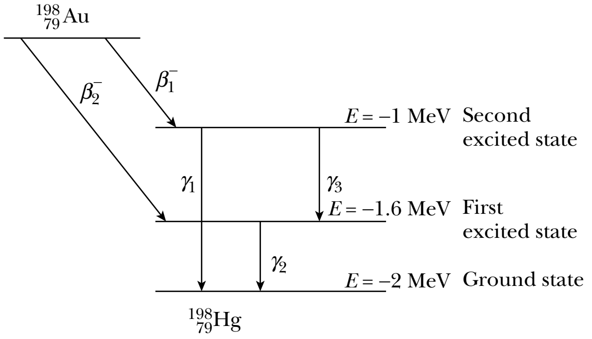
89. Both nuclear fission and nuclear fusion release enormous energy. When a uranium nucleus fission, the released energy is mainly kinetic energy of the repelling fragments. When a pair of hydrogen isotopes fuses, the energy initially released is in the form of
- **(a)** Gamma radiation
- **(b)** Kinetic energy of recoiling particles
- **(c)** Potential energy of the helium nucleus that is formed
- **(d)** Heat

90. The total binding energy of an α particle (^4_2He) is 24.4 MeV whereas the total binding energy of a deutron (^2_1H) is merely 2.2 MeV. When two deutrons are made to combine
- **(a)** An α particle will be obtained
- **(b)** An amount of 22.2 MeV energy will be released
- **(c)** An amount of 20.0 MeV energy will be released
- **(d)** An amount of 22.2 MeV energy will be consumed

91. Which of the following(s) is(are) correct?
- **(a)** The rest mass of a stable nucleus is less than the sum of the rest masses of its separated nucleons
- **(b)** The rest mass of a stable nucleus is greater than the sum of the rest masses of its separated nucleons
- **(c)** In nuclear fusion, energy is released by fusion of two nuclei of medium mass (approximately 100 amu)
- **(d)** In nuclear fission, energy is released by fragmentation of a very heavy nucleus

Linked Comprehension

Paragraph for Questions 92–94: Gold nucleus ($^{198}_{79}Au$) can decay into mercury nucleus ($^{198}_{80}Hg^8$) by two decay schemes shown in figure below, (i) it can emit a β-particle (β_1) and come to ground state by either emitting one γ-ray (γ_1) or emitting two γ-rays (γ_3 and γ_4) (ii) it can emit one β-particle (β_2) and come to ground state by emitting γ_2 ray. Atomic masses: ^{198}Au = 197.9652 amu and ^{198}Hg = 197.9662 amu. Given that 1 amu = 930 MeV/c^2. The energy levels of the nucleus are shown in figure below


$^{198}_{79}Au$
β_2^-
β_1^-
$E = -1$ MeV Second excited state
γ_1
γ_3
$E = -1.6$ MeV First excited state
γ_2
$E = -2$ MeV Ground state
$^{198}_{79}Hg$

92. What is the maximum kinetic energy of emitted β_2 particle?
- **(a)** 1.44 MeV
- **(b)** 0.59 MeV
- **(c)** 1.86 MeV
- **(d)** 1.46 MeV

93. What is the maximum kinetic energy of emitted β_1 particle?
- **(a)** 1.28 MeV
- **(b)** 0.77 MeV
- **(c)** 1.86 MeV
- **(d)** 0.86 MeV

94. The wavelength of emitted γ-rays is in the order
- **(a)** $\lambda_2 > \lambda_3 > \lambda_1$
- **(b)** $\lambda_3 > \lambda_2 > \lambda_1$
- **(c)** $\lambda_1 > \lambda_2 > \lambda_3$
- **(d)** $\lambda_3 = \lambda_2 = \lambda_1$

Paragraph for Questions 95–97: We have two radioactive nuclei A and B. Both convert into a stable nucleus C.

Nucleus A converts into C after emitting two α-particles and three β-particles. Nucleus B converts into C after emitting one α-particle and five β particles. At time $t = 0$, nuclei of A are $4N$ and that of B are N. Half-life of A (into the conversion of C) is 1 min and that of B is 2 min. Initially number of nuclei of C are zero.

95. If atomic numbers and mass numbers of A and B are Z_1, Z_2, A_1 and A_2, respectively. Then

(a) $Z_1 - Z = 6$

(b) $A_1 - A_2 = 4$

(c) both (b) and (d) are correct

(d) both (a) and (c) are wrong

96. What are number of nuclei of C, when number of nuclei of A and B are equal?

(a) $2N_0$ **(b)** $3N_0$

(c) $9N_0/2$ **(d)** $5N_0/2$

97. At what time rate of disintegrations of A and B are equal?

(a) 4 minutes **(b)** 6 minutes

(c) 8 minutes **(d)** 2 minutes

Paragraph for Questions 98–100: In a nuclear fusion reactor, the reaction occurs in two stages: (i) Two deuterium $({}_1^2\mathrm{D})$ nuclei fuse to form a tritium $({}_1^3\mathrm{T})$ nucleus with a proton as a byproduct. The reaction may be represented as D (D, p)T. (ii) A tritium nucleus fuses with another deuterium nucleus to form a helium $({}_2^4\mathrm{He})$ nucleus with a neutron as a byproduct. The reaction is represented as T(D, n)α. Given that $m({}_1^2\mathrm{D}) = 2.014102$ u(atom)

(a) $m({}_1^3\mathrm{T}) = 3.016049$u (atom)

(b) $m({}_2^4\mathrm{He}) = 4.002603$ u (atom)

(c) $m({}_1^1\mathrm{H}) = 1.007825$ u (atom)

(d) $m({}_0^1\mathrm{n}) = 1.008665$ u (atom)

98. The energy released in 1st stage of fusion reaction, is

(a) 4.033 MeV **(b)** 17.587 MeV

(c) 40.33 MeV **(d)** 1.7587 MeV

99. The energy released in the second stage of fusion reaction

(a) 4.033 MeV **(b)** 17.587 MeV

(c) 40.33 MeV **(d)** 1.7587 MeV

100. What percentage of the mass energy of the initial deuterium is released?

(a) 0.184% **(b)** 0.284%

(c) 0.384% **(d)** 0.484%

Paragraph for Questions 101–103: Consider the following nuclear decay: ${}_{92}^{236}\mathrm{U} \rightarrow {}_{90}^{232}\mathrm{Th} + X$

101. What is X?

(a) α **(b)** p

(c) β^+ **(d)** β^-

102. Determine the amount of energy released in this decay. Use the following atomic masses:

${}_{92}^{236}\mathrm{U} = 236.045\ 562$ u; ${}_{90}^{232}\mathrm{Th} = 232.038\ 054$ u;
${}_2^4\mathrm{He} = 4.002\ 603$ u; ${}_0^1\mathrm{n} = 1.008\ 665$ u; and ${}_1^1\mathrm{p} = 1.007\ 277$ u

Conversion factors: 1 u = 931.5 MeV; 1 eV = 1.602×10^{-19} J

(a) 3.5×10^{-8} J **(b)** 4.6×10^{-12} J

(c) 6.0×10^{-10} J **(d)** 7.3×10^{-13} J

103. If the uranium nucleus is at rest before its decay, which one of the following statements is true concerning the final nuclei?

(a) They have equal kinetic energies, but the thorium nucleus has much more momentum.

(b) They have momenta of equal magnitudes, but the thorium nucleus has much more kinetic energy.

(c) They have equal kinetic energies and momenta of equal magnitudes.

(d) They have momenta of equal magnitudes, but X has much more kinetic energy.

Matrix-Match

Match the statements in Column I labeled as (a), (b), (c), and (d) with those in Column II labeled as (p), (q), (r), and (s). Any given statement in Column I can have correct matching with *one or more* statements in Column II.

104.

Column I	Column II
(a) α-ray	**(p)** lowest ionizing power
(b) β-ray	**(q)** lowest ionizing power
(c) β^+ plus ray	**(r)** high ionizing power
(d) γ-ray	**(s)** low penetrating power

105. Symbol Q stands for energy released.

Column I	Column II
(a) α-decay	**(p)** ${}_{92}^{235}\mathrm{U} + {}_0^1\mathrm{n} \rightarrow {}_{56}^{141}\mathrm{Ba} + {}_{36}^{92}\mathrm{Kr} + ({}_0^1\mathrm{n}) + Q$
(b) β-decay	**(q)** ${}_1^3\mathrm{H} + {}_1^2\mathrm{H} \rightarrow {}_2^4\mathrm{He} + Q$
(c) Nuclear fission	**(r)** ${}_{90}^{230}\mathrm{Th} \rightarrow {}_{90}^{226}\mathrm{Ra} + {}_2^4\mathrm{He} + Q$
(d) Nuclear fusion	**(s)** ${}_{55}^{137}\mathrm{Cs} \rightarrow {}_{56}^{137}\mathrm{Ba} + e^- + \bar{\nu} + Q$

Directions for Questions 106–109: In each question, there is a table having 3 columns and 4 rows. Based on the table, there are 3 questions. Each question has 4 options (a), (b), (c) and (d), ONLY ONE of these four options is correct.

106. In the given table, Column I shows different types of nuclei, Column II shows their characteristics and Column III shows the examples of different types of nuclei.

Column I	Column II	Column III
(I) Atoms of element having same atomic number but different mass number	**(i)** Atomic number differ by 1	**(J)** ${}_1^1\mathrm{H}, {}_1^2\mathrm{H}$
(II) Nuclei having equal number of neutrons	**(ii)** Both the atomic number Z and mass number A are different	**(K)** ${}_1^3\mathrm{H}, {}_2^3\mathrm{He}$

Column I	Column II	Column III
(III) Nuclei which have the same mass number *A* but different atomic number *Z*	**(iii)** Have the same chemical properties.	**(L)** $^{9}_{4}$Be, $^{10}_{5}$B
(IV) Nuclei having the same mass number *A* but with the proton number *Z* and neutron number interchanged	**(iv)** Have different chemical properties	**(M)** $^{7}_{3}$Li, $^{7}_{4}$Be

(1) Determine the characteristics of isobars.

(a) (III) (ii) (L) **(b)** (III) (iv) (K)
(c) (II) (i) (M) **(d)** (II) (i) (K)

(2) Determine the characteristics of mirror nuclei.

(a) (I) (iii) (K) **(b)** (IV) (iii) (J)
(c) (I) (ii) (M) **(d)** (IV) (i) (M)

(3) Determine the characteristics of isotones.

(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (II) (ii) (L) **(d)** (I) (iv) (M)

107. In the given table, Column I shows the different types of decays in nuclear fission, Column II shows the change in the element which undergoes decay and Column III shows properties of different types of decays.

Column I	Column II	Column III
(I) Process in which a parent nucleus decays into the daughter nucleus by ejecting helium particle	**(i)** The original element becomes a new chemical element in a process known as nuclear transmutation.	(J) The most common form of cluster decay
(II) Process in which a parent nucleus decays into the daughter nucleus by ejecting an electron	**(ii)** It has a charge of + 2e and a mass of 4 u.	**(K)** An electron at the far right of the curve would have the maximum possible kinetic energy, leaving the energy of the neutrino to be only its small rest mass.
(III) The daughter nucleus is left in the excited state. It decays into any other lower state or ground state.	**(iii)** Conserves a quantum number known as the lepton number.	**(L)** Mode of relaxation of many excited states of atomic nuclei
(IV) Process in which a parent nucleus decays into the daughter nucleus by ejecting a neutron	**(iv)** Formation of fluorescent gamma rays	**(M)** The total energy of the decay process is divided between the electron, the antineutrino, and the recoiling nuclide

(1) What are the characteristics of the below nuclear equation?

$$^{A}_{Z}\text{X(mother nucleus)} \rightarrow {}^{A-4}_{Z-2}\text{Y(daughter nucleus)} + {}^{4}_{2}\text{He}$$

(a) (I) (ii) (J) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (I) (ii) (J)

(2) What are the characteristics of the below nuclear equation?

$$^{A}_{Z}\text{X(mother nucleus)} \rightarrow {}_{Z+1}^{A}\text{Y(daughter nucleus)} + {}^{0}_{-1}\text{e}$$

(a) (III) (ii) (J) **(b)** (II) (iii) (M)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) What are the characteristics of the below nuclear equation?

$$^{A}_{Z}\text{X*(mother nucleus)} \rightarrow {}^{A}_{Z}\text{Y(daughter nucleus)} + hf$$

(a) (III) (iv) (L) **(b)** (I) (i) (K)
(c) (III) (iii) (J) **(d)** (I) (ii) (M)

108. There are different types of radioactive series that radioactive decay follows. In the given table, Column I shows the end or stable nuclei which any one of the radioactive series produces, Column II shows the mass number of different types of radioactive series and Column III shows different values of *n* for different radioactive series.

Column I	Column II	Column III
(I) Stable end product $^{208}_{82}$Pb	**(i)** Mass number $4n + 1$	**(J)** $n = 51$
(II) Stable end product $^{209}_{83}$Bi	**(ii)** Mass number $4n + 3$	**(K)** $n = 52$
(III) Stable end product $^{206}_{82}$Pb	**(iii)** Mass number $4n + 2$	**(L)** $n = 50$
(IV) Stable end product $_{82}$Pb $^{207}_{82}$Pb	**(iv)** Mass number $4n$	**(M)** $n = 53$

(1) Determine the characteristics of thorium.

(a) (III) (ii) (L) **(b)** (I) (iv) (K)
(c) (II) (i) (M) **(d)** (II) (i) (K)

(2) Determine the characteristics of neptunium.

(a) (I) (iii) (K) **(b)** (IV) (iii) (J)
(c) (I) (ii) (M) **(d)** (II) (i) (K)

(3) Determine the characteristics of actinium.

(a) (III) (i) (K) **(b)** (I) (i) (L)
(c) (IV) (ii) (J) **(d)** (I) (iv) (M)

109. Many side particles are produced when radioactive decay takes place. In the given table, Column I shows the charge of the different side particles, Column II shows the mass and Column III shows the ionization power of these side particles.

Column I	Column II	Column III
(I) Charge: $+2e$	**(i)** Mass: $4m_p$	**(J)** Ionization power 100
(II) Charge: $-e$	**(ii)** Mass: m_p	**(K)** Ionization power 10000
(III) Charge: $-2e$	**(iii)** Massless	**(L)** Ionization power 1
(IV) Charge: 0	**(iv)** Mass: m_e	**(M)** Ionization power 1000

(1) What are the characteristics of α- particles?

(a) (I) (ii) (J) **(b)** (IV) (i) (M)
(c) (II) (iv) (K) **(d)** (I) (i) (K)

(2) What the characteristics of β- particles?

(a) (III) (ii) (J) **(b)** (II) (iv) (J)
(c) (II) (iii) (K) **(d)** (I) (i) (M)

(3) What are the characteristics of γ-particles?

(a) (IV) (iii) (L) **(b)** (I) (i) (K)
(c) (III) (iii) (J) **(d)** (I) (ii) (M)

Integer Type

110. A neutron with kinetic energy $K = 10$ MeV activates an endoergic nuclear reaction $n + {}^{12}_{6}\text{C} \rightarrow {}^{9}_{4}\text{Be} + {}^{4}_{2}\text{He}$. Initially ${}^{12}_{6}\text{C}$ was at rest. The threshold energy of this reaction is 6.5 MeV. Find the kinetic energy of the α-particle (in MeV) going at right angle to the direction of the incoming neutron.

111. Two radioactive nuclei X and Y initially contain equal number of atoms. The half-life is 1 h and 2 h, respectively. Calculate the ratio of rates of their disintegration after 2 h.

112. There are two radioactive substances A and B. Decay constant of B is two times of A. Initially both have equal number of nuclei. After n half-lives of A, rate of disintegration of both are equal. What is the value of n?

113. In the uranium radioactive series the initial nucleus is ${}^{238}_{92}\text{U}$ and the final nucleus is ${}^{206}_{82}\text{Pb}$. When the uranium nucleus decays to lead, what is the number of emitted α-particles?

114. A freshly prepared sample of a radioisotope of half-life 1386 s has activity 10^3 disintegrations per second. Given that $\ln 2 = 0.693$, the fraction of the initial number of nuclei (expressed in nearest integer percentage) that will decay in the first 80 s after preparation of the sample is ______.

115. A nuclear power plant supplying electrical power to a village uses a radioactive material of half-life T years as the fuel. The amount of fuel at the beginning is such that the total power requirement of the village is 12.5% of the electrical power available from the plant at that time. If the plant is able to meet the total power needs of the village for a maximum period of nT years, then the value of n is _____.

116. ${}^{131}\text{I}$ is an isotope of Iodine that β-decays to an isotope of xenon with a half-life of 8 days. A small amount of a serum labeled with ${}^{131}\text{I}$ is injected into the blood of a person. The activity of the amount of ${}^{131}\text{I}$ injected was 2.4×10^5 Becquerel (Bq). It is known that the injected serum will get distributed uniformly in the blood stream in less than half an hour. After 11.5 hours, 2.5 ml of blood is drawn from the person's body, and gives an activity of 115 Bq. The total volume of blood in the person's body, in liters is approximately (you may use $e^x \approx 1 + x$ for $|x| \ll 1$ and $\ln 2 \approx 0.7$).

ANSWER KEY

Checkpoints

1. ^{90}As and ^{158}Nd
2. A little more than 75 Bq (elapsed time is little less than three half-lives)
3. ^{206}Pb
4. c and d
5. (e)

Problems

1. 4.269 MeV
2. 4.1×10^{19}
3. (a) 25.4 MeV; (b) 12.80 MeV; (c) 25.0 MeV
4. 73.1×10^{-3} g
5. 2.40 g
6. 1.02 mg
7. 49×10^{13} Bq
8. (a) 1.2×10^{-17}; (b) 3.31×10^{-3983}
9. (a) 4.25 MeV; (b) –24.1 MeV; (c) 28.3 MeV
10. 1.12×10^{11} y
11. (a) 6.6×10^{6} eV; (b) no
12. 0.783 MeV
13. 0.49
14. (b) 0.961 MeV
15. (a) 2.03×10^{20}; (b) $279 \times 10^{9}\ \text{s}^{-1}$
16. 3.09×10^{10} K
17. (a) $7.5 \times 10^{16}\ \text{s}^{-1}$; (b) $4.9 \times 10^{16}\ \text{s}^{-1}$
18. (a) 31.8 MeV; (b) 5.98 MeV; (c) 86 MeV
20. 1.7 mg
21. (a) ^{18}O, ^{60}Ni, ^{92}Mo, ^{144}Sm, and ^{207}Pb each have a filled shell for either the protons or the neutrons; (b) ^{40}K, ^{91}Zr, ^{121}Sb, and ^{143}Nd; (c) ^{13}C, ^{40}K, ^{49}Ti, ^{205}Tl, and ^{207}Pb
22. 9.0×10^{8} Bq
23. (a) –9.50 MeV; (b) 4.66 MeV; (c) –1.30 MeV
24. (a) yttrium; (b) iodine; (c) 50 neutrons; (d) 74 neutrons; (e) 19
25. 4.28×10^{9} y

26. (a) 3.2×10^{12} Bq; (b) 86 Ci
27. (a) 158 Bq; (b) 4.27×10^{-9} Ci
28. 1.21 MeV
29. 8.23 MeV
30. (a) 0.007825 u; (b) + 7.290 MeV/c^2; (c) 0.008665 u; (d) + 8.071 MeV/c^2; (e) – 0.09780 u; (f) – 91.10 MeV/c^2
31. 1.3×10^{-13} m or 130 fm
32. 1.6×10^{25} MeV
33. (a) 8.88×10^{10} s^{-1}; (b) 1.19×10^{15}; (c) 1.11×10^{-7} g
34. (a) 64.2 h; (b) 0.125; (c) 0.0749
35. (b) 7.92 MeV
36. 0.265 g
38. 1.81×10^{-14} m = 18.1 fm
39. 5.3×10^{22} nuclei
40. 209 d
41. (a) 6.30×10^{18}; (b) 5.71×10^{6} Bq
42. (a) 0.390 MeV; (b) 4.61MeV
43. (a) 19.8 MeV; (b) 6.26 MeV; (c) 2.23 MeV; (d) 28.3 MeV; (e) 7.07 MeV; (f) No, the answers do not match
44. 28.3×10^{6} eV
45. 1.0087 u
46. 7.31 MeV per nucleon
47. (a) 6; (b)8
48. 170 keV
49. (a) 10; (b) 226 MeV
50. 8.33×10^{-13} m ≈ 1 pm
51. 1.7×10^{9} y
52. 5.49 MeV
53. 1.41 MeV
54. (a) 6.4×10^{14} J; (b) 8.2×10^{13} J
55. 2.74×10^{-14} J
56. 14.4 kW
57. 7.27 MeV
58. 4.54×10^{26} MeV
59. (a) 4.3×10^{9} kg/s; (b) 3.1×10^{-4}
62. 4.8 MeV
63. – 23.0 MeV
64. 181 MeV
65. (a) ^{95}Sr; (b)^{95}Y; (c)^{134}Te; (d) 3
66. (a) 16 fissions per day; (b) 4.3×10^{8}
67. ^{14}C is the decay product of ^{14}N
68. 0.58
69. 200 Bq
70. 11.44 MeV

Practice Questions

Single Correct Choice Type

1. (d)
2. (c)
3. (a)
4. (c)
5. (c)
6. (b)
7. (d)
8. (b)
9. (c)
10. (a)
11. (b)
12. (b)
13. (a)
14. (a)
15. (a)
16. (d)
17. (a)
18. (c)
19. (b)
20. (c)
21. (d)
22. (d)
23. (c)
24. (b)
25. (d)
26. (b)
27. (d)
28. (d)
29. (d)
30. (c)
31. (b)
32. (b)
33. (a)
34. (d)
35. (c)
36. (b)
37. (b)
38. (c)
39. (b)
40. (a)
41. (b)
42. (c)
43. (c)
44. (c)
45. (c)
46. (c)
47. (c)
48. (a)
49. (d)
50. (d)
51. (b)
52. (b)
53. (a)
54. (b)
55. (d)
56. (b)
57. (b)
58. (a)
59. (b)
60. (c)
61. (b)
62. (d)
63. (b)
64. (b)
65. (d)
66. (a)
67. (b)
68. (c)
69. (d)

More than One Correct Choice Type

70. (b), (d)
71. (c), (d)
72. (b), (c), (d)
73. (a), (c)
74. (a), (b), (d)
75. (a), (b), (d)
76. (b), (d)
77. (a), (b), (c)
78. (a), (d)
79. (c), (d)
80. (a), (c)
81. (a), (c), (d)
82. (a), (d)
83. (a), (c), (d)
84. (a), (b), (c)
85. (a), (b), (c)
86. (a), (b), (c)
87. (c), (d)
88. (a), (b), (c)
89. (a), (b)
90. (a), (c)
91. (a), (d)

Linked Comprehension

92. (d)
93. (d)
94. (a)
95. (b)
96. (c)
97. (b)
98. (a)
99. (b)
100. (d)
101. (a)
102. (d)
103. (d)

Matrix-Match

104. (a) → (s),(r); (b) → (q); (c) → (q); (d) → (p)

105. (a) → (r); (b) → (s); (c) → (p); (d) → (q)

106. (1) → (b); (2) → (d); (3) → (c)

107. (1) → (d); (2) → (b); (3) → (a)

108. (1) → (b); (2) → (d); (3) → (c)

109. (1) → (d); (2) → (b); (3) → (a)

Integer Type

110. 2

111. 1

112. 1

113. 8

114. 4

115. 3

116. 5

Unacademyplusdiscounts

APPENDIX A

The International System of Units (SI)*

Table 1 The SI Base Units**

Quantity	Name	Symbol	Definition
length	meter	m	"… the length of the path traveled by light in vacuum in 1/299,792,458 of a second." (1983)
mass	kilogram	kg	"… this prototype [a certain platinum–iridium cylinder] shall henceforth be considered to be the unit of mass." (1889)
time	second	s	"… the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom." (1967)
electric current	ampere	A	"… that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length." (1946)
thermodynamic temperature	kelvin	K	"… the fraction 1/273.16 of the thermodynamic temperature of the triple point of water." (1967)
amount of substance	mole	mol	"… the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12." (1971)
luminous intensity	candela	cd	"… the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian." (1979)

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*Adapted from "The International System of Units (SI)," National Bureau of Standards Special Publication 330, 1972 edition.The definitions above were adopted by the General Conference of Weights and Measures, an international body, on the dates shown. In this book we do not use the candela.

**In November 2018, the 26th General Conference on Weights and Measures has adopted a resolution to redefine four of the seven base units—the kilogram, kelvin, mole and ampere.

Table 2 Some SI Derived Units

Quantity	Name of Unit	Symbol	
area	square meter	m^2	
volume	cubic meter	m^3	
frequency	hertz	Hz	s^{-1}
mass density (density)	kilogram per cubic meter	kg/m^3	
speed, velocity	meter per second	m/s	
angular velocity	radian per second	rad/s	
acceleration	meter per second per second	m/s^2	
angular acceleration	radian per second per second	rad/s^2	
force	newton	N	$kg \cdot m/s^2$
pressure	pascal	Pa	N/m^2
work, energy, quantity of heat	joule	J	$N \cdot m$
power	watt	W	J/s
quantity of electric charge	coulomb	C	$A \cdot s$
potential difference, electromotive force	volt	V	W/A
electric field strength	volt per meter (or newton per coulomb)	V/m	N/C
electric resistance	ohm	Ω	V/A
capacitance	farad	F	$A \cdot s/V$
magnetic flux	weber	Wb	$V \cdot s$
inductance	henry	H	$V \cdot s/A$
magnetic flux density	tesla	T	Wb/m^2
magnetic field strength	ampere per meter	A/m	
entropy	joule per kelvin	J/K	
specific heat	joule per kilogram kelvin	$J/(kg \cdot K)$	
thermal conductivity	watt per meter kelvin	$W/(m \cdot K)$	
radiant intensity	watt per steradian	W/sr	

Table 3 The SI Supplementary Units

Quantity	Name of Unit	Symbol
plane angle	radian	rad
solid angle	steradian	sr

APPENDIX

B

Some Fundamental Constants of Physics*

			Best (1998) Value	
Constant	**Symbol**	**Computational Value**	**Value**[a]	**Uncertainty**[b]
Speed of light in a vacuum	c	3.00×10^{8} m/s	2.997 924 58	exact
Elementary charge	e	1.60×10^{-19} C	1.602 176 487	0.025
Gravitational constant	G	6.67×10^{-11} $\text{m}^3/\text{s}^2 \cdot \text{kg}$	6.674 28	100
Universal gas constant	R	8.31 $\text{J/mol} \cdot \text{K}$	8.314 472	1.7
Avogadro constant	N_A	6.02×10^{23} mol^{-1}	6.022 141 79	0.050
Boltzmann constant	k	1.38×10^{-23} J/K	1.380 650 4	1.7
Stefan–Boltzmann constant	σ	5.67×10^{-8} $\text{W/m}^2 \cdot \text{K}^4$	5.670 400	7.0
Molar volume of ideal gas at STP[d]	V_m	2.27×10^{-2} m^3/mol	2.271 098 1	1.7
Permittivity constant	ε_0	8.85×10^{-12} F/m	8.854 187 817 62	exact
Permeability constant	μ_0	1.26×10^{-6} H/m	1.256 637 061 43	exact
Planck constant	h	6.63×10^{-34} $\text{J} \cdot \text{s}$	6.626 068 96	0.050
Electron mass[c]	m_e	9.11×10^{-31} kg	9.109 382 15	0.050
		5.49×10^{-4} u	5.485 799 094 3	4.2×10^{-4}
Proton mass[c]	m_p	1.67×10^{-27} kg	1.672 621 637	0.050
		1.0073 u	1.007 276 466 77	1.0×10^{-4}
Ratio of proton mass to electron mass	m_p/m_e	1840	1836.152 672 47	4.3×10^{-4}
Electron charge-to-mass ratio	e/m_e	1.76×10^{11} C/kg	1.758 820 150	0.025
Neutron mass[c]	m_n	1.68×10^{-27} kg	1.674 927 211	0.050
		1.0087 u	1.008 664 915 97	4.3×10^{-4}
Hydrogen atom mass[c]	$m_{^1\text{H}}$	1.0078 u	1.007 825 031 6	0.0005
Deuterium atom mass[c]	$m_{^2\text{H}}$	2.0136 u	2.013 553 212 724	3.9×10^{-5}
Helium atom mass[c]	$m_{^4\text{He}}$	4.0026 u	4.002 603 2	0.067
Muon mass	m_μ	1.88×10^{-28} kg	1.883 531 30	0.056

(Contd.)

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*The values in this table were selected from the 1998 CODATA recommended values (www.physics.nist.gov).

			Best (1998) Value	
Constant	**Symbol**	**Computational Value**	**Value[a]**	**Uncertainty[b]**
Electron magnetic moment	μ_e	9.28×10^{-24} J/T	9.284 763 77	0.025
Proton magnetic moment	μ_p	1.41×10^{-26} J/T	1.410 606 662	0.026
Bohr magneton	μ_B	9.27×10^{-24} J/T	9.274 009 15	0.025
Nuclear magneton	μ_N	5.05×10^{-27} J/T	5.050 783 24	0.025
Bohr radius	a	5.29×10^{-11} m	5.291 772 085 9	6.8×10^{-4}
Rydberg constant	R	1.10×10^{7} m^{-1}	1.097 373 156 852 7	6.6×10^{-6}
Electron Compton wavelength	λ_C	2.43×10^{-12} m	2.426 310 217 5	0.0014

[a] Values given in this column should be given the same unit and power of 10 as the computational value.

[b] Parts per million.

[c] Masses given in u are in unified atomic mass units, where 1 u = $1.660\,538\,782 \times 10^{-27}$ kg.

[d] STP means standard temperature and pressure: 0°C and 1.0 atm (0.1 MPa).

APPENDIX C

Some Astronomical Data

Some Distances from Earth

To the Moon*	3.82×10^{8} m	To the center of our galaxy	2.2×10^{20} m
To the Sun*	1.50×10^{11} m	To the Andromeda Galaxy	2.1×10^{22} m
To the nearest star (Proxima Centauri)	4.04×10^{16} m	To the edge of the observable universe	$\sim 10^{26}$ m

*Mean distance.

The Sun, Earth, and the Moon

Property	Unit	Sun	Earth	Moon
Mass	kg	1.99×10^{30}	5.98×10^{24}	7.36×10^{22}
Mean radius	m	6.96×10^{8}	6.37×10^{6}	1.74×10^{6}
Mean density	kg/m^3	1410	5520	3340
Free-fall acceleration at the surface	m/s^2	274	9.81	1.67
Escape velocity	km/s	618	11.2	2.38
Period of rotation[a]	—	37 d at poles[b] 26 d at equator[b]	23 h 56 min	27.3 d
Radiation power[c]	W	3.90×10^{26}		

[a] Measured with respect to the distant stars.
[b] The Sun, a ball of gas, does not rotate as a rigid body.
[c] Just outside Earth's atmosphere solar energy is received, assuming normal incidence, at the rate of 1340 W/m^2.

Some Properties of the Planets

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto[d]
Mean distance from Sun, 10^6 km	57.9	108	150	228	778	1430	2870	4500	5900
Period of revolution, y	0.241	0.615	1.00	1.88	11.9	29.5	84.0	165	248
Period of rotation,[a] d	58.7	–243[b]	0.997	1.03	0.409	0.426	–0.451[b]	0.658	6.39
Orbital speed, km/s	47.9	35.0	29.8	24.1	13.1	9.64	6.81	5.43	4.74
Inclination of axis to orbit	<28°	≈3°	23.4°	25.0°	3.08°	26.7°	97.9°	29.6°	57.5°
Inclination of orbit to Earth's orbit	7.00°	3.39°		1.85°	1.30°	2.49°	0.77°	1.77°	17.2°
Eccentricity of orbit	0.206	0.0068	0.0167	0.0934	0.0485	0.0556	0.0472	0.0086	0.250

(Contd.)

Some Properties of the Planets (Contd.)

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto[d]
Equatorial diameter, km	4880	12 100	12 800	6790	143 000	120 000	51 800	49 500	2300
Mass (Earth = 1)	0.0558	0.815	1.000	0.107	318	95.1	14.5	17.2	0.002
Density (water = 1)	5.60	5.20	5.52	3.95	1.31	0.704	1.21	1.67	2.03
Surface value of g,[c] m/s^2	3.78	8.60	9.78	3.72	22.9	9.05	7.77	11.0	0.5
Escape velocity,[c] km/s	4.3	10.3	11.2	5.0	59.5	35.6	21.2	23.6	1.3
Known satellites	0	0	1	2	67 + ring	62 + rings	27 + rings	13 + rings	4

[a] Measured with respect to the distant stars.

[b] Venus and Uranus rotate opposite their orbital motion.

[c] Gravitational acceleration measured at the planet's equator.

[d] Pluto is now classified as a dwarf planet.

APPENDIX

D

Conversion Factors

Conversion factors may be read directly from these tables. For example, 1 degree = 2.778×10^{-3} revolutions, so $16.7° = 16.7 \times 2.778 \times 10^{-3}$ rev. The SI units are fully capitalized. Adapted in part from G. Shortley and D. Williams, *Elements of Physics*, 1971, Prentice-Hall, Englewood Cliffs, NJ.

Plane Angle

°	′	″	RADIAN	rev
1 degree = 1	60	3600	1.745×10^{-2}	2.778×10^{-3}
1 minute = 1.667×10^{-2}	1	60	2.909×10^{-4}	4.630×10^{-5}
1 second = 2.778×10^{-4}	1.667×10^{-2}	1	4.848×10^{-6}	7.716×10^{-7}
1 RADIAN = 57.30	3438	2.063×10^{5}	1	0.1592
1 revolution = 360	2.16×10^{4}	1.296×10^{6}	6.283	1

Solid Angle

1 sphere = 4π steradians = 12.57 steradians

Length

cm	METER	km	in.	ft	mi
1 centimeter = 1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 METER = 100	1	10^{-3}	39.37	3.281	6.214×10^{-4}
1 kilometer = 10^{5}	1000	1	3.937×10^{4}	3281	0.6214
1 inch = 2.540	2.540×10^{-2}	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot = 30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile = 1.609×10^{5}	1609	1.609	6.336×10^{4}	5280	1

1 angström = 10^{-10} m
1 nautical mile = 1852 m
= 1.151 miles = 6076 ft
1 fermi = 10^{-15} m
1 light-year = 9.461×10^{12} km
1 parsec = 3.084×10^{13} km
1 fathom = 6 ft
1 Bohr radius = 5.292×10^{-11} m
1 yard = 3 ft
1 rod = 16.5 ft
1 mil = 10^{-3} in.
1 nm = 10^{-9} m

Area

METER^2	cm^2	ft^2	in.^{2}
1 SQUARE METER = 1	10^4	10.76	1550
1 square centimeter = 10^{-4}	1	1.076×10^{-3}	0.1550
1 square foot = 9.290×10^{-2}	929.0	1	144
1 square inch = 6.452×10^{-4}	6.452	6.944×10^{-3}	1

1 square mile = 2.788×10^7 ft^2 = 640 acres

1 barn = 10^{-28} m^2

1 acre = 43 560 ft^2

1 hectare = 10^4 m^2 = 2.471 acres

Volume

METER^3	cm^3	L	ft^3	in.^{3}
1 CUBIC METER = 1	10^6	1000	35.31	6.102×10^4
1 cubic centimeter = 10^{-6}	1	1.000×10^{-3}	3.531×10^{-5}	6.102×10^{-2}
1 liter = 1.000×10^{-3}	1000	1	3.531×10^{-2}	61.02
1 cubic foot = 2.832×10^{-2}	2.832×10^4	28.32	1	1728
1 cubic inch = 1.639×10^{-5}	16.39	1.639×10^{-2}	5.787×10^{-4}	1

1 U.S. fluid gallon = 4 U.S. fluid quarts = 8 U.S. pints = 128 U.S. fluid ounces = 231 in.^{3}

1 British imperial gallon = 277.4 in.^{3} = 1.201 U.S. fluid gallons

Mass

Quantities in the colored areas are not mass units but are often used as such. For example, when we write 1 kg "=" 2.205 lb, this means that a kilogram is a *mass* that *weighs* 2.205 pounds at a location where *g* has the standard value of 9.80665 m/s^2.

g	KILOGRAM	slug	u	oz	lb	ton
1 gram = 1	0.001	6.852×10^{-5}	6.022×10^{23}	3.527×10^{-2}	2.205×10^{-3}	1.102×10^{-6}
1 KILOGRAM = 1000	1	6.852×10^{-2}	6.022×10^{26}	35.27	2.205	1.102×10^{-3}
1 slug = 1.459×10^4	14.59	1	8.786×10^{27}	514.8	32.17	1.609×10^{-2}
1 atomic mass unit = 1.661×10^{-24}	1.661×10^{-27}	1.138×10^{-28}	1	5.857×10^{-26}	3.662×10^{-27}	1.830×10^{-30}
1 ounce = 28.35	2.835×10^{-2}	1.943×10^{-3}	1.718×10^{25}	1	6.250×10^{-2}	3.125×10^{-5}
1 pound = 453.6	0.4536	3.108×10^{-2}	2.732×10^{26}	16	1	0.0005
1 ton = 9.072×10^5	907.2	62.16	5.463×10^{29}	3.2×10^4	2000	1

1 metric ton = 1000 kg

Density

Quantities in the colored areas are weight densities and, as such, are dimensionally different from mass densities. See the note for the mass table.

slug/ft^3	KILOGRAM/METER^3	g/cm^3	lb/ft^3	lb/in.^{3}
1 slug per foot^3 = 1	515.4	0.5154	32.17	1.862×10^{-2}
1 KILOGRAM per METER^3 = 1.940×10^{-3}	1	0.001	6.243×10^{-2}	3.613×10^{-5}
1 gram per centimeter^3 = 1.940	1000	1	62.43	3.613×10^{-2}
1 pound per foot^3 = 3.108×10^{-2}	16.02	16.02×10^{-2}	1	5.787×10^{-4}
1 pound per inch^3 = 53.71	2.768×10^4	27.68	1728	1

Time

y	d	h	min	SECOND
1 year = 1	365.25	8.766×10^{3}	5.259×10^{5}	3.156×10^{7}
1 day = 2.738×10^{-3}	1	24	1440	8.640×10^{4}
1 hour = 1.141×10^{-4}	4.167×10^{-2}	1	60	3600
1 minute = 1.901×10^{-6}	6.944×10^{-4}	1.667×10^{-2}	1	60
1 SECOND = 3.169×10^{-8}	1.157×10^{-5}	2.778×10^{-4}	1.667×10^{-2}	1

Speed

ft/s	km/h	METER/SECOND	mi/h	cm/s
1 foot per second = 1	1.097	0.3048	0.6818	30.48
1 kilometer per hour = 0.9113	1	0.2778	0.6214	27.78
1 METER per SECOND = 3.281	3.6	1	2.237	100
1 mile per hour = 1.467	1.609	0.4470	1	44.70
1 centimeter per second = 3.281×10^{-2}	3.6×10^{-2}	0.01	2.237×10^{-2}	1

1 knot = 1 nautical mi/h = 1.688 ft/s

1 mi/min = 88.00 ft/s = 60.00 mi/h

Force

Force units in the colored areas are now little used. To clarify: 1 gram-force (= 1 gf) is the force of gravity that would act on an object whose mass is 1 gram at a location where g has the standard value of 9.80665 m/s^2.

dyne	NEWTON	lb	pdl	gf	kgf
1 dyne = 1	10^{-5}	2.248×10^{-6}	7.233×10^{-5}	1.020×10^{-3}	1.020×10^{-6}
1 NEWTON = 10^{5}	1	0.2248	7.233	102.0	0.1020
1 pound = 4.448×10^{5}	4.448	1	32.17	453.6	0.4536
1 poundal = 1.383×10^{4}	0.1383	3.108×10^{-2}	1	14.10	1.410×10^{2}
1 gram-force = 980.7	9.807×10^{-3}	2.205×10^{-3}	7.093×10^{-2}	1	0.001
1 kilogram-force = 9.807×10^{5}	9.807	2.205	70.93	1000	1

1 ton = 2000 lb

Pressure

atm	dyne/cm^2	inch of water	cm Hg	PASCAL	lb/in.^{2}	lb/ft^2
1 atmosphere = 1	1.013×10^{6}	406.8	76	1.013×10^{5}	14.70	2116
1 dyne per centimeter^2 = 9.869×10^{-7}	1	4.015×10^{-4}	7.501×10^{-5}	0.1	1.405×10^{-5}	2.089×10^{-3}
1 inch of water[a] at 4°C = 2.458×10^{-3}	2491	1	0.1868	249.1	3.613×10^{-2}	5.202
1 centimeter of mercury[a] at 0°C = 1.316×10^{-2}	1.333×10^{4}	5.353	1	1333	0.1934	27.85
1 PASCAL = 9.869×10^{-6}	10	4.015×10^{-3}	7.501×10^{-4}	1	1.450×10^{-4}	2.089×10^{-2}
1 pound per inch^2 = 6.805×10^{-2}	6.895×10^{4}	27.68	5.171	6.895×10^{3}	1	144
1 pound per foot^2 = 4.725×10^{-4}	478.8	0.1922	3.591×10^{-2}	47.88	6.944×10^{-3}	1

[a] Where the acceleration of gravity has the standard value of 9.80665 m/s^2.

1 bar = 10^{6} dyne/cm^2 = 0.1 MPa

1 millibar = 10^{3} dyne/cm^2 = 10^{2} Pa

1 torr = 1 mm Hg

Energy, Work, Heat

Quantities in the colored areas are not energy units but are included for convenience. They arise from the relativistic mass–energy equivalence formula $E = mc^2$ and represent the energy released if a kilogram or unified atomic mass unit (u) is completely converted to energy (bottom two rows) or the mass that would be completely converted to one unit of energy (rightmost two columns).

Btu	erg	ft · lb	hp · h	JOULE	cal	kW · h	eV	MeV	kg	u
1 British thermal unit = 1	1.055×10^{10}	777.9	3.929×10^{-4}	1055	252.0	2.930×10^{-4}	6.585×10^{21}	6.585×10^{15}	1.174×10^{-14}	7.070×10^{12}
1 erg = 9.481×10^{-11}	1	7.376×10^{-8}	3.725×10^{-14}	10^{-7}	2.389×10^{-8}	2.778×10^{-14}	6.242×10^{11}	6.242×10^{5}	1.113×10^{-24}	670.2
1 foot-pound = 1.285×10^{-3}	1.356×10^{7}	1	5.051×10^{-7}	1.356	0.3238	3.766×10^{-7}	8.464×10^{18}	8.464×10^{12}	1.509×10^{-17}	9.037×10^{9}
1 horsepower-hour = 2545	2.685×10^{13}	1.980×10^{6}	1	2.685×10^{6}	6.413×10^{5}	0.7457	1.676×10^{25}	1.676×10^{19}	2.988×10^{-11}	1.799×10^{16}
1 JOULE = 9.481×10^{-4}	10^{7}	0.7376	3.725×10^{-7}	1	0.2389	2.778×10^{-7}	6.242×10^{18}	6.242×10^{12}	1.113×10^{-17}	6.702×10^{9}
1 calorie = 3.968×10^{-3}	4.1868×10^{7}	3.088	1.560×10^{-6}	4.1868	1	1.163×10^{-6}	2.613×10^{19}	2.613×10^{13}	4.660×10^{-17}	2.806×10^{10}
1 kilowatt-hour = 3413	3.600×10^{13}	2.655×10^{6}	1.341	3.600×10^{6}	8.600×10^{5}	1	2.247×10^{25}	2.247×10^{19}	4.007×10^{-11}	2.413×10^{16}
1 electron-volt = 1.519×10^{-22}	1.602×10^{-12}	1.182×10^{-19}	5.967×10^{-26}	1.602×10^{-19}	3.827×10^{-20}	4.450×10^{-26}	1	10^{-6}	1.783×10^{-36}	1.074×10^{-9}
1 million electron-volts = 1.519×10^{-16}	1.602×10^{-6}	1.182×10^{-13}	5.967×10^{-20}	1.602×10^{-13}	3.827×10^{-14}	4.450×10^{-20}	10^{-6}	1	1.783×10^{-30}	1.074×10^{-3}
1 kilogram = 8.521×10^{13}	8.987×10^{23}	6.629×10^{16}	3.348×10^{10}	8.987×10^{16}	2.146×10^{16}	2.497×10^{10}	5.610×10^{35}	5.610×10^{29}	1	6.022×10^{26}
1 unified atomic mass unit = 1.415×10^{-13}	1.492×10^{-3}	1.101×10^{-10}	5.559×10^{-17}	1.492×10^{-10}	3.564×10^{-11}	4.146×10^{-17}	9.320×10^{8}	932.0	1.661×10^{-27}	1

Power

Btu/h	ft · lb/s	hp	cal/s	kW	WATT
1 British thermal unit per hour = 1	0.2161	3.929×10^{-4}	6.998×10^{-2}	2.930×10^{-4}	0.2930
1 foot-pound per second = 4.628	1	1.818×10^{-3}	0.3239	1.356×10^{-3}	1.356
1 horsepower = 2545	550	1	178.1	0.7457	745.7
1 calorie per second = 14.29	3.088	5.615×10^{-3}	1	4.186×10^{-3}	4.186
1 kilowatt = 3413	737.6	1.341	238.9	1	1000
1 WATT = 3.413	0.7376	1.341×10^{-3}	0.2389	0.001	1

Magnetic Field

gauss	TESLA	milligauss
1 gauss = 1	10^{-4}	1000
1 TESLA = 10^{4}	1	10^{7}
1 milligauss = 0.001	10^{-7}	1

1 tesla = 1 weber/meter^2

Magnetic Flux

maxwell	WEBER
1 maxwell = 1	10^{-8}
1 WEBER = 10^{8}	1

APPENDIX

E

Mathematical Formulas

Geometry

Circle of radius r: circumference = $2\pi r$; area = πr^2.
Sphere of radius r: area = $4\pi r^2$; volume = $4/3\pi r^3$.
Right circular cylinder of radius r and height h:
area = $2\pi r^2 + 2\pi rh$; volume = $\pi r^2 h$.
Triangle of base a and altitude h: area = $1/2ah$.

Quadratic Formula

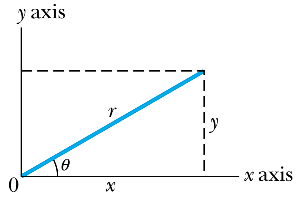
If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Trigonometric Functions of Angle θ

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

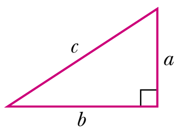
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

$$\sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{r}{y}$$



Pythagorean Theorem

In this right triangle,
$a^2 + b^2 = c^2$

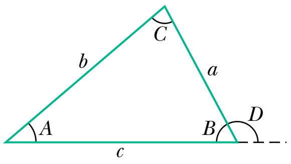


Triangles

Angles are A, B, C
Opposite sides are a, b, c
Angles $A + B + C = 180°$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$c^2 = a^2 + b^2 - 2ab\cos C$
Exterior angle $D = A + C$



Mathematical Signs and Symbols

$=$ equals
$\approx$ equals approximately
$\sim$ is the order of magnitude of
$\neq$ is not equal to
$\equiv$ is identical to, is defined as
$>$ is greater than ($\gg$ is much greater than)
$<$ is less than ($\ll$ is much less than)
$\geq$ is greater than or equal to (or, is no less than)
$\leq$ is less than or equal to (or, is no more than)
$\pm$ plus or minus
$\propto$ is proportional to
Σ the sum of
x_{avg} the average value of x

Trigonometric Identities

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$
$\sin\theta/\cos\theta = \tan\theta$
$\sin^2\theta + \cos^2\theta = 1$
$\sec^2\theta - \tan^2\theta = 1$
$\csc^2\theta - \cot^2\theta = 1$
$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha \pm \beta)\cos\frac{1}{2}(\alpha \pm \beta)$$

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$$

$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta)$$

Binomial Theorem

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots \quad (x^2 < 1)$$

Exponential Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Logarithmic Expansion

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \quad (|x| < 1)$$

Trigonometric Expansions (θ in radians)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

Cramer's Rule

Two simultaneous equations in unknowns x and y,

$$a_1x + b_1y = c_1 \quad \text{and} \quad a_2x + b_2y = c_2,$$

have the solutions

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Products of Vectors

Let $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ be unit vectors in the x, y, and z directions. Then

$$\hat{\mathrm{i}}\cdot\hat{\mathrm{i}} = \hat{\mathrm{j}}\cdot\hat{\mathrm{j}} = \hat{\mathrm{k}}\cdot\hat{\mathrm{k}} = 1, \quad \hat{\mathrm{i}}\cdot\hat{\mathrm{j}} = \hat{\mathrm{j}}\cdot\hat{\mathrm{k}} = \hat{\mathrm{k}}\cdot\hat{\mathrm{i}} = 0,$$

$$\hat{\mathrm{i}}\times\hat{\mathrm{i}} = \hat{\mathrm{j}}\times\hat{\mathrm{j}} = \hat{\mathrm{k}}\times\hat{\mathrm{k}} = 0,$$

$$\hat{\mathrm{i}}\times\hat{\mathrm{j}} = \hat{\mathrm{k}}, \quad \hat{\mathrm{j}}\times\hat{\mathrm{k}} = \hat{\mathrm{i}}, \quad \hat{\mathrm{k}}\times\hat{\mathrm{i}} = \hat{\mathrm{j}}$$

Any vector $\vec{a}$ with components a_x, a_y, and a_z along the x, y, and z axes can be written as

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}.$$

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be arbitrary vectors with magnitudes a, b, and c. Then

$$\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$$

$$(s\vec{a})\times\vec{b} = \vec{a}\times(s\vec{b}) = s(\vec{a}\times\vec{b}) \quad (s = \text{a scalar}).$$

Let θ be the smaller of the two angles between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = a_xb_x + a_yb_y + a_zb_z = ab\cos\theta$$

$$\begin{aligned}
\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} &= \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= \hat{\mathrm{i}}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathrm{j}}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathrm{k}}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\
&= (a_yb_z - b_ya_z)\hat{\mathrm{i}} + (a_zb_x - b_za_x)\hat{\mathrm{j}} \\
&\quad + (a_xb_y - b_xa_y)\hat{\mathrm{k}} \\
|\vec{a}\times\vec{b}| &= ab\sin\theta
\end{aligned}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

Derivatives and Integrals

In what follows, the letters u and v stand for any functions of x, and a and m are constants. To each of the indefinite integrals should be added an arbitrary constant of integration. The *Handbook of Chemistry and Physics* (CRC Press Inc.) gives a more extensive tabulation.

1. $\frac{dx}{dx} = 1$
2. $\frac{d}{dx}(au) = a\frac{du}{dx}$
3. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$
4. $\frac{d}{dx}x^m = mx^{m-1}$
5. $\frac{d}{dx}\ln x = \frac{1}{x}$
6. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
7. $\frac{d}{dx}e^x = e^x$
8. $\frac{d}{dx}\sin x = \cos x$
9. $\frac{d}{dx}\cos x = -\sin x$
10. $\frac{d}{dx}\tan x = \sec^2 x$
11. $\frac{d}{dx}\cot x = -\csc^2 x$
12. $\frac{d}{dx}\sec x = \tan x \sec x$
13. $\frac{d}{dx}\csc x = -\cot x \csc x$
14. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$
15. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$
16. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$

1. $\int dx = x$
2. $\int au\,dx = a\int u\,dx$
3. $\int (u + v)\,dx = \int u\,dx + \int v\,dx$
4. $\int x^m dx = \frac{x^{m+1}}{m+1}\,(m \neq -1)$
5. $\int \frac{dx}{x} = \ln|x|$
6. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
7. $\int e^x\,dx = e^x$
8. $\int \sin x\,dx = -\cos x$
9. $\int \cos x\,dx = \sin x$
10. $\int \tan x\,dx = \ln|\sec x|$
11. $\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$
12. $\int e^{-ax}\,dx = -\frac{1}{a}e^{-ax}$
13. $\int xe^{-ax}\,dx = -\frac{1}{a^2}(ax + 1)\,e^{-ax}$
14. $\int x^2e^{-ax}\,dx = -\frac{1}{a^3}(a^2x^2 + 2ax + 2)e^{-ax}$
15. $\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$
16. $\int_0^\infty x^{2n}e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$
17. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})$
18. $\int \frac{x\,dx}{(x^2 + a^2)^{3/2}} = -\frac{1}{(x^2 + a^2)^{1/2}}$
19. $\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2(x^2 + a^2)^{1/2}}$
20. $\int_0^\infty x^{2n+1}\,e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}(a > 0)$
21. $\int \frac{x\,dx}{x + d} = x - d\ln(x + d)$

APPENDIX F

Properties of the Elements

All physical properties are for a pressure of 1 atm unless otherwise specified

Element	Symbol	Atomic Number Z	Molar Mass, g/mol	Density, g/cm^3 at 20°C	Melting Point, °C	Boiling Point, °C	Specific Heat, J/(g·°C) at 25°C
Actinium	Ac	89	(227)	10.06	1323	(3473)	0.092
Aluminum	Al	13	26.9815	2.699	660	2450	0.900
Americium	Am	95	(243)	13.67	1541	—	—
Antimony	Sb	51	121.75	6.691	630.5	1380	0.205
Argon	Ar	18	39.948	1.6626 × 10^{-3}	–189.4	–185.8	0.523
Arsenic	As	33	74.9216	5.78	817 (28 atm)	613	0.331
Astatine	At	85	(210)	—	(302)	—	—
Barium	Ba	56	137.34	3.594	729	1640	0.205
Berkelium	Bk	97	(247)	14.79	—	—	—
Beryllium	Be	4	9.0122	1.848	1287	2770	1.83
Bismuth	Bi	83	208.980	9.747	271.37	1560	0.122
Bohrium	Bh	107	262.12	—	—	—	—
Boron	B	5	10.811	2.34	2030	—	1.11
Bromine	Br	35	79.909	3.12 (liquid)	–7.2	58	0.293
Cadmium	Cd	48	112.40	8.65	321.03	765	0.226
Calcium	Ca	20	40.08	1.55	838	1440	0.624
Californium	Cf	98	(251)	—	—	—	—
Carbon	C	6	12.01115	2.26	3727	4830	0.691
Cerium	Ce	58	140.12	6.768	804	3470	0.188
Cesium	Cs	55	132.905	1.873	28.40	690	0.243
Chlorine	Cl	17	35.453	3.214 × 10^{-3} (0°C)	–101	–34.7	0.486
Chromium	Cr	24	51.996	7.19	1857	2665	0.448
Cobalt	Co	27	58.9332	8.85	1495	2900	0.423
Copernicium	Cn	112	(285)	—	—	—	—
Copper	Cu	29	63.54	8.96	1083.40	2595	0.385
Curium	Cm	96	(247)	13.3	—	—	—

(Contd.)

Element	Symbol	Atomic Number *Z*	Molar Mass, g/mol	Density, g/cm^3 at 20°C	Melting Point, °C	Boiling Point, °C	Specific Heat, J/(g·°C) at 25°C
Darmstadtium	Ds	110	(271)	—	—	—	—
Dubnium	Db	105	262.114	—	—	—	—
Dysprosium	Dy	66	162.50	8.55	1409	2330	0.172
Einsteinium	Es	99	(254)	—	—	—	—
Erbium	Er	68	167.26	9.15	1522	2630	0.167
Europium	Eu	63	151.96	5.243	817	1490	0.163
Fermium	Fm	100	(237)	—	—	—	—
Flerovium*	Fl	114	(289)	—	—	—	—
Fluorine	F	9	18.9984	1.696×10^{-3} (0°C)	–219.6	–188.2	0.753
Francium	Fr	87	(223)	—	(27)	—	—
Gadolinium	Gd	64	157.25	7.90	1312	2730	0.234
Gallium	Ga	31	69.72	5.907	29.75	2237	0.377
Germanium	Ge	32	72.59	5.323	937.25	2830	0.322
Gold	Au	79	196.967	19.32	1064.43	2970	0.131
Hafnium	Hf	72	178.49	13.31	2227	5400	0.144
Hassium	Hs	108	(265)	—	—	—	—
Helium	He	2	4.0026	0.1664×10^{-3}	–269.7	–268.9	5.23
Holmium	Ho	67	164.930	8.79	1470	2330	0.165
Hydrogen	H	1	1.00797	0.08375×10^{-3}	–259.19	–252.7	14.4
Indium	In	49	114.82	7.31	156.634	2000	0.233
Iodine	I	53	126.9044	4.93	113.7	183	0.218
Iridium	Ir	77	192.2	22.5	2447	(5300)	0.130
Iron	Fe	26	55.847	7.874	1536.5	3000	0.447
Krypton	Kr	36	83.80	3.488×10^{-3}	–157.37	–152	0.247
Lanthanum	La	57	138.91	6.189	920	3470	0.195
Lawrencium	Lr	103	(257)	—	—	—	—
Lead	Pb	82	207.19	11.35	327.45	1725	0.129
Lithium	Li	3	6.939	0.534	180.55	1300	3.58
Livermorium*	Lv	116	(293)	—	—	—	—
Lutetium	Lu	71	174.97	9.849	1663	1930	0.155
Magnesium	Mg	12	24.312	1.738	650	1107	1.03
Manganese	Mn	25	54.9380	7.44	1244	2150	0.481
Meitnerium	Mt	109	(266)	—	—	—	—
Mendelevium	Md	101	(256)	—	—	—	—
Mercury	Hg	80	200.59	13.55	–38.87	357	0.138
Molybdenum	Mo	42	95.94	10.22	2617	5560	0.251
Neodymium	Nd	60	144.24	7.007	1016	3180	0.188
Neon	Ne	10	20.183	0.8387×10^{-3}	–248.597	–246.0	1.03
Neptunium	Np	93	(237)	20.25	637	—	1.26
Nickel	Ni	28	58.71	8.902	1453	2730	0.444
Niobium	Nb	41	92.906	8.57	2468	4927	0.264
Nitrogen	N	7	14.0067	1.1649×10^{-3}	–210	–195.8	1.03
Nobelium	No	102	(255)	—	—	—	—

Element	Symbol	Atomic Number Z	Molar Mass, g/mol	Density, g/cm^3 at 20°C	Melting Point, °C	Boiling Point, °C	Specific Heat, J/(g·°C) at 25°C
Osmium	Os	76	190.2	22.59	3027	5500	0.130
Oxygen	O	8	15.9994	1.3318×10^{-3}	−218.80	−183.0	0.913
Palladium	Pd	46	106.4	12.02	1552	3980	0.243
Phosphorus	P	15	30.9738	1.83	44.25	280	0.741
Platinum	Pt	78	195.09	21.45	1769	4530	0.134
Plutonium	Pu	94	(244)	19.8	640	3235	0.130
Polonium	Po	84	(210)	9.32	254	—	—
Potassium	K	19	39.102	0.862	63.20	760	0.758
Praseodymium	Pr	59	140.907	6.773	931	3020	0.197
Promethium	Pm	61	(145)	7.22	(1027)	—	—
Protactinium	Pa	91	(231)	15.37 (estimated)	(1230)	—	—
Radium	Ra	88	(226)	5.0	700	—	—
Radon	Rn	86	(222)	9.96×10^{-3} (0°C)	(−71)	−61.8	0.092
Rhenium	Re	75	186.2	21.02	3180	5900	0.134
Rhodium	Rh	45	102.905	12.41	1963	4500	0.243
Roentgenium	Rg	111	(280)	—	—	—	—
Rubidium	Rb	37	85.47	1.532	39.49	688	0.364
Ruthenium	Ru	44	101.107	12.37	2250	4900	0.239
Rutherfordium	Rf	104	261.11	—	—	—	—
Samarium	Sm	62	150.35	7.52	1072	1630	0.197
Scandium	Sc	21	44.956	2.99	1539	2730	0.569
Seaborgium	Sg	106	263.118	—	—	—	—
Selenium	Se	34	78.96	4.79	221	685	0.318
Silicon	Si	14	28.086	2.33	1412	2680	0.712
Silver	Ag	47	107.870	10.49	960.8	2210	0.234
Sodium	Na	11	22.9898	0.9712	97.85	892	1.23
Strontium	Sr	38	87.62	2.54	768	1380	0.737
Sulfur	S	16	32.064	2.07	119.0	444.6	0.707
Tantalum	Ta	73	180.948	16.6	3014	5425	0.138
Technetium	Tc	43	(99)	11.46	2200	—	0.209
Tellurium	Te	52	127.60	6.24	449.5	990	0.201
Terbium	Tb	65	158.924	8.229	1357	2530	0.180
Thallium	Tl	81	204.37	11.85	304	1457	0.130
Thorium	Th	90	(232)	11.72	1755	(3850)	0.117
Thulium	Tm	69	168.934	9.32	1545	1720	0.159
Tin	Sn	50	118.69	7.2984	231.868	2270	0.226
Titanium	Ti	22	47.90	4.54	1670	3260	0.523
Tungsten	W	74	183.85	19.3	3380	5930	0.134
Unnamed	Uut	113	(284)	—	—	—	—
Unnamed	Uup	115	(288)	—	—	—	—
Unnamed	Uus	117	—	—	—	—	—
Unnamed	Uuo	118	(294)	—	—	—	—
Uranium	U	92	(238)	18.95	1132	3818	0.117

(Contd.)

Element	Symbol	Atomic Number Z	Molar Mass, g/mol	Density, g/cm^3 at 20°C	Melting Point, °C	Boiling Point, °C	Specific Heat, J/(g·°C) at 25°C
Vanadium	V	23	50.942	6.11	1902	3400	0.490
Xenon	Xe	54	131.30	5.495×10^{-3}	–111.79	–108	0.159
Ytterbium	Yb	70	173.04	6.965	824	1530	0.155
Yttrium	Y	39	88.905	4.469	1526	3030	0.297
Zinc	Zn	30	65.37	7.133	419.58	906	0.389
Zirconium	Zr	40	91.22	6.506	1852	3580	0.276

The values in parentheses in the column of molar masses are the mass numbers of the longest-lived isotopes of those elements that are radioactive. Melting points and boiling points in parentheses are uncertain.

The data for gases are valid only when these are in their usual molecular state, such as H_2, He, O_2, Ne, etc. The specific heats of the gases are the values at constant pressure.

Source: Adapted from J. Emsley, *The Elements*, 3rd ed., 1998, Clarendon Press, Oxford. See also www.webelements.com for the latest values and newest elements.

*The names and symbols for elements 114 (Flerovium, Fl) and 116 (Livermorium, Lv) have been suggested but are not official.

APPENDIX

G

JEE Advanced 2019

PAPER 1

One Option Correct Type

This section contains **FOUR (04)** questions. Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer. For each question, choose the option corresponding to the correct answer.

1. Consider a spherical gaseous cloud of mass density $\rho(r)$ in free space where r is the radial distance from its centre. The gaseous cloud is made of particles of equal mass m moving in circular orbits about the common centre with the same kinetic energy K. The force acting on the particles is their mutual gravitational force. If $\rho(r)$ is constant in time, the particle number density $n(r) = \rho(r)/m$ is [G is universal gravitational constant]

(A) $\dfrac{K}{2\pi r^2 m^2 G}$ (B) $\dfrac{K}{\pi r^2 m^2 G}$

(C) $\dfrac{3K}{\pi r^2 m^2 G}$ (D) $\dfrac{K}{6\pi r^2 m^2 G}$

2. A thin spherical insulating shell of radius R carries a uniformly distributed charge such that the potential at its surface is V_0. A hole with a small area $\alpha 4\pi R^2 (\alpha << 1)$ is made on the shell without affecting the rest of the shell. Which one of the following statements is correct?

(A) The potential at the centre of the shell is reduced by $2\alpha V_0$.

(B) The magnitude of electric field at the centre of the shell is reduced by $\dfrac{\alpha V_0}{2R}$.

(C) The ratio of the potential at the centre of the shell to that of the point at $\dfrac{1}{2}R$ from centre towards the hole will be $\dfrac{1-\alpha}{1-2\alpha}$.

(D) The magnitude of electric field at a point, located on a line passing through the hole and shell's centre, on a distance $2R$ from the centre of the spherical shell will be reduced by $\dfrac{\alpha V_0}{2R}$.

3. A current carrying wire heats a metal rod. The wire provides a constant power (P) to the rod. The metal rod is enclosed in an insulted container. It is observed that the temperature (T) in the metal rod changes with time (t) as

$$T(t) = T_0(1+\beta t^{1/4}),$$

where β is a constant with appropriate dimension while T_0 is a constant with dimension of temperature. The heat capacity of the metal is,

(A) $\dfrac{4P(T(t)-T_0)^3}{\beta^4 T_0^4}$ (B) $\dfrac{4P(T(t)-T_0)^4}{\beta^4 T_0^5}$

(C) $\dfrac{4P(T(t)-T_0)^2}{\beta^4 T_0^3}$ (D) $\dfrac{4P(T(t)-T_0)}{\beta^4 T_0^2}$

4. In a radioactive sample, ${}^{40}_{19}$K nuclei either decay into stable ${}^{40}_{20}$Ca nuclei with decay constant 4.5×10^{-10} per year or into stable ${}^{40}_{18}$Ar nuclei with decay constant 0.5×10^{-10} per year. Given that in this sample all the stable ${}^{40}_{20}$Ca and ${}^{40}_{18}$Ar nuclei are produced by the ${}^{40}_{19}$K nuclei only. In time $t\times10^9$ years, if the ratio of the sum of stable ${}^{40}_{20}$Ca and ${}^{40}_{18}$Ar nuclei to the radioactive ${}^{40}_{19}$K nuclei is 99, the value of t will be,

[Given: ln 10 = 2.3]

(A) 1.15 (B) 9.2

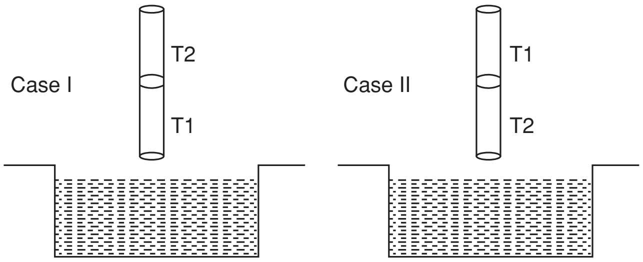
(C) 2.3 (D) 4.6

One or More Than One Option Correct Type

This section contains **EIGHT (08)** questions. Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) the correct answer(s). For each question, choose the option(s) corresponding to (all) the correct answer(s).

5. A cylindrical capillary tube of 0.2 mm radius is made by joining two capillaries T1 and T2 of different materials having water contact angles of 0º and 60º, respectively. The capillary tube is dipped vertically in water in two different configurations, case I and II as shown in figure. Which of the following option(s) is(are) correct?

[Surface tension of water = 0.075 N/m, density of water = 1000 kg/m^3, take g = 10 m/s^2]



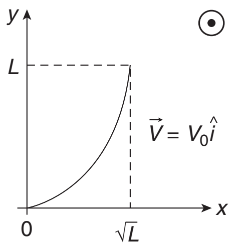
(A) The correction in the height of water column raised in the tube, due to weight of water contained in the meniscus, will be different for both cases.

(B) For case II, if the capillary joint is 5 cm above the water surface, the height of water column raised in the tube will be 3.75 cm. (Neglect the weight of the water in the meniscus)

(C) For case I, if the joint is kept at 8 cm above the water surface, the height of water column in the tube will be 7.5 cm. (Neglect the weight of the water in the meniscus)

(D) For case I, if the capillary joint is 5 cm above the water surface, the height of water column raised in the tube will be more than 8.75 cm. (Neglect the weight of the water in the meniscus)

6. A conducting wire of parabolic shape, initially $y = x^2$, is moving with velocity $\vec{V} = V_0\hat{i}$ in a non-uniform magnetic field $\vec{B} = B_0\left(1+\left(\frac{y}{L}\right)^{\beta}\right)\hat{k}$, as shown in figure. If V_0, B_0, L and β are positive constants and $\Delta\phi$ is the potential difference developed between the ends of the wire, then the correct statement(s) is/are:



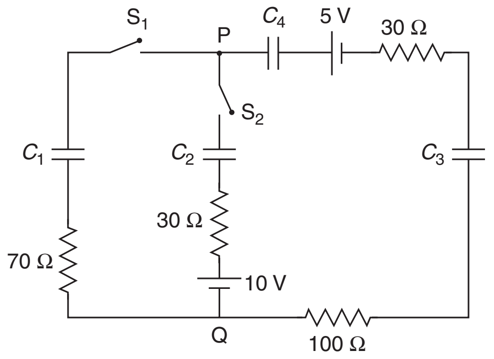
(A) $|\Delta\phi| = \frac{1}{2}B_0V_0L$ for $\beta = 0$

(B) $|\Delta\phi| = \frac{4}{3}B_0V_0L$ for $\beta = 2$

(C) $|\Delta\phi|$ remains the same if the parabolic wire is replaced by a straight wire, $y = x$ initially, of length $\sqrt{2}L$.

(D) $|\Delta\phi|$ is proportional to the length of the wire projected on the y-axis.

7. In the circuit shown, initially there is no charge on capacitors and keys S_1 and S_2 are open. The values of the capacitors are $C_1 = 10\ \mu\text{F}$, $C_2 = 30\ \mu\text{F}$ and $C_3 = C_4 = 80\ \mu\text{F}$.



Which of the statement(s) is/are correct?

(A) At time $t = 0$, the key S_1 is closed, the instantaneous current in the closed circuit will be 25 mA.

(B) If key S_1 is kept closed for long time such that capacitors are fully charged, the voltage across the capacitor C_1 will be 4 V.

(C) The key S_1 is kept closed for long time such that capacitors are fully charged. Now key S_2 is closed, at this time, the instantaneous current across 30 Ω resistor (between points P and Q) will be 0.2 A (round off to 1st decimal place).

(D) If key S_1 is kept closed for long time such that capacitors are fully charged, the voltage difference between points P and Q will be 10 V.

8. A charged shell of radius R carries a total charge Q. Given Φ as the flux of electric field through a closed cylindrical surface of height h, radius r and with its centre same as that of the shell. Here, centre of the cylinder is a point on the axis of the cylinder which is equidistant from its top and bottom surfaces. Which of the following option(s) is/are correct?
[ε_0 is the permittivity of free space]

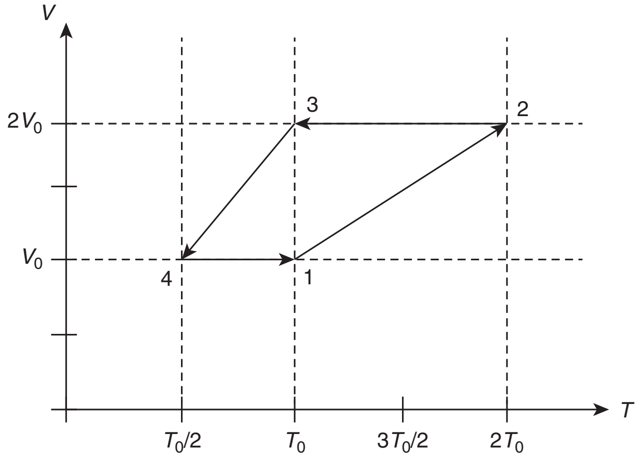
(A) If $h > 2R$ and $r > R$ then $\Phi = Q/\varepsilon_0$

(B) If $h < 8R/5$ and $r = 3R/5$ then $\Phi = 0$

(C) If $h > 2R$ and $r = 3R/5$ then $\Phi = Q/5\varepsilon_0$

(D) If $h > 2R$ and $r = 4R/5$ then $\Phi = Q/5\varepsilon_0$

9. One mole of a monatomic ideal gas goes through a thermodynamic cycle, as shown in the volume versus temperature (V–T) diagram. The correct statement(s) is/are:
[R is the gas constant]



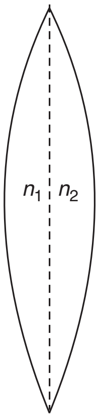
(A) Work done in this thermodynamic cycle ($1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$) is $|W| = \frac{1}{2}RT_0$

(B) The above thermodynamic cycle exhibits only isochoric and adiabatic processes.

(C) The ratio of heat transfer during processes $1 \rightarrow 2$ and $2 \rightarrow 3$ is $\left|\frac{Q_{1\rightarrow 2}}{Q_{2\rightarrow 3}}\right| = \frac{5}{3}$.

(D) The ratio of heat transfer during processes $1 \rightarrow 2$ and $3 \rightarrow 4$ is $\left|\frac{Q_{1\rightarrow 2}}{Q_{3\rightarrow 4}}\right| = \frac{1}{2}$.

10. A thin convex lens is made of two materials with refractive indices n_1 and n_2, as shown in figure. The radius of curvature of the left and right spherical surfaces are equal. f is the focal length of the lens when $n_1 = n_2 = n$. The focal length is $f + \Delta f$ when $n_1 = n$ and $n_2 = n + \Delta n$. Assuming $\Delta n << (n-1)$ and $1 < n < 2$, the correct statement(s) is/are,



(A) $\left|\frac{\Delta f}{f}\right| < \left|\frac{\Delta n}{n}\right|$

(B) For $n = 1.5$, $\Delta n = 10^{-3}$ and $f = 20$ cm, the value of $|\Delta f|$ will be 0.02 cm (round off to 2nd decimal place).

(C) If $\frac{\Delta n}{n} < 0$ then $\frac{\Delta f}{f} > 0$

(D) The relation between $\frac{\Delta f}{f}$ and $\frac{\Delta n}{n}$ remains unchanged if both the convex surfaces are replaced by concave surfaces of the same radius of curvature.

11. Let us consider a system of units in which mass and angular momentum are dimensionless. If length has dimension of L, which of the following statement(s) is/are correct?

(A) The dimension of linear momentum is L^{-1}

(B) The dimension of energy is L^{-2}

(C) The dimension of force is L^{-3}

(D) The dimension of power is L^{-5}

12. Two identical moving coil galvanometers have 10 Ω resistance and full scale deflection at 2 μA current. One of them is converted into a voltmeter of 100 mV full scale reading and the other into an Ammeter of 1 mA full scale current using appropriate resistors. These are then used to measure the voltage and current in the Ohm's law experiment with $R = 1000\ \Omega$ resistor by using an ideal cell. Which of the following statement(s) is/are correct?

(A) The resistance of the Voltmeter will be 100 kΩ.

(B) The resistance of the Ammeter will be 0.02 Ω (round off to 2nd decimal place).

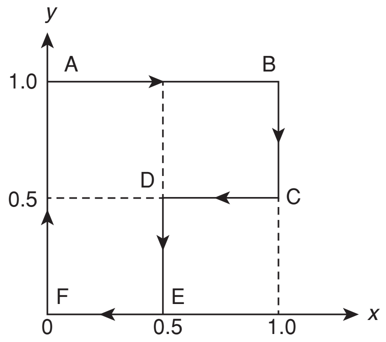
(C) The measured value of R will be $978\ \Omega < R < 982\ \Omega$.

(D) If the ideal cell is replaced by a cell having internal resistance of 5 Ω then the measured value of R will be more than 1000 Ω.

Integer Answer Type

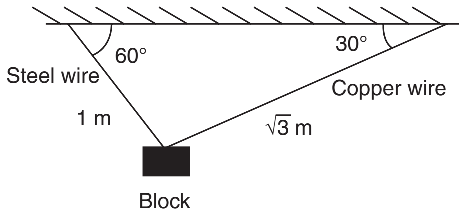
This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**. If the numeral value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.

13. A particle is moved along a path AB-BC-CD-DE-EF-FA, as shown in figure, in presence of a force $\vec{F} = (\alpha y\hat{i} + 2\alpha x\hat{j})$ N, where x and y are in metre and $\alpha = -1\ \text{N m}^{-1}$. The work done on the particle by this force $\vec{F}$ will be _______Joule.



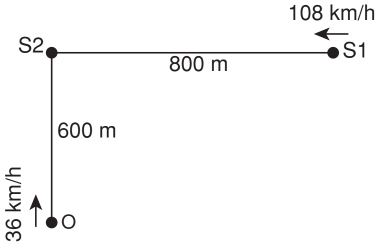
14. A block of weight 100 N is suspended by copper and steel wires of same cross-sectional area 0.5 cm^2 and, length $\sqrt{3}$ m and 1 m, respectively. Their other ends are fixed on a ceiling as shown in figure. The angles subtended by copper and steel wires with ceiling are 30° and 60°, respectively. If elongation in copper wire is (Δl_C) and elongation in steel wire is (Δl_S), then the ratio $\frac{\Delta l_C}{\Delta l_S}$ is ______.

[Young's modulus for copper and steel are 1×10^{11} N/m^2 and 2×10^{11} N/m^2, respectively.]



15. A train S1, moving with a uniform velocity of 108 km/h, approaches another train S2 standing on a platform. An observer O moves with a uniform velocity of 36 km/h towards S2, as shown in figure. Both the trains are blowing whistles of same frequency 120 Hz. When O is 600 m away from S2 and distance between S1 and S2 is 800 m, the number of beats heard by O is ________.

[Speed of the sound = 330 m/s]



16. A parallel plate capacitor of capacitance C has spacing d between two plates having area A. The region between the plates is filled with N dielectric layers, parallel to its plates, each with thickness $\delta = \frac{d}{N}$. The dielectric constant of the mth layer is $K_m = K\left(1+\frac{m}{N}\right)$. For a very large $N(>10^3)$, the capacitance C is $\alpha\left(\frac{K\varepsilon_0 A}{d\ln 2}\right)$. The value of α will be ______.

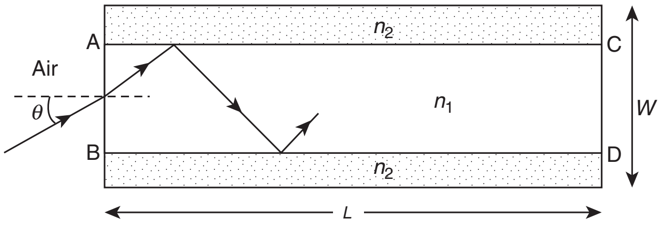
[ε_0 is the permittivity of free space]

17. A liquid at 30 °C is poured very slowly into a Calorimeter that is at temperature of 110 °C. The boiling temperature of the liquid is 80 °C. It is found that the first 5 g of the liquid completely evaporates. After pouring another 80 g of the liquid the equilibrium temperature is found to be 50 °C. The ratio of the Latent heat of the liquid to its specific heat will be ________ °C.

[Neglect the heat exchange with surrounding]

18. A planar structure of length L and width W is made of two different optical media of refractive indices $n_1 = 1.5$ and $n_2 = 1.44$ as shown in figure. If $L >> W$, a ray entering from end AB will emerge from end CD only if the total internal reflection condition is met inside the structure. For $L = 9.6$ m, if the incident angle θ is varied, the maximum time taken by a ray to exit the plane CD is $t\times10^{-9}$ s, where t is _____.

[Speed of light $c = 3\times10^8$ m/s]



PAPER 2

One or More Than One Option Correct Type

This section contains **EIGHT (08)** questions. Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) the correct answer(s). For each question, choose the option(s) corresponding to (all) the correct answer(s).

1. A thin and uniform rod of mass M and length L is held vertical on a floor with large friction. The rod is released from rest so that it falls by rotating about its contact-point with the floor without slipping. Which of the following statement(s) is/are correct, when the rod makes an angle 60º with vertical?

[g is the acceleration due to gravity]

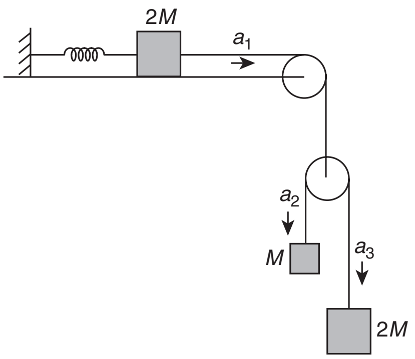
(A) The angular speed of the rod will be $\sqrt{\frac{3g}{2L}}$.

(B) The angular acceleration of the rod will be $\frac{2g}{L}$.

(C) The radial acceleration of the rod's centre of mass will be $\frac{3g}{4}$.

(D) The normal reaction force from the floor on the rod will be $\frac{Mg}{16}$.

2. A block of mass $2M$ is attached to a massless spring with spring-constant k. This block is connected to two other blocks of masses M and $2M$ using two massless pulleys and strings. The accelerations of the blocks are a_1, a_2 and a_3 as shown in the figure. The system is released from rest with the spring in its unstretched state. The maximum extension of the spring is x_0. Which of the following option(s) is/are correct? [g is the acceleration due to gravity. Neglect friction]



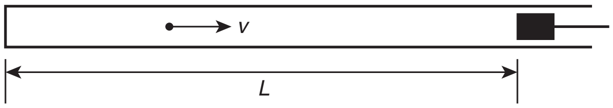
(A) $x_0 = \frac{4Mg}{k}$

(B) When spring achieves an extension of $\frac{x_0}{2}$ for the first time, the speed of the block connected to the spring is $3g\sqrt{\frac{M}{5k}}$.

(C) At an extension of $\frac{x_0}{4}$ of the spring, the magnitude of acceleration of the block connected to the spring is $\frac{3g}{10}$.

(D) $a_2 - a_1 = a_1 - a_3$

3. A small particle of mass m moving inside a heavy, hollow and straight tube along the tube axis undergoes elastic collision at two ends. The tube has no friction and it is closed at one end by a flat surface while the other end is fitted with a heavy movable flat piston as shown in figure. When the distance of the piston from closed end is $L = L_0$ the particle speed is $v = v_0$. The piston is moved inward at a very low speed V such that $V << \frac{dL}{L}v_0$, where dL is the infinitesimal displacement of the piston. Which of the following statement(s) is/are correct?



(A) The rate at which the particle strikes the piston is v/L.

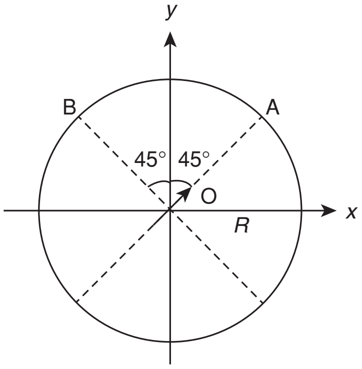
(B) After each collision with the piston, the particle speed increases by $2V$.

(C) If the piston moves inward by dL, the particle speed increases by $2v\frac{dL}{L}$.

(D) The particle's kinetic energy increases by a factor of 4 when the piston is moved inward from L_0 to $\frac{1}{2}L_0$.

4. An electric dipole with dipole moment $\frac{p_0}{\sqrt{2}}(\hat{i}+\hat{j})$ is held fixed at the origin O in the presence of a uniform electric field of magnitude E_0. If the potential is constant on a circle of radius R centered at the origin as shown in figure, then the correct statement(s) is/are:

(ε_0 is permittivity of free space. $R >>$ dipole size)



(A) $R = \left(\frac{p_0}{4\pi\varepsilon_0 E_0}\right)^{1/3}$

(B) Total electric field at point A is $\vec{E}_A = \sqrt{2}E_0(\hat{i}+\hat{j})$

(C) Total electric field at point B is $\vec{E}_B = 0$

(D) The magnitude of total electric field on any two points of the circle will be same.

5. A mixture of ideal gas containing 5 moles of monatomic gas and 1 mole of rigid diatomic gas is initially at pressure P_0, volume V_0, and temperature T_0. If the gas mixture is adiabatically compressed to a volume $V_0/4$, then the correct statement(s) is/are,

(Given $2^{1.2} = 2.3$; $2^{3.2} = 9.2$; R is gas constant)

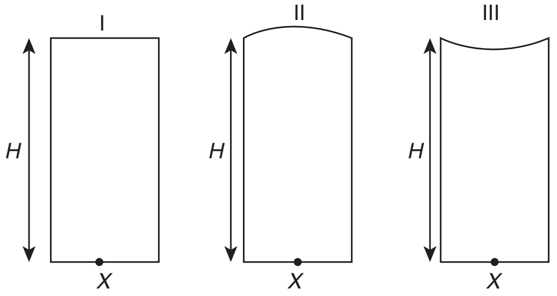
(A) The work $|W|$ done during the process is $13RT_0$

(B) The average kinetic energy of the gas mixture after compression is in between $18RT_0$ and $19RT_0$

(C) The final pressure of the gas mixture after compression is in between $9P_0$ and $10P_0$

(D) Adiabatic constant of the gas mixture is 1.6

6. Three glass cylinders of equal height H = 30 cm and same refractive index n = 1.5 are placed on a horizontal surface as shown in figure. Cylinder I has a flat top, cylinder II has a convex top and cylinder III has a concave top. The radii of curvature of the two curved tops are same (R = 3 m). If H_1, H_2, and H_3 are the apparent depths of a point X on the bottom of the three cylinders, respectively, the correct statement(s) is/are:



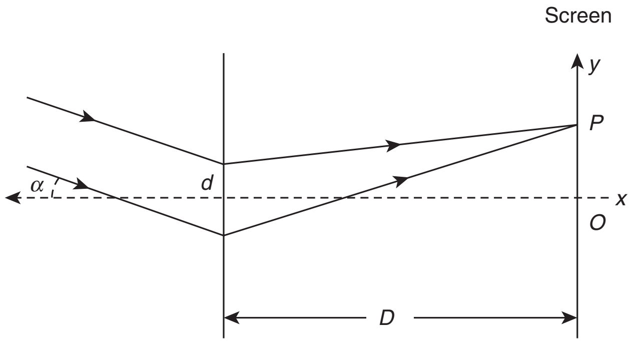
(A) $H_2 > H_1$

(B) $H_3 > H_1$

(C) $H_2 > H_3$

(D) $0.8 \text{ cm} < (H_2 - H_1) < 0.9 \text{ cm}$

7. In a Young's double slit experiment, the slit separation d is 0.3 mm and the screen distance D is 1 m. A parallel beam of light of wavelength 600 nm is incident on the slits at angle α as shown in figure. On the screen, the point O is equidistant from the slits and distance PO is 11.0 mm. Which of the following statement(s) is/are correct?



(A) For $\alpha = \frac{0.36}{\pi}$ degree, there will be destructive interference at point O.

(B) For $\alpha = 0$, there will be constructive interference at point P.

(C) For $\alpha = \frac{0.36}{\pi}$ degree, there will be destructive interference at point P.

(D) Fringe spacing depends on α.

8. A free hydrogen atom absorbing a photon of wavelength λ_a gets excited from the state $n = 1$ to the state $n = 4$. Immediately after that the electron jumps to $n = m$ state by emitting a photon of wavelength λ_e. Let the change in momentum of atom due to the absorption and the emission are Δp_a and Δp_e, respectively. If $\lambda_a/\lambda_e = \frac{1}{5}$, which of the option(s) is/are correct?

[Use hc = 1242 eV nm; 1 nm = 10^{-9} m, h and c are Planck's constant and speed of light, respectively]

(A) $m = 2$

(B) $\lambda_e = 418\,\text{nm}$

(C) $\Delta p_a / \Delta p_e = \frac{1}{2}$

(D) The ratio of kinetic energy of the electron in the state $n = m$ to the state $n = 1$ is $\frac{1}{4}$

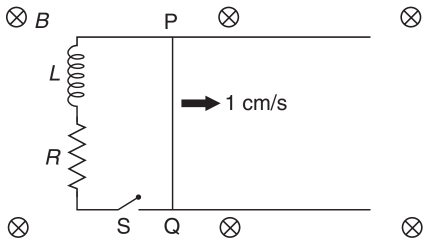
Integer Answer Type

This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**. If the numeral value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.

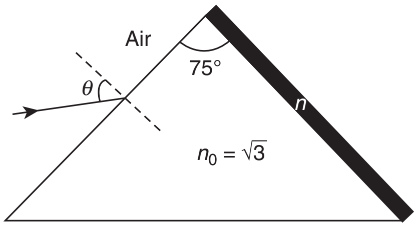
9. A ball is thrown from ground at an angle θ with horizontal and with an initial speed u_0. For the resulting projectile motion, the magnitude of average velocity of the ball up to the point when it hits the ground for the first time is V_1. After hitting the ground, the ball rebounds at the same angle θ but with a reduced speed of u_0/α. Its motion continues for a long time as shown in figure. If the magnitude of average velocity of the ball for entire duration of motion is 0.8 V_1, the value of α is__________.

10. A 10 cm long perfectly conducting wire PQ is moving with a velocity 1 cm/s on a pair of horizontal rails of zero resistance. One side of the rails is connected to an inductor L = 1 mH and a resistance R = 1 Ω. As shown in figure the horizontal rails, L and R lie in the same plane with a uniform magnetic field B = 1 T perpendicular to the plane. If the key S is closed at certain instant, the current in the circuit after 1 millisecond is $x \times 10^{-3}$ A, where the value of x is ______.

[Assume the velocity of wire PQ remains constant (1 cm/s) after key S is closed. Given: $e^{-1} = 0.37$, where e is base of the natural logarithm]

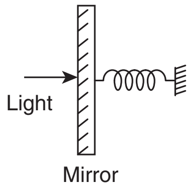


11. A monochromatic light is incident from air on a refracting surface of a prism of angle 75° and refractive index $n_0 = \sqrt{3}$. The other refracting surface of the prism is coated by a thin film of material of refractive index n as shown in figure. The light suffers total internal reflection at the coated prism surface for an incidence angle of $\theta \le 60°$. The value of n^2 is ______.



12. A perfectly reflecting mirror of mass M mounted on a spring constitutes a spring-mass system of angular frequency Ω such that $\frac{4\pi M\Omega}{h} = 10^{24}\ \text{m}^{-2}$ with h as Planck's constant. N photons of wavelength $\lambda = 8\pi \times 10^{-6}$ m strike the mirror simultaneously at normal incidence such that the mirror gets displaced by 1 µm. If the value of N is $x \times 10^{12}$, then the value of x is ______.

[Consider the spring as massless]



13. Suppose a $^{226}_{88}\text{Ra}$ nucleus at rest and in ground state undergoes α-decay to a $^{222}_{86}\text{Rn}$ nucleus in its excited state. The kinetic energy of the emitted α particle is found to be 4.44 MeV. $^{222}_{86}\text{Rn}$ nucleus then goes to its ground state by γ-decay. The energy of the emitted γ photon is ______ keV.

[Given: atomic mass of $^{226}_{88}\text{Ra} = 226.005\,\text{u}$, atomic mass of $^{222}_{86}\text{Rn} = 222.000$ u, atomic mass of α particle = 4.000 u, 1 u = 931 MeV/c^2, c is speed of the light]

14. An optical bench has 1.5 m long scale having four equal divisions in each cm. While measuring the focal length of a convex lens, the lens is kept at 75 cm mark of the scale and the object pin is kept at 45 cm mark. The image of the object pin on the other side of the lens overlaps with image pin that is kept at 135 cm mark. In this experiment, the percentage error in the measurement of the focal length of the lens is ______.

Matrix-Match Type

This section contains **TWO (02)** List-Match sets. Each List-match set has **TWO (02)** Multiple Choice Questions. Each List-Match set has two lists: **List-I** and **List-II**. **List-I** has Four entries (I), (II), (III) and (IV) and **List-II** has **Six** entries (P), (Q), (R), (S), (T) and (U). **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.

Directions for Questions 15 and 16: Answer Q.15 and Q.16 by appropriately matching the lists based on the information given in the paragraph.

A musical instrument is made using four different metal strings, 1, 2, 3 and 4 with mass per unit length μ, 2μ, 3μ and 4μ, respectively. The instrument is played by vibrating the strings by varying the free length in between the range L_0 and $2L_0$. It is found that in string-1 (μ) at free length L_0 and tension T_0 the fundamental mode frequency is f_0.

List-I gives the above four strings while list-II lists the magnitude of some quantity.

List-I	**List-II**
(I) String-1 (μ)	**(P)** 1
(II) String-2 (2μ)	**(Q)** $\frac{1}{2}$
(III) String-3 (3μ)	**(R)** $\frac{1}{\sqrt{2}}$
(IV) String-4 (4μ)	**(S)** $\frac{1}{\sqrt{3}}$
	(T) $\frac{3}{16}$
	(U) $\frac{1}{16}$

15. If the tension in each string is T_0, the correct match for the highest fundamental frequency in f_0 units will be,

(A) I → P, II → Q, III → T, IV → S
(B) I → P, II → R, III → S, IV → Q
(C) I → Q, II → S, III → R, IV → P
(D) I → Q, II → P, III → R, IV → T

16. The length of the strings 1, 2, 3 and 4 are kept fixed at $L_0, \frac{3L_0}{2}, \frac{5L_0}{4}$, and $\frac{7L_0}{4}$, respectively. Strings 1, 2, 3, and 4 are vibrated at their 1st, 3rd, 5th, and 14th harmonics, respectively such that all the strings have same frequency. The correct match for the tension in the four strings in the units of T_0 will be,

(A) I → P, II → R, III → T, IV → U
(B) I → P, II → Q, III → T, IV → U
(C) I → P, II → Q, III → R, IV → T
(D) I → T, II → Q, III → R, IV → U

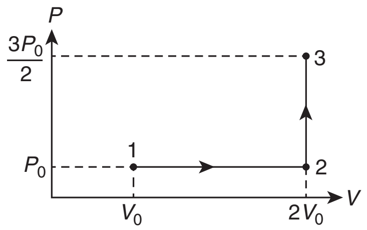
Directions for Questions 17 and 18: Answer Q.17 and Q.18 by appropriately matching the list based on the information given in the paragraph.

In a thermodynamic process on an ideal monatomic gas, the infinitesimal heat absorbed by the gas is given by $T\Delta X$, where T is temperature of the system and ΔX is the infinitesimal change in a thermodynamic quantity X of the system. For a mole of monatomic ideal gas, $X = \frac{3}{2}R\ln\left(\frac{T}{T_A}\right) + R\ln\left(\frac{V}{V_A}\right)$. Here, R is gas constant, V is volume of gas, T_A and V_A are constants.

The List-I below gives some quantities involved in a process and List-II given some possible values of these quantities.

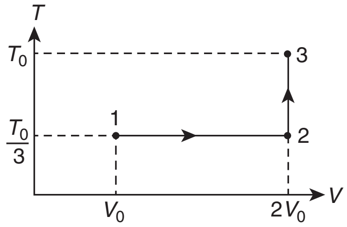
List-I	**List-II**
(I) Work done by the system in process 1 → 2 → 3	**(P)** $\frac{1}{3}RT_0\ln 2$
(II) Change in internal energy in process 1 → 2 → 3	**(Q)** $\frac{1}{3}RT_0$
(III) Heat absorbed by the system in process 1 → 2 → 3	**(R)** RT_0
(IV) Heat absorbed by the system in process 1 → 2	**(S)** $\frac{4}{3}RT_0$
	(T) $\frac{1}{3}RT_0(3+\ln 2)$
	(U) $\frac{5}{6}RT_0$

17. If the process carried out on one mole of monatomic ideal gas is as shown in figure in the PV-diagram with $P_0V_0 = \frac{1}{3}RT_0$, the correct match is,



(A) I → Q, II → R, III → P, IV → U
(B) I → Q, II → R, III → S, IV → U
(C) I → S, II → R, III → Q, IV → T
(D) I → Q, II → S, III → R, IV → U

18. If the process on one mole of monatomic ideal gas is as shown in the TV-diagram with $P_0V_0 = \frac{1}{3}RT_0$, the correct match is,



(A) I → P, II → R, III → T, IV → S
(B) I → P, II → T, III → Q, IV → T
(C) I → P, II → R, III → T, IV → P
(D) I → S, II → T, III → Q, IV → U

ANSWER KEY

Paper 1

1. (A)	**2.** (C)	**3.** (A)	**4.** (B)	**5.** (A), (B), (C)
6. (B), (C), (D)	**7.** (A), (B)	**8.** (A), (B), (C)	**9.** (A), (C)	**10.** (B), (C), (D)
11. (A), (B), (C)	**12.** (B), (C)	**13.** (0.75)	**14.** (2.0)	**15.** (8.1)
16. (1)	**17.** (270)	**18.** (50)		

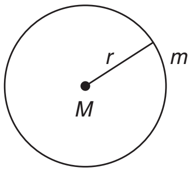
Paper 2

1. (A), (C), (D)	**2.** (D)	**3.** (B), (D)	**4.** (A), (C)	**5.** (A), (C), (D)
6. (A), (C)	**7.** (C)	**8.** (A), (D)	**9.** (4.0)	**10.** (0.632)
11. (1.5)	**12.** (1)	**13.** (135)	**14.** (1.388)	**15.** (B)
16. (B)	**17.** (B)	**18.** (C)		

ANSWERS WITH EXPLANATION

PAPER 1

1. Spherical gaseous cloud is moving in circular orbit about common centre.



Gaseous cloud consists of particles of equal mass = m, same kinetic energy = K and mass density of gaseous cloud = $\rho(r)$. Mutual gravitational force acts on particles and mass density $\rho(r)$ is constant in time. Since the cloud is rotating in circular orbit under the action of gravitational force, the centripetal force is being balanced by the gravitational force. That is,
Gravitational force = Centripetal force

$$\Rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r}$$

where M is mass of inner cloud and v is velocity of cloud.

$$\Rightarrow \frac{GM}{r} = v^2$$

$$\Rightarrow M = \frac{v^2 r}{G} \qquad (1)$$

Now, $K = \frac{1}{2}mv^2 \Rightarrow v^2 = \frac{2K}{m}$

Substitute value of v^2 in (1), we get

$$M = \frac{2Kr}{Gm}$$

Differentiating it w.r.t 'r', we get

$$\frac{dM}{dr} = \frac{d}{dr}\left(\frac{2Kr}{Gm}\right) = \frac{2K}{Gm}$$

$$\Rightarrow \frac{dM}{dr} = \frac{2K}{Gm}$$

We know that mass element $\frac{dM}{dr} = 4\pi r^2 \cdot \rho(r)$. Therefore,

$$4\pi r^2 \rho(r) = \frac{2K}{Gm}$$

$$\Rightarrow \rho(r) = \frac{2K}{4\pi r^2 Gm} = \frac{K}{2\pi r^2 Gm}$$

So, number density $n(r) = \frac{\rho(r)}{m} = \frac{K}{2\pi r^2 m^2 G}$

2. Statement (A) is incorrect. The potential at the centre of the shell is reduced by αV_0 and not $2\alpha V_0$. We know that for uniformly distributed charged shell surface charge density,

$$\sigma = \frac{Q}{4\pi R^2}$$

where Q is charge on thin spherical shell. Now, charge on small area $\alpha 4\pi R^2$ is

$$dq = \frac{Q}{4\pi R^2} \times \alpha 4\pi R^2$$

$$\Rightarrow dq = \alpha Q$$

Initial potential, $V_{\text{initial}} = \frac{KQ}{R} = V_0$

Final potential, $V_{\text{final}} = \frac{KQ}{R} - \frac{kdq}{R} = \frac{KQ}{R} - \frac{\alpha KQ}{R}$

$$\Rightarrow V_{\text{final}} = \frac{KQ}{R}(1-\alpha)$$

Change in potential, $\Delta V = V_{\text{initial}} - V_{\text{final}} = \frac{KQ}{R} - \frac{KQ}{R}(1-\alpha)$

$$= \frac{KQ}{R}(1-1+\alpha)$$

$$= \alpha\frac{KQ}{R} = \alpha V_0$$

Therefore, change in potential $= \alpha V_0$

Statements (B) is incorrect. The magnitude of electric field at the centre of shell is increased by $\dfrac{\alpha V_0}{2e}$ and not decreased by $\dfrac{\alpha V_0}{2R}$.

Statement (D) is incorrect. Initial electric field at the given point,

$$E_{\text{initial}} = \frac{KQ}{(2R)^2} \Rightarrow E_{\text{initial}} = \frac{KQ}{4R^2}$$

Final electric field, $E_{\text{final}} = \dfrac{KQ}{4R^2} - \dfrac{KQ\alpha}{R^2}$

Therefore, change in electric field $= E_{\text{initial}} - E_{\text{final}}$

$$= \frac{KQ}{4R^2} - \left(\frac{KQ}{4R^2} - \frac{KQ\alpha}{R^2}\right) = \frac{KQ\alpha}{R^2}$$

Using $\dfrac{KQ}{R} = V_0$, we get

Change in electric field $= \dfrac{\alpha V_0}{R}$

Statement (C) is correct. The ratio of the potential at the centre of the shell to that of the point at $\dfrac{1}{2}R$ from the centre towards the hole will be $\dfrac{1-\alpha}{1-2\alpha}$.

As calculated earlier potential at centre of shell $= \dfrac{KQ}{R}(1-\alpha)$

Potential at point $\dfrac{1}{2}R$ from centre $= V_0 - \dfrac{kdq}{R/2}$

$$= \frac{KQ}{R} - \frac{2KQ\alpha}{R} = \frac{KQ}{R}(1-2\alpha)$$

Therefore, the ratio is $\dfrac{\dfrac{KQ}{R}(1-\alpha)}{\dfrac{KQ}{R}(1-2\alpha)} = \dfrac{(1-\alpha)}{(1-2\alpha)}$

3. Power at equilibrium is equal to rate of heat transfer through metal rod. Therefore,

$$P = \frac{dQ}{dt} \text{ or } P = \frac{cdT}{dt} \quad (\because dQ = cdT) \qquad (1)$$

where c is heat capacity.

Given, $T(t) = T_0(1 - \beta t^{1/4})$ (2)

Differentiating it, we get

$$\frac{dT}{dt} = \frac{d}{dt}[T_0(1+\beta t^{1/4})] = T_0\beta\frac{1}{4}t^{\frac{1}{4}-1}$$

$$\Rightarrow \frac{dT}{dt} = \frac{1}{4}T_0\beta t^{-3/4}$$

From Eq. (1), $P = c \times \dfrac{1}{4}T_0\beta t^{-3/4}$

$$\Rightarrow c = \frac{4P}{T_0\beta t^{-3/4}} = \frac{4P}{T_0\beta}t^{3/4} \qquad (3)$$

From Eq. (2),

$$T = T_0 + T_0\beta t^{1/4} \Rightarrow T - T_0 = T_0\beta t^{1/4}$$

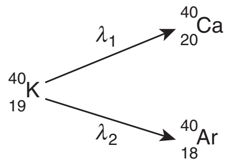
$$\Rightarrow t^{1/4} = \frac{T - T_0}{T_0\beta} \Rightarrow t^{3/4} = \frac{(T-T_0)^3}{T_0^3\beta^3}$$

Put value of $t^{3/4}$ in Eq. (3), we get

$$c = \frac{4P}{T_0\beta}\frac{(T-T_0)^3}{T_0^3\beta^3} = \frac{4P(T-T_0)^3}{T_0^4\beta^4}$$

Thus, heat capacity of metal is $\dfrac{4P(T(t)-T_0)^3}{T_0^4\beta^4}$.

4. Given, radioactive sample ${}^{40}_{19}\text{K}$ can decay either to ${}^{40}_{20}\text{Ca}$ or ${}^{40}_{18}\text{Ar}$.



Decay constant for decay to ${}^{40}_{20}\text{Ca}$ is 4.5×10^{-10} per year. Decay constant for decay to ${}^{40}_{18}\text{Ar}$ is 0.5×10^{-10} per year. In time $t \times 10^9$ years, ratio of sum of ${}^{40}_{20}\text{Ca}$ and ${}^{80}_{18}\text{Ar}$ to ${}^{40}_{19}\text{K}$ is 99.

$\Rightarrow$ 99% of ${}^{40}_{19}\text{K}$ has decayed.

Now, $\dfrac{dN}{dt} = -\lambda_1 N - \lambda_2 N = -(\lambda_1 + \lambda_2)N$

Now $N = N_0 e^{-(\lambda_1+\lambda_2)t} \Rightarrow \dfrac{N}{N_0} = e^{-(\lambda_1+\lambda_2)t}$ (1)

Given $\dfrac{N_0 - N}{N} = 99 \Rightarrow \dfrac{N_0}{N} - 1 = 99 \Rightarrow \dfrac{N_0}{N} = 100$

So $\dfrac{N}{N_0} = \dfrac{1}{100}$

Substitute all values in Eq. (1), we get

$$\frac{1}{100} = e^{-(4.5\times10^{-10} + 0.5\times10^{-10})t\times10^9}$$

$$\Rightarrow \frac{1}{100} = e^{-5.0\times10^{-10}\times t\times10^9} \Rightarrow \frac{1}{100} = e^{-5.0\times t\times10^{-1}}$$

$$\Rightarrow 100 = e^{\frac{5\times t}{10}}$$

Taking log on both sides, we obtain

$$\log(100) = \frac{5\times t}{10} \Rightarrow \log 10^2 = \frac{5\times t}{10}$$

$$\Rightarrow 2\log 10 = \frac{5\times t}{10} \Rightarrow \frac{2\times 2.3\times 10}{5} = t$$

$$\Rightarrow t = 9.2 \text{ year}$$

5. Statement (A) is correct, the correction in the height of water column raised in the tube, due to weight of water contained in the meniscus, will be different for both cases since the angle of contact are different in both the cases and height depends on angle of contact.

Height of water column is defined as

$$h = \frac{2T\cos\theta}{\rho g R} \Rightarrow h \alpha \cos\theta$$

where T is surface tension of water; θ is contact angle; ρ is density of water; g is acceleration due to gravity; R radius of capillary tube.

For case I, where tube T1 is in contact with water,

$$h = \frac{2T\cos\theta}{\rho g R} = \frac{2\times 0.075\times\cos 0^\circ}{1000\times 10\times 0.2\times 10^{-3}}$$

$$\Rightarrow h = \frac{2\times 0.075\times 1}{1000\times 10\times 0.2\times 10^{-3}} = 0.075 \text{ m}$$

or $\quad h = 7.5 \text{ m}$

Therefore, height of water column cannot be more than 7.5 cm so statements in option (D) is incorrect and option (C) is correct.

Similarly, for case II when tube T2 is in contact with water,

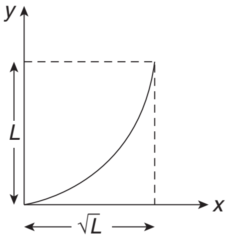
$$h = \frac{2T\cos\theta}{\rho g R} = \frac{2\times 0.075\times\cos 60^\circ}{1000\times 10\times 0.2\times 10^{-3}}$$

$$h = \frac{2\times 0.075}{1000\times 10\times 0.2\times 10^{-3}}\times\frac{1}{2} = 3.75\text{cm}$$

Hence, statement in option (B) is correct.

Therefore, option (A), (B), (C) are correct and (D) is incorrect.

6. Given, conducting wire of parabolic shape is moving in non-uniform magnetic field.



Equation of parabola, $y = x^2$

Velocity of wire, $\vec{V} = V_0 i$

Non-uniform magnetic field, $\vec{B} = B_0\left(1-\left(\frac{y}{L}\right)^\beta\right)\hat{k}$

Potential difference developed is

$$\Delta\phi = \int_0^L BV_0 dy = \int_0^L V_0 B_0\left(1+\left(\frac{y}{L}\right)^\beta\right)dy$$

$$\Rightarrow \Delta\phi = V_0 B_0\int_0^L\left(1+\frac{y\beta}{L\beta}\right)dy = V_0 B_0\left(y+\frac{y^{\beta+1}}{L^\beta(\beta+1)}\Bigg|_0^L\right)$$

$$\Rightarrow \phi = V_0 B_0\left[(L-0)+\left(\frac{L^{\beta+1}-0^{\beta+1}}{L^\beta(\beta+1)}\right)\right] = V_0 B_0\left[L+\frac{L^{\beta+1}}{L^\beta(\beta+1)}\right]$$

$$\Rightarrow \Delta\phi = V_0 B_0\left(L+\frac{L}{\beta+1}\right) = V_0 B_0\left(1+\frac{1}{\beta+1}\right)L$$

$$\Rightarrow \Delta\phi = V_0 B_0\left(1+\frac{1}{\beta+1}\right)L \qquad (1)$$

If $\beta = 0, \quad \Delta\phi = V_0 B_0\left(1+\frac{1}{1}\right)L = 2V_0 B_0 L$

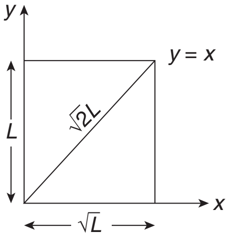
So, option (A) is incorrect.

If $\beta = 2, \quad \Delta\phi = V_0 B_0\left(1+\frac{1}{2+1}\right)L = V_0 B_0\left(1+\frac{1}{3}\right)L$

$$\Rightarrow \Delta\phi = \frac{4}{3}V_0 B_0 L$$

Hence, option (B) is correct.

$\Delta\phi$ will remain same if wire is replaced by straight wire of length $\sqrt{2}L$ and $y = x$.

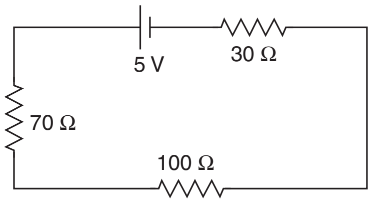


Hence, option (C) is correct.

From Eq. (1), $\Delta\phi \propto L$, So option (D) is correct.

Thus, options (B), (C), (D) are correct and option (A) is incorrect.

7. At $t = 0$, when S_1 is closed the equivalent circuit is



All capacities will be short circuit here. The equivalent resistance of circuit will be

$$R_{eq} = 30+70+100 = 200\ \Omega$$

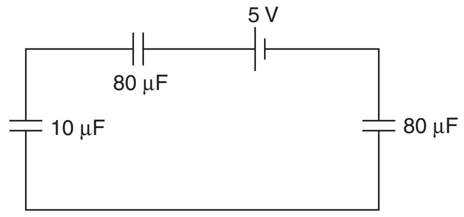
So current through the circuit will be obtained using Ohm's law as

$$I = \frac{V}{R_{\text{eq}}} = \frac{5}{200} = \frac{1}{40} = 0.025\,\text{A} = 25\times10^{-3}\,\text{A}$$

or $\quad I = 25$ mA

Hence, option (A) is correct.

If S_1 is kept closed for long time, charge on each capacitor will be same. Therefore,



$$\frac{1}{C_{\text{eq}}} = \frac{1}{10} + \frac{1}{80} + \frac{1}{80} = \frac{1}{10} + \frac{2}{80}$$

$$= \frac{1}{40} + \frac{1}{40} = \frac{4+1}{40} = \frac{5}{40}$$

$$\Rightarrow C_{\text{eq}} = \frac{40}{5} = 8\,\mu\text{F}$$

Using, $Q = CV$, $Q = (8\,\mu\text{F})(5) = 40\,\mu\text{C}$

Therefore, voltage across the capacitor C_1 will be

$$V_{C_1} = \left(\frac{Q}{Q+C_1}\right)\times V = \left(\frac{40}{40+10}\right)\times 5 = \frac{40}{50}\times 5$$

$$V_{C_1} = 4\,\text{V}$$

Hence, option (B) is correct.

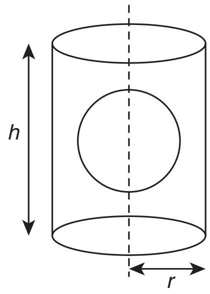
Also, $V_P - \frac{40}{10} = V_Q \Rightarrow V_P - V_Q = 4\,\text{V}$

Hence, option (D) is incorrect.

Just after closing S_2, charge on each capacitor remain same. Therefore, option (A) and (B) are correct and (C) and (D) are incorrect.

8. Radius of charged shell = R. Total charge = Q. Flux through cylinder of height h and radius r, is ϕ.

Option (A): $h > 2R$ and $r > R$ then $\phi = \frac{Q}{\varepsilon_0}$

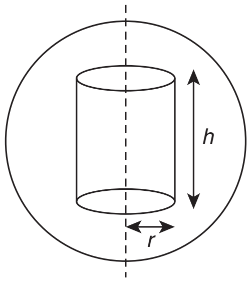


Total charge enclosed in cylinder is Q. Therefore,

$$\phi = \frac{Q}{\varepsilon_0}$$

Hence, option (A) is correct.

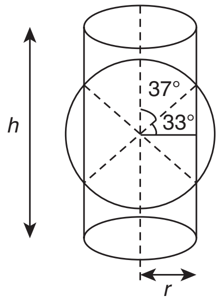
Option (B): $h < \frac{8R}{5}$ and $r = \frac{3R}{5}$ then $\phi = 0$



Total charge enclosed in cylinder is zero, so $\phi = 0$.

Hence, option (B) is correct.

Option (C): $h > 2R$ and $r = \frac{3R}{5}$ then $\phi = \frac{Q}{5\varepsilon_0}$



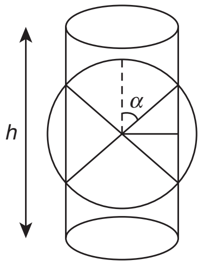
Total charge enclosed in cylinder is

$$Q_{\text{en}} = (1-\cos 37°)Q = 0.2Q$$

$$\Rightarrow \phi = \frac{Q_{\text{en}}}{\varepsilon_0} = \frac{0.2Q}{\varepsilon_0} = \frac{Q}{5\varepsilon_0}$$

Hence, option (C) is correct.

Option (D): $h > 2R$ and $r = \frac{4R}{5}$ then $\phi = \frac{Q}{5\varepsilon_0}$



Total charge enclosed is

$$Q_{\text{en}} = (1-\cos\theta)Q = \left(1-\frac{3}{5}\right)Q \quad \left(\because \sin\theta = \frac{r}{R} = \frac{3}{5} \Rightarrow \cos\theta = \frac{3}{5}\right)$$

$$Q_{\text{en}} = \frac{2}{5}Q$$

$$\Rightarrow \phi = \frac{Q_{\text{en}}}{\varepsilon_0} = \frac{2}{5}\frac{Q}{\varepsilon_0}$$

Hence, option (D) is incorrect.

9. From the given V–T diagram:

Process 1 $\rightarrow$ 2 is isobaric where pressure is constant,

$$P = \frac{RT_0}{V_0}$$

Process 2 $\rightarrow$ 3 is isochoric where volume is constant,

$$V = 2V_0$$

Process 3 $\rightarrow$ 4 is isobaric where pressure is constant,

$$P = \frac{RT_0}{2V_0}$$

Process 4 $\rightarrow$ 1 is isochoric where volume is constant, $V = V_0$. So, work done in thermodynamic cycle ($1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$) is

$$|W| = P_{1\rightarrow 2}V_0 - P_{3\rightarrow 4}V_0 = \frac{RT_0}{V_0}V_0 - \frac{RT_0}{2V_0}V_0$$

$$|W| = \frac{1}{2}RT_0$$

Hence, option (A) is correct.
The thermodynamic cycle exhibits only isochoric and isobaric process. Hence, option (B) is incorrect.
Now, the ratio of heat transfer during processes $1 \rightarrow 2$ and $2 \rightarrow 3$ is

$$\left|\frac{Q_{1\rightarrow 2}}{Q_{2\rightarrow 3}}\right| = \left|\frac{nC_P\Delta T_{12}}{nC_V\Delta T_{23}}\right| = \left|\frac{\frac{5R}{2}(2T_0 - T_0)}{\frac{3R}{2}(T_0 - 2T_0)}\right|$$

$$\Rightarrow \left|\frac{Q_{1\rightarrow 2}}{Q_{2\rightarrow 3}}\right| = \frac{5}{3}$$

Hence, option (C) is correct.
Ratio of heat transfer during process $1 \rightarrow 2$ and $3 \rightarrow 4$ is

$$\left|\frac{Q_{1\rightarrow 2}}{Q_{3\rightarrow 4}}\right| = \left|\frac{nC_P\Delta T_{12}}{nC_P\Delta T_{34}}\right| = \left|\frac{\frac{5R}{2}(2T_0 - T_0)}{\frac{5R}{2}\left(\frac{T_0}{2} - T_0\right)}\right|$$

$$= \left|\frac{T_0}{-T_{0/2}}\right| = 2$$

$$\left|\frac{Q_{1\rightarrow 2}}{Q_{3\rightarrow 4}}\right| = 2$$

Thus, option (A) and (C) are correct and option (B) and (D) are incorrect.

10. Given, thin convex lens made of two materials.

Radius of curvature of left spherical surface = radius of curvature of right spherical surface

When $n_1 = n_2 = n$, focal length $= f$

When $n_1 = n$ and $n_2 = n + \Delta n$, focal length $= f + \Delta f$

Here, $\Delta n \ll (n-1)$ and $1 < n < 2$

Now, when $n_1 = n_2 = n$, from lens maker's formula

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (n-1)\left(\frac{1}{R} - \frac{1}{(-R)}\right)$$

$$\Rightarrow \frac{1}{f} = (n-1)\left(\frac{2}{R}\right) \tag{1}$$

When $n_1 = n$ and $n_2 = n + \Delta n$

$$\frac{1}{f+\Delta f} = (n-1)\frac{1}{R_1} - (n+\Delta n - 1)\frac{1}{+R_2}$$

$$= (n-1)\frac{1}{R} - (n+\Delta n-1)\frac{1}{(-R)}$$

$$\Rightarrow \frac{1}{f+\Delta f} = \frac{1}{R}(n-1+n+\Delta n-1) = \frac{1}{R}(2n+\Delta n-2)$$

$$\Rightarrow \frac{1}{f+\Delta f} = \frac{1}{R}(2n+\Delta n-2) \tag{2}$$

Subtracting Eq. (2) from Eq. (1), we get

$$\frac{1}{f} - \frac{1}{f+\Delta f} = (n-1)\left(\frac{2}{R}\right) - \frac{1}{R}(2n+\Delta n-2)$$

$$= \frac{1}{R}(2n-2-2n-\Delta n+2) = -\frac{\Delta n}{R}$$

$$\Rightarrow \frac{1}{f} - \frac{1}{f+\Delta f} = -\frac{\Delta n}{R} \Rightarrow \frac{(f+\Delta f)-f}{f(f+\Delta f)} = -\frac{\Delta n}{R}$$

$$\Rightarrow \frac{\Delta f}{f(f+\Delta f)} = -\frac{\Delta n}{R} \sim \frac{\Delta f}{f} = -\frac{\Delta n}{R}$$

$$\Rightarrow \frac{\Delta f}{f} = -\frac{\Delta n}{R} \times f$$

Using Eq. (1), we get $f = \dfrac{R}{2(n-1)}$

$$\Rightarrow \frac{\Delta f}{f} = -\frac{\Delta n}{R} \times \frac{R}{2(n-1)} \Rightarrow \frac{\Delta f}{f} = -\frac{\Delta n}{2(n-1)}$$

$$\Rightarrow \left|\frac{\Delta f}{f}\right| = \frac{1}{2}\left|\frac{\Delta n}{n-1}\right|$$

$$\Rightarrow \left|\frac{\Delta f}{f}\right| > \left|\frac{\Delta n}{n}\right|$$

Hence, option (A) is incorrect.

Now, for $n = 1.5$, $\Delta n = 10^{-3}$ and $f = 20$ cm. Using relation $\dfrac{\Delta f}{f(f+\Delta f)} = -\dfrac{\Delta n}{R}$ and Eq. (1), we get

$$\frac{\Delta f}{f(f+\Delta f)} = -\frac{\Delta n}{2(n-1)f} \Rightarrow \frac{\Delta f}{f+\Delta f} = -\frac{\Delta n}{2n-2}$$

$$\Rightarrow \Delta f(2n-2) = -\Delta n(f+\Delta f) \Rightarrow \Delta f(2n-2) = -\Delta nf - \Delta n\Delta f$$

$$\Rightarrow \Delta f(2n-2+\Delta n) = -\Delta nf$$

$$\Rightarrow \Delta f = \frac{-\Delta n \times f}{2n-2+\Delta n}$$

$$\Rightarrow |\Delta f| = \frac{\Delta n + f}{2n-2+\Delta n}$$

Substituting all values, we get

$$|\Delta f| = \frac{10^{-3}\times 20}{2\times 1.5 - 2 + 10^{-3}} = 0.02\,\text{cm}$$

Hence, option (B) is correct.

Since $\frac{\Delta f}{f} = -\frac{\Delta n}{2(n-1)}$, so if $\frac{\Delta n}{n} < 0$ then $\frac{\Delta f}{f} > 0$.

Hence, option (C) is correct.

Since the relation between $\frac{\Delta f}{f}$ and $\frac{\Delta n}{n}$ is independent of radius of curvature so the relation remains unchanged only the sign of focal length changes. Hence, option (D) is correct.

Thus option (A) is incorrect and (B), (C), (D) are correct.

11. Given, Mass and angular momentum are dimensionless.

Dimension of length = L

Angular momentum = mvr

$$\Rightarrow mvr = \text{M}^0\text{L}^0\text{T}^0$$

Since v = m s^{-1}, therefore,

$$\Rightarrow \text{M}^0\frac{\text{L}}{\text{T}}\text{L} = \text{M}^0\text{L}^0\text{T}^0 \Rightarrow \text{M}^0\text{L}^0\text{T}^{-1} = \text{M}^0\text{L}^0\text{T}^0$$

$$\Rightarrow \text{L}^2 = \text{T} \qquad (1)$$

Dimension of linear momentum:

Linear momentum, $p = mv$

$\Rightarrow$ Dimension of $p = \text{MLT}^{-1}$

Using Eq. (1), dimension of $p = \text{MLT}^{-1} = \text{MLL}^{-2}$

But mass is dimensionless, so dimension of $p = \text{L}^{-1}$

Hence, option (A) is correct.

Dimension of energy:

Energy, $E = \frac{1}{2}mv^2$

$\Rightarrow$ Dimension of $E = \text{M}(\text{LT}^{-1})^2 = \text{ML}^2\text{T}^{-2}$

Using Eq. (1), dimension of $E = \text{ML}^2\text{L}^{-4} = \text{ML}^{-2}$

Mass is dimensionless, so dimension of $E = \text{L}^{-2}$

Hence, Option (B) is correct.

Dimension of force:

Force $F = ma$

Dimension of force = MLT^{-2}

Using Eq. (1) and fact that mass is dimensionless, we have

Dimension of force $= \text{M}^0\text{LL}^{-4} = \text{L}^{-3}$

Hence, option (C) is correct.

Dimension of power:

$$\text{Power, } P = \frac{\text{Energy } E}{\text{Time } t}$$

$$\Rightarrow \text{Dimension of power} = \frac{\text{ML}^2\text{T}^{-2}}{\text{T}} = \text{ML}^2\text{T}^{-3}$$

Using Eq. (1) and fact that mass is dimensionless, dimension of power = L^{-4}

Hence, option (D) is incorrect.

Therefore, option (A), (B) and (C) are correct and option (D) is incorrect.

12. Given, Resistance of galvanometer $R_g = 10\ \Omega$

Full scale defection at current $I_g = 2\ \mu\text{A} = 2\times 10^{-6}$ A

For galvanometer converted to voltmeter of range 100 mV = 100×10^{-3} V, we have

$$V = (R_g + R)I_g$$

$$\Rightarrow R = \frac{V}{I_g} - R_g$$

Substituting values, we get

$$R = \frac{100\times 10^{-3}}{2\times 10^{-6}} - 10$$

$$= 50\times 10^3 - 10$$

$$= 5\times 10^4 - 10$$

$$R \sim 5\times 10^4\,\Omega \text{ or } 50\,\text{k}\Omega$$

Thus, resistance of voltmeter is $\sim 50\,\text{k}\Omega$, so option (A) is incorrect.

For galvanometer converted to ammeter of range 1mA = 1×10^{-3} A, we have

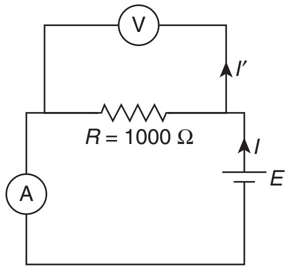
$$R = \frac{R_g \times I_g}{I - I_g} = \frac{10\times 2\times 10^{-6}}{1\times 10^{-3} - 2\times 10^{-6}} = \frac{2\times 10\times 10^{-6}}{10^{-6}(1\times 10^3 - 2)}$$

$$R = \frac{2\times 10}{1000-2} \sim \frac{2\times 10}{1000} \sim 0.02\,\Omega$$

Therefore, resistance of ammeter will be 0.02 Ω.

Hence, option (B) is correct.

Now the ammeter and voltmeter are used to measure voltage and current in Ohm's law experiment with R = 1000 Ω.



Current $I = \dfrac{E}{R_{\text{eq}}}$

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R} + \frac{1}{R_{\text{voltmeter}}}$$

$$\frac{1}{R_{\text{eq}}} = \frac{R_{\text{voltmeter}} + R}{R \times R_{\text{voltmeter}}}$$

Therefore,

$$I = \frac{E(R + R_{\text{voltmeter}})}{(R \times R_{\text{voltmeter}})} = \frac{(1000 + 5\times 10^4)E}{1000 \times 5 \times 10^4}$$

$$I = \frac{51 \times 10^3 E}{5 \times 10^7} = \frac{51}{5} \times 10^{-4} E$$

Now, $I' = I\left(\dfrac{R + R_{\text{voltmeter}}}{R}\right) = \dfrac{1000 + 5\times 10^4}{1000} I$

$$I' = \frac{51 \times 10^3}{1000} I = 51 I$$

So $R_{\text{measured}} = \dfrac{I'}{I} \times 1000 = 980.4\ \Omega$

Hence, option (C) is correct.
Therefore, option (B) and (C) are correct, (A) and (D) are incorrect.

13. Given, Force on particle, $\vec{F} = (\alpha y i + 2\alpha j)\,\text{N}$; $\alpha = -1\,\text{N m}^{-1}$

Work done on particle from A to B is

$$W_{\text{AB}} = \vec{F} \cdot d\vec{x} = \alpha y \cdot \Delta x = (-1)(1.0)(1.0 - 0)$$
$$W_{\text{AB}} = -1.0$$

Work done on particle from B to C is

$$W_{\text{BC}} = \vec{F} \cdot d\vec{y} = 2\alpha x \cdot \Delta y = (2)(-1)(1.0)(0.5 - 1.0)$$
$$W_{\text{BC}} = 1.0$$

Work done on particle from C to D

$$W_{\text{CD}} = \vec{F} \cdot d\vec{x} = \alpha y \Delta x = (-1)(0.5)(0.5 - 1.0)$$
$$W_{\text{CD}} = (-1)(0.5)(-0.5) = 0.25$$

Work done on particle from D to E

$$W_{\text{DE}} = \vec{F} \cdot d\vec{y} = 2\alpha x \Delta y = (2)(-1)(0.5)(0 - 0.5)$$
$$W_{\text{DE}} = (2)(-1)(0.5)(-0.5) = 0.5$$

Work done on particle from E to F

$$W_{\text{EF}} = \vec{F} \cdot d\vec{x} = \alpha y \Delta x = (-1)(0)(0 - 0.5)$$
$$W_{\text{EF}} = 0$$

Work done on particle from F to A

$$W_{\text{FA}} = \vec{F} \cdot d\vec{y} = 2\alpha x \Delta y = (2)(-1)(0)(1.0 - 0)$$
$$W_{\text{FA}} = 0$$

Total work done,

$$W = W_{\text{AB}} + W_{\text{BC}} + W_{\text{CA}} + W_{\text{DE}} + W_{\text{EF}} + W_{\text{FA}}$$
$$= -1.0 + 1.0 + 0.25 + 0.5 + 0 + 0 = 0.75\,\text{J}$$

Therefore, the work done on the particle by the force $\vec{F}$ will be 0.75 joule.

14. We know,

Elongation of wire, $\Delta l = \dfrac{Fl}{A_{\text{c}}}$

where F is force on wire or tension; L is length of wire; A is area of cross-section; Y is Young's modulus. Therefore, elongation in copper wire is

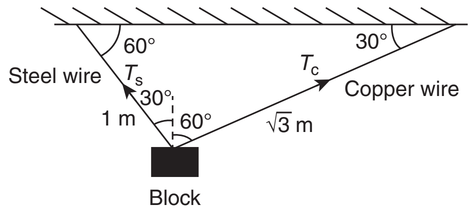
$$\Delta l_{\text{c}} = \frac{F_{\text{c}} l_{\text{c}}}{A_{\text{c}} Y_{\text{c}}} = \frac{T_{\text{c}} \times \sqrt{3}}{0.5 \times 1 \times 10^{11}}$$

Elongation in steel wire is

$$\Delta l_{\text{s}} = \frac{F_{\text{s}} l_{\text{s}}}{A_{\text{s}} Y_{\text{s}}} = \frac{T_{\text{s}} \times 1}{0.5 \times 2 \times 10^{11}} = \frac{T_{\text{s}}}{1 \times 10^{11}}$$

$$\Rightarrow \frac{\Delta l_{\text{c}}}{\Delta l_{\text{s}}} = \frac{T_{\text{c}} \times \sqrt{3}}{0.5 \times 1 \times 10^{11}} \times \frac{1 \times 10^{11}}{T_{\text{s}}} = \frac{\sqrt{3}}{0.5} \frac{T_{\text{c}}}{T_{\text{s}}} = 2\sqrt{3} \frac{T_{\text{c}}}{T_{\text{s}}}$$

Now

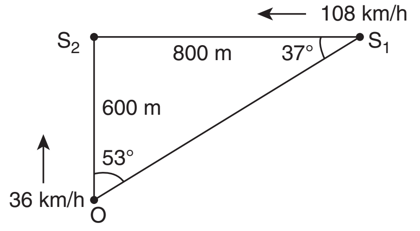


$$T_{\text{s}} \sin 30° = T_{\text{c}} \sin 60° \Rightarrow \frac{T_{\text{c}}}{T_{\text{s}}} = \frac{\sin 30°}{\sin 60°} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{\Delta l_{\text{c}}}{\Delta l_{\text{s}}} = 2\sqrt{3} \times \frac{1}{\sqrt{3}} = 2$$

Thus, the ratio $\dfrac{\Delta l_{\text{c}}}{\Delta l_{\text{s}}}$ is 2.0.

15. Velocity of train S_1 = 108 km/h; Velocity of observer O = 36 km/h; Frequency of both the whistles = 120 Hz; Distance between O and S_2 = 600 m; Distance between S_1 and S_2 = 800 m; Velocity of sound, v = 330 m/s.



Frequency observed by observer O from S_1 is

$$f_1 = f\left(\frac{v + v_0 \cos 53°}{v - v_{s_1} \cos 31°}\right) = 120\left(\frac{330 - 10\cos 53°}{330 - 30\cos 37°}\right)$$

$$f_1 = 120\left(\frac{330 + 6}{330 - 24}\right) = 120 \times \frac{336}{306}$$

Frequency observed by observe O from S_2 is

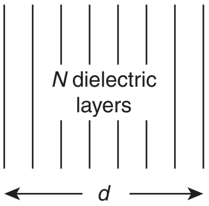
$$f_2 = f\left(\frac{v + v_0}{v}\right) = 120 \times \frac{(330 + 10)}{330} = \frac{120 \times 340}{330}$$

Number of beats $= n = f_1 - f_2$

$$n = 120 \times \frac{336}{306} - \frac{120 \times 340}{330}$$
$$= 131.76 - 123.63 = 8.13$$

Therefore, number of beats heard by O is 8.13 (~ 8) Hz.

16. We have Capacitance $= C$; Area of plates $= A$.

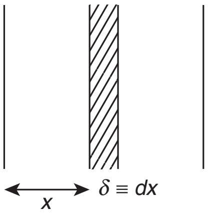


Thickness of dielectric layer is $\delta = \dfrac{d}{N}$

Dielectric constant of mth layer is $K_m = K\left(1 + \dfrac{m}{N}\right)$

For $N > 10^3$, $C = \alpha\left(\dfrac{k\varepsilon_0 A}{d \ln 2}\right)$ (1)

Let the mth layer is at distance x from plate



$$\Rightarrow \delta \equiv dx \text{ and } \frac{m}{N} = \frac{x}{d}$$
$$\Rightarrow K_m = K\left(1 + \frac{x}{d}\right)$$

Now, these dielectric layers are parallel to the plates so we can think of it as N capacitors connected in series with capacitance of each layer be

$$dC = \frac{A\varepsilon_0 K_m}{dx} = \frac{A\varepsilon_0}{dx} k\left(1 + \frac{x}{d}\right)$$

Equivalent capacitance will be

$$\frac{1}{C_{eq}} = \int_0^d \frac{1}{dC} = \int_0^d \frac{1}{A\varepsilon_0 K\left(1 + \frac{x}{d}\right)} \cdot dx$$

$$\frac{1}{C_{eq}} = \frac{1}{A\varepsilon_0 K} \int_0^d \frac{1}{\left(1 + \frac{x}{d}\right)} dx = \frac{1}{A\varepsilon_0 K}\left(\ln\left(1 + \frac{x}{d}\right) \times d \Big|_0^d\right)$$

$$\frac{1}{C_{eq}} = \frac{d}{A\varepsilon_0 K}\left[\ln\left(1 + \frac{d}{d}\right) - \ln\left(1 + \frac{0}{d}\right)\right] = \frac{d}{A\varepsilon_0 K}(\ln(1+1) - \ln(1))$$

$$\frac{1}{C_{eq}} = \frac{d}{A\varepsilon_0 K}(\ln(2) - 0) = \frac{d}{A\varepsilon_0 K}\ln 2$$

or $C_{eq} = \dfrac{A\varepsilon_0 K}{d \ln 2}$

Comparing with equation (1), we get $\alpha = 1$.

17. Given,
Temperature of pouring liquid = 30 °C
Temperature of calorimeter = 110 °C
First 5 g of liquid vapourised.
80 g of liquid addition gives equilibrium temperature = 50 °C
Boiling point of liquid = 80 °C
Now, we know that

$$msdT + mL = WdT$$

where m is mass of calorimeter; s is specific heat and L is latent heat of the liquid.
When first 5 g of liquid is poured:

$$5 \times s \times (80 - 30) + 5L = W(110 - 80)$$
$$5 \times 50 \times s + 5L = 30W$$
$$250s + 5L = 30W \quad (1)$$

When 80 g of liquid is poured:

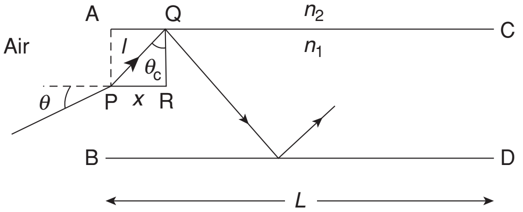
$$80 \times s(50 - 30) + 80 \times 0 = W(80 - 50)$$
$$80 \times s \times 20 = 30W$$
$$1600s = 30W \quad (2)$$

From Eqs. (1) and (2), we get
$1600s = 250s + 5L$

$$\Rightarrow 1600s - 250s = 5L$$
$$\Rightarrow 1350s = 5L$$
$$\frac{L}{s} = \frac{1350}{5} = 270\ °\text{C}$$

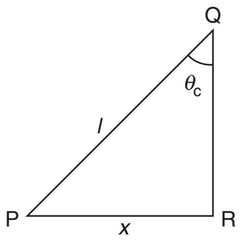
Therefore, the ratio of latent heat of liquid to specific heat is 270 °C.

18. Given, $n_1 = 1.5$; $n_2 = 1.44$; $L >> W$, $L = 9.6$ m.



Maximum time taken by ray to exit the plane CD $= t \times 10^{-9}$ s.
For this particular case, the light ray undergoes total internal reflection at surfaces.
For total internal reflection, we have

$$n_1 \sin\theta_c = n_2$$
$$\Rightarrow \sin\theta_c = \frac{n_2}{n_1} = \frac{1.44}{1.5}$$



Now in ΔPQR, $\sin\theta_c = \dfrac{x}{l}$

$$\Rightarrow l = \frac{x}{\sin\theta_c} \Rightarrow l = \frac{1.5}{1.44}x$$

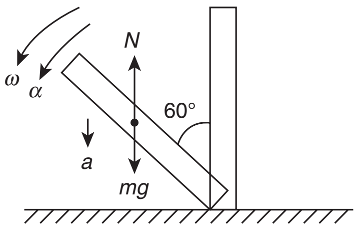
Therefore, time taken by ray to exit, $t = \dfrac{l}{c/n_1} = \dfrac{n_1 l}{c}$

Time, $t = \dfrac{1.5}{3\times10^8}\times\dfrac{1.5}{1.44}\times 9.6 = 5\times10^{-8} = 50\times10^{-9}$

$\Rightarrow t = 50$

PAPER 2

1. Given: Mass of the rod is M; Length of rod is L.



- **Option (A):** Applying work–energy theorem, we get the change in kinetic energy of the rod as

$$\frac{MgL}{2}(1-\cos\theta) = \frac{1}{2}I\omega^2$$

where I is the moment of inertia $\left(=\dfrac{ML^2}{3}\right)$ and ω is the angular speed of the rod. Therefore,

$$\frac{MgL}{2}(1-\cos 60^\circ) = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

$$\Rightarrow \frac{MgL}{2}\left(1-\frac{1}{2}\right) = \frac{ML^2}{6}\omega^2$$

$$\Rightarrow \frac{MgL}{2}\times\frac{1}{2} = \frac{ML^3}{6}\omega^2$$

$$\Rightarrow \frac{g}{2} = \frac{L}{3}\omega^2 \Rightarrow \omega^2 = \frac{3g}{2L} \Rightarrow \omega = \sqrt{\frac{3g}{2L}}$$

That is, the angular speed of the rod is obtained as $\sqrt{\dfrac{3g}{2L}}$.

Hence, option (A) is correct.

- **Option (B):** Now, using $\tau = I\alpha$, where $\alpha = \dfrac{\tau}{I}$ and τ is the torque, we get

$$\alpha = \frac{\left(Mg\dfrac{L}{2}\sin\theta\right)}{\left(\dfrac{1}{3}ML^2\right)} = \frac{3g\sin\theta}{2L} = \frac{3g\sin 60^\circ}{2L}$$

$$\Rightarrow \alpha = \frac{3\sqrt{3}}{4L}g$$

Therefore, the angular acceleration of the rod is obtained as $\dfrac{3\sqrt{3}}{4L}g$.

Hence, option (B) is incorrect.

- **Option (C):** The radial acceleration of the rod is

$$\frac{L}{2}\omega^2 = \frac{L}{2}\times\frac{3g}{2L}$$

That is, the radial acceleration of the rod's centre of mass is $\dfrac{3}{4}g$.

Hence, option (C) is correct.

- **Option (D):** The net acceleration of centre of mass of rod is given by

a_{net} = Radial acceleration × $\cos\theta$ + Tangential acceleration × $\sin\theta$

$$\Rightarrow a = \frac{L}{2}\alpha\sin\theta + \omega^2\frac{L}{2}\cos\theta$$

Substituting the values in this equation, we get

$$a = \frac{L}{2}\frac{3\sqrt{3}}{4}\frac{g}{L}\sin 60^\circ + \frac{3g}{2L}\times\frac{L}{2}\cos 60^\circ$$

$$a = \frac{3\sqrt{3}}{8}g\frac{\sqrt{3}}{2} + \frac{3}{4}g\frac{1}{2} = \frac{9}{16}g + \frac{3}{8}g = \left(\frac{9}{16}+\frac{3}{8}\right)g$$

$$\Rightarrow a = \frac{9+3\times2}{16}g = \frac{9+6}{16}g = \frac{15}{16}g$$

Now, from Newton's second law of motion, we have

$$Mg - N = Ma$$

$$\Rightarrow Mg - N = \frac{15}{16}g\times M$$

$$\Rightarrow N = Mg - \frac{15}{16}Mg$$

$$\Rightarrow N = Mg\left(1-\frac{15}{16}\right) = Mg\frac{1}{16}$$

Thus, the normal reaction force from the floor on the rod is found to be $\dfrac{Mg}{16}$.

Hence, option (D) is correct.

2. Mass M has acceleration $a_2 - a_1$. Mass $2M$ has acceleration $a_3 - a_1$.

- **Option (D):** As the upward acceleration of mass M is equal to downward acceleration of mass $2M$, we have

$$a_2 - a_1 = -(a_3 - a_1)$$

or $$a_2 - a_1 = a_1 - a_3 \qquad (1)$$

Hence, option (D) is correct.

We shall check other three options:

- **Option (A):** We have

$$2Mg - T - 2Ma_3 \qquad (2)$$

$$Mg - T = Ma_2 \qquad (3)$$

$$2T - kx = 2Ma_1 \qquad (4)$$

and $\dfrac{T - M(g - a_1)}{M} = \dfrac{2M(g - a_1) - T}{2M}$

$$\Rightarrow T = \frac{4M}{3}(g - a_1)$$

Substituting the value of T is Eq. (4), we get

$$2 \times \frac{4}{3} M(g - a_1) - kx = 2Ma_1$$

$$\Rightarrow \frac{8}{3} Mg - \frac{8}{3} Ma_1 - kx = 2Ma_1$$

$$\Rightarrow 2Ma_1 + \frac{8}{3} Ma_1 = \frac{8}{3} Mg - kx$$

$$\Rightarrow \frac{14M}{3} a_1 = \frac{8}{3} Mg - kx \qquad (5)$$

If we compare this equation to the equation of simple harmonic motion, we get the amplitude as $\dfrac{8Mg}{3K}$ and the maximum extension is

$$x_0 = 2 \times \frac{8Mg}{3K} = \frac{16}{3}\frac{Mg}{K}$$

Hence, option (A) is incorrect.

- **Option (C):** When we have the extension of the spring as

$$\frac{x_0}{4} = \frac{1}{4} \times \frac{16}{3}\frac{Mg}{K} = \frac{4}{3}\frac{Mg}{K}$$

Let us use Eq. (5):

$$\frac{14}{3} Ma_1 = \frac{8}{3} Mg - \frac{4}{3}\frac{Mg}{K} \times K = \frac{8}{3} Mg - \frac{4}{3} Mg = \frac{4}{3} Mg$$

$$\Rightarrow \frac{14}{3} Ma_1 = \frac{4}{3} Mg \Rightarrow a_1 = \frac{2}{7} g$$

This, the magnitude of acceleration of block connected to spring $a_1 = \dfrac{2}{7} g$.

Hence, option (C) is incorrect.

- **Option (B):** When we have the extension of the spring as $\dfrac{x_0}{2}$, we can write as

$$\text{Speed of block} = \omega \times \text{Amplitude}$$

From Eq. (5), we have

$$\omega^2 = \frac{3K}{14M} \Rightarrow \omega = \sqrt{\frac{3K}{14M}}$$

Thus, the speed of the block is

$$\sqrt{\frac{3K}{14M}} \times \frac{8Mg}{3K} = \frac{8}{3} g\sqrt{\frac{3M}{14K}}$$

Hence, option (B) is incorrect.

3. Given: The distance of the piston from closed end is $L = L_0$; Speed of particle is $v = v_0$.

- **Option (A):** The rate of collision of particle is

$$\frac{1}{2L/V} = \frac{v}{2L} = \frac{v_0}{2L_0}$$

Hence, option (A) is incorrect.

- **Option (B):** The speed of particle after collision is

$$2V + v = 2V + v_0$$

Thus, the change in speed is

$$(2V + v_0) - v_0 = 2V$$

Therefore, after each collision with piston, the particle speed increases by $2V$.

Hence, option (B) is correct.

- **Option (C):** If the piston moves inward by dL, the speed of particle increases by

$$dv = 2V \times \frac{dL}{V} \times \frac{v}{2L} = v\frac{dL}{L}$$

$$\Rightarrow dv = \frac{dL}{L} v$$

Thus, the particle speed increases by $v\dfrac{dL}{L}$.

Hence, option (C) is incorrect.

- **Option (D):** We know that the kinetic energy is expressed as

$$(\text{KE}) = \frac{1}{2} Mv^2$$

At L_0, we have

$$(\text{KE})_1 = \frac{1}{2} mv_0^2$$

At $\dfrac{L_0}{2}$, we have

$$(\text{KE})_2 = \frac{1}{2} m(2v_0)^2 = 4\left(\frac{1}{2} mv_0^2\right) = 4\,(\text{KE})_1$$

That is, $(\text{KE})_2 = 4(\text{KE})_1$

Thus, the particle's kinetic energy is increased by a factor of 4 when the piston is moved inward from L_0 to $\dfrac{1}{2}L_0$.

Hence, option (D) is correct.

4. The dipole moment of electric dipole is $\vec{p} = \dfrac{p_0}{\sqrt{2}}(\hat{i} + \hat{j})$.

The magnitude of uniform electric field is E_0.

- **Option (C):** Given: The potential is constant on a circle of radius R; thus, the tangential electric field at all points of circle must be zero.

Therefore, the total electric field at point B, which is the tangential electric field, is zero. That is, $\vec{E}_B = 0$.

Hence, option (C) is correct.

- **Option (A):** Now, we have

$$E_0 = \frac{1}{4\pi\varepsilon_0}\frac{|p|}{R^3} = \frac{1}{4\pi\varepsilon_0}\frac{p_0}{R^3}$$

$$\Rightarrow R^3 = \frac{p_0}{4\pi\varepsilon_0 E_0} \Rightarrow R = \left(\frac{p_0}{4\pi\varepsilon_0 E_0}\right)^{1/3}$$

Hence, option (A) is correct.

- **Option (B):** The total electric field at point A is given by

$$E_A = E_0 + 2 \times \frac{1}{4\pi\varepsilon_0}\frac{p}{R^3} = E_0 + 2E_0 = 3E_0$$

Hence, option (B) is incorrect.

Now since E_0 is uniform, the electric field due to dipole is different at different points.
Thus, the magnitude of total electric field on any two points of the circle is different.
Hence, option (D) is incorrect.

5. The given ideal gas is a mixture of 5 moles of monatomic gas and 1 mole of rigid diatomic gas.
Initial pressure of the ideal gas is P_0. Initial volume is V_0. Initial temperature is T_0.
The gas mixture is adiabatically compressed to $\frac{V_0}{4}$.

- **Option (D):** γ value (adiabatic constant) of the ideal gas mixture is expressed as

$$\gamma = \frac{n_1 C_{P_1} + n_2 C_{P_2}}{n_1 C_{V_1} + n_2 C_{V_2}}$$

For 5 moles of monatomic gas, we have $n_1 = 5$. Therefore,

$$C_{P_1} = \frac{5R}{2}$$

$$C_{V_1} = \frac{3R}{2}$$

For 1 mole of diatomic gas, we have $n_2 = 1$. Therefore,

$$C_{P_2} = \frac{7R}{2}$$

$$C_{V_2} = \frac{5R}{2}$$

Hence, the γ value is given as

$$\gamma = \frac{\left(5\times\frac{5R}{2} + 1\times\frac{7R}{2}\right)}{\left(5\times\frac{3R}{2} + 1\times\frac{5R}{2}\right)} = \frac{\left(\frac{25R}{2} + \frac{7R}{2}\right)}{\left(\frac{15R}{2} + \frac{5R}{2}\right)}$$

$$\gamma = \frac{(25+7)R/2}{(15+5)R/2} = \frac{32}{20} = \frac{8}{5} = 1.6$$

That is, the adiabatic constant of the gas mixture is 1.6.
Hence, option (D) is correct.

- **Option (C):** For an adiabatic process, we have

$$PV^{\gamma} = \text{constant}$$

Therefore,

$$P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$$

$$\Rightarrow P_0 V_0^{\gamma} = P\left(\frac{V_0}{4}\right)^{\gamma} = P_0 V_0^{\gamma} = \frac{PV_0^{\gamma}}{4^{\gamma}}$$

$$\Rightarrow P = P_0 4^{\gamma} = P_0 4^{8/5} = P_0 (2)^{2\times(8/5)}$$

$$\Rightarrow P = P_0 2^{3.2} = 9.2 P_0$$

Hence, option (C) is correct.

- **Option (A):** Now, the work done is

$$W = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1} = \frac{P_0 V_0 - P_0 \times 9.2 \times \frac{V_0}{4}}{\frac{8}{5} - 1}$$

$$W = \frac{P_0 V_0\left(1 - \frac{9.2}{4}\right)}{\frac{3}{5}} = \frac{P_0 V_0 (4 - 9.2)\times 5}{3\times 4}$$

$$= \frac{P_0 V_0 (-5.2)\times 5}{12} = -2.16\, P_0 V_0$$

Now, using gas equation, $PV = nRT$, we have

$$P_0 V_0 = 5RT_0$$

Therefore, work done is

$$W = -2.16 \times 6RT_0 = -12.96(RT_0)$$

$$\Rightarrow W \sim -13RT_0$$

$$|W| = 13RT_0$$

Hence, option (A) is correct.

- **Option (B):** Average kinetic energy of gas is given by $nC_V T_0$.

After compression, we have

$$T_2 = \left(\frac{P_2 V_2}{P_1 V_1}\right) T_1 = \left(\frac{P_2 V_2}{P_0 V_0}\right) T_0$$

$$T_2 = \left(\frac{9.2 P_0 \times \frac{V_0}{4}}{P_0 V_0}\right) T_0 = 2.3 T_0$$

or $T = 2.3T_0$, and

$$C_V = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2} = \frac{\left(5\times\frac{3R}{2} + 1\times\frac{5R}{2}\right)}{5+1}$$

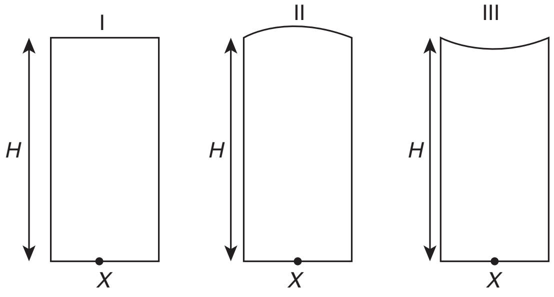
$$C_V = \frac{\left(\frac{15R}{2} + \frac{5R}{2}\right)}{6} = \frac{\left(20\times\frac{R}{2}\right)}{6} = \frac{10}{6}R = \frac{5}{3}R$$

Thus, the average kinetic energy of gas is

$$6\times\frac{5}{3}R\times 2.3T_0 = 23\, RT_0$$

Hence, option (B) is incorrect.

6. Given, $H = 30$ cm; $n = 1.5$; $R = 3$ m (for gas cylinders II and III).



Lens maker's formula: $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$.

Here, $R_1 = \infty$ and $R_2 = R$. Therefore,

$$\frac{1}{f} = \frac{(1-n)}{R}$$

Also, $$\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$$

Therefore, $\frac{1-n}{R}=\frac{n_0}{v}-\frac{n}{u}$

- **For gas cylinder I:** $n = 1.5$; $n_a = 1$; $R=\infty$; $v = H_1$; $H = u = 30$ cm. Therefore,

$$\frac{1-1.5}{\infty}=\frac{1}{-H_1}-\frac{1.5}{-30}$$

$$\Rightarrow 0=\frac{-1}{H_1}+\frac{1.5}{30}\Rightarrow\frac{1}{H_1}=\frac{1.5}{30}\Rightarrow H_1=\frac{30}{1.5}=\frac{300}{15}=20\text{ cm}$$

- **For gas cylinder II:** $n = 1.5$; $n_a = 1$; $R = 3$ m $= 300$ cm; $v = H_2$; $u = H_1 = 30$ cm. Therefore,

$$\frac{1-1.5}{-300}=\frac{1}{-H_2}-\frac{1.5}{-30}$$

$$\Rightarrow\frac{0.5}{-300}=\frac{-1}{H_2}+\frac{1.5}{30}$$

$$\Rightarrow\frac{1}{H_2}=\frac{15}{300}-\frac{0.5}{300}=\frac{15-0.5}{300}=\frac{14.5}{300}=\frac{300}{14.5}\text{ cm}$$

- **For gas cylinder III:** $n = 1.5$; $n_a = 1$; $R = 3$ m $= 300$ cm; $v = H_3$; $u = H_1 = 30$ cm. Therefore,

$$\frac{1-1.5}{+300}=\frac{1}{-H_3}-\frac{1.5}{-30}$$

$$\Rightarrow\frac{-0.5}{300}=\frac{-1}{H_3}+\frac{1.5}{30}$$

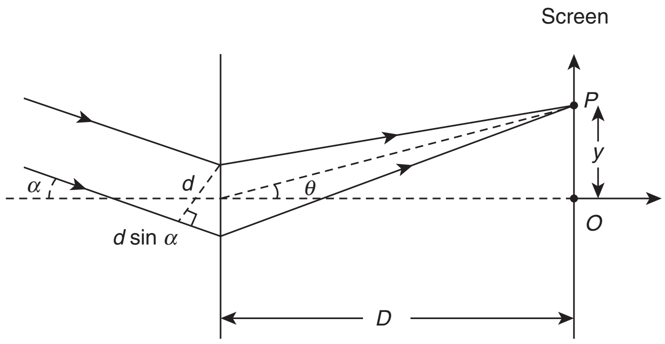
$$\Rightarrow\frac{1}{H_3}=\frac{15}{300}+\frac{0.5}{300}=\frac{15+0.5}{300}=\frac{15.5}{300}=\frac{300}{15.5}\text{ cm}$$

From these findings, we have the following:

- $H_2 > H_1$. Hence, option (A) is correct.
- $H_3 < H_1$. Hence, option (B) is incorrect.
- $H_2 > H_3$. Hence, option (C) is correct.
- $H_2 - H_1 = \frac{300}{145} - 20 = 20.69 - 20 = 0.7$ cm.

Hence, option (D) is incorrect.

7. Given, $d = 0.3\text{ mm} = 0.3\times10^{-3}$ m; $D = 1$ m; $\lambda = 600\text{ nm} = 600\times10^{-9}$ m; $PO = y = 11.0\text{ mm} = 11\times10^{-3}$ m. Now, the path difference is given as



$$\Delta x = d\sin\alpha+\frac{yd}{D}$$

Option (C): For $\alpha=\frac{0.36}{\pi}$, the path difference is

$$\Delta x = d\sin\left(\frac{0.36}{\pi}\right)+\frac{yd}{D}$$

$$\Delta x = 0.3\times10^{-3}\sin\left(\frac{0.36}{\pi}\right)+\frac{11\times10^{-3}\times0.3\times10^{-3}}{1}$$

$$\Delta x = 3900\times10^{-9}\text{ m}$$

For destructive interference, we have

$$\Delta x=(2n-1)\frac{\lambda}{2}$$

$$\Rightarrow\left(\frac{2\times\Delta x}{\lambda}+1\right)\times\frac{1}{2}=n$$

$$\Rightarrow\left(\frac{2\times3900\times10^{-9}}{600\times10^{-9}}+1\right)\times\frac{1}{2}=n$$

$$\Rightarrow(13+1)\times\frac{1}{2}=n\quad\Rightarrow\frac{14}{2}=n\Rightarrow n=7$$

Therefore, for $\alpha=\frac{0.36}{\pi}$ degree, there will be destructive interference at point P.

Hence, option (C) is correct.

Option (B): For $\alpha = 0$, path difference is

$$\Delta x=\frac{yd}{D}=\frac{11\times10^{-3}\times0.3\times10^{-3}}{1}=33\times10^{-7}$$

or $\Delta x = 3300\times10^{-9}$ m

Now, for constructive interference, we have

$$\Delta x=n\lambda\Rightarrow n=\frac{\Delta x}{\lambda}=\frac{3300\times10^{-9}}{600\times10^{-9}}=5.5$$

Therefore, there will be destructive interference at point P.

Hence, option (B) is incorrect.

Option (A): At point O, $y = 0$, $\sin\alpha\approx\alpha$. For $\alpha=\frac{0.36}{\pi}$, the path difference is

$$\Delta x = d\alpha+0$$

$$\Rightarrow\Delta x=0.3\times10^{-3}\times\frac{0.36}{\pi}\times\frac{\pi}{180}$$

Now, for destructive interference, we have

$$\Delta x=(2n-1)\frac{\lambda}{2}$$

$$\Rightarrow n=\frac{1}{2}\times\left[\frac{2\Delta x}{\lambda}+1\right]$$

$$\Rightarrow n=\frac{1}{2}\left[\frac{2\times0.3\times10^{-3}}{600\times10^{-9}}\times\frac{0.36}{\pi}\times\frac{\pi}{180}+1\right]$$

$$=\frac{1}{2}(2+1)=\frac{3}{2}$$

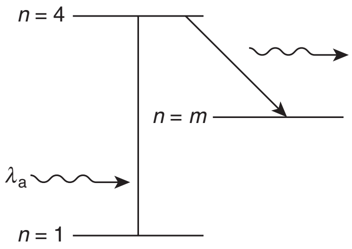
Therefore, there will be constructive interference at point O. Hence, option (A) is incorrect.

Option (D): Fringe width will remain same in all cases. Hence, option (D) is incorrect.

8. **Option (A):** We know that $E = \frac{hc}{\lambda}$

For absorption:

$$(E_4 - E_1) = \frac{hc}{\lambda_a} \qquad (1)$$



For emission:

$$(E_4 - E_m) = \frac{hc}{\lambda_e} \qquad (2)$$

Divide Eq. (2) by Eq. (1), we get

$$\frac{hc/\lambda_e}{\lambda c/\lambda_a} = \frac{E_4 - E_m}{E_4 - E_1}$$

$$\Rightarrow \frac{\lambda_a}{\lambda_e} = \frac{E_4 - E_m}{E_4 - E_1}$$

Given, $\frac{\lambda_a}{\lambda_e} = \frac{1}{5}$, that is,

$$\frac{E_4 - E_m}{E_4 - E_1} = \frac{1}{5}$$

Now, $E_n \propto \frac{1}{n^2}$

$$\Rightarrow \frac{\left(\frac{1}{4^2} - \frac{1}{m^2}\right)}{\left(\frac{1}{4^2} - \frac{1}{1^2}\right)} = \frac{1}{5} \Rightarrow \frac{\left(\frac{1}{16} - \frac{1}{m^2}\right)}{\left(\frac{1}{16} - \frac{1}{1}\right)} = \frac{1}{5}$$

$$\Rightarrow \frac{\left(\frac{1}{16} - \frac{1}{m^2}\right)}{\left(\frac{-15}{16}\right)} = \frac{1}{5} \Rightarrow \frac{1}{16} - \frac{1}{m^2} = \frac{-1}{5} \times \frac{15}{16}$$

$$\Rightarrow \frac{1}{16} + \frac{15}{5 \times 16} = \frac{1}{m^2} \Rightarrow \frac{5+15}{5 \times 16} = \frac{1}{m^2}$$

$$\Rightarrow \frac{20}{5 \times 16} = \frac{1}{m^2} \Rightarrow \frac{1}{4} = \frac{1}{m^2} \Rightarrow m = 2$$

Hence, option (A) is correct.

Option (C): Using $\lambda = \frac{h}{mv}$ and $E = \frac{hc}{\lambda}$, we get

$E = mv.c$ or $E = pc$

$$\Rightarrow p = \frac{E}{c}$$

Therefore, $\Delta p_e = \frac{E_4 - E_m}{c}$ and $\Delta p_a = \frac{E_4 - E_1}{c}$

Then $\frac{\Delta p_e}{\Delta p_a} = \frac{E_4 - E_m}{E_4 - E_1} = \frac{\left(\frac{1}{4^2} - \frac{1}{2^2}\right)}{\left(\frac{1}{4^2} - \frac{1}{1^2}\right)} = \frac{\frac{1}{16} - \frac{1}{4}}{\frac{1}{16} - 1}$

$$\Rightarrow \frac{\Delta p_e}{\Delta p_a} = \frac{\frac{1-4}{16}}{\frac{1-16}{16}} = \frac{(-3/16)}{(-15/16)} = \frac{3}{15} = \frac{1}{5}$$

or $$\frac{\Delta p_a}{\Delta p_e} = 5$$

Hence, option (C) is incorrect.

Option (D): Now, kinetic energy $\propto \frac{1}{n^2}$

Therefore, ratio of kinetic energies of electron in state $n = m$ to state $n = 1$ is

$$\frac{(\text{KE})_{n=m}}{(\text{KE})_{n=1}} = \frac{1}{m^2} = \frac{1}{2^2} = \frac{1}{4}$$

Hence, option (D) is correct.

Option (B): Since $E \propto \frac{1}{n^2}$; or change in energy for a hydrogen atom, we have

$$\Delta E = 13.6\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

Equating the above expression with $E = \frac{hc}{\lambda}$, we obtain

$$\frac{hc}{\lambda} = 13.6\left(\frac{1}{m^2} = \frac{1}{n^2}\right) \Rightarrow \lambda = \frac{hc}{13.6} \times \left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

$$\lambda_e = \frac{1240\text{ nm}}{13.6} \times \frac{1}{\left(\frac{1}{2^2} - \frac{1}{4^2}\right)}$$

$$= \frac{1240\text{ nm}}{13.6} \frac{1}{\left(\frac{1}{4} - \frac{1}{16}\right)}$$

$$\lambda_e = \frac{1240\text{ nm}}{13.6} \times \frac{16}{(4-1)} = \frac{1240\text{ nm}}{13.6} \times \frac{16}{3}\text{nm}$$

$$= 486.27\text{ nm}$$

Hence, option (B) is incorrect.

9. Given, initial speed of ball $= u_0$; angle of projection $= \theta$; average velocity of ball when it hits ground first $= V_1$; reduction in velocity $= \frac{u_0}{\alpha}$; average velocity for entire motion $= 0.8\ V_1$.

Since Average velocity, $v_{\text{avg}} = \dfrac{\text{Total displacement}}{\text{Total time taken}}$

Now, average velocity of the ball when it hits ground for the first time is

$$V_1 = u_0 \cos\theta$$

Time $$T = \frac{2u_0 \sin\theta}{g}$$

Therefore,

$$v_{\text{avg}} = \frac{u_0 \cos\theta . T + \frac{u_0}{\alpha}\cos\theta . \frac{T}{\alpha} + \frac{u_0}{\alpha^2}\cos\theta . \frac{T}{\alpha^2} + \cdots}{T + \frac{T}{\alpha} + \frac{T}{\alpha^2} + \cdots}$$

$$= \frac{u_0 \cos\theta . T\left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^4} + \cdots\right)}{T\left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots\right)}$$

$$= \frac{u_0 \cos\theta\left(1 + \frac{1}{\alpha^2} + \frac{1}{\alpha^4} + \cdots\right)}{\left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots\right)} = u_0 \cos\theta \frac{\left(\dfrac{1}{1 - \frac{1}{\alpha^2}}\right)}{\left(\dfrac{1}{1 - \frac{1}{\alpha}}\right)}$$

$$= u_0 \cos\theta \frac{\left(\dfrac{\alpha^2}{\alpha^2 - 1}\right)}{\left(\dfrac{\alpha}{\alpha - 1}\right)} = u_0 \cos\theta \frac{\alpha^2(\alpha - 1)}{\alpha(\alpha^2 - 1)}$$

$$v_{\text{avg}} = u_0 \cos\theta \frac{\alpha(\alpha - 1)}{(\alpha - 1)(\alpha + 1)} = u_0 \cos\theta \frac{\alpha}{\alpha + 1}$$

or $$v_{\text{avg}} = V_1 \frac{\alpha}{\alpha + 1}$$

It is given that for the entire motion magnitude of average velocity is

$$v_{\text{avg}} = 0.8 V_1$$

$$\Rightarrow 0.8\, V_1 = V_1 . \frac{\alpha}{\alpha + 1} \Rightarrow 0.8 = \frac{\alpha}{\alpha + 1}$$

$$\Rightarrow 0.8(\alpha + 1) = \alpha \Rightarrow 0.8\alpha + 0.8 = \alpha$$

$$\Rightarrow \alpha - 0.8\alpha = 0.8 \Rightarrow 0.2\alpha = 0.8$$

or $$\alpha = 4.0$$

10. Given $B = 1$ T; $l = 10$ cm $= 10 \times 10^{-2}$ m; $v = 1$ cm/s $= 1 \times 10^{-2}$ m/s. Since we know that

Motional emf or induced emf, $\varepsilon = Blv$

where B is magnetic field; l is length of wire; v is velocity of wire. Therefore,

$$\varepsilon = 1 \times 10 \times 10^{-2} \times 1 \times 10^{-2} = 10^{-3}\ \text{V}$$

Hence, emf induced in circuit is $\varepsilon = 10^{-3}$ V

Now, current, $I = \dfrac{\varepsilon}{R}(1 - e^{-Rt/L})$

Given, $R = 1\ \Omega$; $L = 1$ mH $= 1 \times 10^{-3}$ H; $t = 1$ ms $= 1 \times 10^{-3}$ s. Therefore,

$$I = \frac{10^{-3}}{1}\left(1 - e^{\frac{-1\times1\times10^{-3}}{1\times10^{-3}}}\right)$$

$$\Rightarrow I = 10^{-3}(1 - e^{-1}) = 10^{-3}\left(1 - \frac{1}{e}\right)$$

Using $e = 2.718$, we obtain

$$I = 10^{-3}\left(1 - \frac{1}{2.718}\right) = 0.632 \times 10^{-3}$$

Thus, value of x is 0.632.

11. Given, angle of incidence $= \theta$; refractive index, $n_0 = \sqrt{3}$; refractive index of thin film $= n$;

total internal reflection occurs for $\theta \le 60°$ and angle of prism, $A = 75°$.

When light suffers refraction from surface of prism at angle $\theta = 60°$, then by Snell's law

$$\frac{\sin i}{\sin r} = n$$

$$\Rightarrow \frac{\sin\theta}{\sin r} = n_0 \Rightarrow \frac{\sin 60}{\sin r} = \sqrt{3} \Rightarrow \frac{\sqrt{3}}{2} \times \frac{1}{\sin r} = \sqrt{3}$$

$$\Rightarrow \sin r = \frac{1}{2} \quad \text{or } r = 30°$$

After refraction from air into prism surface, the light incident on second surface at an angle of incidence $\theta' = A - r = 75° - 30° = 45°$.

Now, since $\theta' < 60°$, therefore total internal reflection takes place. For total internal reflection at the other refracting surface, using Snell's law, we have

$$\Rightarrow n_0 \sin\theta' = n \sin 90°$$

$$\Rightarrow \frac{\sqrt{3}\sin 45°}{\sin 90°} = n \Rightarrow \sqrt{3} \times \frac{1}{\sqrt{2}} = n$$

$$\Rightarrow n = \sqrt{\frac{3}{2}}$$

$$\Rightarrow n^2 = \frac{3}{2} = 1.5$$

12. Given, $\dfrac{4\pi M\Omega}{h} = 10^{24}\ \text{m}^{-2}$ (1)

where M is mass of reflecting mirror; Ω is angular frequency; h is Planck's constant.

Wavelength of photons that strike mirror, $\lambda = 8\pi \times 10^{-6}$ m

Displacement in mirror = 1 μm

Momentum transferred in mirror, $\Delta p = \dfrac{2hN}{\lambda}$

Speed obtained by mirror, $v = \dfrac{2hN}{\lambda} \times \dfrac{1}{M}$ (2)

As we know that the system oscillates in simple harmonic motion, therefore,

$$v = A\,\Omega; \tag{3}$$

where A is amplitude of displacement.
Now, substituting Eq. (1) in Eq. (3) and equating with Eq. (2), we get

$$\frac{2hN}{\lambda M} = A\frac{10^{24}\times h}{4\pi M} \Rightarrow \frac{2N}{\lambda} = A\times\frac{10^{24}}{4\pi}$$

$$\Rightarrow N = \frac{A\times\lambda\times 10^{24}}{2\times 4\pi}$$

Substituting all values, we get

$$N = \frac{1\times 10^{-6}\times 8\pi\times 10^{-6}\times 10^{24}}{8\pi} = 10^{12}$$

Comparing with $N = x\times 10^{12}$, we obtain $x = 1$.

13. The given reaction is represented as

$${}^{226}_{88}\text{Ra} \xrightarrow{\alpha\text{-decay}} {}^{222}_{86}\text{Rn} + \alpha\text{-particle} \xrightarrow{\gamma\text{-decay}}$$

Kinetic energy of emitted α-particle, $E_\alpha = 4.44$ MeV
We know that total energy emitted in any decay is

$$Q = (\Delta m)c^2$$

where Δm is mass defect; c is speed of light. Now,
Mass defect, Δm = (Mass of Ra) – (Mass of Rn + Mass of α-particle)
Therefore, $\Delta m = (226.005) - (222.000 + 4.000)$
$= (226.005 - 226.000)$
or $\Delta m = 0.005\ u$
Therefore, $Q = 0.005 \times 931.5 = 4.655$ MeV
($\because$ 1 u = 931 MeV/c^2)

Since we know that $E \propto \frac{1}{m}$.

So $$\frac{E_{\text{Rn}}}{E_\alpha} = \frac{m_\alpha}{m_{\text{Rn}}}$$

$$\Rightarrow E_{\text{Rn}} = \frac{m_\alpha E_\alpha}{m_{\text{Rn}}} = \frac{4.000}{222.00}\times 4.44\text{ MeV}$$

$$\Rightarrow E_{\text{Rn}} = 0.08\text{ MeV}$$

Hence, energy of γ photon is

$$E_\gamma = Q - (E_{\text{Rn}} + E_\alpha) = 4.655 - (4.44 + 0.08)$$
$$= 4.655 - 4.520$$
$$= 0.135\text{ MeV or }135\text{ keV}$$

Energy of emitted γ photon = 135 keV

14. Lens formula is given by

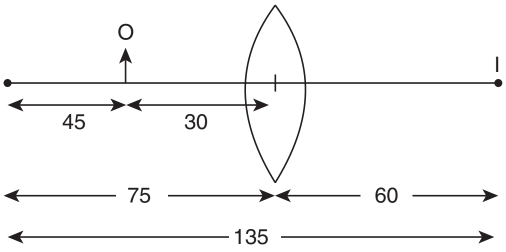
$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u} \tag{1}$$

On differentiating the lens formula, we get

$$\frac{\Delta f}{f^2} = \frac{\Delta v}{v^2} + \frac{\Delta u}{u^2}$$

Thus, percentage error in measurement of focal length is

$$\frac{\Delta f}{f}\times 100 = \left(\frac{\Delta v}{v^2} + \frac{\Delta u}{u^2}\right)\times f\times 100 \tag{2}$$



Given, least count of optical bench $= \frac{1}{4}$ cm;

Object distance, $u = 75 - 45 = 30$ cm and error using Eq. (1) is

$$\Delta u = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Image distance, $v = 135 - 75 = 60$ cm and error using Eq. (1) is

$$\Delta v = \frac{1}{2}$$

Using Eq. (1), we get

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u} \Rightarrow \frac{1}{f} = \frac{1}{60} + \frac{1}{30} = \frac{1+2}{60} = \frac{3}{60}$$
$$\Rightarrow f = 20\text{ cm}$$

Substitute all values in (2), we get

$$\frac{\Delta f}{f}\times 100 = \left(\frac{1/2}{60^2} + \frac{1/2}{30^2}\right)\times 20\times 100$$
$$= \frac{1}{2}\left(\frac{1}{3600} + \frac{1}{900}\right)\times 20\times 100$$
$$= 1000\left(\frac{1+4}{3600}\right) = \frac{5}{3600}\times 1000$$
$$= 1.388$$

Hence, percentage error in measurement of focal length of lens is 1.3888.

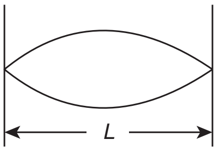
15. Mass per unit length of 4 strings is μ, 2μ, 3μ, 4μ.
For string-1, free length = L_0; tension = T_0; fundamental mode frequency = f_0.
We know that fundamental mode frequency is

$$f = \frac{v}{\lambda}$$

where v is speed of sound and λ is wavelength produced in string-1. Also,

$$L = \frac{\lambda}{2} \Rightarrow \lambda = 2L$$

$$\Rightarrow f = \frac{v}{2L}$$



and velocity can also given as

$$v = \sqrt{\frac{T}{\mu}}$$

where T is tension and μ is mass per unit length of the string. Therefore,

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

For string-1: $f_0 = \frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}}$

For string-2: $f_1 = \frac{1}{2L_0}\sqrt{\frac{T_0}{2\mu}} = \frac{1}{\sqrt{2}}\frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}}$

$$\Rightarrow f_1 = \frac{1}{\sqrt{2}} f_0$$

For string-3: $f_2 = \frac{1}{2L_0}\sqrt{\frac{T_0}{3\mu}} = \frac{1}{\sqrt{3}}\frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}}$

$$\Rightarrow f_2 = \frac{1}{\sqrt{3}} f_0$$

For string-4: $f_3 = \frac{1}{2L_0}\sqrt{\frac{T_0}{4\mu}} = \frac{1}{2}\frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}}$

$$\Rightarrow f_3 = \frac{1}{2} f_0$$

Therefore, magnitude of fundamental mode frequency of each string is

String-1: $f_0 \to 1$ [i.e., option (P)]

String-2: $f_1 = \frac{1}{\sqrt{2}} f_0 \to \frac{1}{\sqrt{2}}$ [i.e., option (R)]

String-3: $f_2 = \frac{1}{\sqrt{3}} f_0 \to \frac{1}{\sqrt{3}}$ [i.e., option (S)]

String-4: $f_3 = \frac{1}{2} f_0 \to \frac{1}{2}$ [i.e., option (Q)]

16. Length of string-1 = L_0

Length of string-2 $= \frac{3L_0}{2}$

Length of string-3 $= \frac{5L_0}{4}$

Length of string-4 $= \frac{7L_0}{4}$

String 1, 2, 3 and 4 are vibrated at their 1st, 3rd, 5th and 14th harmonics having same frequency.
Now, for string-1, vibrated in 1st harmonics frequency is

$$f_1 = \frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}}$$

For string-2, vibrated in 3rd harmonics, frequency is

$$f_2 = \frac{3}{2L}\sqrt{\frac{T_2}{2\mu}} = \frac{3}{2\times\left(\frac{3L_0}{2}\right)}\sqrt{\frac{T_2}{2\mu}} = \frac{2}{2L_0}\sqrt{\frac{T_2}{2\mu}}$$

Since frequency of all the strings is same, therefore, we have

$$\Rightarrow f_1 = f_2$$

$$\Rightarrow \frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}} = \frac{2}{2L_0}\sqrt{\frac{T_2}{2\mu}}$$

$$\Rightarrow \sqrt{T_0} = 2\sqrt{\frac{T_2}{2}} \Rightarrow T_0 = 4\times\frac{T_2}{2}$$

$$\Rightarrow T_2 = \frac{T_0}{2}$$

For string-3, vibrated in 5th harmonics, frequency is

$$f_3 = 5\times\frac{1}{2L}\sqrt{\frac{T_3}{3\mu}} = 5\times\frac{1}{2\left(\frac{5L_0}{4}\right)}\sqrt{\frac{T_3}{3\mu}}$$

$$f_3 = \frac{4}{2L_0}\sqrt{\frac{T_3}{3\mu}}$$

Now, $f_1 = f_3$

$$\Rightarrow \frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}} = \frac{4}{2L_0}\sqrt{\frac{T_3}{3\mu}} \Rightarrow \sqrt{T_0} = 4\sqrt{\frac{T_3}{3}}$$

$$\Rightarrow T_0 = 16\times\frac{T_3}{3}$$

$$\Rightarrow T_3 = \frac{3T_0}{16}$$

For string-4, vibrated in 14th harmonics, frequency is

$$f_4 = 14\times\frac{1}{2L}\sqrt{\frac{T_4}{4\mu}} = 14\times\frac{1}{2\times\left(\frac{7L_0}{4}\right)}\sqrt{\frac{T_4}{4\mu}} = 8\times\frac{1}{2L_0}\sqrt{\frac{T_4}{4\mu}}$$

$$\Rightarrow f_4 = 4\times\frac{1}{2L_0}\sqrt{\frac{T_4}{\mu}}$$

Again, $f_1 = f_4$

$$\Rightarrow \frac{1}{2L_0}\sqrt{\frac{T_0}{\mu}} = 4\times\frac{1}{2L_0}\sqrt{\frac{T_4}{\mu}} \Rightarrow \sqrt{T_0} = 4\sqrt{T_4}$$

$$\Rightarrow T_0 = 16\times T_4$$

$$\Rightarrow T_4 = \frac{T_0}{16}$$

Therefore, tension of each of the 4 strings in units of T_0 is

String-1: $T_0 \to 1$ [i.e., option (P)]

String-2: $T_2 = \frac{T_0}{2} \to \frac{1}{2}$ [i.e., option (Q)]

String-3: $T_3 = \frac{3T_0}{16} \to \frac{3}{16}$ [i.e., option (T)]

String-4: $T_4 = \frac{T_0}{16} \to \frac{1}{16}$ [i.e., option (U)]

17. Infinitesimal heat absorbed by gas is $T\Delta X$.

For one mole of ideal gas,

$$X = \frac{3}{2}R\ln\left(\frac{T}{T_\text{A}}\right) + R\ln\left(\frac{V}{V_\text{A}}\right)$$

For (I): Work done by system in process $1 \to 2 \to 3$ is given by

$$W = W_{12} + W_{23} \tag{1}$$

Using definition of work done, we have

$$W = P\Delta V = P(V_2 - V_1)$$
$$W_{12} = P_0(2V_0 - V_0) = P_0V_0$$
$$W_{23} = P_0(2V_0 - 2V_0) = 0$$

Therefore, Eq. (1) becomes

$$W = P_0V_0$$

Now, given that

$$P_0V_0 = \frac{1}{3}RT_0$$

Therefore, work done

$$W = P_0V_0 = \frac{1}{3}RT_0 \qquad \text{[i.e., option (Q)]}$$

For (II): Now change in internal energy in process $1 \to 2 \to 3$ is given by

$$\Delta U = nC_V\Delta T$$

where C_V is specific heat at constant volume.

$$\Delta U = \frac{f}{2}nR\Delta T$$

where f is degree of freedom. Using $PV = nRT$ in above equation, we get

$$\Delta U = \frac{f}{2}\Delta(PV) = \frac{f}{2}(P_\text{f}V_\text{f} - P_\text{i}V_\text{i})$$

$$\Rightarrow \Delta U = \frac{3}{2}\left(\frac{3}{2}P_0 \times 2V_0 - P_0V_0\right) = \frac{3}{2} \times 2P_0V_0$$

$$\Rightarrow \Delta U = 3P_0V_0$$

Using $P_0V_0 = \frac{1}{3}RT_0$, we have

$$\Delta U = 3P_0V_0 = RT_0 \qquad \text{[i.e., option (R)]}$$

For (III): Now for heat absorbed by system in process $1 \to 2 \to 3$, we have

According to first law of thermodynamics,

$$\Delta Q = \Delta W + \Delta U \tag{2}$$

where ΔQ is change in heat or heat absorbed by system.

Using $\Delta W = \frac{1}{3}RT_0$ and $\Delta U = RT_0$ Eq. (2), we obtain

$$\Delta Q = \frac{1}{3}RT_0 + RT_0 = \frac{4}{3}RT_0$$

Therefore, heat absorbed by system in process $1 \to 2 \to 3$ is given by

$$\Delta Q = \frac{4}{3}RT_0 \qquad \text{[i.e., option (S)]}$$

For (IV): Heat absorbed by system is process $1 \to 2$ is

$$\Delta U_{12} = nC_P\Delta T$$

where C_P is specific heat at constant pressure. Therefore,

$$\Delta U_{12} = \frac{f}{2}nR\Delta T$$

Again, using ideal gas equation $PV = nRT$, we get

$$\Delta U_{12} = \frac{f}{2}\Delta(PV) = \frac{f}{2}(P_\text{f}V_\text{f} - P_\text{i}V_\text{i})$$

$$\Rightarrow \Delta U_{12} = \frac{3}{2}(P_0(2V_0) - P_0V_0) = \frac{3}{2}P_0V_0$$

From Eq. (2), we have

$$\Delta Q_{12} = \Delta U_{12} + \Delta W_{12}$$
$$= \frac{3}{2}P_0V_0 + P_0V_0 \qquad (\because \Delta W_{12} = P_0V_0)$$
$$\Rightarrow \Delta Q_{12} = \frac{5}{2}P_0V_0$$
$$\Rightarrow \Delta Q_{12} = \frac{5}{2} \times \frac{1}{3}RT_0 = \frac{5}{6}RT_0 \qquad \left(\because P_0V_0 = \frac{1}{3}RT_0\right)$$

Therefore, heat absorbed in process $1 \to 2$ is given by

$$\Delta Q = \frac{5}{6}RT_0 \qquad \text{[i.e., option (U)]}$$

Hence, the correct match is I → Q, II → R, III → S, IV → U.

18. **For (I):** Work done by system in process $1 \to 2 \to 3$ is

$$\Delta W = W_{12} + W_{23}$$

Now $W_{12} = nRT\ln\left(\frac{V_\text{f}}{V_\text{i}}\right)$ and $W_{23} = 0$, therefore, we have

$$\Delta W = nRT\ln\left(\frac{V_\text{f}}{V_\text{i}}\right) = \frac{RT_0}{3}\ln\left(\frac{2V_0}{V_0}\right) = \frac{RT_0}{3}\ln 2$$

Therefore, work done by system in process $1 \to 2 \to 3$ is given by

$$\Delta W = \frac{1}{3}RT_0\ln 2 \qquad \text{[i.e., Option (P)]}$$

For (II): Change in internal energy in process $1 \to 2 \to 3$ is

$$\Delta U = nC_V\Delta T$$

where C_V is specific heat at constant volume. Therefore,

$$\Delta U = \frac{f}{2}nR(T_\text{f} - T_\text{i}) = \frac{3}{2}R\left(T_0 - \frac{T_0}{3}\right) = \frac{3}{2}R\frac{2}{3}T_0$$

$$\Rightarrow \Delta U = RT_0$$

Thus, change in internal energy in process $1 \to 2 \to 3$ is

$$\Delta U = RT_0 \qquad \text{[i.e., Option (R)]}$$

For (III): For heat absorbed by system in process 1 → 2 → 3, from first law of thermodynamics, we have

$$\Delta Q = \Delta U + \Delta W \qquad (1)$$

Now, using $\Delta U = RT_0$ and $\Delta W = \frac{1}{3}RT_0 \ln 2$, we obtain

$$\Delta Q = RT_0 + \frac{1}{3}RT_0 \ln 2 = \frac{1}{3}RT_0(3 + \ln 2)$$

Therefore, heat absorbed by system in process 1 → 2 → 3 is given by

$$\Delta Q = \frac{1}{3}RT_0(3 + \ln 2) \qquad \text{[i.e., Option (T)]}$$

For (IV): Heat absorbed in process 1 → 2 is

$$\Delta Q_{12} = \Delta U_{12} + \Delta W_{12} \qquad \text{(Using Eq. (1): } \Delta Q = \Delta U + \Delta W\text{)}$$

We know that

$$\Delta W_{12} = nRT \ln\left(\frac{V_f}{V_i}\right) = \frac{RT_0}{3} n\left(\frac{2V_0}{V_0}\right)$$

$$\Rightarrow \Delta W_{12} = \frac{1}{3}RT_0 \ln 2$$

Now, $\Delta U_{12} = \frac{f}{2}nR(T_f - T_i) = \frac{3}{2}R\left(\frac{T_0}{3} - \frac{T_0}{3}\right)$

$$\Rightarrow \Delta U_{12} = 0$$

Therefore, $\Delta Q_{12} = \Delta U_{12} + \Delta W_{12} = 0 + \frac{1}{3}RT_0 \ln 2$

$$\Rightarrow \Delta Q_{12} = \frac{1}{3}RT_0 \ln 2$$

Thus, heat absorbed by system in process 1 → 2 is given by

$$\Delta Q_{12} = \frac{1}{3}RT_0 \ln 2 \qquad \text{[i.e., Option (P)]}$$

Hence, the correct match is I → P, II → R, III → T, IV → P.

FOR JEE (MAIN & ADVANCED)

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- **Complete coverage** and **alignment** of text as per JEE syllabus.
- **Effective pedagogy** to support conceptual learning
  - ❖ ***Detailed figures*** in the text to help visualize concepts.
  - ❖ ***Checkpoints*** as stopping points to check understanding of concepts explained.
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